Incompressible limit and non-isentropic fluids

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Introduction

Generalities: Incompressible flows equations justification:

- From compressible Navier–Stokes or Euler equations (shallow-water system)
- Flow velocity field small compared to sound velocity

Limit = incompressible equations.
Correction = acoustic waves, gravity waves.....

Small parameter = Mach number, Froude number
For instance $\varepsilon = \text{Mach} = \frac{\text{fluid velocity}}{\text{sound velocity}}$

- Car: $50 \text{ km/h} / 120 \text{ km/h} = 1/20$
- Plane: $800 \text{ km/h} / 1200 \text{ km/h} = 0.66$

velocity motions $< 150 \text{ km}$ are essentially incompressible

Difference = Noise (waves..)
Compressible isentropic Euler equations:

\[ \partial_t \rho + \text{div}(\rho u) = 0 \]
\[ \partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p(\rho) = 0 \]

Scaling:

\[ u(t, x) = \varepsilon U(\varepsilon t, x) \]
gives

\[ \partial_t \rho + \text{div}(\rho U) = 0 \]
\[ \partial_t (\rho U) + \text{div}(\rho U \otimes U) + \frac{\nabla p(\rho)}{\varepsilon^2} = 0 \]

Then limit \( \varepsilon \to 0 \) provides

\[ \nabla p(\rho) = 0. \]

Thus, using the mass equation, \( \rho \) is a constant \( \rho = 1 \quad \Rightarrow \quad \text{divergence free condition} \)
\[ \text{div} U = 0 \ (U \text{ denoted } u \text{ in the sequel}). \]
Introduction

Wave equation:

\[ \psi = \frac{\rho - 1}{\varepsilon} \]

gives

\[ \partial_t \psi + \text{div}(\psi u) + \frac{\text{div}u}{\varepsilon} = 0 \]

\[ \partial_t u + \text{div}(u \otimes u) + h(\psi) + p'(1) \frac{\nabla \psi}{\varepsilon} = 0 \]

Combinaison of a wave equation

\[ \partial_t \sigma + \text{div}v = 0 \]

\[ \partial_t v + p'(1) \nabla \sigma = 0 \]

with a nonlinear equation.

Time scales:
* \( O(1) \): Fluid evolution
* \( O(\varepsilon) \): Wave evolution (wave propagation velocity \( = 1/\varepsilon \)).

Attended result: If we look the incompressible part of \( u \)

\[ \implies \text{convergence to incompressible Euler} \]
Introduction

Non-exhaustive bibliography:

- S. Klainerman, A. Majda: Existence on a time interval independent on Mach number.
- S. Klainerman, A. Majda: Convergence with well prepared data ($\psi = O(\varepsilon)$).
- S. Ukai: Whole space and waves going to infinity in times $O(\varepsilon)$.
- S. Schochet: Incompressible limit, general initial data.
- E. Grenier: Rotating fluids, general initial data.
- I. Gallagher: Oscillating limit parabolic systems.
- B. Desjardins, E. Grenier: Incompressible limit with Strichartz on weak solutions.
Introduction

Main ideas: Periodic and whole space case

Step 1: wave group
\( \mathcal{L}(t)(\sigma_0, v_0) \) group solutions of

\[
\begin{align*}
\partial_t \sigma + \text{div} v &= 0 \\
\partial_t v + \nabla \sigma &= 0
\end{align*}
\]

with initial data \((\sigma_0, v_0)\).
\( \mathcal{L}(t) \) is an isometry from \( H^s \) into \( H^s \) for

- periodic box
- whole space

Step 2: conjugate process
We conjugate \( \mathcal{L}(t) \) posing

\[
(\bar{\psi}, \bar{u}) = \mathcal{L}(-t/\varepsilon)(\psi, u)
\]

and we get the equation under the form

\[
\partial_t (\bar{\psi}, \bar{u}) + \mathcal{L}(-t/\varepsilon)Q(\mathcal{L}(t/\varepsilon)(\bar{\psi}, \bar{u})) = 0
\]
Introduction

Step 3: Limit process
\( \partial_t (\bar{\rho}, \bar{u}) \) is bounded (No problem with compactness in space).
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The equations read:

\[ \partial_t \rho + \text{div}(\rho u) = 0, \]

\[ \rho(\partial_t u + u \cdot \nabla u) + \nabla p = 0, \]

with

\[ \partial_t S + u \cdot \nabla S = 0. \]

where \( S \) entropy, \( p \) given by the state law \( \rho = R(p, S) \).

Example: \( \rho = p^{1/\gamma} e^{-S/\gamma} \).

Change of variable: Let \((p, u)\) then denoting \( p = \bar{p} \exp^{\varepsilon q} \), we get

\[ a(\partial_t q + u \cdot \nabla q) + \frac{1}{\varepsilon} \text{div} u = 0, \]

\[ r(\partial_t u + u \cdot \nabla u) + \frac{1}{\varepsilon} \nabla q = 0 \]

\[ \partial_t S + u \cdot \nabla S = 0 \]
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Formal limit

From mass and momentum equations:
\( \text{div} u = 0 \) and \( \nabla q = 0 \) then

\[
\text{div} u = 0, \\
r(\partial_t u + u \cdot \nabla u) + \nabla \Pi = 0 \\
\partial_t S + u \cdot \nabla S = 0
\]

with \( \rho = R(\bar{\rho}, S) \) and \( r(S) \).

Wave equation:

\[
\partial_t (\sigma, v) = \mathcal{A}(\sigma, v)
\]

with

\[
\mathcal{A} = \begin{pmatrix}
0 & a^{-1}(S)\nabla \\
r^{-1}(S)\nabla & 0
\end{pmatrix}.
\]

which gives

\[
\partial_{tt} \sigma - \text{div}(S(t, x)^{-1} \nabla \sigma) = 0.
\]

Remark: \( \partial_t S \) is bounded but wave equation with variable coefficients.
Step 1: Wave equation

\[ \partial_t^2 \sigma - \varepsilon^{-2} \text{div}(S(t, x)^{-1} \nabla \sigma) = 0. \]

Let \( \mathcal{L}(t) \) the solutions group. We want that \( \mathcal{L}(t) \) is bounded uniformly from \( H^s \) into \( H^s \).

Energy estimates:

* \( L^2 \): Energy gives uniform bound in \( L^2 \).
* \( H^1 \): \( \partial_t \) satisfies a wave equation with unbounded source term with respect to \( \varepsilon \).

Spectral decomposition

Problem: Variable coefficients with respect to time!

Problem: Multi-eigenvalues possibility!

Problem: Crossing eigenvalues possibility!

\[ \implies \text{bad behavior possibility} \ldots \ldots \]

Generic results: "for almost all initial data"
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Two questions:

- Solve equations on some time interval which is independent of Mach number?
- Characterization of the limit when Mach goes to zero?

First question:


Second question:

- G. Métivier and S. Schochet: Whole space and Euler.
- T. Alazard: Exterior domain and Euler; whole space and full CNS eqs.
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Relies on theorem:

\[ \varepsilon^2 \partial_t (a^\varepsilon(t, x) \partial_t \phi^\varepsilon) - \text{div}(b^\varepsilon(t, x) \nabla \phi^\varepsilon) = \varepsilon f^\varepsilon(t, x) \]

where

\( \phi^\varepsilon \) is bounded in \( C^0([0, T]; H^2(\mathcal{R}^d)) \), \( f^\varepsilon \) is bounded in \( L^2([0, T]; L^2(\mathcal{R}^d)) \),

\( a^\varepsilon \) and \( b^\varepsilon \) decay to zero at spatial infinity in same similar manner:

\( a^\varepsilon(t, x) \geq c \), \( |a^\varepsilon(t, x) - a| = O(|x|^{-1-\delta}) \), \( |\nabla a^\varepsilon(t, x)| = O(|x|^{-2-\delta}) \),

Then \( \phi^\varepsilon \) converges strongly to 0 in \( L^2_{\text{loc}}([0, T] \times \mathcal{R}^d) \) to \( (0, 0) \).
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Singular limit and nonisentropic Euler or NS systems.


Non-isentropic fluids

Averaged equation for non-isentropic NS equations:


\[
\begin{align*}
\partial_t \bar{\rho} + \text{div} (\bar{\rho} \bar{u}) &= 0, \\
\text{div} \bar{u} &= 0, \\
\bar{\rho} \bar{a} &= 1,
\end{align*}
\]

\[
\partial_t (\bar{\rho} \bar{u}) + \text{div} (\bar{\rho} \bar{u} \otimes \bar{u}) + \nabla \bar{P} - \mu \Delta \bar{u}
\]

\[
= \sum_{\ell, m} \frac{\alpha_\ell^+ \alpha_m^- + \alpha_\ell^- \alpha_m^+}{2} \left( \nabla (\Psi_m \Psi_\ell) - \frac{\bar{a}}{\lambda_\ell^2} \nabla (\nabla \Psi_\ell \cdot \nabla \Psi_m) \right)
\]

with \( (\lambda_j^2, \Psi_j) \) denote the eigenvectors of the nonlinear wave equation

\[
- \text{div} (\bar{a} \nabla \Psi_j) = \lambda_j^2 \Psi_j \quad \text{and} \quad \varphi_j(t) = \int_0^t \lambda_j(s) \, ds.
\]
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The coefficients $\alpha_{k}^{\sigma_k}$ with $\sigma_k \in \{+, -\}$ denote the components of the acoustic waves on a basis depending on $\{\Psi_j\}_{j \in \mathbb{N}}$. They are governed by the dynamical system

$$
\frac{d\alpha_{k}^{\sigma_k}}{dt} + \frac{\lambda_k^2 (\lambda + 2\mu)}{2} \alpha_{k}^{\sigma_k} + \sum_{\ell} \mu \frac{\alpha_{\ell}^{\sigma_k}}{2\lambda_k^2} \int \text{curl}(\bar{a} \nabla \Psi_k) \cdot \text{curl}(\bar{a} \nabla \Psi_{\ell}) \, dx
$$

$$
= \sum_{\ell} \frac{\alpha_{\ell}^{\sigma_k}}{2} \int \left\{ \Psi_{\ell} \partial_t \Psi_k + \frac{\nabla \Psi_{\ell}}{\lambda_k} \partial_t \left( \frac{\bar{a} \nabla \Psi_k}{\lambda_k} \right) \right\} \, dx
$$

$$
+ \frac{(\gamma - 1)}{4\sqrt{2}} \sum_{\ell, m, \sigma_\ell, \sigma_m} \frac{i\sigma_k \lambda_k \alpha_{\ell}^{\sigma_{\ell}} \alpha_{m}^{\sigma_{m}}}{\sigma_\ell \varphi_{\ell} + \sigma_m \varphi_{m} = \sigma_k \varphi_k} \int \Psi_k \Psi_m \Psi_{\ell} \, dx
$$

$$
- \sum_{\ell} \frac{\alpha_{\ell}^{\sigma_k}}{2\lambda_k^2} \int \bar{a} \ \text{div} \ (\bar{u} \otimes \nabla \Psi_{\ell} + \nabla \Psi_{\ell} \otimes \bar{u}) \cdot \nabla \Psi_k \, dx
$$

$$
- \sum_{\ell, m, \sigma_\ell, \sigma_m} \frac{i\alpha_{\ell}^{\sigma_{\ell}} \alpha_{m}^{\sigma_{m}}}{2\sqrt{2}} \frac{1}{\sigma_k \lambda_k \sigma_\ell \lambda_\ell \sigma_m \lambda_m} \int \bar{a} \ \text{div} \ (\bar{a} \nabla \Psi_{\ell} \otimes \nabla \Psi_m) \cdot \nabla \Psi_k \, dx.
$$
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Some comments:

- Nonhomogeneity $\implies$ extra streaming term $\bar{a} \nabla (\nabla \Psi_l \cdot \nabla \Psi_m)$
- Energy exchange between main contribution and waves
- More complex than anelastic limit
- For weak and local process (difficulties): regularity and vanishing properties on $\bar{a}$?

References: anelastic limit with non-vanishing heterogeneity profile

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Non-constant coefficients limit...... Some examples in environmental problems.

Rigid lid approximation for bilayers models

- bi-layers shallow water systems.
- Sedimentation.
- Fluide-Structure interaction.

⇒ time dependent parameter.

An example.

\[
\begin{aligned}
\partial_t h + \text{div}(hv) &= 0, \\
\partial_t (hv) + \text{div}(hv \otimes v) + h \frac{\nabla (h + z_b)}{\text{Fr}^2} &= 0, \\
\partial_t z_b + \text{div}(q_b(h, v)) &= 0
\end{aligned}
\]

where \( z_b \) is the movable bed thickness. Formulas in literature for \( q_b \): Grass equation, Meyer-Peter and Muller equation, formulas of Nielsen, Fernández Luque and Van Beek.....

- Grass model: Solid transport given by

\[
q_b(h, v) = A_g |v|^{m_g} v, \quad 0 \leq m_g \leq 3
\]
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Transversality and crossing of eigenvalues.


Several papers: C. Fermanian, P. Gérard, Y. Colin de Verdière.... etc..
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Spectral decomposition

Let

\[ \partial_t^2 \sigma - \varepsilon^{-2} \text{div}(S(x)^{-1} \nabla \sigma) = 0 \]

forgetting time dependency

Spectrum:

\(-\text{div}(S(x)^{-1} \nabla \cdot)\) is a self-adjoint operator

Eigenvalues \(\lambda_j\) (with eventual multiplicity)

\(\Pi_j\) its corresponding eigenspace and \(\psi_j\) orthonormal basis.

Eigenspaces geometry:

Double eigenvalues

\[ \Sigma_{j,k} = \left\{ \lambda_j(S) = \lambda_k(S) \right\}. \]

In a neighborhood of a double eigenvalue,

\[ \Pi_j + \Pi_k \]

is continuous, but not \(\psi_j\), nor \(\psi_k\).
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Is $\Sigma_{j,k}$ of codimension 2?

A matrix model:

Symmetric matrices with eigenvalue at least double are of co-dimension 2 in the symmetric matrices set.

In dimension 2 ....

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Characteristic polynomial

$$X^2 - (a + c)X + ac - b^2$$

Eigenvalues:

$$\frac{a + c}{2} \pm \frac{\sqrt{(a - c)^2 + b^2}}{2}$$

Then

$$\Sigma_{j,k} = \{ b = 0, a = c \}$$

line in a three dimensional space.

The eigenvectors do not depend on $x - \Pi x$ where $\Pi$ is the projection on $\Sigma_{j,k}$. 
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Is $\Sigma_{j,k}$ of codimension 2?

It seems that all have to be done!!

Question:

$$
\mu \left( S \mid \lambda_j(S) - \lambda_k(S) < \varepsilon \right) \leq C\varepsilon^2
$$

Difficulties:

- Definition of the measure $\mu$ in infinite dimension space?
- Uniformity with respect to the approximation?

Let $\Pi_N$ projection on finite dimension space (Galerkin)

Let

$$
\Sigma_{j,k} = \{ S = \Pi_N(S) \mid |\lambda_j(S) - \lambda_k(S)| < \varepsilon \}
$$

On $\mathbb{R}^N$ the measure of Besov type

$$
\mu_N = \bigotimes_{k=1}^N \frac{k^s}{2} 1_{[-1/k^s,1/k^s]}
$$
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Measure of neighborhoods of $\Sigma_{j,k}$

$$\Sigma_{j,k}^{N,\varepsilon} = \{ S = \Pi_N(S) \mid |\lambda_j(S) - \lambda_k(S)| < \varepsilon \}$$

$$\mu_N = \bigotimes_{k=1}^{N} \frac{k^s}{2} \left[ -\frac{1}{k^s}, \frac{1}{k^s} \right]$$

**Theorem 0.** Under hypothesis of non degeneracy, there exists a constant $C_0$ such that

$$\mu_N(\Sigma_{j,k}^{N,\varepsilon}) \leq C_0 \varepsilon^2$$

for all $N$ and all $\varepsilon$.

**Proof**

Effect of regularity: $\Sigma_{j,k}$ is a graph with respect to the first components $\Pi_N x$.

**Remarks:**

- Codimension 2 notion "in the measure $\mu_N$ sense".
- $\Sigma_{j,k}$ has a null measure too, but what is important is its approximation.
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Measure of neighborhoods of $\Sigma_{j,k}$

- Approximate diagonalisation
- Ansatz on $(\psi_j, \psi_k)$:

If $S_0 \in \Sigma_{j,k}$ then $\lambda_j(S_0 + S)$ and $\lambda_k(S_0 + S)$ are given by

$$
\frac{\lambda_j(S_0) + \lambda_k(S_0)}{2} + \frac{1}{2} \left( \int S|\nabla \psi_j|^2 + \int S|\nabla \psi_k|^2 \right)
$$

$$
\pm \frac{1}{2} \sqrt{\left( \int S|\nabla \psi_j|^2 - \int S|\nabla \psi_l|^2 \right)^2 + 4 \left( \int S \nabla \psi_j \nabla \psi_k \right)^2} + O(|S|_{H^s}^2).
$$

gives informations locally.

- Simple eigenvalues are Lipschitzian

$$
\nabla_S \lambda_j(S_0) . S = - \int S|\nabla \psi_j|^2.
$$

- Eigenvalues cannot be closed too quickly.
- When they are closed ... Above ansatz.
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Outside $\Sigma_{j,k}$

$$\partial_t^2 \sigma - \varepsilon^{-2} \text{div}(S(t,x)^{-1} \nabla \sigma) = 0$$

We decompose

$$\sigma(t) = \sum_j \alpha_j(t) \psi_j(S(t)) \exp\left(\varepsilon^{-2} \int_0^t \lambda_j(S(t))\right).$$

We get

$$\partial_t \alpha_j = -\left(\sum_k \alpha_k(t) \nabla \psi_k(S(t)) \cdot S'(t) \mid \psi_j(S(t))\right).$$

This is correctly bounded from above!

As soon as $S(t)$ avoids double eigenvalues, $\mathcal{L}$ is bounded.
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Is it possible to avoid $\Sigma_{j,k}$?

Geometry of the problem:

Find initial data which avoid a codimension 2 subset.

Regular flow case in finite dimension

$\Theta(t_1, t_2)$ flow, $\Sigma$ of codimension 2 to be avoided

We have to evaluate

$$A_\varepsilon = \{x \mid \exists 0 \leq t \leq T \Theta(0, t)x \in \Sigma_\varepsilon\}.$$ 

$$= \bigcup_t \{x \mid \Theta(0, t)x \in \Sigma_\varepsilon\}.$$

Two hypothesis:

- Flow with bounded divergence
- Bounded flow

$$\mu(A_\varepsilon) \leq C\varepsilon T.$$ 

Problem: The flow is not regular!!!
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Limit equation

Well prepared data:

Waves with $O(\varepsilon)$ size. Limit = incompressible non-homogeneous Euler equations

Ill prepared data:

- Waves with $O(1)$ size.
- Limit = Euler with a source term: wave interactions.
- Source term = combinaison of terms involving $\psi_j(S) \implies$ singular around to $\Sigma_{j,k}$.

Type equation

ODE of the form

$$\partial_t \phi + Q(\phi) = R\left(\frac{x - \Pi x}{\|x - \Pi x\|}\right)$$

with $\Pi$ projection on a codimension 2 variety.
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Dimension 2 example

\[ \dot{x} = \phi\left(\frac{x}{|x|}\right) \]

with \( \phi \) continuous defined from the unit circle to \( \mathbb{R}^2 \).

Polar coordinates:

\[ x(t) = \rho(t)e^{i\theta(t)} \]

with

\[ \rho \dot{\theta} = \chi(\theta) \]
\[ \dot{\rho} = \psi(\theta) \]

with \( \chi(\theta) = \text{Im}(\phi(e^{i\theta})e^{-i\theta}) \). Change of time gives

\[ \dot{\theta} = \chi(\theta) \]
\[ \dot{\rho} = \psi(\theta)\rho. \]
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Discussion

- Possible asymptots: $\theta$ with $\chi(\theta) = \theta$.
- Stability depends on $\chi'$.
- Multiple possibility in function of sign of $\psi$.

Flow:

- The flow is discontinuous: We pass on the left or on the right of the singularity
- or we enter directly in the singularity in finite time.

Divergence:

Through calculation, if $A$ set

$$\mu(\Theta(t)(A)) \leq C\mu(A)$$

with $C'$ independent on $t$ and on $A$. 
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Vector field with a homogeneous degree 0 singularity

\[ \dot{x} = \phi \left( x, \frac{x_h}{|x_h|} \right) \]

with \( x_h = (x_1, x_2) \).

- Perturbative arguments with respect to the dimension 2.
- Under geometrical hypothesis: Existence except for a codimension 1 subset.
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Resonances

\[ \Sigma_{j,k,l} = \{ S \mid \lambda_j(S) + \lambda_k(S) = \lambda_l(S) \}. \]

- Heuristically \( \Sigma_{j,k,l} \) is of codimension 1.
- Codimension 1 in the measure sense

\[ \mu \{ S \mid |\lambda_j(S) + \lambda_k(S) - \lambda_l(S)| < \varepsilon \} \leq C\varepsilon. \]

More precisely

**Theorem 0.** 2 Under non degeneracy hypothesis,

\[ \mu_N \left( \Sigma_{j,k,l}^{N,\varepsilon} \right) \leq C\varepsilon. \]
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Proof of resonance theorem

Differential calculus

\[ d(\lambda_j + \lambda_k - \lambda_l) = \left( |\nabla \psi_j|^2 + |\nabla \psi_k|^2 - |\nabla \psi_l|^2 \right) \]

The differential does not vanishes if

\[ |\nabla \psi_j|^2 + |\nabla \psi_k|^2 - |\nabla \psi_l|^2 \neq 0. \]

Differential belongs to all \( H^s \): eigenvalues vary slowly when perturbate high frequencies.

Differential depends essentially of the first \( N \) components...

\( \Sigma_{j,k,l} \) is a graph with respect to its first \( N \) components if \( N \) is large enough.
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In progress: non-homogeneous incompressible limit

- First step: Check that the limit system has a solution for almost all initial data.
- Check that almost all initial data avoids \( \Sigma_{j,k} \).
- Conjugate nonhomogeneous incompressible NS equation with \( \mathcal{L} \).
- Pass to the limit
- Pass to the limit in the resonances.

Objective: Convergence for almost all initial data......