THE LONG WAY OF A VISCOUS VORTEX DIPOLE

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Abstract. We consider the evolution of a viscous vortex dipole in $\mathbb{R}^2$ originating from a pair of point vortices with opposite circulations. At high Reynolds number $\mathrm{Re} \gg 1$, the dipole can travel a very long way, compared to the distance between the vortex centers, before being slowed down and eventually destroyed by diffusion. In this regime we construct an accurate approximation of the solution in the form of a two-parameter asymptotic expansion involving the aspect ratio of the dipole and the inverse Reynolds number. We then show that the exact solution of the Navier-Stokes equations remains close to the approximation on a time interval of length $\mathcal{O}(\mathrm{Re}^\sigma)$, where $\sigma < 1$ is arbitrary. This improves upon previous results which were essentially restricted to $\sigma = 0$. As an application, we provide a rigorous justification of an existing formula which gives the leading order correction to the translation speed of the dipole due to finite size effects.

1. Introduction

Numerical investigations of freely decaying two-dimensional turbulence show that the distribution of vorticity tends to concentrate in a relatively small fraction of the spatial domain, so as to form a collection of coherent vortices which interact over a long period of time and grow in size due to diffusion and merging [23, 10]. In contrast, the small scales of the flow correspond to thin filaments of vorticity, which may be created during rare events such as vortex mergers. A precise description of vortex interactions thus appears as a necessary step toward a better understanding of the dynamics of two-dimensional flows at high Reynolds number [19, 24].

As long as the distances between the vortex centers remain substantially larger than the sizes of the vortex cores, the dynamics of a finite collection of vortices in $\mathbb{R}^2$ is well approximated by the *point vortex system*:

$$z_i'(t) \,=\, \sum_{j\neq i} \frac{\Gamma_j}{2\pi}\,\frac{\big(z_i(t)-z_j(t)\big)^\perp}{|z_i(t)-z_j(t)|^2}\,, \qquad i,j = 1,\dots,N\,, \tag{1.1}$$

where $z_1(t),\dots,z_N(t) \in \mathbb{R}^2$ denote the positions of the vortex centers and $\Gamma_1,\dots,\Gamma_N \in \mathbb{R}$ are the circulations of the corresponding vortices. The ODE system (1.1), which was already studied by Helmholtz and Kirchhoff, can be rigorously derived as an asymptotic model for the evolution of sharply concentrated solutions of the two-dimensional Euler or Navier-Stokes equations [22, 21, 12, 6]. Note that (1.1) only makes sense if $z_i(t) \neq z_j(t)$ when $i \neq j$, but this condition is not always preserved under evolution. Indeed, if $N \geq 3$ and if the circulations Γ_i do not have the same sign, there are examples of solutions of (1.1) for which three vortices collide in finite time [22]. However, the set of initial data leading to such collisions is negligible in the sense of Lebesgue's measure.

To formulate more precisely the relation between the point vortex system (1.1) and the fundamental equations of fluid motion, we start from the two-dimensional vorticity equation

$$\partial_t\omega(x,t) + \boldsymbol{u}(x,t)\cdot\nabla\omega(x,t) \,=\, \nu\Delta\omega(x,t)\,, \qquad x\in\mathbb{R}^2\,, \quad t>0\,, \tag{1.2}$$

where $\boldsymbol{u}(x,t) = \big(u_1(x,t),u_2(x,t)\big) \in \mathbb{R}^2$ denotes the velocity of the fluid at point $x \in \mathbb{R}^2$ and time $t > 0$, and $\omega := \partial_1 u_2 - \partial_2 u_1$ is the associated vorticity. The constant parameter $\nu > 0$ represents the kinematic viscosity of the fluid. The velocity field $\boldsymbol{u}$ is divergence-free, and can be expressed in terms of the vorticity ω by the *Biot-Savart formula* $\boldsymbol{u}(\cdot,t) = \mathrm{BS}[\omega(\cdot,t)]$, where

$$\mathrm{BS}[\omega](x) \,:=\, \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,\mathrm{d}y\,, \qquad x\in\mathbb{R}^2\,. \tag{1.3}$$

As in (1.1), if $x = (x_1, x_2) \in \mathbb{R}^2$, we denote $x^\perp = (-x_2, x_1)$ and $|x|^2 = x_1^2 + x_2^2$. The most important conserved quantities for the dynamics of (1.2) are the *mean* $\mathrm{M}[\omega]$ and the *linear momentum* $\mathrm{m}[\omega] = (\mathrm{m}_1[\omega], \mathrm{m}_2[\omega])$ defined by

$$\mathrm{M}[\omega] = \int_{\mathbb{R}^2} \omega(x)\,\mathrm{d}x\,, \qquad \mathrm{m}_i[\omega] = \int_{\mathbb{R}^2} x_i \omega(x)\,\mathrm{d}x, \qquad i = 1, 2\,. \tag{1.4}$$

It is known that the Cauchy problem for the evolution equation (1.2) is globally well-posed if the initial vorticity ω^{in} is a finite Radon measure on $\mathbb{R}^2$ [11, 2]. In particular, given any integer $N \geq 1$, we can consider the discrete measure

$$\omega^{in} = \sum_{i=1}^{N} \Gamma_i\, \delta(\cdot - x_i)\,, \tag{1.5}$$

which represents a finite collection of point vortices of circulations $\Gamma_1, \dots, \Gamma_N \in \mathbb{R}$ located at the positions $x_1, \dots, x_N \in \mathbb{R}^2$. Without loss of generality, we assume that $\Gamma_i \neq 0$ and that $x_i \neq x_j$ when $i \neq j$. The point vortex system (1.1) with initial data $z_i(0) = x_i$ is then well posed on the time interval $[0, T]$ for some $T > 0$, and we denote

$$d := \min_{t \in [0,T]} \min_{i \neq j} |z_i(t) - z_j(t)| > 0\,, \qquad |\Gamma| := |\Gamma_1| + \cdots + |\Gamma_N| > 0\,. \tag{1.6}$$

The following statement is the starting point of our analysis.

Theorem 1.1. [12] *Assume that the point vortex system* (1.1) *with circulations* Γ_i *and initial positions* x_i *is well posed on the time interval* $[0, T]$. *There exists a constant* $C_0 > 0$ *such that the unique solution* ω^ν *of the vorticity equation* (1.2) *with initial data* (1.5) *satisfies*

$$\frac{1}{|\Gamma|} \int_{\mathbb{R}^2} \Big| \omega^\nu(x,t) - \sum_{i=1}^{N} \frac{\Gamma_i}{4\pi\nu t}\, e^{-\frac{|x - z_i(t)|^2}{4\nu t}} \Big|\,\mathrm{d}x \leq C_0 \frac{\nu t}{d^2}\,, \qquad \text{for all } t \in (0, T)\,, \tag{1.7}$$

where $z_1(t) \dots, z_N(t)$ *is the solution of* (1.1) *such that* $z_i(0) = x_i$ *for* $i = 1, \dots, N$.

This result shows that the solution ω^ν of the vorticity equation (1.2) with initial data (1.5) can be approximated by a superposition of Lamb-Oseen vortices whose centers follow the evolution defined by the point vortex system. The approximation is accurate as long as the radius $\sqrt{\nu t}$ of the vortex cores is much smaller than the distance d between the centers. Importantly, the constant C_0 in (1.7) depends on the normalized time $|\Gamma| T / d^2$, but not on the viscosity parameter $\nu > 0$. In particular, for any fixed $t \in (0, T]$, we can take the limit $\nu \to 0$ in (1.7) so as to obtain the weak convergence result

$$\omega^\nu(\cdot, t) \xrightharpoonup[\nu \to 0]{} \sum_{i=1}^{N} \Gamma_i\, \delta\big(\cdot - z_i(t)\big)\,, \quad \text{for all } t \in [0, T]\,, \tag{1.8}$$

which provides a natural link between the dynamics of the vorticity equation (1.2) and the point vortex system (1.1).

Interesting questions arise when trying to extend the approximation result in Theorem 1.1 to longer time scales. The first one is related to the collapse of point vortices. Assume for instance that the solution of system (1.1) is defined on the maximal time interval $[0, T_*)$, with three vortices collapsing at the origin as $t \to T_*$. In that case estimate (1.7) is valid on the time interval $(0, T)$ for any $T < T_*$, but the constant $C_0 > 0$ blows up as $T \to T_*$ because the distance d converges to zero in this limit. So it is not clear at all if the weak convergence (1.8) holds up to collision time, although this is certainly a reasonable conjecture. A fortiori, we do not know if a limiting procedure as in (1.8) can be used to define a continuation of the point vortex dynamics after collapse.

In a different direction, one may consider global solutions of the point vortex system, for which $|z_i(t) - z_j(t)| \geq d > 0$ for all $t \geq 0$ if $i \neq j$. This is the case, for instance, if $N = 2$ or if the circulations Γ_i are all positive [22]. Here again the approximation result (1.7) can be applied on any time interval $(0, T)$, but the constant C_0 depends on T and increases rapidly as $T \to +\infty$. In particular, it is not clear if the solution $\omega^\nu(x, t)$ is well approximated by a superposition of

Lamb-Oseen vortices as long as $\nu t \ll d^2$, namely when the radius of the vortex cores is small compared to the distance d between the centers.

The present paper is a contribution to the study of the latter question in the particular case where $N = 2$ and $\Gamma_1 + \Gamma_2 = 0$, which corresponds to a *vortex dipole*. To be specific, given $\Gamma > 0$ and $d > 0$, we consider initial data of the form

$$\omega^{in} = \Gamma\delta(\cdot - x_\ell) - \Gamma\delta(\cdot - x_{\mathrm{r}})\,, \qquad \text{where} \quad x_\ell = \Big(-\frac{d}{2}\,, 0\Big)\,, \quad x_{\mathrm{r}} = \Big(\frac{d}{2}\,, 0\Big)\,, \tag{1.9}$$

so that ω^{in} represents a pair of point vortices of circulations $\pm\Gamma$ separated by a distance d. In that situation, the point vortex dynamics predicts a uniform translation at speed $V = \Gamma/(2\pi d)$ in a direction normal to the line segment joining the vortex centers. More precisely, the positions $z_\ell(t), z_{\mathrm{r}}(t)$ of the point vortices with circulation Γ, $-\Gamma$ (respectively) are given by

$$z_\ell(t) = \Big(-\frac{d}{2}\,, Z_2(t)\Big)\,, \qquad z_{\mathrm{r}}(t) = \Big(\frac{d}{2}\,, Z_2(t)\Big)\,, \tag{1.10}$$

where $Z_2(t) = Vt$. Theorem 1.1 thus asserts that the solution ω^ν of (1.2) with initial data (1.9) is well approximated, on any fixed time interval $(0, T)$, by the viscous vortex dipole

$$\omega^\nu_{\mathrm{app}}(x,t) = \frac{\Gamma}{4\pi\nu t}\Big(e^{-\frac{|x-z_\ell(t)|^2}{4\nu t}} - e^{-\frac{|x-z_{\mathrm{r}}(t)|^2}{4\nu t}}\Big)\,, \qquad x \in \mathbb{R}^2\,, \quad t > 0\,, \tag{1.11}$$

provided $\nu T \ll C_0^{-1} d^2$ where the constant C_0 depends on T.

To discuss the ultimate validity of such an approximation, we introduce as in [14] the following quantities, which play an important role in our analysis:

$$\varepsilon(t) = \frac{\sqrt{\nu t}}{d}\,, \qquad \delta = \frac{\nu}{\Gamma}\,, \qquad T_{\mathsf{adv}} = \frac{d^2}{\Gamma}\,, \qquad T_{\mathsf{diff}} = \frac{d^2}{\nu}\,. \tag{1.12}$$

Here $\varepsilon(t)$ is the *aspect ratio* of the viscous vortex dipole (1.11) at time $t > 0$, namely the ratio of the size of the vortex cores to the distance between the vortex centers. The dimensionless parameter δ, which measures the relative strength of the viscous and the advection effects, is usually called the *inverse Reynolds number*. In view of (1.10), the *advection time* T_{adv} is the time needed for the vortex dipole to cover the distance $VT_{\mathsf{adv}} = d/(2\pi)$. Finally, the *diffusion time* T_{diff} is the time at which the aspect ratio $\varepsilon(t)$ becomes equal to one, so that the support of both vortices strongly overlap. We are interested in the high Reynolds number regime $\delta \ll 1$, for which $T_{\mathsf{adv}} \ll T_{\mathsf{diff}}$. In that situation, the viscous vortex dipole can travel over a very long distance in the vertical direction before being destroyed by diffusion.

Our main result shows that the solution of (1.2) can be approximated by a viscous vortex dipole over a long time interval of size $T_{\mathsf{adv}}\delta^{-\sigma}$, where $0 < \sigma < 1$, provided that the position $Z_2(t)$ of the vortex centers (1.10) is chosen in a suitable way.

Theorem 1.2. *Fix $\sigma \in [0,1)$. There exist positive constants C and δ_0 such that, for any $\Gamma > 0$, any $d > 0$ and any $\nu > 0$ such that $\delta := \nu/\Gamma \le \delta_0$, the solution ω^ν of* (1.2) *with initial data* (1.9) *satisfies*

$$\frac{1}{\Gamma}\int_{\mathbb{R}^2} \big|\,\omega^\nu(x,t) - \omega^\nu_{\mathrm{app}}(x,t)\big|\,\mathrm{d}x \le C\,\frac{\nu t}{d^2}\,, \qquad \text{for all } t \in \big(0, T_{\mathsf{adv}}\delta^{-\sigma}\big)\,, \tag{1.13}$$

where $\omega^\nu_{\mathrm{app}}$ is the viscous vortex dipole (1.11), (1.10) *and the vertical position $Z_2(t)$ of the vortex centers is a smooth function satisfying $Z_2(0) = 0$ and*

$$Z_2'(t) = \frac{\Gamma}{2\pi d}\Big(1 - 2\pi\alpha\,\varepsilon^4 + \mathcal{O}\big(\varepsilon^5 + \delta^2\varepsilon + \delta\varepsilon^2\big)\Big)\,, \qquad \text{with } \alpha \approx 22.24\,. \tag{1.14}$$

Remark 1.3. The formula (1.14) shows that the leading order correction to the constant velocity V predicted by the point vortex system is negative and proportional to ε^4, where ε is as in (1.12). This fact is known in the literature, see [17] for a detailed discussion. The constant $\alpha > 0$ has an explicit expression in terms of the solution of a linear differential equation on $\mathbb{R}_+$, see Section 3.4 below. While Theorem 1.2 asserts the existence of *some* function $Z_2(t)$ satisfying (1.14) such that estimate (1.13) holds, it is important to mention that $Z_2(t)$ is not entirely characterized by (1.14), unless σ is sufficiently small.

If $\sigma = 0$, Theorem 1.2 is merely a particular case of Theorem 1.1, and instead of (1.14) we can simply take $Z_2'(t) = V = \Gamma/(2\pi d)$. The situation is different when $\sigma \in (0,1)$, because we have to control the solution of (1.2) on a much longer time scale. According to (1.14), the total distance D covered by the vortex dipole over the time interval $(0, T_{\mathsf{adv}}\delta^{-\sigma})$ can be estimated as

$$D = Z_2\big(T_{\mathsf{adv}}\delta^{-\sigma}\big) \approx \frac{\Gamma}{2\pi d}\, T_{\mathsf{adv}}\delta^{-\sigma} = \frac{d}{2\pi}\,\delta^{-\sigma},$$

so that $D \gg d$ if $\delta > 0$ is sufficiently small. To prove the validity of (1.13), it is therefore necessary to have a very accurate expression of the vertical velocity $Z_2'(t)$, as can be seen from the following back-of-the-envelope calculation. As is easy to verify, changing the vertical position $Z_2(t)$ of the vortex dipole (1.11) by a small quantity $z(t)$ produces an error proportional to $|z(t)|/(d\varepsilon)$ when computing the left-hand side of (1.13). This means that we have to know $Z_2(t)$ with a precision of order $d\varepsilon^3$ to ensure that the right-hand side of (1.13) remains $\mathcal{O}(\varepsilon^2)$. Since the length of the time interval is $T_{\mathsf{adv}}\delta^{-\sigma}$, it is sufficient to control the vertical velocity $Z_2'(t)$ up to an error of order $(\Gamma/d)\varepsilon^3\delta^\sigma$. To relate this to (1.14), we observe that

$$\varepsilon(t)^2 = \frac{\nu t}{d^2} = \frac{\delta t}{T_{\mathsf{adv}}}, \qquad \text{so that} \quad \varepsilon(t)^2 \le \varepsilon\big(T_{\mathsf{adv}}\delta^{-\sigma}\big)^2 = \delta^{1-\sigma}. \tag{1.15}$$

Given $M \in \mathbb{N}$, we thus have $\varepsilon^{3+M} \le \varepsilon^3\,\delta^{M(1-\sigma)/2} \le \varepsilon^3\,\delta^\sigma$ if $\sigma \le M/(M+2)$. So, for instance, if we know that $Z_2'(t) = \Gamma/(2\pi d) + \mathcal{O}(\varepsilon^4)$, that information is sufficient to obtain estimate (1.13) provided $\sigma \le 1/3$. Similarly, the improved formula (1.14) which includes the leading order correction to the point vortex dynamics is good enough for our purposes provided that $\sigma \le 1/2$, but for larger values of σ we need to derive a higher order approximation of the vertical speed.

As in the previous works [12, 14], the proof of Theorem 1.2 relies on the construction of an approximate solution of (1.2), in the form of a power series expansion in the small parameters ε and δ. Our expansion involves the vorticity distribution in suitable self-similar variables, as well as the vertical speed of the vortex dipole. A linear approximation in δ turns out to be sufficient, but the order M of the expansion in ε must satisfy $M > (3+\sigma)/(1-\sigma)$. As σ can be chosen arbitrarily close to 1, this means that we have to construct an asymptotic expansion to *arbitrarily high order* in ε, which can be done by an iterative procedure that is described in Section 3. Note that a fourth-order expansion was sufficient in the situations considered in [12, 14], because it was assumed that $\sigma = 0$ in [12] and $0 < \sigma \ll 1$ in [14]. As an aside we mention that, in the inviscid case $\delta = 0$, our approximate solution shares many similarities with exact traveling wave solutions of the two-dimensional Euler equation describing vortex pairs, which were constructed by variational methods or perturbation arguments, see [3, 4, 5, 8, 18].

The second step in the proof consists in showing that the exact solution of the vorticity equation (1.2) remains close to the approximation constructed in the first step. This is far from obvious because the linearized equation at the approximate solution contains very large advection terms that could potentially trigger violent instabilities, and also because we need a control over the long time interval $(0, T_{\mathsf{adv}}\delta^{-\sigma})$. We follow here the earlier study [14] which considers the related, but more complicated, problem of a an axisymmetric vortex ring in the low viscosity limit. The idea is to control the difference between the solution and its approximation using suitable energy functionals, which incorporate the necessary information about the stability of the approximate solution. Unlike in [12], it does not seem possible to rely on weighted enstrophy estimates only, so we follow the general approach introduced by Arnold to study the stability of steady states for the 2D Euler equations, see [13]. This method gives a clear roadmap to design a nonlocal energy-like functional whose evolution is barely affected by the most dangerous terms in the linearized equation at the approximate solution.

There are numerous previous works devoted to the study of localized solutions of the viscous vorticity equation (1.2). Most of them consider initial data that are contained in a disjoint union of balls of radius $\varepsilon_0 \ll 1$, and show that the corresponding solutions remain essentially concentrated in small regions that evolve according to the point vortex dynamics, see [21, 7]. The effect of the viscosity is often treated perturbatively, under the assumption that $\nu \le C\varepsilon_0^\alpha$ for some positive constants C and α. The solution is usually controlled on a time interval $[0, T]$

that is fixed independently of ε_0 [21], but if the solution of the point vortex system is globally defined it is possible to take $T \ge c\,|\log \varepsilon_0|$ [7]. To our knowledge, the only previous work where the viscosity can be chosen independently of ε_0, so that initial data of the form (1.5) are allowed, is [6]. Here also, under the assumption that the point vortices do not collapse in finite time, the solution can be controlled on the time interval $(0,T)$ with $T \ge c\,|\log(\varepsilon_0^2+\nu)|$. Taking $\varepsilon_0 = 0$ we obtain $T \ge c\,|\log \nu|$ for the initial data (1.5), whereas Theorem 1.2 describes the solution over a much longer time scale $T = \mathcal{O}(\nu^{-\sigma})$, but only in the simple example of a vortex dipole.

2. The rescaled vorticity equation

In this section, we introduce the framework needed to prove Theorem 1.2. Following [12] we first define self-similar variables which allow us to desingularize the dynamics of (1.2) at initial time, when point vortices are considered as initial data. We thus obtain a rescaled system describing a "zoomed-in" evolution for the vorticity distribution, with the property that the initial data are smooth in the new variables. We next make an appropriate choice for the positions of the vortex cores, which ensures that the rescaled vorticity distribution has vanishing first order moments.

2.1. **Self-similar variables.** Our goal is to study the solution ω of the vorticity equation (1.2) with initial data $\omega^{in} = \Gamma\delta(\cdot - x_\ell) - \Gamma\delta(\cdot - x_{\rm r})$, where $\Gamma > 0$ is the circulation parameter and the initial positions $x_{\rm r}, x_\ell$ are given by (1.9). For symmetry reasons, the solution $\omega(x_1,x_2,t)$ is an odd function of x_1 for all times. It will be convenient to use the notation

$$\widetilde{x} \,=\, (-x_1, x_2)\,, \qquad \text{for any } x \,=\, (x_1,x_2) \in \mathbb{R}^2\,. \tag{2.1}$$

We introduce self-similar variables $\xi_{\rm r}$ (for the right vortex) and ξ_ℓ (for the left vortex) defined by

$$\xi_{\rm r} \,=\, \frac{x - Z(t)}{\sqrt{\nu t}}\,, \qquad \xi_\ell \,=\, \frac{\widetilde{x} - Z(t)}{\sqrt{\nu t}}\,, \tag{2.2}$$

where $Z(t) \in \mathbb{R}^2$ is the time-dependent location of the right vortex, which is a free parameter at this stage. In view of the Helmholtz-Kirchhoff dynamics, we anticipate that

$$Z(t) \,=\, \biggl(\frac{d}{2}\,,\, Z_2(t)\biggr)\,, \qquad \text{where} \quad Z_2(t) \,\approx\, \frac{\Gamma t}{2\pi d}\,. \tag{2.3}$$

Using the conserved quantities of (1.2), we verify below that the assumption $Z_1(t) = d/2$ is indeed a natural one, and we also make an appropriate choice for the vertical position $Z_2(t)$.

For the time being, we look for a solution of equation (1.2) in the form

$$\omega(x,t) \,=\, -\frac{\Gamma}{\nu t}\,\Omega(\xi_{\rm r},t) + \frac{\Gamma}{\nu t}\,\Omega(\xi_\ell,t)\,, \tag{2.4}$$

where the self-similar variables $\xi_{\rm r}, \xi_\ell$ are defined in (2.2), and the rescaled vorticity $\Omega(\xi,t)$ is a new dependent variable to be determined. It is important to realize that the same function Ω is used to describe the vorticity distribution of both the right and the left vortex, which reflects the fact that the solution ω of (1.2) is an odd function of the variable x_1. The corresponding decompositions for the velocity field $\boldsymbol{u} = (u_1,u_2)$ and the stream function ψ are found to be

$$\boldsymbol{u}(x,t) \,=\, -\frac{\Gamma}{\sqrt{\nu t}}\,U(\xi_{\rm r},t) - \frac{\Gamma}{\sqrt{\nu t}}\,\widetilde{U}(\xi_\ell,t), \qquad \psi(x,t) = -\Gamma\,\Psi(\xi_{\rm r},t) + \Gamma\,\Psi(\xi_\ell,t)\,, \tag{2.5}$$

where it is understood that

$$U \,=\, \nabla^\perp\Psi \quad \text{and} \quad \Psi = \Delta^{-1}\Omega\,, \quad \text{with} \quad (\Delta^{-1}\Omega)(\xi) \,:=\, \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|\xi-\eta|\,\Omega(\eta)\,{\rm d}\eta\,. \tag{2.6}$$

In what follows we write $U = \mathrm{BS}[\Omega]$ and $\Psi = \Delta^{-1}\Omega$ when (2.6) holds. Using (2.5) it is easy to verify that u_1, ψ are odd functions of the variable x_1 whereas u_2 is an even function of x_1.

To formulate the equation satisfied by $\Omega(\xi,t)$, we introduce the rescaled diffusion operator

$$\mathcal{L} \,:=\, \Delta_\xi + \frac{1}{2}\xi\cdot\nabla_\xi + 1\,. \tag{2.7}$$

If f,g are C^1 functions on $\mathbb{R}^2$, we also define the Poisson bracket

$$\{f,g\} \,=\, \nabla^\perp f\cdot\nabla g \,=\, (\partial_1 f)(\partial_2 g) - (\partial_2 f)(\partial_1 g)\,. \tag{2.8}$$

Lemma 2.1. *The vorticity distribution ω defined by (2.4) is a solution of (1.2) provided the function Ω satisfies the evolution equation*

$$t\partial_t\Omega \,=\, \mathcal{L}\Omega + \frac{1}{\delta}\Big\{\Psi - \mathcal{T}_\varepsilon\Psi + \frac{\varepsilon d\xi_1}{\Gamma}\,Z_2'\,,\,\Omega\Big\}\,,\quad t>0\,, \tag{2.9}$$

where $\varepsilon = \sqrt{\nu t}/d$, $\delta = \nu/\Gamma$, and $\mathcal{T}_\varepsilon$ is the translation/reflection operator defined by

$$(\mathcal{T}_\varepsilon\Psi)(\xi) \,=\, \Psi\big(\widetilde{\xi} - \varepsilon^{-1}e_1\big)\,,\quad e_1 = (1,0)\,. \tag{2.10}$$

Proof. We write equation (1.2) in the form $\partial_t\omega + \{\psi,\omega\} = \nu\Delta\omega$, where $\psi = \Delta^{-1}\omega$ is the stream function. Using (2.4) it is easy to verify that

$$\partial_t\omega - \nu\Delta\omega \,=\, \frac{\Gamma}{\nu t^2}\bigg[\Big(-t\partial_t\Omega + \mathcal{L}\Omega + \sqrt{\frac{t}{\nu}}\,Z'\cdot\nabla\Omega\Big)(\xi_{\rm r},t) + \Big(t\partial_t\Omega - \mathcal{L}\Omega - \sqrt{\frac{t}{\nu}}\,Z'\cdot\nabla\Omega\Big)(\xi_\ell,t)\bigg]\,.$$

On the other hand, in view of (2.1)–(2.3), the self-similar variables $\xi_{\rm r},\xi_\ell$ satisfy the relations

$$\xi_\ell \,=\, \widetilde{\xi}_{\rm r} - \varepsilon^{-1}e_1\,,\qquad \xi_{\rm r} \,=\, \widetilde{\xi}_\ell - \varepsilon^{-1}e_1\,,$$

so that $\Psi(\xi_\ell,t) = (\mathcal{T}_\varepsilon\Psi)(\xi_{\rm r},t)$ and $\Psi(\xi_{\rm r},t) = (\mathcal{T}_\varepsilon\Psi)(\xi_\ell,t)$ by (2.10). In view of (2.5), (2.4), we thus have

$$\begin{aligned}\big\{\psi,\omega\big\}_x \,&=\, \frac{\Gamma^2}{\nu t}\bigg[\Big\{\Psi(\xi_{\rm r},t) - \mathcal{T}_\varepsilon\Psi(\xi_{\rm r},t)\,,\,\Omega(\xi_{\rm r},t)\Big\}_x + \Big\{\Psi(\xi_\ell,t) - \mathcal{T}_\varepsilon\Psi(\xi_\ell,t)\,,\,\Omega(\xi_\ell,t)\Big\}_x\bigg]\\ &=\, \frac{\Gamma^2}{\nu^2 t^2}\bigg[\Big\{\Psi - \mathcal{T}_\varepsilon\Psi\,,\,\Omega\Big\}_\xi(\xi_{\rm r},t) - \Big\{\Psi - \mathcal{T}_\varepsilon\Psi\,,\,\Omega\Big\}_\xi(\xi_\ell,t)\bigg]\,.\end{aligned}$$

Here we write $\{\cdot,\cdot\}_x$ or $\{\cdot,\cdot\}_\xi$ according to whether the Poisson bracket is computed with respect to the original variable x or the rescaled variable ξ. Recalling that $\delta = \nu/\Gamma$, we deduce from the equalities above that (1.2) is satisfied provided

$$t\partial_t\Omega \,=\, \mathcal{L}\Omega + \frac{1}{\delta}\Big\{\Psi - \mathcal{T}_\varepsilon\Psi\,,\,\Omega\Big\} + \sqrt{\frac{t}{\nu}}\,Z'\cdot\nabla\Omega\,,\quad t>0\,. \tag{2.11}$$

Since $\sqrt{t/\nu} = (\varepsilon d)/(\Gamma\delta)$ and $Z'\cdot\nabla\Omega = Z_2'\partial_2\Omega = Z_2'\{\xi_1,\Omega\}$, we deduce (2.9) from (2.11). □

Remark 2.2. The evolution equation (2.9) can be written in the equivalent form

$$t\partial_t\Omega \,=\, \mathcal{L}\Omega + \frac{1}{\delta}\Big(U + \mathbf{T}_\varepsilon U + \frac{\varepsilon d}{\Gamma}\,Z'(t)\Big)\cdot\nabla\Omega\,,\quad t>0\,, \tag{2.12}$$

where $U = \nabla^\perp\Psi$ is the velocity field obtained from Ω via the usual Biot-Savart law in $\mathbb{R}^2$, and $\mathbf{T}_\varepsilon U = -\nabla^\perp\mathcal{T}_\varepsilon\Psi$ denotes the velocity generated by the mirror vortex, namely

$$(\mathbf{T}_\varepsilon U)(\xi,t) \,=\, \widetilde{U}\big(\widetilde{\xi} - \varepsilon^{-1}e_1,t\big)\,. \tag{2.13}$$

Note that both velocity fields U and $\mathbf{T}_\varepsilon U$ are divergence free.

It is essential to realize that the evolution equation (2.9) or (2.12) is singular at initial time $t=0$, because the term $t\partial_t\Omega$ involving the time derivative vanishes. In particular, the initial data Ω_0 cannot be chosen arbitrarily, as can be seen by taking the limit $t\to 0$ (hence also $\varepsilon\to 0$) in (2.9). Using the fact that $\nabla\mathcal{T}_\varepsilon\Psi \to 0$ in that limit, which is established in Lemma 3.8 below, we obtain the relation

$$0 \,=\, \mathcal{L}\Omega_0 + \frac{1}{\delta}\big\{\Psi_0,\Omega_0\big\}\,,\quad\text{where}\quad \Delta\Psi_0 \,=\, \Omega_0\,. \tag{2.14}$$

This is exactly the equation satisfied by the profile of a self-similar solution of the Navier-Stokes equation (1.2). It thus follows from [16, Prop. 1.3] that any solution of (2.14) that is integrable over $\mathbb{R}^2$ is necessarily a multiple of the *Lamb-Oseen vortex*

$$\Omega_0(\xi) \,=\, G(\xi),\qquad\text{where}\quad G(\xi) \,=\, \frac{1}{4\pi}\,e^{-|\xi|^2/4}\,. \tag{2.15}$$

A direct calculation shows that $\mathcal{L}\Omega_0 = 0$, and since Ω_0, Ψ_0 are both radially symmetric one has $\{\Psi_0, \Omega_0\} = 0$ too. Note that Ω_0 is normalized so that $\int_{\mathbb{R}^2} \Omega_0 \,\mathrm{d}\xi = 1$, which ensures that the vorticity distribution (2.4) satisfies $\omega(\cdot,t) \rightharpoonup \omega^{in}$ as $t \to 0$. For later use, we recall that the velocity field U_0 and the stream function Ψ_0 associated with Ω_0 have the following expressions

$$U_0(\xi) \,=\, U^G(\xi) \,:=\, \frac{1}{2\pi}\frac{\xi^\perp}{|\xi|^2}\Big(1 - e^{-|\xi|^2/4}\Big)\,, \qquad \Psi_0(\xi) \,=\, \frac{1}{4\pi}\mathrm{Ein}\big(|\xi|^2/4\big) - \frac{\gamma_E}{4\pi}\,, \tag{2.16}$$

where $\mathrm{Ein}(s) = \int_0^s (1-\mathrm{e}^{-\tau})/\tau \,\mathrm{d}\tau$ is the exponential integral, and $\gamma_E = 0,5772\ldots$ is the Euler-Mascheroni constant.

It follows from the previous works [11, 12, 2] that the rescaled vorticity equation (2.9) has a unique global solution with initial data Ω_0 given by (2.15). This solution is smooth for positive times, and is continuous up to $t = 0$ in the weighted L^2 space defined by (3.6) below. Our purpose here is to give an accurate description of that solution in the regime where the viscosity parameter is small.

2.2. **Formula for the vertical speed.** There is no canonical definition of the center of a viscous vortex. Reasonable possibilities are the center of vorticity, the maximum point of the vorticity distribution, or the stagnation point of the flow, but these notions do not coincide in general. In our approach the vortex center is defined as the point $Z(t) \in \mathbb{R}^2$ where the self-similar variable ξ_r is centered, see (2.2). We choose it to be the center of vorticity, which means that the function $\Omega(\xi,t)$ satisfies the moment conditions

$$\int_{\mathbb{R}^2} \xi_1 \Omega(\xi,t)\,\mathrm{d}\xi \,=\, 0\,, \quad \text{and} \quad \int_{\mathbb{R}^2} \xi_2 \Omega(\xi,t)\,\mathrm{d}\xi \,=\, 0\,. \tag{2.17}$$

In fact the first condition in (2.17) is fulfilled if $Z_1(t) = d/2$, which we already assumed in (2.3). The second condition requires a specific choice of the vertical position $Z_2(t)$, which we now describe.

Lemma 2.3. *Let Ω be the solution of (2.9) with initial data (2.15), and $U = \nabla^\perp \Psi$ be the associated velocity field. If the vertical velocity is chosen so that*

$$Z_2'(t) \,=\, -\frac{\Gamma}{\varepsilon d}\int_{\mathbb{R}^2} U_2\big(\widetilde{\xi} - \varepsilon^{-1}e_1, t\big)\Omega(\xi,t)\,\mathrm{d}\xi\,, \quad t > 0\,, \tag{2.18}$$

then the following moment identities hold true

$$\mathrm{M}[\Omega(t)] \,=\, 1\,, \qquad \mathrm{m}_1[\Omega(t)] \,=\, \mathrm{m}_2[\Omega(t)] \,=\, 0\,, \qquad t > 0\,. \tag{2.19}$$

Here and in what follows we use the notation (1.4) for the moments of Ω.

Proof. It is easy to verify that $t\partial_t \int_{\mathbb{R}^2} \Omega \,\mathrm{d}\xi = 0$, because the diffusion operator (2.7) is mass preserving and the velocity field $U + \mathbf{T}_\varepsilon U + (\varepsilon d/\Gamma)Z'(t)$ in (2.12) is divergence free. Since Ω_0 is normalized so that $\mathrm{M}[\Omega_0] = 1$, it follows that $\mathrm{M}[\Omega(t)] = 1$ for all $t \ge 0$.

We next consider the evolution of the moment $\mathrm{m}_j[\Omega]$ for $j = 1, 2$. We observe that

$$\int_{\mathbb{R}^2} \xi_j \mathcal{L}\Omega \,\mathrm{d}\xi \,=\, -\frac{1}{2}\int_{\mathbb{R}^2} \xi_j \Omega \,\mathrm{d}\xi\,, \quad \text{and} \quad \int_{\mathbb{R}^2} \xi_j U\cdot\nabla\Omega\,\mathrm{d}\xi \,=\, -\int_{\mathbb{R}^2} U_j \Omega\,\mathrm{d}\xi \,=\, 0\,.$$

Indeed the first equality is easily obtained using the definition (2.7) and integrating by parts. The second one is a well-known property of the Biot-Savart kernel in $\mathbb{R}^2$, related to the conservation of the moment $\mathrm{m}_j[\omega]$ for the original Navier-Stokes equation (1.2). It follows that

$$t\partial_t \mathrm{m}_j[\Omega] \,=\, -\frac{1}{2}\mathrm{m}_j[\Omega] - \frac{1}{\delta}\int_{\mathbb{R}^2} \big(\mathbf{T}_\varepsilon U\big)_j\,\Omega\,\mathrm{d}\xi - \frac{\varepsilon d}{\delta\Gamma}\,Z_j'(t)\,, \quad t > 0\,. \tag{2.20}$$

At this point, the idea is to choose $Z_j'(t)$ so that the last two terms of (2.20) cancel, namely

$$Z_j'(t) \,=\, -\frac{\Gamma}{\varepsilon d}\int_{\mathbb{R}^2} \big(\mathbf{T}_\varepsilon U\big)_j(\xi,t)\,\Omega(\xi,t)\,\mathrm{d}\xi\,, \quad t > 0\,. \tag{2.21}$$

Equation (2.20) then reduces to $t\partial_t \mathrm{m}_j[\Omega] = -\frac{1}{2}\mathrm{m}_j[\Omega]$, which means that $t^{1/2}\mathrm{m}_j[\Omega(t)]$ is independent of time. That quantity obviously converges to zero as $t \to 0$, and we conclude that $\mathrm{m}_j[\Omega(t)] = 0$ for all times, which gives (2.19).

To understand the precise meaning of (2.21), we write (2.13) in the more explicit form

$$\big(\mathbf{T}_\varepsilon U\big)_1(\xi_1,\xi_2,t) \,=\, -U_1\big(-\xi_1-\varepsilon^{-1},\xi_2,t\big)\,, \qquad \big(\mathbf{T}_\varepsilon U\big)_2(\xi_1,\xi_2,t) \,=\, U_2\big(-\xi_1-\varepsilon^{-1},\xi_2,t\big)\,.$$

In particular, using the Biot-Savart formula, we find for $j = 1$:

$$\begin{aligned}\int_{\mathbb{R}^2}\big(\mathbf{T}_\varepsilon U\big)_1\,\Omega\,\mathrm{d}\xi \,&=\, -\int_{\mathbb{R}^2} U_1\big(-\xi_1-\varepsilon^{-1},\xi_2\big)\Omega(\xi)\,\mathrm{d}\xi \\ &=\, \frac{1}{2\pi}\iint_{\mathbb{R}^4}\frac{\xi_2-\eta_2}{|\xi_1+\varepsilon^{-1}+\eta_1|^2+|\xi_2-\eta_2|^2}\,\Omega(\xi)\,\Omega(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \,=\, 0\,,\end{aligned}$$

where in the last equality we used the fact that the integrand is odd with respect to the change of variables $\xi \leftrightarrow \eta$. According to (2.21) we thus have $Z_1'(t) = 0$, hence $Z_1(t) = d/2$ for all $t \ge 0$, in agreement with (2.3). In the case where $j = 2$, the right-hand side of (2.21) does not vanish in general, and gives the formula (2.18) for the vertical speed. □

In the particular case where the vorticity Ω is given by (2.15) and the velocity field U by (2.16), the vertical speed Z_2' defined by (2.18) only depends on the small parameter ε, except for the prefactor Γ/d. It is then instructive to note that the speed Z_2' agrees *to all orders* with the prediction of the point vortex system.

Lemma 2.4. *If* $\Omega = G$ *and* $U = U^G$, *the vertical velocity* (2.18) *satisfies*

$$Z_2' \,=\, -\frac{\Gamma}{\varepsilon d}\int_{\mathbb{R}^2} U_2^G\big(\widetilde{\xi}-\varepsilon^{-1}e_1\big)G(\xi)\,\mathrm{d}\xi \,=\, \frac{\Gamma}{2\pi d} + O(\varepsilon^\infty)\,. \tag{2.22}$$

Proof. Since the Gaussian function G decays very rapidly at infinity whereas the velocity field U^G is bounded, it is clear that integrating over the region $|\xi| \ge 1/(2\varepsilon)$ gives a contribution to (2.22) that is smaller than any power of ε as $\varepsilon \to 0$. We can thus write

$$\begin{aligned}\frac{d}{\Gamma}\,Z_2' \,&=\, -\frac{1}{\varepsilon}\int_{\{|\xi|<1/(2\varepsilon)\}} U_2^G\big(\widetilde{\xi}-\varepsilon^{-1}e_1\big)G(\xi)\,\mathrm{d}\xi + O(\varepsilon^\infty) \\ &=\, \frac{1}{2\pi\varepsilon}\int_{\{|\xi|<1/(2\varepsilon)\}}\frac{\xi_1+1/\varepsilon}{|\xi_1+1/\varepsilon|^2+|\xi_2|^2}\,G(\xi)\,\mathrm{d}\xi + O(\varepsilon^\infty)\,,\end{aligned} \tag{2.23}$$

where in the second line we used the explicit expression (2.16) of the velocity field U^G, and the fact that $|\xi_1+1/\varepsilon|^2+|\xi_2|^2 \ge 1/(4\varepsilon^2)$ on the domain of integration. We next expand the fraction in (2.23) as follows:

$$\frac{1}{\varepsilon}\,\frac{\xi_1+1/\varepsilon}{|\xi_1+1/\varepsilon|^2+|\xi_2|^2} \,=\, \frac{1+\varepsilon\xi_1}{|1+\varepsilon\xi_1|^2+|\varepsilon\xi_2|^2} \,=\, \sum_{n=0}^{\infty}(-1)^n\varepsilon^n Q_n^c(\xi)\,, \tag{2.24}$$

where Q_n^c is the homogeneous polynomial on $\mathbb{R}^2$ defined by $Q_n^c\big(r\cos\theta, r\sin\theta\big) = r^n\cos(n\theta)$, see Section A.1. Note that the series converges uniformly for $|\xi| \le 1/(2\varepsilon)$, so that we can exchange the sum with the integral in (2.23). Since G is a radial function, only the term corresponding to $n = 0$ gives a nonzero contribution, and we conclude that

$$\frac{d}{\Gamma}\,Z_2' \,=\, \frac{1}{2\pi}\int_{\{|\xi|<1/(2\varepsilon)\}} G(\xi)\,\mathrm{d}\xi + O(\varepsilon^\infty) \,=\, \frac{1}{2\pi} + O(\varepsilon^\infty)\,,$$

which is the desired result. □

3. The approximate solution

The purpose of this section is to construct an approximate solution of the rescaled vorticity equation (2.9) by performing an asymptotic expansion in the small parameter $\varepsilon = \sqrt{\nu t}/d$. Given an integer $M \geq 2$, our approximate solution takes the form

$$\Omega_{\rm app}(\xi,t) \,=\, \Omega_0(\xi) + \sum_{k=2}^{M} \varepsilon(t)^k\, \Omega_k(\xi)\,, \qquad \Psi_{\rm app}(\xi,t) \,=\, \Psi_0(\xi) + \sum_{k=2}^{M} \varepsilon(t)^k\, \Psi_k(\xi)\,, \tag{3.1}$$

where the vorticity profiles Ω_k have to be determined, and the stream function profiles are given by $\Psi_k = \Delta^{-1}\Omega_k$ as in (2.6). The corresponding expansion for the vertical speed is

$$Z_2'(t) \,=\, \frac{\Gamma}{2\pi d}\,\zeta_{\rm app}(t)\,, \quad \text{where} \quad \zeta_{\rm app}(t) \,=\, 1 + \sum_{k=1}^{M-1} \varepsilon(t)^k\, \zeta_k\,, \tag{3.2}$$

for some $\zeta_k \in \mathbb{R}$. It is important to note that $\Omega_{\rm app}$, $\Psi_{\rm app}$ and $\zeta_{\rm app}$ depend on time only through the aspect ratio $\varepsilon = \sqrt{\nu t}/d$. Since $\varepsilon \to 0$ as $t \to 0$, the leading order terms Ω_0, Ψ_0 in (3.1) are also the initial data of the approximate solution $\Omega_{\rm app}, \Psi_{\rm app}$. For consistency we must choose Ω_0 to be the Lamb-Oseen vortex (2.15), and $\Psi_0 = \Delta^{-1}\Omega_0$ is the associated stream function given by (2.16). Similarly, Lemma 2.4 implies that $\zeta_{\rm app}(0) = 1$.

To measure by how much our approximate solution fails to satisfy (2.9), we introduce the remainder

$$\mathcal{R}_M \,:=\, \delta\big(\mathcal{L}\Omega_{\rm app} - t\partial_t\Omega_{\rm app}\big) + \Big\{\Psi_{\rm app} - \mathcal{T}_\varepsilon\Psi_{\rm app} + \frac{\varepsilon\xi_1}{2\pi}\,\zeta_{\rm app}\,,\, \Omega_{\rm app}\Big\}\,. \tag{3.3}$$

Using Lemma 3.8 below and the important fact that $t\partial_t\varepsilon = \varepsilon/2$, we deduce from (3.1), (3.2) that the remainder (3.3) can be expanded into a power series in ε. As we shall see in Section 3.2 below, the remainder of the trivial approximation $(\Omega_{\rm app}, \Psi_{\rm app}) = (\Omega_0, \Psi_0)$ already satisfies $\mathcal{R}_0 = \mathcal{O}(\varepsilon^2)$, and this is the reason for which the expansions (3.1) start at $k = 2$ instead of $k = 1$. Our goal is to choose the profiles $\Omega_k, \Psi_k, \zeta_k$ in such a way that $\mathcal{R}_M = \mathcal{O}(\varepsilon^{M+1})$ in an appropriate topology. In fact we can require a little bit less if we observe that the quantity (3.3) involves another small parameter $\delta = \nu/\Gamma$, which (unlike ε) does not depend on time. A priori all profiles $\Omega_k, \Psi_k, \zeta_k$ depend on δ when $k > 0$, but it turns out that contributions of order $\mathcal{O}(\delta^2)$ are negligible for our purposes. So we can assume that

$$\Omega_k \,=\, \Omega_k^E + \delta\Omega_k^{NS}\,, \qquad \Psi_k \,=\, \Psi_k^E + \delta\Psi_k^{NS}\,, \qquad \zeta_k \,=\, \zeta_k^E + \delta\zeta_k^{NS}\,, \tag{3.4}$$

where the Euler profiles $\Omega_k^E, \Psi_k^E, \zeta_k^E$ and the viscous corrections $\Omega_k^{NS}, \Psi_k^{NS}, \zeta_k^{NS}$ are now independent of δ, and can be chosen so that $\mathcal{R}_M = \mathcal{O}(\varepsilon^{M+1} + \delta^2\varepsilon^2)$.

A final observation is that the profiles (3.4) are not uniquely determined unless additional conditions are imposed. For instance, as was already discussed in Section 2.2, the vorticity Ω has a vanishing linear moment with respect to the ξ_2 variable only if an appropriate choice is made for the vertical speed Z_2'. Using the notation (1.4) the hypotheses we make on the vorticity profiles can be formulated as follows:

Hypotheses 3.1. The vorticity profiles Ω_k in (3.4) satisfy:

H1) $\mathrm{M}[\Omega_0] = 1$ and $\mathrm{M}[\Omega_k] = 0$ for all $k \geq 1$;

H2) $\mathrm{m}_1[\Omega_k] = \mathrm{m}_2[\Omega_k] = 0$ for all $k \geq 0$;

H3) Ω_k^E is an *even* function of ξ_2 for all $k \geq 0$.

It is important to note that hypotheses H1, H2 apply to both Ω_k^E and Ω_k^{NS}, whereas the third assumption H3 only concerns the Euler profiles Ω_k^E. As a matter of fact, experimental observations and numerical simulations of counter-rotating vortex pairs in viscous fluids clearly show that the full vorticity distribution is not symmetric with respect to the line joining the vortex centers, see [9].

We are now in position to state the main result of this section.

Proposition 3.2. *Given any integer $M \geq 2$ there exists an approximate solution of the form (3.1), (3.2), (3.4) satisfying Hypotheses 3.1 such that the remainder (3.3) satisfies the estimate*

$$\mathcal{R}_M \,=\, \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{M+1} + \delta^2\varepsilon^2\big)\,, \tag{3.5}$$

where the notation $\mathcal{O}_{\mathcal{Z}}$ is introduced in Definition 3.4 below.

Remark 3.3. We do not claim that the approximate solution is uniquely determined by the properties listed in Proposition 3.2, but there is a canonical choice that makes it unique, see Remark 3.13 below for a discussion of this question.

The choice of the integer M, which determines the accuracy of the approximate solution, depends on the intended purpose. The leading order deformation of the stream lines and of the level sets of vorticity is already obtained for $M = 2$, whereas the first correction to the vertical speed Z_2' only appears when $M = 5$, see Section 3.4. In particular, this means that $\zeta_k = 0$ for $k = 1, 2, 3$. As we shall see in Section 4, we need to take $M > 3$ if we want to control the solution of (1.2) over a time interval $[0, T_{\mathsf{adv}}]$ that is independent of the viscosity parameter. More generally, if $0 \le \sigma < 1$, we need $M > (3 + \sigma)/(1 - \sigma)$ to reach the time $T_{\mathsf{adv}}\delta^{-\sigma}$.

3.1. **Function spaces and operators.** We first define the function spaces in which our approximate solution will be constructed. Following [16, 12], we introduce the weighted L^2 space

$$\mathcal{Y} \,=\, \left\{ f \in L^2(\mathbb{R}^2) \,:\, \int_{\mathbb{R}^2} |f(\xi)|^2 \, \mathrm{e}^{|\xi|^2/4} \, \mathrm{d}\xi < \infty \right\}, \tag{3.6}$$

which is a Hilbert space equipped with the scalar product

$$\langle f, g \rangle_{\mathcal{Y}} \,=\, \int_{\mathbb{R}^2} f(\xi) \, g(\xi) \, \mathrm{e}^{|\xi|^2/4} \, \mathrm{d}\xi \,, \qquad \forall \, f, g \in \mathcal{Y}. \tag{3.7}$$

Using polar coordinates (r, θ) defined by $\xi = \big(r\cos\theta, r\sin\theta\big)$, we can expand any $f \in \mathcal{Y}$ in a Fourier series with respect to the angular variable θ. This leads to the direct sum decomposition

$$\mathcal{Y} \,=\, \bigoplus_{n=0}^{\infty} \mathcal{Y}_n \,, \tag{3.8}$$

where $\mathcal{Y}_n = \big\{ f \in \mathcal{Y} \,:\, f = a(r)\cos(n\theta) + b(r)\sin(n\theta) \text{ with } a, b : \mathbb{R}_+ \to \mathbb{R} \big\}$. Note that $\mathcal{Y}_n \perp \mathcal{Y}_{n'}$ if $n \neq n'$, so that the decomposition (3.8) is orthogonal. We also consider the dense subset $\mathcal{Z} \subset \mathcal{Y}$ defined by

$$\mathcal{Z} \,=\, \big\{ f : \mathbb{R}^2 \to \mathbb{R} \,:\, \xi \mapsto \mathrm{e}^{|\xi|^2/4} f(\xi) \in \mathcal{S}_*(\mathbb{R}^2) \big\} \,, \tag{3.9}$$

where $\mathcal{S}_*(\mathbb{R}^2)$ denotes the space of smooth functions with at most polynomial growth at infinity. More precisely, a function $g : \mathbb{R}^2 \to \mathbb{R}$ belongs to $\mathcal{S}_*(\mathbb{R}^2)$ if for any multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$ there exists $C > 0$ and $N \in \mathbb{N}$ such that $|\partial^\alpha g(\xi)| \le C(1 + |\xi|)^N$ for all $\xi \in \mathbb{R}^2$. As an aside we note that $\mathcal{S}_*(\mathbb{R}^2)$ is the multiplier space of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ and of the space $\mathcal{S}'(\mathbb{R}^2)$ of tempered distributions. Although we do not need to equip $\mathcal{S}_*(\mathbb{R}^2)$ with a precise topology, the following notation will be useful.

Definition 3.4. Let $M \in \mathbb{N}$ be a positive integer.

1) If $g_\varepsilon \in \mathcal{S}_*(\mathbb{R}^2)$ depends on a small parameter $\varepsilon > 0$, we say that $g_\varepsilon = \mathcal{O}_{\mathcal{S}_*}\big(\varepsilon^M\big)$ if

$$\forall \, \alpha \in \mathbb{N}^2 \; \exists \, C > 0 \; \exists \, N \in \mathbb{N} \quad \text{such that} \quad |\partial^\alpha g_\varepsilon(\xi)| \le C(1 + |\xi|)^N \varepsilon^M \;\; \forall \, \xi \in \mathbb{R}^2 \,.$$

2) Similarly, if $f_\varepsilon \in \mathcal{Z}$, we write $f_\varepsilon = \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^M\big)$ if $\mathrm{e}^{|\xi|^2/4} f_\varepsilon = \mathcal{O}_{\mathcal{S}_*}\big(\varepsilon^M\big)$.

We next study three linear operators which play an important role in the construction of the approximate solution.

A) *The diffusion operator.* We consider the rescaled diffusion operator $\mathcal{L}$ defined by (2.7) as a linear operator in $\mathcal{Y}$ with (maximal) domain

$$D(\mathcal{L}) \,=\, \big\{ f \in \mathcal{Y} \,:\, \Delta f \in \mathcal{Y}, \, \xi \cdot \nabla f \in \mathcal{Y} \big\} \,. \tag{3.10}$$

It is well known that $\mathcal{L}$ is *self-adjoint* in the Hilbert space $\mathcal{Y}$ with compact resolvent and purely discrete spectrum:

$$\sigma(\mathcal{L}) \,=\, \left\{ -\frac{n}{2} \,:\, n \in \mathbb{N} \right\}, \tag{3.11}$$

see for instance [15, Appendix A]. The one-dimensional kernel of $\mathcal{L}$ is spanned by the Gaussian function G defined in (2.15), whereas the first-order derivatives $\partial_1 G, \partial_2 G$ span the two-dimensional eigenspace corresponding to the eigenvalue $-1/2$. More generally, the eigenvalue $-n/2$ has multiplicity $n+1$ and the eigenfunctions are Hermite functions of degree n.

It is easy to verify that the operator $\mathcal{L}$ is invariant under rotations about the origin in $\mathbb{R}^2$, so that it commutes with the direct sum decomposition (3.8): if $f \in \mathcal{Y}_n \cap D(\mathcal{L})$, then $\mathcal{L}f \in \mathcal{Y}_n$. It is also clear that $\mathcal{L}\mathcal{Z} \subset \mathcal{Z}$, where $\mathcal{Z}$ is defined by (3.9). In the same spirit, the following result will be established in the Appendix:

Lemma 3.5. *For any $\kappa > 0$ and any $f \in \mathcal{Z}$ one has $(\kappa - \mathcal{L})^{-1} f \in \mathcal{Z}$.*

B) *The advection operator.* Another important operator, denoted by $\Lambda : D(\Lambda) \to \mathcal{Y}$, arises when linearizing the quadratic term in (1.2) at the Lamb-Oseen vortex. The operator is defined by

$$\Lambda f \,=\, U^G \cdot \nabla f + \mathrm{BS}[f] \cdot \nabla G\,, \qquad f \in D(\Lambda)\,, \tag{3.12}$$

where the functions G, U^G are given by (2.15), (2.16) and the Biot-Savart operator by (1.3). Equivalently, we have

$$\Lambda f \,=\, \big\{\Psi_0\,,\, f\big\} + \big\{\Delta^{-1} f\,,\, \Omega_0\big\}\,, \qquad f \in D(\Lambda)\,, \tag{3.13}$$

where Ω_0, Ψ_0 are also defined in (2.15), (2.16) and $\{\cdot,\cdot\}$ is the Poisson bracket (2.8). The operator Λ is considered as acting on the maximal domain

$$D(\Lambda) \,=\, \big\{f \in \mathcal{Y} \,:\, U^G \cdot \nabla f \in \mathcal{Y}\big\}\,. \tag{3.14}$$

It is not difficult to verify that Λ is also invariant under rotations about the origin, so that it commutes with the the direct sum decomposition (3.8). Moreover, it is clear that $\Lambda f \in \mathcal{Z}$ if $f \in \mathcal{Z}$, because the velocity fields U^G and $\mathrm{BS}[f]$ belong to the multiplier space $\mathcal{S}_*(\mathbb{R}^2)^2$. Finally we recall the following properties established in [16, 20, 12, 14]:

Proposition 3.6. *The operator Λ is skew-adjoint in the Hilbert space $\mathcal{Y}$ with kernel*

$$\mathrm{Ker}(\Lambda) \,=\, \mathcal{Y}_0 \,\oplus\, \big\{\beta_1 \partial_1 G + \beta_2 \partial_2 G \,:\, \beta_1, \beta_2 \in \mathbb{R}\big\}\,. \tag{3.15}$$

In addition, if $g \in \mathrm{Ker}(\Lambda)^\perp \cap \mathcal{Z}$, the equation $\Lambda f = g$ has a unique solution $f \in \mathrm{Ker}(\Lambda)^\perp \cap \mathcal{Z}$, and f is an even function of the variable ξ_2 if g is an odd function of ξ_2.

Remark 3.7. Since Λ is skew-adjoint in $\mathcal{Y}$ we have $\mathrm{Ker}(\Lambda)^\perp = \overline{\mathrm{Ran}(\Lambda)}$, where $\mathrm{Ran}(\Lambda)$ is the range of Λ. So a necessary condition for the equation $\Lambda f = g$ to have a solution is that $g \perp \mathrm{Ker}(\Lambda)$, which according to (3.15) is equivalent to

$$\mathcal{P}_0\, g \,=\, 0\,, \quad \text{and} \quad \mathrm{m}_1[g] \,=\, \mathrm{m}_2[g] \,=\, 0\,, \tag{3.16}$$

where $\mathcal{P}_0$ is the orthogonal projection in $\mathcal{Y}$ onto the subspace $\mathcal{Y}_0$ of radially symmetric functions, and $\mathrm{m}_1, \mathrm{m}_2$ are the first-order moments defined in (1.4). Note that the solvability conditions (3.16) are not sufficient in general to ensure that $g \in \mathrm{Ran}(\Lambda)$, but under the additional assumption that $g \in \mathcal{Z}$ Proposition 3.15 shows that $g = \Lambda f$ for some $f \in \mathcal{Z}$.

C) *The translation/reflection operator.* Finally we study the action of the operator $\mathcal{T}_\varepsilon$ defined by (2.10) on functions $\Psi \in \mathcal{S}_*(\mathbb{R}^2)$ such that $\Delta\Psi \in \mathcal{Z}$.

Lemma 3.8. *Assume that $\Omega \in \mathcal{Z}$ and let $\Psi = \Delta^{-1}\Omega$ as in (2.6). For each integer $n \ge 1$, let P_n be the polynomial of degree n given by*

$$P_n(\xi) \,=\, \frac{(-1)^{n-1}}{n}\,\frac{1}{2\pi}\int_{\mathbb{R}^2} Q_n^c\big(\xi_1 + \eta_1, \xi_2 - \eta_2\big)\,\Omega(\eta)\,\mathrm{d}\eta\,, \tag{3.17}$$

where Q_n^c is the homogeneous polynomial on $\mathbb{R}^2$ defined by $Q_n^c(r\cos\theta, r\sin\theta) = r^n\cos(n\theta)$, see Section A.1. Then for all $N \in \mathbb{N}$ one has the expansion

$$\big(\mathcal{T}_\varepsilon \Psi\big)(\xi) \,=\, C\,\log\frac{1}{\varepsilon} + \sum_{n=1}^{N} \varepsilon^n P_n(\xi) + \mathcal{O}_{\mathcal{S}_*}\big(\varepsilon^{N+1}\big)\,, \tag{3.18}$$

where $C = \mathrm{M}[\Omega]/(2\pi)$ with $\mathrm{M}[\Omega]$ given by (1.4).

Proof. Using the representation (2.6) and the definition (2.10) of the operator $\mathcal{T}_\varepsilon$, we find

$$\begin{aligned}\big(\mathcal{T}_\varepsilon\Psi\big)(\xi) \,&=\, \Psi(-\xi_1 - 1/\varepsilon, \xi_2) \,=\, \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\sqrt{|\xi_1{+}\eta_1{+}1/\varepsilon|^2 + |\xi_2{-}\eta_2|^2}\,\Omega(\eta)\,\mathrm{d}\eta \\ &=\, \frac{1}{2\pi}\,\log\frac{1}{\varepsilon}\int_{\mathbb{R}^2}\Omega(\eta)\,\mathrm{d}\eta + \frac{1}{4\pi}\int_{\mathbb{R}^2}\log\Big(|1{+}\varepsilon(\xi_1{+}\eta_1)|^2 + \varepsilon^2|\xi_2{-}\eta_2|^2\Big)\,\Omega(\eta)\,\mathrm{d}\eta\,.\end{aligned}$$

The first term in the right-hand side is $C\log(1/\varepsilon)$, so we need only consider the last integral. We can also assume that $\xi \in B_{R_\varepsilon}$, where B_{R_ε} is the ball of radius $R_\varepsilon := 1/(4\varepsilon)$ centered at the origin. Indeed, any function in $S_*(\mathbb{R}^2)$ restricted to the complementary region $B^c_{R_\varepsilon}$ is already of order ε^N for all N, and we clearly have $\mathcal{T}_\varepsilon\Psi \in S_*(\mathbb{R}^2)$. Similarly we can restrict the domain of integration so that $\eta \in B_{R_\varepsilon}$, up to negligible errors. Now, if $\xi, \eta \in B_{R_\varepsilon}$, we denote $x = \varepsilon(\xi_1 + \eta_1, \xi_2 - \eta_2) \in \mathbb{R}^2$ and we use the expansion

$$\frac{1}{2}\log\big(1 + 2x_1 + |x|^2\big) \,=\, \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,Q^c_n(x)\,, \qquad |x| < 1\,, \tag{3.19}$$

which is justified in Section A.1. This leads to the formula

$$\big(\mathcal{T}_\varepsilon\Psi\big)(\xi) \,=\, C\,\log\frac{1}{\varepsilon} + \sum_{n=1}^{N}\frac{(-1)^{n-1}}{n}\frac{\varepsilon^n}{2\pi}\int_{B_{R_\varepsilon}} Q^c_n(\xi_1+\eta_1, \xi_2-\eta_2)\,\Omega(\eta)\,\mathrm{d}\eta + \mathcal{O}_{\mathcal{S}_*}\big(\varepsilon^{N+1}\big)\,.$$

Since $\Omega \in \mathcal{Z}$, we can replace the integral in B_{R_ε} with the integral in $\mathbb{R}^2$ up to an error of size $\mathcal{O}_{\mathcal{S}_*}(\varepsilon^{N+1})$. Thus (3.18) is proved in view of (3.17). □

Remark 3.9. For $n = 1$ and 2, it follows from (3.17) that

$$P_1(\xi) \,=\, \frac{1}{2\pi}\int_{\mathbb{R}^2}(\xi_1+\eta_1)\,\Omega(\eta)\,\mathrm{d}\eta\,, \quad P_2(\xi) \,=\, -\frac{1}{4\pi}\int_{\mathbb{R}^2}\Big[(\xi_1{+}\eta_1)^2 - (\xi_2{-}\eta_2)^2\Big]\,\Omega(\eta)\,\mathrm{d}\eta\,.$$

In particular, if $\mathrm{M}[\Omega] = 0$, we have $\nabla P_1 = 0$ so that $\nabla\mathcal{T}_\varepsilon\Psi = \mathcal{O}_{\mathcal{S}_*}(\varepsilon^2)$. If in addition $\mathrm{m}_1[\Omega] = \mathrm{m}_2[\Omega] = 0$, then $\nabla P_2 = 0$ so that $\nabla\mathcal{T}_\varepsilon\Psi = \mathcal{O}_{\mathcal{S}_*}(\varepsilon^3)$.

3.2. **The second order approximation.**

Before starting the construction of our approximate solution, we compute the error generated by the naive approximation $\Omega_{\mathrm{app}} = \Omega_0$, $\zeta_{\mathrm{app}} = 1$, which corresponds to setting $M = 0$ in (3.1), (3.2). In that case, since $\mathcal{L}\Omega_0 = t\partial_t\Omega_0 = 0$ and $\{\Psi_0, \Omega_0\} = 0$, the remainder (3.3) reduces to

$$\mathcal{R}_0 \,:=\, \Big\{-\mathcal{T}_\varepsilon\Psi_0 + \frac{\varepsilon\xi_1}{2\pi}\,,\,\Omega_0\Big\} \,=\, \Big(\mathbf{T}_\varepsilon U^G + \frac{\varepsilon}{2\pi}\,e_2\Big)\cdot\nabla G\,, \tag{3.20}$$

where the operators $\mathcal{T}_\varepsilon$ and $\mathbf{T}_\varepsilon$ are defined in (2.10) and (2.13), respectively. The following statement is a particular case of the results established in [12, Section 3.1]. We give a short proof for the reader's convenience.

Proposition 3.10. *For any integer $N \ge 2$ the remainder (3.20) satisfies*

$$\mathcal{R}_0(\xi) \,=\, \frac{1}{4\pi}\sum_{n=2}^{N}(-1)^{n-1}\varepsilon^n Q^s_n(\xi)\,G(\xi) + \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{N+1}\big)\,, \tag{3.21}$$

where Q^s_n is the homogeneous polynomial on $\mathbb{R}^2$ defined by $Q^s_n(r\cos\theta, r\sin\theta) = r^n\sin(n\theta)$, see Section A.1.

Proof. We introduce the notation $\eta = \varepsilon^{-1}e_1$. Using the definition of U^G in (2.16), we find

$$\big(\mathbf{T}_\varepsilon\,U^G\big)(\xi) \,=\, \widetilde{U^G}\big(\widetilde{\xi} - \eta\big) \,=\, -U^G(\xi+\eta)\,, \quad \text{and} \quad \frac{\varepsilon}{2\pi}e_2 \,=\, \frac{1}{2\pi}\frac{\eta^\perp}{|\eta|^2}\,, \tag{3.22}$$

hence

$$\mathcal{R}_0(\xi) \,=\, \frac{1}{2\pi}\bigg(\frac{\eta^\perp}{|\eta|^2} - \frac{(\xi+\eta)^\perp}{|\xi+\eta|^2} + \frac{(\xi+\eta)^\perp}{|\xi+\eta|^2}\,e^{-|\xi+\eta|^2/4}\bigg)\cdot\nabla G(\xi)\,. \tag{3.23}$$

To prove (3.21), it is sufficient to estimate (3.23) for $|\xi| \le 1/(2\varepsilon)$, because in the complementary region the right-hand side of (3.23) is of order ε^∞ in $\mathcal{Z}$. If $|\xi| \le 1/(2\varepsilon)$, then $|\xi+\eta|^2 \ge 1/(4\varepsilon^2)$

so that the exponential factor in (3.23) is $\mathcal{O}(\varepsilon^\infty)$. Therefore, since $\nabla G(\xi) = -(\xi/2)G(\xi)$, we obtain

$$\mathcal{R}_0(\xi) \,=\, \frac{1}{4\pi}\,\xi\cdot\biggl(\frac{(\xi+\eta)^\perp}{|\xi+\eta|^2} - \frac{\eta^\perp}{|\eta|^2}\biggr)G(\xi)\,\chi\bigl(\varepsilon|\xi|\bigr) + \mathcal{O}_{\mathcal{Z}}\bigl(\varepsilon^\infty\bigr)\,, \tag{3.24}$$

where χ is a smooth function on $\mathbb{R}_+$ such that $\chi(r) = 1$ for $r \le 1/4$ and $\chi(r) = 0$ for $r \ge 1/2$. To conclude the proof we observe that, for $|\xi| \le 1/(2\varepsilon)$,

$$\xi\cdot\biggl(\frac{(\xi+\eta)^\perp}{|\xi+\eta|^2} - \frac{\eta^\perp}{|\eta|^2}\biggr) \,=\, \varepsilon\xi_2\biggl(\frac{1}{|1+\varepsilon\xi_1|^2+|\varepsilon\xi_2|^2} - 1\biggr) \,=\, \sum_{n=2}^{\infty}(-1)^{n-1}\varepsilon^n Q_n^s(\xi)\,,$$

where the last equality follows from (A.3). Thus (3.24) implies (3.21). □

Remark 3.11. It is important to notice that, according to (3.21), the remainder $\mathcal{R}_0$ is already of order ε^2. As is clear from the above proof, the cancellation of the first order in ε is due to the choice $\zeta_{\text{app}} = 1$, which in turn is equivalent to $Z_2' = \Gamma/(2\pi d)$. In other words, in our approach the translation speed given the Helmholtz-Kirchhoff system can be recovered by minimizing the error of the naive approximation. This is a general feature of interacting vortices in the plane, see [12]. In the case of axisymmetric vortex rings, which share many similarities with counter-rotating vortex pairs, the remainder of the naive approximation is $\mathcal{O}(\varepsilon)$ and not better, even if the translation speed is chosen appropriately, see [14].

Since $\mathcal{R}_0 = \mathcal{O}_{\mathcal{Z}}(\varepsilon^2)$, the first nontrivial step in our construction is the second order approximation, corresponding to $M = 2$, for which $\Omega_{\text{app}} = \Omega_0 + \varepsilon^2\Omega_2$ and $\Psi_{\text{app}} = \Psi_0 + \varepsilon^2\Psi_2$. Although this is not immediately obvious, we anticipate that $\zeta_1 = 0$, which means that there is no correction to the translation speed at this level of the approximation. To determine the unknown profile Ω_2, the strategy is again to minimize the remainder (3.3), which takes the form

$$\begin{aligned}\mathcal{R}_2 \,&=\, \delta\varepsilon^2(\mathcal{L}-1)\Omega_2 + \Bigl\{\Psi_0 - \mathcal{T}_\varepsilon\Psi_0 + \frac{\varepsilon\xi_1}{2\pi} + \varepsilon^2\Psi_2 - \varepsilon^2\mathcal{T}_\varepsilon\Psi_2\,,\,\Omega_0 + \varepsilon^2\Omega_2\Bigr\}\\ &=\, \delta\varepsilon^2(\mathcal{L}-1)\Omega_2 + \mathcal{R}_0 + \varepsilon^2\Lambda\Omega_2 + \varepsilon^2\mathcal{N}_2\,,\end{aligned} \tag{3.25}$$

where Λ is the linear operator (3.13) and

$$\mathcal{N}_2 \,=\, \Bigl\{-\mathcal{T}_\varepsilon\Psi_0 + \frac{\varepsilon\xi_1}{2\pi} + \varepsilon^2\Psi_2 - \varepsilon^2\mathcal{T}_\varepsilon\Psi_2\,,\,\Omega_2\Bigr\} - \Bigl\{\mathcal{T}_\varepsilon\Psi_2\,,\,\Omega_0\Bigr\}\,. \tag{3.26}$$

Invoking Proposition 3.10 with $N = 2$, we obtain

$$\mathcal{R}_0 \,=\, \varepsilon^2\mathcal{H}_2 + \mathcal{O}_{\mathcal{Z}}\bigl(\varepsilon^3\bigr)\,, \qquad \text{where} \quad \mathcal{H}_2(\xi) \,=\, -\frac{1}{2\pi}\,\xi_1\xi_2 G(\xi)\,, \tag{3.27}$$

and we can thus write the remainder (3.28) in the form

$$\mathcal{R}_2 \,=\, \varepsilon^2\Bigl[\delta(\mathcal{L}-1)\Omega_2 + \Lambda\Omega_2 + \mathcal{H}_2\Bigr] + \varepsilon^2\mathcal{N}_2 + \mathcal{O}_{\mathcal{Z}}\bigl(\varepsilon^3\bigr)\,. \tag{3.28}$$

The properties recalled in Section 3.1 imply that the linear operator $\delta(\mathcal{L}-1)+\Lambda$ is maximal dissipative in the Hilbert space $\mathcal{Y}$, hence invertible for any $\delta > 0$. Since $\mathcal{H}_2 \in \mathcal{Z} \subset \mathcal{Y}$, there exists a unique profile $\Omega_2 \in \mathcal{Y}$ such that the quantity inside brackets in (3.28) vanishes exactly. However, we need to verify that $\Omega_2 \in \mathcal{Z}$ and that Ω_2 does not blow up in the limit $\delta \to 0$, which is not immediately obvious. For these reasons, we find it simpler to solve the problem approximately, using only the information given by Proposition 3.6. We set $\Omega_2 = \Omega_2^E + \delta\Omega_2^{NS}$, where

i) $\Omega_2^E \in \mathcal{Y}_2 \cap \mathcal{Z}$ is the unique solution of $\Lambda\Omega_2^E + \mathcal{H}_2 = 0$;

ii) $\Omega_2^{NS} \in \mathcal{Y}_2 \cap \mathcal{Z}$ is the unique solution of $\Lambda\Omega_2^{NS} + (\mathcal{L}-1)\Omega_2^E = 0$.

We recall that $\mathcal{Y}_2$ is the subspace of $\mathcal{Y}$ corresponding to the angular Fourier mode $n = 2$, see (3.8). The explicit expression (3.27) shows that $\mathcal{H}_2 \in \mathcal{Y}_2 \cap \mathcal{Z}$, hence $\mathcal{H}_2 \in \text{Ker}(\Lambda)^\perp$ in view of (3.15). Applying Proposition 3.6 we conclude that the equation $\Lambda\Omega_2^E + \mathcal{H}_2 = 0$ has indeed a unique solution $\Omega_2^E \in \mathcal{Y}_2 \cap \mathcal{Z}$, which is an even function of ξ_2 because $\mathcal{H}_2$ is obviously odd with respect to ξ_2. Similarly, since $(\mathcal{L}-1)\Omega_2^E \in \mathcal{Y}_2 \cap \mathcal{Z}$, it follows from Proposition 3.6 that the equation $\Lambda\Omega_2^{NS} + (\mathcal{L}-1)\Omega_2^E = 0$ has a unique solution $\Omega_2^{NS} \in \mathcal{Y}_2 \cap \mathcal{Z}$. We conclude that the full profile Ω_2 belongs to $\mathcal{Y}_2 \cap \mathcal{Z}$, which implies in particular that the moment conditions in Hypotheses 3.1 are automatically satisfied.

Remark 3.12. It is possible to obtain a more explicit expression of the profile Ω_2, in terms of solutions of linear ODEs on $\mathbb{R}_+$, see Section 3.4. In particular $\Omega_2^E(\xi) = (\xi_2^2 - \xi_1^2)w_2(|\xi|)$ for some smooth and *nonnegative* function w_2 with Gaussian decay at infinity. The profile Ω_2^E, which represents the leading order correction to the radially symmetric vortex Ω_0 in the expansion (3.1), is responsible for the deformation of the stream lines and of the level lines of the vorticity, which are nearly elliptical at this order of approximation, see Fig. 1.

With the above choice of Ω_2 the remainder (3.25) takes the form

$$\mathcal{R}_2 \,=\, \varepsilon^2\delta^2(\mathcal{L}-1)\Omega_2^{NS} + \varepsilon^2\mathcal{N}_2 + \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^3\big) \,=\, \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^3 + \delta^2\varepsilon^2\big)\,, \tag{3.29}$$

because it is easy to verify using (3.26), Lemma 3.8 and Remark 3.9 that $\mathcal{N}_2 = \mathcal{O}_{\mathcal{Z}}(\varepsilon^2)$. This concludes the proof of Proposition 3.2 in the particular case where $M = 2$.

Remark 3.13. A minor drawback of solving the linear equation $\delta(\mathcal{L}-1)\Omega_2 + \Lambda\Omega_2 + \mathcal{H}_2 = 0$ perturbatively in δ is that the solution is not unique. Indeed, since the subspace $\mathcal{Y}_0$ of radially symmetric functions is contained in $\mathrm{Ker}(\Lambda)$ by (3.15), we can add to Ω_2^{NS} any element of $\mathcal{Y}_0 \cap \mathcal{Z}$ without affecting the remainder estimate (3.29), and Hypotheses 3.1 are still satisfied as well. Uniqueness is restored if one assumes that $\mathcal{P}_0\Omega_2^{NS} = 0$, where $\mathcal{P}_0$ is the orthogonal projection in $\mathcal{Y}$ onto $\mathcal{Y}_0$. Note that we must always impose $\mathcal{P}_0\Omega_2^E = 0$, otherwise equation ii) above has no solution.

3.3. **The induction step.** We now use an induction argument to complete the proof of Proposition 3.2. Assume that the conclusion holds for some integer $M \ge 2$ (the case $M = 2$ being settled in Section 3.2.) We consider a refined approximate solution of the form

$$\tilde\Omega_{\rm app} = \Omega_{\rm app} + \varepsilon^{M+1}\Omega_{M+1}\,, \quad \tilde\Psi_{\rm app} = \Psi_{\rm app} + \varepsilon^{M+1}\Psi_{M+1}\,, \quad \tilde\zeta_{\rm app} = \zeta_{\rm app} + \varepsilon^M\zeta_M\,, \tag{3.30}$$

where $\Omega_{\rm app}, \Psi_{\rm app}, \zeta_{\rm app}$ are as in (3.1), (3.2), and where $\Omega_{M+1} \in \mathcal{Z}$, $\Psi_{M+1} \in \mathcal{S}_*(\mathbb{R}^2)$, $\zeta_M \in \mathbb{R}$ have to be determined so that $\mathcal{R}_{M+1} = \mathcal{O}_{\mathcal{Z}}(\varepsilon^{M+2} + \delta^2\varepsilon^2)$.

We first study the remainder $\mathcal{R}_M$ given by (3.3), which is a quadratic polynomial in the parameter δ in view of (3.4). Using in particular Lemma 3.8, we can expand the right-hand side of (3.3) in powers of ε and, by induction hypothesis, the expansion starts at order $\mathcal{O}_{\mathcal{Z}}(\varepsilon^{M+1})$ for the terms that are proportional to δ^0 or δ^1. In other words, there exist $\mathcal{H}_0, \mathcal{H}_1 \in \mathcal{Z}$ such that

$$\mathcal{R}_M \,=\, \varepsilon^{M+1}\mathcal{H}_0 + \delta\varepsilon^{M+1}\mathcal{H}_1 + \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{M+2} + \delta^2\varepsilon^2\big)\,. \tag{3.31}$$

The idea is of course to choose $\Omega_{M+1}, \Psi_{M+1}, \zeta_M$ so as to cancel the terms $\mathcal{H}_0, \mathcal{H}_1$ in (3.31). To do that, we need some information on the first order moments. Using (3.3), one can check by a direct calculation that $\mathrm{M}[\mathcal{R}_M] = \mathrm{m}_1[\mathcal{R}_M] = 0$, so that $\mathrm{M}[\mathcal{H}_j] = \mathrm{m}_1[\mathcal{H}_j] = 0$ for $j = 0, 1$. However, we have $\mathrm{m}_2[\mathcal{R}_M] \neq 0$ in general. In addition, it follows from Hypotheses 3.1 that $\mathcal{R}_M$ is an odd function of ξ_2 when $\delta = 0$, which implies that $\mathcal{H}_0$ is an odd function of ξ_2.

We next consider the remainder of the refined approximation (3.30), which reads

$$\begin{aligned}
\mathcal{R}_{M+1} \,&:=\, \delta\big(\mathcal{L}\tilde\Omega_{\rm app} - t\partial_t\tilde\Omega_{\rm app}\big) + \Big\{\tilde\Psi_{\rm app} - \mathcal{T}_\varepsilon\tilde\Psi_{\rm app} + \frac{\varepsilon\xi_1}{2\pi}\,\tilde\zeta_{\rm app}\,,\, \tilde\Omega_{\rm app}\Big\} \\
&= \mathcal{R}_M + \delta\varepsilon^{M+1}\big(\mathcal{L} - \tfrac{M+1}{2}\big)\,\Omega_{M+1} + \varepsilon^{M+1}\Big\{\Psi_{\rm app} - \mathcal{T}_\varepsilon\Psi_{\rm app} + \frac{\varepsilon\xi_1}{2\pi}\,\zeta_{\rm app}\,,\, \Omega_{M+1}\Big\} \\
&\quad + \varepsilon^{M+1}\Big\{\Psi_{M+1} - \mathcal{T}_\varepsilon\Psi_{M+1} + \frac{\xi_1}{2\pi}\,\zeta_M\,,\, \Omega_{\rm app} + \varepsilon^{M+1}\Omega_{M+1}\Big\}\,.
\end{aligned} \tag{3.32}$$

Using the expansion (3.31) and the identity

$$\big\{\Psi_0\,,\,\Omega_{M+1}\big\} + \Big\{\Psi_{M+1} + \frac{\xi_1}{2\pi}\,\zeta_M\,,\,\Omega_0\Big\} \,=\, \Lambda\Omega_{M+1} + \frac{\zeta_M}{2\pi}\,\partial_2\Omega_0\,,$$

where Λ is the differential operator (3.12), we can write the quantity $\mathcal{R}_{M+1}$ in the form

$$\mathcal{R}_{M+1} \,=\, \varepsilon^{M+1}\mathcal{A}_{M+1} + \varepsilon^{M+1}\mathcal{N}_{M+1} + \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{M+2} + \delta^2\varepsilon^2\big)\,, \tag{3.33}$$

where $\mathcal{A}_{M+1}$ is the collection of the principal terms:

$$\mathcal{A}_{M+1} \,=\, \delta\bigl(\mathcal{L}-\tfrac{M+1}{2}\bigr)\Omega_{M+1} + \Lambda\Omega_{M+1} + \mathcal{H}_0 + \delta\mathcal{H}_1 + \frac{\zeta_M}{2\pi}\,\partial_2\Omega_0\,, \tag{3.34}$$

whereas $\mathcal{N}_{M+1}$ gathers higher order corrections:

$$\begin{aligned}\mathcal{N}_{M+1} \,=\, &\Bigl\{\Psi_{\rm app} - \Psi_0 - \mathcal{T}_\varepsilon\Psi_{\rm app} + \frac{\varepsilon\xi_1}{2\pi}\,\zeta_{\rm app}\,,\,\Omega_{M+1}\Bigr\}\\ &+ \Bigl\{\Psi_{M+1} + \frac{\xi_1}{2\pi}\,\zeta_M\,,\,\Omega_{\rm app} - \Omega_0 + \varepsilon^{M+1}\Omega_{M+1}\Bigr\} - \Bigl\{\mathcal{T}_\varepsilon\Psi_{M+1}\,,\,\Omega_{\rm app} + \varepsilon^{M+1}\Omega_{M+1}\Bigr\}\,.\end{aligned}$$

We now determine $\Omega_{M+1}, \Psi_{M+1}, \zeta_M$ so as to minimize the quantity $\mathcal{A}_{M+1}$. We first define the correction to the vertical speed:

$$\zeta_M \,=\, \zeta_M^E + \delta\zeta_M^{NS}\,, \quad \text{where} \quad \frac{\zeta_M^E}{2\pi} \,=\, \int_{\mathbb{R}^2}\xi_2\mathcal{H}_0(\xi)\,{\rm d}\xi\,, \quad \frac{\zeta_M^{NS}}{2\pi} \,=\, \int_{\mathbb{R}^2}\xi_2\mathcal{H}_1(\xi)\,{\rm d}\xi\,. \tag{3.35}$$

We thus have $\mathcal{A}_{M+1} = \delta\bigl(\mathcal{L}-\frac{M+1}{2}\bigr)\Omega_{M+1} + \Lambda\Omega_{M+1} + \tilde{\mathcal{H}}_0 + \delta\tilde{\mathcal{H}}_1$, where

$$\tilde{\mathcal{H}}_0 \,:=\, \mathcal{H}_0 + \frac{\zeta_M^E}{2\pi}\,\partial_2\Omega_0\,, \qquad \tilde{\mathcal{H}}_1 \,:=\, \mathcal{H}_1 + \frac{\zeta_M^{NS}}{2\pi}\,\partial_2\Omega_0\,, \tag{3.36}$$

and the choice (3.35) ensures that $\mathrm{m}_2[\tilde{\mathcal{H}}_0] = \mathrm{m}_2[\tilde{\mathcal{H}}_1] = 0$. We next define

$$\Omega_{M+1} \,=\, \Omega_{M+1}^{E,0} + \Omega_{M+1}^{E,1} + \delta\Omega_{M+1}^{NS}\,, \tag{3.37}$$

where the vorticity profiles $\Omega_{M+1}^{E,0}, \Omega_{M+1}^{E,1}, \Omega_{M+1}^{NS}$ are determined in the following way:

(1) The radially symmetric function $\Omega_{M+1}^{E,0} \in \mathcal{Y}_0\cap\mathcal{Z}$ is the unique solution, given by Lemma 3.5, of the elliptic equation

$$\bigl(\mathcal{L}-\tfrac{M+1}{2}\bigr)\Omega_{M+1}^{E,0} + \mathcal{P}_0\tilde{\mathcal{H}}_1 \,=\, 0\,, \tag{3.38}$$

where $\mathcal{P}_0$ is the orthogonal projection in $\mathcal{Y}$ onto the radial subspace $\mathcal{Y}_0$.

(2) The function $\Omega_{M+1}^{E,1} \in \mathrm{Ker}(\Lambda)^\perp\cap\mathcal{Z}$ is the unique solution, given by Proposition 3.6, of

$$\Lambda\Omega_{M+1}^{E,1} + \tilde{\mathcal{H}}_0 \,=\, 0\,. \tag{3.39}$$

Remark that $\tilde{\mathcal{H}}_0 \in \mathrm{Ker}(\Lambda)^\perp$ because $\tilde{\mathcal{H}}_0(\xi)$ is an odd function of ξ_2, which implies that $\mathcal{P}_0\tilde{\mathcal{H}}_0 = 0$ and $\mathrm{m}_1[\tilde{\mathcal{H}}_0] = 0$, and because $\mathrm{m}_2[\tilde{\mathcal{H}}_0] = 0$ by our choice of ζ_M^E. Note also that $\Omega_{M+1}^{E,1}$ is an even function of ξ_2, as asserted in Proposition 3.6.

(3) The function $\Omega_{M+1}^{NS} \in \mathrm{Ker}(\Lambda)^\perp\cap\mathcal{Z}$ is the unique solution, given by Proposition 3.6, of

$$\Lambda\Omega_{M+1}^{NS} + (1-P_0)\tilde{\mathcal{H}}_1 + \bigl(\mathcal{L}-\tfrac{M+1}{2}\bigr)\Omega_{M+1}^{E,1} \,=\, 0\,, \tag{3.40}$$

where the last two terms belong to $\mathrm{Ker}(\Lambda)^\perp\cap\mathcal{Z}$ by construction.

In view of (3.35)–(3.40) we have $\mathcal{A}_{M+1} = \delta^2\bigl(\mathcal{L}-\frac{M+1}{2}\bigr)\Omega_{M+1}^{NS}$, and the profile Ω_{M+1} satisfies Hypotheses 3.1. Returning to (3.33) we thus find

$$\begin{aligned}\mathcal{R}_{M+1} \,&=\, \varepsilon^{M+1}\delta^2\bigl(\mathcal{L}-\tfrac{M+1}{2}\bigr)\Omega_{M+1}^{NS} + \varepsilon^{M+1}\mathcal{N}_{M+1} + \mathcal{O}_{\mathcal{Z}}\bigl(\varepsilon^{M+2}+\delta^2\varepsilon^2\bigr)\\ &=\, \mathcal{O}_{\mathcal{Z}}\bigl(\varepsilon^{M+2}+\delta^2\varepsilon^2\bigr)\,,\end{aligned}$$

because using Lemma 3.8 it is easy to verify that $\mathcal{N}_{M+1} = \mathcal{O}_{\mathcal{Z}}(\varepsilon^2)$. This concludes the induction step, and the proof of Proposition 3.2 is now complete. □

3.4. **Leading order correction to the vertical speed.** The goal of this section is to compute the leading order correction to the vertical speed Z_2' in the approximate solution (3.1), (3.2). It turns out that this correction occurs for $M = 5$, which means that $\zeta_k = 0$ for $k = 1, 2, 3$, see [17]. As is explained in Section 3.3, the coefficient ζ_4 is chosen so as to ensure the solvability of the "elliptic" equation for the vorticity profile Ω_5, as in (3.35). Fortunately, it turns out that the expression of ζ_4 only involves the leading order correction Ω_2 to the vorticity distribution. No information on Ω_3 and Ω_4 is needed at this stage.

Lemma 3.14. *Using polar coordinates* $\xi = (r\cos\theta, r\sin\theta)$, *the leading order correction* Ω_2 *in the approximate solution* (3.1) *takes the form*

$$\Omega_2(\xi) \;=\; -\mathsf{w}_2(r)\cos(2\theta) + \delta\,\hat{\mathsf{w}}_2(r)\sin(2\theta)\,, \tag{3.41}$$

for some $\mathsf{w}_2, \hat{\mathsf{w}}_2 : \mathbb{R}_+ \to \mathbb{R}$ *with* $\mathsf{w}_2 > 0$.

Proof. We already know that $\Omega_2 = \Omega_2^E + \delta\Omega_2^{NS}$ where $\Lambda\Omega_2^E + \mathcal{H}_2 = 0$ and $\Lambda\Omega_2^{NS} + (\mathcal{L}-1)\Omega_2^E = 0$, see Section 3.2. According to (3.27) we have $-\mathcal{H}_2 = b(r)\sin(2\theta)$ where $b(r) = r^2 g(r)/(2\pi)$ with g as in (A.8). In particular $\mathcal{H}_2 \in \mathcal{Y}_2 \cap \mathcal{Z}$, so that we can apply Lemma A.1 in Section A.3. We thus obtain the formulas $\Omega_2^E = w(r)\cos(2\theta)$, $\Psi_2^E = \varphi(r)\cos(2\theta)$, where

$$w(r) \;=\; -\varphi(r)h(r) - \frac{b(r)}{2v_0(r)} \;=\; -h(r)\Big(\varphi(r) + \frac{r^2}{4\pi}\Big)\,, \tag{3.42}$$

and φ is the unique solution of the ODE (A.11) with $n = 2$ such that $\varphi(r) = \mathcal{O}(r^2)$ as $r \to 0$ and $\varphi(r) = \mathcal{O}(r^{-2})$ as $r \to +\infty$. It follows from the the maximum principle that φ is a positive function, so that $w(r) < 0$ for all $r > 0$ in view of (3.42). We thus have $\Omega_2^E = -\mathsf{w}_2(r)\cos(2\theta)$ with $\mathsf{w}_2(r) = -w(r) > 0$. We deduce that $(\mathcal{L}-1)\Omega_2^E = a(r)\cos(2\theta)$ for some $a : \mathbb{R}_+ \to \mathbb{R}$, and applying Lemma A.1 again we conclude that $\Omega_2^{NS} = \hat{\mathsf{w}}_2(r)\sin(2\theta)$ for some $\hat{\mathsf{w}}_2 : \mathbb{R}_+ \to \mathbb{R}$. □

Remark 3.15. Similarly the next correction $\Omega_3 = \Omega_3^E + \delta\Omega_3^{NS}$ is determined by the relations $\Lambda\Omega_3^E + \mathcal{H}_3 = 0$ and $\Lambda\Omega_3^{NS} + (\mathcal{L} - \frac{3}{2})\Omega_3^E = 0$, where $\mathcal{H}_3$ is the third order term in the expansion (3.21), namely

$$\mathcal{H}_3(\xi) \;=\; \frac{1}{4\pi}\,Q_3^s(\xi)G(\xi) \;=\; \frac{1}{4\pi}\,(3\xi_1^2\xi_2 - \xi_2^3)G(\xi)\,.$$

Thus $\mathcal{H}_3 \in \mathcal{Y}_3 \cap \mathcal{Z}$ and $\mathcal{H}_3 = b(r)\sin(3\theta)$ for some $b : \mathbb{R}_+ \to \mathbb{R}$. Proceeding exactly as before, we thus find that $\Omega_3^E = \mathsf{w}_3(r)\cos(3\theta)$ and $\Omega_3^{NS} = \hat{\mathsf{w}}_3(r)\sin(3\theta)$ for some $\mathsf{w}_3, \hat{\mathsf{w}}_3 : \mathbb{R}_+ \to \mathbb{R}$.

We are now able to formulate the main result of this section.

Proposition 3.16. *If* $M \ge 5$ *the advection speed* (3.2) *satisfies*

$$Z_2'(t) \;=\; \frac{\Gamma}{2\pi d}\Big(1 - 2\pi\alpha\,\varepsilon^4 + \mathcal{O}\big(\varepsilon^5 + \delta^2\varepsilon\big)\Big)\,, \tag{3.43}$$

where

$$\alpha \;=\; \frac{1}{\pi}\int_{\mathbb{R}^2}(\xi_2^2 - \xi_1^2)\Omega_2(\xi)\,\mathrm{d}\xi \;=\; \int_0^\infty r^3\mathsf{w}_2(r)\,\mathrm{d}r \;\approx\; 22.24\,. \tag{3.44}$$

Remark 3.17. The error term $\mathcal{O}(\varepsilon^5 + \delta^2\varepsilon)$ in (3.43) is probably not optimal. We believe that it can be improved to $\mathcal{O}(\varepsilon^5 + \delta^2\varepsilon^4)$, but this requires nontrivial modifications of our arguments. We also recall that $\varepsilon^2 = \delta t/T_{\mathsf{adv}}$, where T_{adv} is defined in (1.12). If $t \gtrsim T_{\mathsf{adv}}$, then $\delta \lesssim \varepsilon^2$ and the error term in (3.43) can be replaced by $\mathcal{O}(\varepsilon^5)$.

Proof. We consider the approximate solution (3.1), (3.2) for some $M \ge 5$. According to Proposition 3.2, the remainder (3.3) satisfies $\mathcal{R}_M = \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{M+1} + \delta^2\varepsilon^2\big)$. To obtain a formula for the vertical speed, we multiply both members of (3.3) by ξ_2 and we integrate over $\xi \in \mathbb{R}^2$. Proceeding as in the proof of Lemma 2.4, and recalling that $U_{2,\mathrm{app}}(\tilde{\xi} - \varepsilon^{-1}e_1, t) = \partial_1\mathcal{T}_\varepsilon\Psi_{\mathrm{app}}$, we find

$$\int_{\mathbb{R}^2}\Big(\partial_1\mathcal{T}_\varepsilon\Psi_{\mathrm{app}} - \frac{\varepsilon}{2\pi}\,\zeta_{\mathrm{app}}\Big)\Omega_{\mathrm{app}}\,\mathrm{d}\xi \;=\; \mathcal{O}\big(\varepsilon^{M+1} + \delta^2\varepsilon^2\big)\,.$$

Since $\mathrm{M}[\Omega_{\mathrm{app}}] = 1$ by Hypotheses 3.1, we deduce the representation formula

$$\zeta_{\mathrm{app}} \;=\; \frac{2\pi}{\varepsilon}\int_{\mathbb{R}^2}(\partial_1\mathcal{T}_\varepsilon\Psi_{\mathrm{app}})\Omega_{\mathrm{app}}\,\mathrm{d}\xi + \mathcal{O}\big(\varepsilon^M + \delta^2\varepsilon\big)\,, \tag{3.45}$$

which is the analogue of (2.18).

In the rest of the proof, we assume that $M = 5$. To compute the integral in (3.45), we apply Lemma 3.8 with $\Omega = \Omega_{\mathrm{app}}$ and $\Psi = \Psi_{\mathrm{app}}$. This gives the expansion

$$\partial_1\big(\mathcal{T}_\varepsilon\Psi_{\mathrm{app}}\big)(\xi) \;=\; \sum_{n=1}^{5}\varepsilon^n\partial_1 P_n(\xi) + \mathcal{O}_{\mathcal{S}_*}\big(\varepsilon^6\big)\,, \tag{3.46}$$

where the polynomials P_n are given by (3.17) with $\Omega = \Omega_{\rm app}$. Using Remark 3.9 as well as the information on the moments of $\Omega_{\rm app}$ contained in Hypotheses 3.1, we find

$$\partial_1 P_1(\xi) \,=\, \frac{1}{2\pi}\,\mathrm{M}[\Omega_{\rm app}] \,=\, \frac{1}{2\pi}\,, \qquad \partial_1 P_2(\xi) \,=\, -\frac{1}{2\pi}\Big(\xi_1 \mathrm{M}[\Omega_{\rm app}] + \mathrm{m}_1[\Omega_{\rm app}]\Big) \,=\, -\frac{\xi_1}{2\pi}\,.$$

Similarly, a direct calculation shows that

$$\begin{aligned}\partial_1 P_3(\xi) \,&=\, \frac{1}{2\pi}\int_{\mathbb{R}^2}\Big((\xi_1+\eta_1)^2 - (\xi_2-\eta_2)^2\Big)\Omega_{\rm app}(\eta)\,\mathrm{d}\eta \\ &=\, \frac{1}{2\pi}\,(\xi_1^2-\xi_2^2) + \frac{1}{2\pi}\int_{\mathbb{R}^2}(\eta_1^2-\eta_2^2)\Omega_{\rm app}(\eta)\,\mathrm{d}\eta \,=\, \frac{1}{2\pi}\,Q_2^c(\xi) - \frac{\alpha\varepsilon^2}{2} + \mathcal{O}_{\mathcal{S}_*}\big(\varepsilon^3\big)\,,\end{aligned}$$

where in the last equality we used the fact that $\Omega_{\rm app} = \Omega_0 + \varepsilon^2\Omega_2 + \mathcal{O}_{\mathcal{Z}}(\varepsilon^3)$ with Ω_2 as in (3.41), together with the definition of α in (3.44). Finally, since $\Omega_{\rm app} = \Omega_0 + \mathcal{O}_{\mathcal{Z}}(\varepsilon^2)$, we also have

$$\partial_1 P_4(\xi) \,=\, -\frac{1}{2\pi}\,Q_3^c(\xi) + \mathcal{O}_{\mathcal{S}_*}\big(\varepsilon^2\big)\,, \qquad \partial_1 P_5(\xi) \,=\, \frac{1}{2\pi}\,Q_4^c(\xi) + \mathcal{O}_{\mathcal{S}_*}\big(\varepsilon^2\big)\,.$$

Note that the homogeneous polynomials Q_n^c already appear in the expansion (2.24). Summarizing, we have shown that

$$\frac{2\pi}{\varepsilon}\,\partial_1\big(\mathcal{T}_\varepsilon\Psi_{\rm app}\big)(\xi) \,=\, \sum_{n=0}^{4}(-1)^n\varepsilon^n Q_n^c(\xi) - \pi\alpha\varepsilon^4 + \mathcal{O}_{\mathcal{S}_*}\big(\varepsilon^5\big)\,. \tag{3.47}$$

To conclude the proof, we multiply (3.47) by $\Omega_{\rm app}(\xi)$ and we integrate over $\xi \in \mathbb{R}^2$. The contribution of the leading order term Ω_0 in $\Omega_{\rm app}$ is $1 - \pi\alpha\varepsilon^4 + \mathcal{O}(\varepsilon^5)$, and using again (3.41) and Hypotheses 3.1 we obtain

$$\frac{2\pi}{\varepsilon}\int_{\mathbb{R}^2}\big(\partial_1\mathcal{T}_\varepsilon\Psi_{\rm app}\big)\big(\Omega_{\rm app}-\Omega_0\big)\,\mathrm{d}\xi \,=\, \varepsilon^4\int_{\mathbb{R}^2}Q_2^c(\xi)\Omega_2(\xi)\,\mathrm{d}\xi + \mathcal{O}\big(\varepsilon^5\big) \,=\, -\pi\alpha\varepsilon^4 + \mathcal{O}\big(\varepsilon^5\big)\,.$$

Altogether we thus find $\zeta_{\rm app} = 1 - 2\pi\alpha\varepsilon^4 + \mathcal{O}\big(\varepsilon^5 + \delta^2\varepsilon\big)$. □

3.5. **Functional relationship in the inviscid case.** In this section, we investigate the properties of our approximate solution (3.1), (3.2) in the limiting situation where $\delta = 0$, which corresponds to the inviscid case. In view of (3.3) and Proposition 3.2, we have

$$\mathcal{R}_M^E \,:=\, \Big\{\Psi_{\rm app}^E - \mathcal{T}_\varepsilon\Psi_{\rm app}^E + \frac{\varepsilon\xi_1}{2\pi}\,\zeta_{\rm app}^E\,,\,\Omega_{\rm app}^E\Big\} \,=\, \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{M+1}\big)\,, \tag{3.48}$$

where, as in (3.4), the letter E in superscript refers to the Euler equation. We consider the stream function in the uniformly translating frame attached to the vortex center, defined as

$$\Phi_{\rm app}^E \,:=\, \Psi_{\rm app}^E - \mathcal{T}_\varepsilon\Psi_{\rm app}^E + \frac{\varepsilon\xi_1}{2\pi}\,\zeta_{\rm app}^E + \frac{1}{2\pi}\log\frac{1}{\varepsilon}\,, \tag{3.49}$$

where the last term in the right-hand side is an irrelevant constant that is included for convenience. The remainder $\mathcal{R}_M^E = \big\{\Phi_{\rm app}^E\,,\,\Omega_{\rm app}^E\big\}$ would vanish identically if we had a functional relationship of the form $\Phi_{\rm app}^E + F(\Omega_{\rm app}^E) = 0$ for some (smooth) function $F : \mathbb{R} \to \mathbb{R}$. In reality, since $\mathcal{R}_M^E = \mathcal{O}(\varepsilon^{M+1})$, the best we can hope for is an approximate functional relationship, which holds up to corrections of order $\mathcal{O}(\varepsilon^{M+1})$.

At leading order ($M = 0$), our approximate solution is $\Omega_{\rm app} = \Omega_0$, $\Phi_{\rm app} = \Psi_{\rm app} = \Psi_0$, where Ω_0 and Ψ_0 are defined in (2.15), (2.16). It follows that $\Phi_0 + F_0(\Omega_0) = 0$ if we define

$$F_0(s) \,=\, \frac{1}{4\pi}\Big(\gamma_E - \mathrm{Ein}\big(\log\frac{1}{4\pi s}\big)\Big)\,, \qquad 0 < s \le \frac{1}{4\pi}\,. \tag{3.50}$$

Note that F_0 is smooth, strictly increasing, and satisfies $F_0(s) \sim -(4\pi)^{-1}\log\log\frac{1}{s}$ as $s \to 0$. For later use we define

$$A(\xi) \,:=\, F_0'\big(\Omega_0(\xi)\big) \,=\, -\frac{\nabla\Psi_0(\xi)}{\nabla\Omega_0(\xi)} \,=\, \frac{4}{|\xi|^2}\Big(e^{|\xi|^2/4} - 1\Big)\,. \tag{3.51}$$

We next investigate the functional relationship for the second order approximation ($M = 2$). As is explained in Section 3.2 we have $\Omega_{\rm app}^E = \Omega_0 + \varepsilon^2\Omega_2^E$ and $\Psi_{\rm app}^E = \Psi_0 + \varepsilon^2\Psi_2^E$, where the

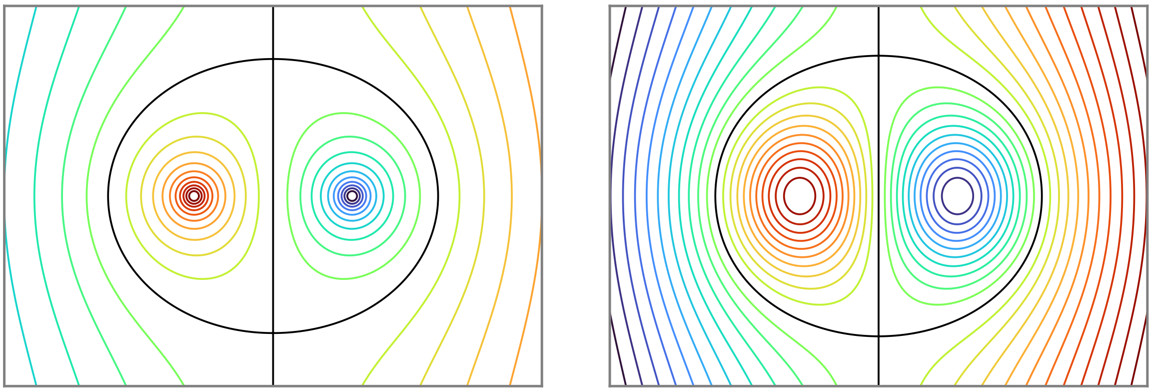

FIGURE 1. The level lines of the function $\Phi^E_{\rm app}$ defined in (3.49), which correspond to the stream lines of the inviscid approximate solution $\Omega^E_{\rm app}$ in the co-moving frame, are represented for $M = 2$ and $\varepsilon = 1/50$ (left) or $\varepsilon = 1/8$ (right). Large positive values of $\Phi^E_{\rm app}$ are depicted in red, and large negative values in blue. The flow has two elliptic stagnation points located at $\xi = 0$ and $\xi = (-1/\varepsilon, 0)$ in our coordinates, as well as two hyperbolic points on the black line which separates the vortex dipole from the exterior flow. Near the vortex centers, the stream lines are nearly elliptical with a major axis in the ξ_2-direction, which reflects the fact that $\mathsf{w}_2 > 0$ in (3.41).

corrections Ω^E_2, Ψ^E_2 are computed in Lemma 3.14. Using Lemma 3.8 with $N = 2$, we obtain the following expansion of the stream function in the moving frame:

$$\Phi^E_{\rm app} \;=\; \Psi_0 + \varepsilon^2\Psi^E_2 + \frac{\varepsilon^2}{4\pi}\left(\xi_1^2 - \xi_2^2\right) + \mathcal{O}_{\mathcal{S}_*}\left(\varepsilon^3\right). \tag{3.52}$$

The level lines of the second order approximation computed above are shown in Fig. 1 (ignoring the ε^3 corrections). We now look for a relationship of the form $\Phi^E_{\rm app} + F(\Omega^E_{\rm app}) = \mathcal{O}_{\mathcal{S}_*}(\varepsilon^3)$, where $F = F_0 + \varepsilon^2 F_2$ for some $F_2 : (0,+\infty) \to \mathbb{R}$. This is problematic, however, because it is not true that $\Omega^E_{\rm app}(\xi) > 0$ for all $\xi \in \mathbb{R}^2$. As is explained in Remark 3.21 below, this difficulty is avoided if we only require that the second order Taylor polynomial in ε of the quantity $\Phi^E_{\rm app} + F(\Omega^E_{\rm app})$ vanishes. This gives the relation

$$\Psi^E_2 + \frac{1}{4\pi}\left(\xi_1^2 - \xi_2^2\right) + F_0'(\Omega_0)\Omega^E_2 + F_2(\Omega_0) \;=\; 0\,, \tag{3.53}$$

which serves as a definition of F_2. It turns out that the first three terms in (3.53) sum up to zero, so that we can actually take $F_2 \equiv 0$. Indeed, this is a consequence of equation (3.42) in Lemma 3.14, because $\Psi^E_2 = \varphi(r)\cos(2\theta)$, $\xi_1^2 - \xi_2^2 = r^2\cos(2\theta)$, $\Omega^E_2 = w(r)\cos(2\theta)$, and $F_0'(\Omega_0) = A = 1/h$ by (3.51).

To establish the functional relationship at any order, we use an induction argument on the integer M, as in the proof of Proposition 3.2. The following definition will be useful:

Definition 3.18. We say that a smooth function $F : (0,+\infty) \to \mathbb{R}$ belongs to the class $\mathcal{K}$ if $F(\Omega_0) \in \mathcal{S}_*(\mathbb{R}^2)$, where Ω_0 is defined in (2.15).

Note that, if $\mathcal{H} \in \mathcal{S}_*(\mathbb{R}^2)$, the equation $F(\Omega_0) = \mathcal{H}$ has a solution $F \in \mathcal{K}$ if and only if $\{\mathcal{H}, \Omega_0\} = 0$, namely if $\mathcal{H}$ is radially symmetric. In that case the function F is uniquely determined on the range of Ω_0, which is the interval $(0,(4\pi)^{-1}]$. Since $\mathcal{H}$ may grow polynomially at infinity, the function F may have a logarithmic singularity at $s = 0$. The following observation is also useful.

Lemma 3.19. *Let $F : (0,+\infty) \to \mathbb{R}$ be in the class $\mathcal{K}$. Then, for all $k \in \mathbb{N}$, the k-th order derivative of F has the property that $F^{(k)}(\Omega_0)\Omega_0^k \in \mathcal{S}_*(\mathbb{R}^2)$.*

Proof. We verify by induction over k that the function $\mathcal{F}_k := F^{(k)}(\Omega_0)\Omega_0^k$ belongs to $\mathcal{S}_*(\mathbb{R}^2)$. This is the case for $k = 0$ because $F \in \mathcal{K}$. Assume now that $\mathcal{F}_k \in \mathcal{S}_*(\mathbb{R}^2)$ for some $k \in \mathbb{N}$. Using the identity $\nabla\Omega_0(\xi) = -(\xi/2)\Omega_0(\xi)$, we obtain by a direct calculation

$$2\nabla\mathcal{F}_k(\xi) + \xi\big(\mathcal{F}_{k+1}(\xi) + k\mathcal{F}_k(\xi)\big) \,=\, 0\,, \qquad \xi \in \mathbb{R}^2\,. \tag{3.54}$$

Since both $\nabla\mathcal{F}_k$ and $\xi\mathcal{F}_k$ belong to $\mathcal{S}_*(\mathbb{R}^2)^2$, so does $\xi\mathcal{F}_{k+1}$ by (3.54). This means that $\mathcal{F}_{k+1}$ and its derivatives have at most a polynomial growth at infinity, so that $\mathcal{F}_{k+1} \in \mathcal{S}_*(\mathbb{R}^2)$. □

Proposition 3.20. *Given any integer $M \ge 2$, let $\Omega^E_{\rm app}, \Psi^E_{\rm app}$ be the approximate solution (3.1) given by Proposition 3.2 with $\delta = 0$. There exists $F \in \mathcal{K}$ of the form $F = F_0 + \varepsilon^2 F_2 + \cdots + \varepsilon^M F_M$ such that the stream function $\Phi^E_{\rm app}$ defined by (3.49) satisfies*

$$\Pi_M\Big(\Phi^E_{\rm app} + F\big(\Omega^E_{\rm app}\big)\Big) \,=\, 0\,, \quad \textit{where} \quad \Pi_M f \,=\, \sum_{k=0}^{M} \frac{\varepsilon^k}{k!}\,\frac{{\rm d}^k f}{{\rm d}\varepsilon^k}\Big|_{\varepsilon=0}\,. \tag{3.55}$$

Remark 3.21. Unfortunately, we cannot write the conclusion of Proposition 3.20 in the seemingly more natural form

$$\Phi^E_{\rm app} + F\big(\Omega^E_{\rm app}\big) \,=\, \mathcal{O}_{\mathcal{S}_*}\big(\varepsilon^{M+1}\big)\,,$$

because the quantity $F\big(\Omega^E_{\rm app}\big)$ is not properly defined. Indeed, our assumptions on the approximate solution do not ensure that $\Omega^E_{\rm app}(\xi) > 0$ for all $\xi \in \mathbb{R}^2$, whereas a function $F \in \mathcal{K}$ is only defined on $(0, +\infty)$. However the Taylor polynomial $\Pi_M F\big(\Omega^E_{\rm app}\big)$ is well defined, because it only involves derivatives of F evaluated at Ω_0, and its coefficients belong to the space $\mathcal{S}_*(\mathbb{R}^2)$ by Lemma 3.19.

Proof. To simplify the notation, we drop the superscript "E" and the subscript "app" everywhere. We proceed by induction over the integer M. Assume that, for some $M \ge 2$, we have constructed $\Phi \in \mathcal{S}_*(\mathbb{R}^2)$, $\Omega \in \mathcal{Z}$, and $F \in \mathcal{K}$ of the form

$$\Phi \,=\, \Phi_0 + \sum_{k=2}^{M} \varepsilon^k \Phi_k\,, \qquad \Omega \,=\, \Omega_0 + \sum_{k=2}^{M} \varepsilon^k \Omega_k\,, \qquad F \,=\, F_0 + \sum_{k=2}^{M} \varepsilon^k F_k\,,$$

such that $\big\{\Phi, \Omega\big\} = \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{M+1}\big)$ and $\Pi_M\big(\Phi + F(\Omega)\big) = 0$. (We have just checked that this holds for $M = 2$.) Suppose now that we have a refined expansion of the form

$$\tilde\Phi \,=\, \Phi + \varepsilon^{M+1}\Phi_{M+1}\,, \quad \tilde\Omega \,=\, \Omega + \varepsilon^{M+1}\Omega_{M+1}\,, \quad \text{where} \quad \big\{\tilde\Phi, \tilde\Omega\big\} = \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{M+2}\big)\,. \tag{3.56}$$

We want to find $F_{M+1} \in \mathcal{K}$ such that $\Pi_{M+1}\big(\tilde\Phi + \tilde F(\tilde\Omega)\big) = 0$ with $\tilde F = F + \varepsilon^{M+1}F_{M+1}$.

First, using the induction hypothesis, we observe that

$$\Pi_{M+1}\big(\Phi + F(\Omega)\big) \,=\, \big(\Pi_{M+1} - \Pi_M\big)\big(\Phi + F(\Omega)\big) \,=\, \varepsilon^{M+1}\mathcal{H}_{M+1}\,, \tag{3.57}$$

for some $\mathcal{H}_{M+1} \in \mathcal{S}_*(\mathbb{R}^2)$. We deduce that $\big\{\Phi\,,\,\Omega\big\} = \varepsilon^{M+1}\big\{\mathcal{H}_{M+1}\,,\,\Omega_0\big\} + \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{M+2}\big)$, because

$$\Pi_{M+1}\big\{\Phi\,,\,\Omega\big\} \,=\, \Pi_{M+1}\big\{\Phi{+}F(\Omega)\,,\,\Omega\big\} \,=\, \Pi_{M+1}\big\{\Pi_{M+1}(\Phi{+}F(\Omega))\,,\,\Omega\big\} \,=\, \varepsilon^{M+1}\big\{\mathcal{H}_{M+1}\,,\,\Omega_0\big\}\,.$$

On the other hand, it follows from the definition of $\tilde\Phi, \tilde\Omega$ in (3.56) that

$$\big\{\tilde\Phi\,,\,\tilde\Omega\big\} \,=\, \big\{\Phi\,,\,\Omega\big\} + \varepsilon^{M+1}\big\{\Phi_{M+1}\,,\,\Omega_0\big\} + \varepsilon^{M+1}\big\{\Phi_0\,,\,\Omega_{M+1}\big\} + \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{M+2}\big)\,,$$

so the assumption that $\big\{\tilde\Phi, \tilde\Omega\big\} = \mathcal{O}_{\mathcal{Z}}\big(\varepsilon^{M+2}\big)$ implies

$$\big\{\mathcal{H}_{M+1}\,,\,\Omega_0\big\} + \big\{\Phi_{M+1}\,,\,\Omega_0\big\} + \big\{\Phi_0\,,\,\Omega_{M+1}\big\} \,=\, 0\,. \tag{3.58}$$

Finally using (3.56), (3.57) we find

$$\begin{aligned}\Pi_{M+1}\big(\tilde\Phi + \tilde F(\tilde\Omega)\big) \,&=\, \Pi_{M+1}\big(\Phi + F(\Omega)\big) + \Pi_{M+1}\big(\tilde\Phi - \Phi\big) + \Pi_{M+1}\big(\tilde F(\tilde\Omega) - F(\Omega)\big)\\ &=\, \varepsilon^{M+1}\Big(\mathcal{H}_{M+1} + \Phi_{M+1} + F_0'(\Omega_0)\Omega_{M+1} + F_{M+1}(\Omega_0)\Big)\,,\end{aligned} \tag{3.59}$$

because at the points where $\tilde\Omega$ and Ω are positive we have the identity

$$\tilde F(\tilde\Omega) - F(\Omega) \,=\, F\big(\Omega + \varepsilon^{M+1}\Omega_{M+1}\big) - F(\Omega) \,+\, \varepsilon^{M+1}F_{M+1}\big(\Omega + \varepsilon^{M+1}\Omega_{M+1}\big)\,.$$

In view of (3.59), we must choose $F_{M+1} \in \mathcal{K}$ so that $\mathcal{A}_{M+1} + F_{M+1}(\Omega_0) = 0$, where

$$\mathcal{A}_{M+1} := \mathcal{H}_{M+1} + \Phi_{M+1} + F'(\Omega_0)\Omega_{M+1} \in \mathcal{S}_*(\mathbb{R}^2)\,.$$

As was mentioned after Definition 3.18, this is possible if and only if $\{\mathcal{A}_{M+1}, \Omega_0\} = 0$, but this solvability condition is implied by (3.58) because

$$\big\{F'(\Omega_0)\Omega_{M+1}\,,\,\Omega_0\big\} \,=\, \big\{\Omega_{M+1}\,,\,F(\Omega_0)\big\} \,=\, \big\{\Omega_{M+1}\,,\,-\Phi_0\big\} \,=\, \big\{\Phi_0\,,\,\Omega_{M+1}\big\}\,.$$

So there exists $F_{M+1} \in \mathcal{K}$ such that $\Pi_{M+1}\big(\tilde\Phi + \tilde F(\tilde\Omega)\big) = 0$ with $\tilde F = F + \varepsilon^{M+1}F_{M+1}$. This concludes the proof. □

Corollary 3.22. *With the same notation as in Proposition 3.20, there exist $\sigma_1 > 0$, $C > 0$, and $N \in \mathbb{N}$ (depending on M) such that, for $\varepsilon > 0$ small enough,*

$$\big|\nabla\big(\Phi^E_{\rm app} + F(\Omega^E_{\rm app})\big)(\xi)\big| \,\le\, C\,\varepsilon^{M+1}\big(1+|\xi|\big)^N\,, \quad \textit{for } |\xi| \,\le\, 2\varepsilon^{-\sigma_1}\,. \tag{3.60}$$

Proof. We recall that $\Omega^E_{\rm app} = \Omega_0 + \varepsilon^2\Omega_2 + \cdots + \varepsilon^M\Omega_M$. For any $k \in \{2,\dots,M\}$, we have $\Omega_k \in \mathcal{Z}$, so that $\varepsilon^k|\Omega_k(\xi)| \le C_k\,\varepsilon^k\,\Omega_0(\xi)(1+|\xi|)^{N_k}$ for some constants $C_k > 0$ and $N_k \in \mathbb{N}$. This means that $\varepsilon^k\Omega_k/\Omega_0$ is very small in the region $\{\xi \in \mathbb{R}^2 \,:\, |\xi| \le 2\varepsilon^{-\sigma_1}\}$ if $\sigma_1 < k/N_k$ and $\varepsilon > 0$ is small enough. As a consequence, if $\sigma_1 > 0$ is small enough, we have the estimate

$$\frac{1}{2}\,\Omega_0(\xi) \,\le\, \Omega^E_{\rm app}(\xi) \,\le\, 2\Omega_0(\xi)\,, \qquad |\xi| \,\le\, 2\varepsilon^{-\sigma_1}\,, \tag{3.61}$$

which implies in particular that $\Omega^E_{\rm app}(\xi)$ is positive when $|\xi| \le 2\varepsilon^{-\sigma_1}$.

We now consider the function $\Theta(\varepsilon,\xi) := \Phi^E_{\rm app}(\xi) + F(\Omega^E_{\rm app}(\xi))$, which is well defined for $\varepsilon > 0$ sufficiently small if $|\xi| \le 2\varepsilon^{-\sigma_1}$. Using a Taylor expansion of order M at $\varepsilon = 0$, and taking into account the fact that $\Pi_M\Theta = 0$ by Proposition 3.20, we obtain the representation formula

$$\Theta(\varepsilon,\xi) \,=\, \frac{\varepsilon^{M+1}}{M!}\int_0^1 (1-\tau)^M\partial_\varepsilon^{M+1}\Theta(\tau\varepsilon,\xi)\,{\rm d}\tau\,, \qquad |\xi| \,\le\, 2\varepsilon^{-\sigma_1}\,. \tag{3.62}$$

The integrand in (3.62) is estimated by straightforward calculations, using the bound (3.61) and the fact that $\Phi^E_{\rm app} \in \mathcal{S}_*(\mathbb{R}^2)$, $\Omega^E_{\rm app} \in \mathcal{Z}$, and $F \in \mathcal{K}$. We find that $|\partial_\varepsilon^{M+1}\Theta(\tau\varepsilon,\xi)| \le C(1+|\xi|)^N$ for some integer $N \in \mathbb{N}$, and a similar estimate holds for the derivatives with respect to ξ. This gives the bound (3.60) after integrating over $\tau \in [0,1]$. □

4. Correction to the approximate solution

In this section, our goal is to show that the exact solution $\Omega(\xi,t)$ of (2.9) with initial data (2.15) remains close to the approximate solution $\Omega_{\rm app}(\xi,t)$ constructed in Section 3. The accuracy of our approximation depends on the integer M in (3.1), which is chosen large enough, and on the inverse Reynolds number $\delta = \nu/\Gamma$, which is taken sufficiently small. We make the decomposition

$$\Omega(\xi,t) \,=\, \Omega_{\rm app}(\xi,t) + \delta w(\xi,t)\,, \quad \Psi(\xi,t) \,=\, \Psi_{\rm app}(\xi,t) + \delta\varphi(\xi,t)\,, \tag{4.1}$$

where w is the vorticity perturbation and $\varphi = \Delta^{-1}w$ is the associated correction of the stream function. We also decompose the vertical speed Z_2' as

$$Z_2'(t) \,=\, \frac{\Gamma}{2\pi d}\big(\zeta_{\rm app}(t) + \zeta(t)\big)\,, \tag{4.2}$$

where $\zeta_{\rm app}$ is the approximation (3.2) and, in agreement with (2.18), the correction ζ is given by the formula

$$\zeta(t) \,=\, \frac{2\pi}{\varepsilon}\int_{\mathbb{R}^2}\Big((\partial_1\mathcal{T}_\varepsilon\Psi)\Omega\Big)(\xi,t)\,{\rm d}\xi \,-\, \zeta_{\rm app}(t)\,. \tag{4.3}$$

Inserting (4.1), (4.2) into (2.9) and using the definition (3.3) of the remainder $\mathcal{R}_M$, we find that the vorticity perturbation w satisfies

$$t\partial_t w - \mathcal{L}w = \frac{1}{\delta}\Big\{\Psi_{\text{app}} - \mathcal{T}_\varepsilon\Psi_{\text{app}} + \frac{\varepsilon\xi_1}{2\pi}\zeta_{\text{app}}\,, w\Big\} + \frac{1}{\delta}\Big\{\varphi - \mathcal{T}_\varepsilon\varphi\,, \Omega_{\text{app}}\Big\} \tag{4.4}$$

$$+ \Big\{\varphi - \mathcal{T}_\varepsilon\varphi\,, w\Big\} + \frac{1}{\delta^2}\mathcal{R}_M + \frac{\varepsilon\,\zeta}{2\pi\delta^2}\Big\{\xi_1, \Omega_{\text{app}} + \delta w\Big\}\,. \tag{4.5}$$

This evolution equation has to be solved with zero initial data at time $t = 0$, because both $\Omega_{\text{app}}(\cdot,t)$ and $\Omega(\cdot,t)$ converge as $t \to 0$ to the same limit Ω_0 given by (2.15), see the discussion after Remark 2.2. Moreover, using the moment conditions (2.19) in Lemma 2.3, which are also fulfilled by the approximate solution Ω_{app} in view of Hypotheses 3.1, we see that the solution of (4.4)–(4.5) satisfies

$$\mathrm{M}[w(\cdot,t)] \,=\, \mathrm{m}_1[w(\cdot,t)] \,=\, \mathrm{m}_2[w(\cdot,t)] \,=\, 0\,, \qquad \text{for all } t > 0\,. \tag{4.6}$$

Note that the last condition $\mathrm{m}_2[w] = 0$ is ensured by our choice (4.3) of the correction ζ to the approximate vertical speed ζ_{app}.

The structure of the evolution equation (4.4)–(4.5) is quite transparent. In the first line we find the linearization of (2.9) at the approximate solution Ω_{app}, in a frame moving with the approximate vertical velocity (3.2). The nonlinear interaction between the vorticity perturbation w and the associated stream function φ is described by the first term in (4.5), whereas the last term takes into account the small correction ζ to the vertical speed. Since $w(\cdot,0) = 0$, the solution of (4.4)–(4.5) is actually driven by the source term $\delta^{-2}\mathcal{R}_M$ in (4.5), which measures by how much Ω_{app} fails to be an exact solution of (2.9). According to Proposition 3.2, this term is small if M is large enough and δ small enough.

We are now in a position to state the main result of this section, which is a refined version of Theorem 1.2.

Theorem 4.1. *Fix $\sigma \in [0,1)$, take $M \in \mathbb{N}$ such that $M > (3+\sigma)/(1-\sigma)$, and let w be the solution of (4.4)–(4.6) with initial data $w|_{t=0} = 0$. There exist positive constants C and δ_0 such that, for any $\Gamma > 0$, any $d > 0$ and any $\nu > 0$ with $\delta := \nu/\Gamma \le \delta_0$, the following holds:*

$$\|(\Omega - \Omega_{\text{app}})(t)\|_{\mathcal{X}_\varepsilon} \,=\, \delta\|w(t)\|_{\mathcal{X}_\varepsilon} \,\le\, C\big(\delta^{-1}\varepsilon^{M+1} + \delta\varepsilon^2\big), \quad \textit{for all } t \in \big(0, T_{\mathsf{adv}}\delta^{-\sigma}\big)\,, \tag{4.7}$$

where ε and T_{adv} are as in (1.12), and the function space $\mathcal{X}_\varepsilon \hookrightarrow L^1(\mathbb{R}^2)$ is defined in (4.23) below. Moreover, the vertical speed satisfies

$$\frac{d}{\Gamma}\,Z_2'(t) \,=\, \frac{1}{\varepsilon}\int_{\mathbb{R}^2}\big(\partial_1\mathcal{T}_\varepsilon\Psi_{\text{app}}\big)\Omega_{\text{app}}\,\mathrm{d}\xi + \mathcal{O}\big(\varepsilon^M + \delta^{-1}\varepsilon^{M+1} + \delta^2\varepsilon + \delta\varepsilon^2\big)\,. \tag{4.8}$$

Remark 4.2. As is explained in Sections 4.1 and 4.2, we need to introduce a carefully designed weighted space $\mathcal{X}_\varepsilon$ to fully exploit the stability properties of our approximate solution Ω_{app} of (2.9). At this stage, however, it is enough to know that $\mathcal{X}_\varepsilon \hookrightarrow L^1(\mathbb{R}^2)$ uniformly in ε. Observing that $\delta^{-1}\varepsilon^{M+1} \le \varepsilon^2$ when $t \le T_{\mathsf{adv}}\delta^{-\sigma}$ and δ is small enough, and recalling that $\Omega_{\text{app}} = \Omega_0 + \mathcal{O}(\varepsilon^2)$ by (3.1), we see that (4.7) readily implies the estimate (1.13) in Theorem 1.2, in view of (1.11) and (2.4). Similarly, assuming that $M \ge 5$ is sufficiently large, the expression (1.14) of the vertical speed follows from (4.8) if we use the relation (3.45) and the expression of ζ_{app} computed in Proposition 3.16.

The rest of this section is devoted to the proof of Theorem 4.1, which is organized as follows. In Section 4.1 we isolate the most dangerous terms in the equation (4.4)–(4.5), and we explain why they are difficult to control. We then discuss how to overcome these issues in a simplified situation where the geometric ideas underlying Arnold's variational principle can be presented without too many technicalities. The functional framework needed to prove Theorem 4.1 is introduced in Section 4.2, where the ε-dependent weighted space $\mathcal{X}_\varepsilon$ appearing in (4.7) is precisely defined. The short Section 4.3 is entirely devoted to the control of the correction ζ to the vertical speed. In Section 4.4 we introduce our main energy functional, inspired from Arnold's theory, and we study its coercivity properties. We also state the key Proposition 4.10, which is the core of the proof of Theorem 4.1. The time derivative of our energy functional is computed in Section 4.5, and consists of various terms that are estimated in the subsequent sections 4.6–4.9. Once this is done, a simple Grönwall-type argument allows us to complete the proof of Proposition 4.10, hence also of Theorem 4.1.

4.1. **Main difficulties and Arnold's variational principle.** Before proceeding with the analysis of the evolution equation (4.4)–(4.5), it is convenient to identify the terms that produce the main contributions. We recall that, according to (3.4), our approximate solution can be decomposed as

$$\Omega_{\mathrm{app}} \,=\, \Omega^E_{\mathrm{app}} + \delta\Omega^{NS}_{\mathrm{app}}\,, \qquad \Psi_{\mathrm{app}} \,=\, \Psi^E_{\mathrm{app}} + \delta\Psi^{NS}_{\mathrm{app}}\,, \qquad \zeta_{\mathrm{app}} \,=\, \zeta^E_{\mathrm{app}} + \delta\zeta^{NS}_{\mathrm{app}}\,,$$

where Ω^E_{app} is the Eulerian approximation already considered in Section 3.5, and $\Omega^{NS}_{\mathrm{app}}$ is a viscous correction. In analogy with the definition of Φ^E_{app} in (3.49), we denote

$$\Phi^{NS}_{\mathrm{app}} \,:=\, \Psi^{NS}_{\mathrm{app}} - \mathcal{T}_\varepsilon \Psi^{NS}_{\mathrm{app}} + \frac{\varepsilon\xi_1}{2\pi}\,\zeta^{NS}_{\mathrm{app}}\,. \tag{4.9}$$

The equation (4.4)–(4.5) can then be written in the equivalent form

$$t\partial_t w - \mathcal{L}w \,=\, \frac{1}{\delta}\big(\{\Phi^E_{\mathrm{app}}, w\} + \{\varphi - \mathcal{T}_\varepsilon\varphi, \Omega^E_{\mathrm{app}}\}\big) + \{\Phi^{NS}_{\mathrm{app}}, w\} + \{\varphi - \mathcal{T}_\varepsilon\varphi, \Omega^{NS}_{\mathrm{app}}\} \tag{4.10}$$

$$+ \{\varphi - \mathcal{T}_\varepsilon\varphi\,, w\} + \frac{1}{\delta^2}\,\mathcal{R}_M + \frac{\varepsilon\,\zeta}{2\pi\delta^2}\{\xi_1, \Omega_{\mathrm{app}} + \delta w\}\,. \tag{4.11}$$

As is explained in the introduction, the main challenge in the proof of Theorem 4.1 is to control the vorticity perturbation w over the long time interval $(0, T_{\mathsf{adv}}\delta^{-\sigma})$, where $\sigma \in (0,1)$. The size of w is measured in a function space $\mathcal{X}_\varepsilon \hookrightarrow L^1(\mathbb{R}^2)$, keeping in mind that $L^1(\mathbb{R}^2)$ is the Lebesgue space whose norm is invariant under the self-similar scaling (2.4). The best we can hope for is to propagate the bounds we have on the forcing term, and thanks to Proposition 3.2 we know that $\delta^{-2}\mathcal{R}_M = \mathcal{O}_{\mathcal{Z}}(\delta^{-2}\varepsilon^{M+1} + \varepsilon^2)$. This explains the right-hand side of estimate (4.7) in Theorem 4.1.

To control the nonlinear terms in (4.11), we have to make sure that w remains small, and in the light of the above we must therefore assume that $\delta^{-2}\varepsilon^{M+1} \ll 1$. Recalling that $\varepsilon \le \delta^{(1-\sigma)/2}$ when $t \le T_{\mathsf{adv}}\delta^{-\sigma}$, we see that this condition is met when $\delta > 0$ is small if we suppose that M is large enough so that $M > (3+\sigma)/(1-\sigma)$. The term involving $\delta^{-2}\varepsilon\zeta\{\xi_1, \Omega_{\mathrm{app}}\}$ is also potentially problematic, because the bound on ζ that will be established in Section 4.3 below does not compensate for the large prefactor δ^{-2}, but due to a subtle cancellation (related to translation invariance in the vertical direction) this term will not seriously affect our energy estimates.

More importantly, we have to handle carefully the linear terms in (4.10), especially the nonlocal ones involving the stream function perturbation φ. The most dangerous terms are multiplied by δ^{-1} and correspond to the linearization of (2.9) at the Eulerian approximate solution Ω^E_{app}. In contrast, the contributions due to the viscous corrections are of size $\mathcal{O}(\varepsilon^2)$ and will be easy to control. As was already observed in [12], the linearized operator at the naive approximation Ω_0 is skew-symmetric in a Gaussian weighted space, so that the leading order terms in (4.10) do not contribute to the energy estimates in that space. Moreover, the corrections due to the difference $\Omega_{\mathrm{app}} - \Omega_0$ are proportional to $\delta^{-1}\varepsilon^2$, a quantity that remains small as long as $t \ll T_{\mathsf{adv}}$.

The real difficulties begin when one tries to control the solution w of (4.10)–(4.11) on a time interval $(0,T)$ with $T \gg T_{\mathsf{adv}}$. If T is independent of $\delta = \nu/\Gamma$, this can still be done using an appropriate modification of the Gaussian weight in the energy estimates, see [12]. However, to reach longer time scales corresponding to $T = T_{\mathsf{adv}}\delta^{-\sigma}$ with $\sigma > 0$, we need a different and more robust approach that fully exploits the stability properties of our approximate solution Ω_{app}. In the particular case under consideration, we know that Ω^E_{app} is very close to a traveling wave solution of the 2D Euler equation, and we even have an approximate functional relationship between Ω^E_{app} and Φ^E_{app}, as shown in Section 3.5. For steady states (or traveling waves) of the 2D Euler equations with a global functional relationship between vorticity and stream function, Arnold [1] introduced a beautiful and general variational principle that can be used to investigate stability. This approach was recently revisited in [13] and successfully applied to the study of the vanishing viscosity limit for axisymmetric vortex rings [14], a problem that has many similarities with the case of vortex dipoles. In the rest of this paragraph, we briefly present Arnold's general strategy and we explain how it can be implemented in our situation.

Assume that we are given a steady state of the two-dimensional Euler equation, with the property that the vorticity ω_* and the associated stream function $\psi_* = \Delta^{-1}\omega_*$ satisfy a global functional relationship of the form $\psi_* = F(\omega_*)$, where F is a smooth function. Following [1] we consider the energy functional

$$\mathcal{E}[\omega] \,:=\, \int \mathcal{F}(\omega)\,\mathrm{d}x - \frac{1}{2}\int \psi\omega\,\mathrm{d}x\,, \tag{4.12}$$

where $\mathcal{F}$ is a primitive of F. Here and in what follows we are vague about the spatial domain under consideration, and we do not perform rigorous calculations. The right-hand side of (4.12) is the sum of the kinetic energy of the fluid and a Casimir functional, so that $\mathcal{E}[\omega]$ is conserved under the evolution defined by the Euler equation. The first variation of $\mathcal{E}$ at ω_* vanishes by construction, and taking the second variation we obtain the quadratic form

$$\mathcal{E}''(\omega_*)[\omega,\omega] \,=\, \int F'(\omega_*)\omega^2\,\mathrm{d}x - \int \psi\omega\,\mathrm{d}x\,. \tag{4.13}$$

A key observation is that this quadratic function of ω is invariant under the evolution defined by the linearized Euler equation at ω_*. Indeed, if $\partial_t\omega = -\{\psi_*,\omega\} - \{\psi,\omega_*\}$, a direct calculation shows that $\mathcal{E}''(\omega_*)[\omega,\omega]$ does not vary in time. As a consequence, if the quadratic form (4.13) has a definite sign, it can be used to prove the stability of the steady state ω_*, not only for the Euler equation but for related systems as well, see [13] for an application to the stability of vortices in the 2D Navier-Stokes equations.

We now explain how to implement Arnold's approach in our situation. We consider a simplified version of the evolution equation (4.10)–(4.11), where we rescale time and only keep the problematic terms that we have just identified. Given a small parameter $\varepsilon > 0$ and an arbitrary real number ζ, our model system reads

$$\partial_\tau w \,=\, \big\{\Phi^E_{\mathrm{app}}\,, w\big\} + \big\{\varphi - \mathcal{T}_\varepsilon\varphi + \zeta\xi_1\,, \Omega^E_{\mathrm{app}}\big\}\,, \qquad \Delta\varphi \,=\, w\,, \tag{4.14}$$

where Ω^E_{app} is the Eulerian approximate solution and Φ^E_{app} is given by (3.49). Extrapolating the conclusions of Proposition 3.20, we assume for simplicity that we have an exact, global relationship of the form $\Phi^E_{\mathrm{app}} = -F(\Omega^E_{\mathrm{app}})$, for some $F : \mathbb{R} \to \mathbb{R}$ (the minus sign is introduced for convenience, just to ensure that F is an increasing function.) Inspired by (4.13), we define the quadratic functional

$$E(w) \,:=\, \frac{1}{2}\int F'(\Omega^E_{\mathrm{app}})w^2\,\mathrm{d}\xi \,+\, \frac{1}{2}\int \big(\varphi - \mathcal{T}_\varepsilon\varphi\big)w\,\mathrm{d}\xi\,, \tag{4.15}$$

where the last term is, up to a sign, the total energy of the fluid, taking into account the mirror vortex located at $\xi = (-1/\varepsilon, 0)$.

As before, we claim that the functional $E(w)$ is invariant under the linear evolution defined by (4.14). To prove this, we first observe that

$$\partial_\tau E[w] \,=\, \int F'(\Omega^E_{\mathrm{app}})w\partial_\tau w\,\mathrm{d}\xi \,+\, \int \big(\varphi - \mathcal{T}_\varepsilon\varphi\big)\partial_\tau w\,\mathrm{d}\xi\,, \tag{4.16}$$

where, to obtain the second term, we used the identities

$$\int(\partial_\tau\varphi)w\,\mathrm{d}\xi \,=\, \int \varphi\partial_\tau w\,\mathrm{d}\xi\,, \qquad \int(\mathcal{T}_\varepsilon\partial_\tau\varphi)w\,\mathrm{d}\xi \,=\, \int(\mathcal{T}_\varepsilon\varphi)\partial_\tau w\,\mathrm{d}\xi\,. \tag{4.17}$$

The first one is established by writing $w = \Delta\varphi$ and integrating by parts, the second one using in addition the fact that the translation-reflection operator $\mathcal{T}_\varepsilon$ defined in (2.10) is self-ajdoint in $L^2(\mathbb{R}^2)$ and commutes with the Laplacian.

It remains to evaluate the right-hand side of (4.16). Starting from (4.14) and using the functional relation $\Phi^E_{\rm app} + F(\Omega^E_{\rm app}) = 0$, we find

$$\begin{aligned}\int F'(\Omega^E_{\rm app}) w\partial_\tau w \,{\rm d}\xi \,&=\, \int F'(\Omega^E_{\rm app}) w\Big(\{w\,,F(\Omega^E_{\rm app})\} + \{\varphi - \mathcal{T}_\varepsilon\varphi + \zeta\xi_1\,,\Omega^E_{\rm app}\}\Big)\,{\rm d}\xi\\ &=\, 0 + \int w\{\varphi - \mathcal{T}_\varepsilon\varphi + \zeta\xi_1\,, F(\Omega^E_{\rm app})\}\,{\rm d}\xi\,,\end{aligned}$$

where we invoke familiar identities involving Poisson brackets, such as $F'(a)\{a,b\} = \{F(a),b\}$, $\{F(a),G(a)\} = 0$, and $\int\{a,b\}c\,{\rm d}\xi = \int a\{b,c\}\,{\rm d}\xi$. Similarly we obtain

$$\begin{aligned}\int(\varphi - \mathcal{T}_\varepsilon\varphi)\partial_\tau w\,{\rm d}\xi \,&=\, \int(\varphi - \mathcal{T}_\varepsilon\varphi)\Big(\{w\,,F(\Omega^E_{\rm app})\} + \{\varphi - \mathcal{T}_\varepsilon\varphi + \zeta\xi_1\,,\Omega^E_{\rm app}\}\Big)\,{\rm d}\xi\\ &=\, 0 + \int(\varphi - \mathcal{T}_\varepsilon\varphi)\Big(\{w\,,F(\Omega^E_{\rm app})\} + \zeta\{\xi_1\,,\Omega^E_{\rm app}\}\Big)\,{\rm d}\xi\,.\end{aligned}$$

So we deduce from (4.16) that

$$\begin{aligned}\partial_\tau E[w] \,&=\, \zeta\int\Big(w\{\xi_1\,,F(\Omega^E_{\rm app})\} + (\varphi - \mathcal{T}_\varepsilon\varphi)\{\xi_1\,,\Omega^E_{\rm app}\}\Big)\,{\rm d}\xi\\ &=\, \zeta\int w\{\xi_1\,,F(\Omega^E_{\rm app}) + \Phi^E_{\rm app}\}\,{\rm d}\xi \,=\, 0\,,\end{aligned}\tag{4.18}$$

which is the desired result. Here, in the last line, we used the identity

$$\int(\varphi - \mathcal{T}_\varepsilon\varphi)\{\xi_1\,,\Omega^E_{\rm app}\}\Big)\,{\rm d}\xi \,=\, \int w\{\xi_1\,,\Psi^E_{\rm app} - \mathcal{T}_\varepsilon\Psi^E_{\rm app}\}\,{\rm d}\xi \,=\, \int w\{\xi_1\,,\Phi^E_{\rm app}\}\,{\rm d}\xi\,,$$

which is established in the same way as (4.17).

The main purpose of the heuristic arguments above is to explain how to construct an energy functional that is naturally adapted to the leading order terms in the evolution equation (4.10)–(4.11). Although the general strategy is clear, many technical difficulties arise when turning these ideas into a rigorous proof. For instance, we can only exploit the functional relationship between $\Phi^E_{\rm app}$ and $\Omega^E_{\rm app}$ in the region where we are able to justify it, namely for $|\xi| \le 2\varepsilon^{-\sigma_1}$ with $\sigma_1 > 0$ sufficiently small. Outside the vortex core, the energy functional has to be substantially modified, but we can exploit the fact that our approximate solution $\Omega_{\rm app}$ is extremely small in that region. In addition, understanding the coercivity properties of the energy functional and its interplay with the dissipation operator $\mathcal{L}$ is highly non trivial. Similar problems were addressed in the previous works [13, 14], but here we have to face the additional difficulty of handling a perturbative expansion to an arbitrary order in the parameter ε.

4.2. **The weighted space and its properties.** After these preliminaries, we construct the weight function that will enter our energy functional. We give ourselves three real numbers σ_1, σ_2, and γ satisfying

$$0 \,<\, \sigma_1 \,<\, \frac{1}{2}\,, \qquad \sigma_2 \,>\, 1\,, \qquad \gamma \,=\, \frac{\sigma_1}{\sigma_2} \,<\, \frac{1}{2}\,. \tag{4.19}$$

In the course of the proof the parameter σ_1 will be chosen small, depending on the order M of the approximate solution (3.1), whereas σ_2 will be large. In particular we assume that σ_1 is small enough to ensure the validity of Corollary 3.22, which asserts the existence of an approximate functional relationship between the vorticity $\Omega^E_{\rm app}$ and the stream function $\Phi^E_{\rm app}$ defined by (3.49). More precisely, we define

$$\Theta(\varepsilon,\xi) \,:=\, \Phi^E_{\rm app}(\xi) + F\big(\Omega^E_{\rm app}(\xi)\big)\,, \tag{4.20}$$

where $F \in \mathcal{K}$ is the function introduced in Proposition 3.20. Then, according to (3.60), there exists an integer $N \in \mathbb{N}$ such that

$$|\nabla_\xi\Theta(\varepsilon,\xi)| \,\lesssim\, \varepsilon^{M+1}(1 + |\xi|)^N, \qquad \text{for } |\xi| \,\le\, 2\varepsilon^{-\sigma_1}\,. \tag{4.21}$$

For later use, we also recall that $\Omega_0(\xi)/2 \le \Omega^E_{\rm app}(\xi) \le 2\Omega_0(\xi)$ when $|\xi| \le 2\varepsilon^{-\sigma_1}$, see (3.61).

We now decompose the space domain $\mathbb{R}^2$ into three disjoint regions:

$$\begin{aligned} \mathrm{I}_\varepsilon &= \bigl\{\xi \in \mathbb{R}^2 \,:\, |\xi| < 2\varepsilon^{-\sigma_1}\,,\ F'\bigl(\Omega^E_{\rm app}(\xi)\bigr) < \exp(\varepsilon^{-2\sigma_1}/4)\bigr\}\,, && \text{(Inner)}\\ \mathrm{II}_\varepsilon &= \bigl\{\xi \in \mathbb{R}^2 \,:\, \xi \notin \mathrm{I}_\varepsilon\,,\ |\xi| \le \varepsilon^{-\sigma_2}\bigr\}\,, && \text{(Intermediate)}\\ \mathrm{III}_\varepsilon &= \bigl\{\xi \in \mathbb{R}^2 \,:\, |\xi| > \varepsilon^{-\sigma_2}\bigr\}\,, && \text{(Outer)} \end{aligned}$$

which depend on time through the parameter $\varepsilon = \sqrt{\nu t}/d$. Our weight function is defined as

$$W_\varepsilon(\xi) \,=\, \begin{cases} F'\bigl(\Omega^E_{\rm app}(\xi)\bigr) & \text{in } \mathrm{I}_\varepsilon\,,\\ \exp(\varepsilon^{-2\sigma_1}/4) & \text{in } \mathrm{II}_\varepsilon\,,\\ \exp(|\xi|^{2\gamma}/4) & \text{in } \mathrm{III}_\varepsilon\,. \end{cases} \tag{4.22}$$

For any fixed $\varepsilon > 0$, the weight W_ε is a positive, locally Lipschitz and piecewise smooth function, but the derivative ∇W_ε has a discontinuity at the boundaries of the regions $\mathrm{I}_\varepsilon, \mathrm{II}_\varepsilon, \mathrm{III}_\varepsilon$, see Fig. 2. Finally we introduce the weighted L^2 space

$$\mathcal{X}_\varepsilon \,=\, \Bigl\{f \in L^2(\mathbb{R}^2) \,:\, \|f\|^2_{\mathcal{X}_\varepsilon} := \int_{\mathbb{R}^2} W_\varepsilon(\xi)|f(\xi)|^2\,\mathrm{d}\xi < \infty\Bigr\}\,. \tag{4.23}$$

The estimates established in Proposition 4.4 below readily imply that

$$\mathcal{Y} \hookrightarrow \mathcal{X}_\varepsilon \hookrightarrow L^p(\mathbb{R}^2), \qquad \text{for all } p \in [1,2]\,,$$

where $\mathcal{Y}$ is the Gaussian space defined in (3.6). Moreover, in the limit where $\varepsilon \to 0$, it is easy to verify that $W_\varepsilon(\xi) \to W_0(\xi) := F_0'(\Omega_0(\xi)) \equiv A(\xi)$, where A is defined in (3.51). We denote by $\mathcal{X}_0$ the space (4.22) with $\varepsilon = 0$.

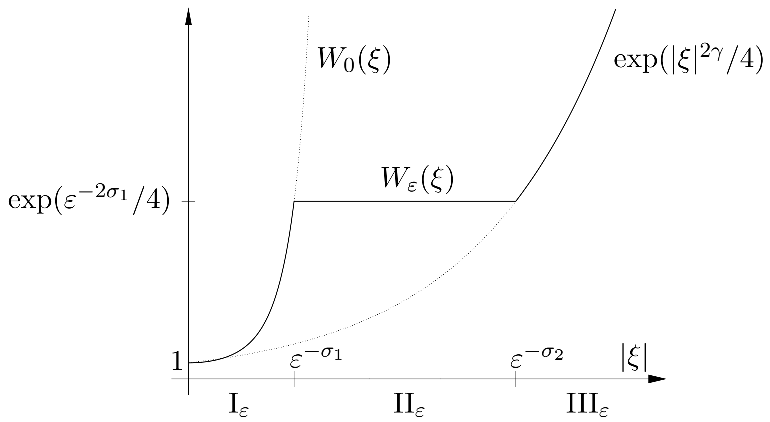

FIGURE 2. A schematic representation of the graph of the weight function W_ε defined in (4.22). In the inner region I_ε, the weight is close for $\varepsilon > 0$ small to the radially symmetric function $W_0(\xi) = 4|\xi|^{-2}\bigl(e^{|\xi|^2/4} - 1\bigr)$. It then takes constant values in the intermediate region II_ε, and grows like $\exp(|\xi|^{2\gamma}/4)$ in the outer region III_ε. The dashed lines illustrate the bounds (4.25), where the constants C_1, C_2 are independent of ε.

Remark 4.3. In the inner region I_ε, the weight (4.22) is constructed following exactly Arnold's approach as discussed in Section 4.1. This is possible because the quantity $\Theta = \Phi^E_{\rm app} + F(\Omega^E_{\rm app})$ is small in that region, which means that we almost have a functional relationship between the vorticity and the stream function. In the intermediate region II_ε the weight is just a constant, so that the dangerous advection terms multiplied by δ^{-1} in (4.10) will disappear after an integration by parts. Finally, in the outer region III_ε, the evolution defined by (4.10)–(4.11) is essentially driven by the diffusion operator $\mathcal{L}$, and it turns out that a radially symmetric weight with moderate growth at infinity is appropriate in that case. Note that our definition of the weight in the intermediate and outer regions is the same as in [12], whereas the Arnold strategy for the inner region was put forward in [14]. There is a minor simplification here with respect to [14]: the weight (4.22) is independent of the inverse Reynolds number δ, because it only involves the Eulerian approximation $\Omega^E_{\rm app}$.

We collect elementary properties of the inner region I_ε and of the weight function W_ε in the following lemma, which is the analogue of [14, Lemmas 4.2 & 4.3].

Proposition 4.4. *If $\varepsilon, \sigma_1 > 0$ are small enough, the following holds true:*

i) The inner region I_ε is diffeomorphic to an open disk, and there exists a constant $\kappa > 0$ such that

$$\big\{|\xi| \le \varepsilon^{-\sigma_1}\big\} \subset \mathrm{I}_\varepsilon \subset \big\{|\xi|^2 \le \varepsilon^{-2\sigma_1} + \kappa|\log(\varepsilon)|\big\}\,. \tag{4.24}$$

ii) There exist $C_1, C_2 > 0$ such that the weight W_ε satisfies the uniform bounds

$$C_1 \exp(|\xi|^{2\gamma}/4) \,\le\, W_\varepsilon(\xi) \,\le\, C_2 W_0(\xi)\,, \qquad \textit{for } \xi \in \mathbb{R}^2\,, \tag{4.25}$$

where $W_0 = A$ is defined in (3.51) and γ in (4.19).

iii) For any $\gamma_2 < 2$, there exists $C_3 > 0$ such that

$$|W_\varepsilon(\xi) - W_0(\xi)| + |\nabla W_\varepsilon(\xi) - \nabla W_0(\xi)| \,\le\, C_3 \varepsilon^{\gamma_2} W_0(\xi)\,, \qquad \textit{for } \xi \in \mathrm{I}_\varepsilon\,. \tag{4.26}$$

Proof. The main step is to show that, if $\sigma_1 > 0$ is small enough,

$$\big|F'(\Omega^E_{\rm app}(\xi)) - W_0(\xi)\big| + \big|\nabla\big(F'(\Omega^E_{\rm app}(\xi)) - W_0(\xi)\big)\big| \,\lesssim\, \varepsilon^{\gamma_2} W_0(\xi)\,, \quad \text{for } |\xi| \le 2\varepsilon^{-\sigma_1}\,, \tag{4.27}$$

where $F = F_0 + \varepsilon^2 F_2 + \cdots + \varepsilon^M F_M$ is the function defined in Proposition 3.20. Since $F \in \mathcal{K}$ we know in particular that $F''(\Omega_0)\Omega_0^2 \in \mathcal{S}_*(\mathbb{R}^2)$, see Lemma 3.19. As is easily verified, this implies that there exists $C > 0$ and $N \in \mathbb{N}$ such that

$$\sup_{1/2 \le \lambda \le 2} \big|F''\big(\lambda\Omega_0(\xi)\big)\big| \,\le\, C\,(1+|\xi|)^N\,\Omega_0(\xi)^{-2}\,, \qquad \xi \in \mathbb{R}^2\,. \tag{4.28}$$

In the region where $|\xi| \le 2\varepsilon^{-\sigma_1}$, we observe that

$$F'(\Omega^E_{\rm app}(\xi)) - F'(\Omega_0(\xi)) \,=\, \big(\Omega^E_{\rm app}(\xi) - \Omega_0(\xi)\big)\int_0^1 F''\big((1-s)\Omega_0(\xi) + s\Omega^E_{\rm app}(\xi)\big)\,\mathrm{d}s\,, \tag{4.29}$$

where the integrand can be estimated using (4.28) since $\Omega_0(\xi)/2 \le \Omega^E_{\rm app}(\xi) \le 2\Omega_0(\xi)$. By definition of F we also have

$$F'(\Omega_0(\xi)) - W_0(\xi) \,=\, F'(\Omega_0(\xi)) - F_0'(\Omega_0(\xi)) \,=\, \sum_{k=2}^M \varepsilon^k F_k'(\Omega_0(\xi))\,, \qquad \xi \in \mathbb{R}^2\,. \tag{4.30}$$

Combining (4.29), (4.30) and using the fact that $\Omega^E_{\rm app} - \Omega_0 = \mathcal{O}_{\mathcal{Z}}(\varepsilon^2)$, we obtain an estimate of the form

$$\big|F'(\Omega^E_{\rm app}(\xi)) - W_0(\xi)\big| \,\le\, C\varepsilon^2(1+|\xi|)^N\,\Omega_0(\xi)^{-1}\,, \qquad |\xi| \le 2\varepsilon^{-\sigma_1}\,,$$

with a possibly larger exponent N. Since $\Omega_0(\xi)^{-1} \lesssim \mathrm{e}^{|\xi|^2/4} \lesssim (1+|\xi|)^2 W_0(\xi)$, we obtain the desired estimate for the first term in the left-hand side of (4.27) by taking σ_1 sufficiently small so that $\varepsilon^{2-\sigma_1(N+2)} \le \varepsilon^{\gamma_2}$ when $\varepsilon \ll 1$. The corresponding bound on $\nabla(F'(\Omega^E_{\rm app}) - W_0)$ is obtained by differentiating (4.29), (4.30) and proceeding similarly.

Estimate (4.27) shows in particular that $F'(\Omega^E_{\rm app}(\xi))$ is very close to $W_0(\xi)$ when $|\xi| \le 2\varepsilon^{-\sigma_1}$, and this implies that the region I_ε satisfies the inclusions (4.24). Also, using the definition (4.22), it is straightforward to verify that the weight W_ε satisfies the bounds (4.25) in all three regions I_ε, II_ε, and III_ε. Finally (4.26) immediately follows from (4.27) since $W_\varepsilon = F'(\Omega^E_{\rm app})$ in region I_ε. □

We next derive useful estimates for the stream function and the velocity field in terms of the vorticity in our function space $\mathcal{X}_\varepsilon$.

Lemma 4.5. *Let $w \in \mathcal{X}_\varepsilon$ and $\varphi = \Delta^{-1}w$ as in (2.6). Then, for all $2 < q < \infty$ there exists a constant $C > 0$ such that*

$$\|(1+|\cdot|)^{-1}\varphi\|_{L^q} + \|\nabla\varphi\|_{L^q} \,\le\, C\,\|w\|_{\mathcal{X}_\varepsilon}\,. \tag{4.31}$$

Moreover

$$\|(1+|\cdot|)\nabla\varphi\|_{L^\infty} \,\le\, C\big(\|\nabla w\|_{\mathcal{X}_\varepsilon}^{1/2} + \|w\|_{\mathcal{X}_\varepsilon}^{1/2}\big)\,\|w\|_{\mathcal{X}_\varepsilon}^{1/2}\,. \tag{4.32}$$

Proof. In view of (2.6) we have, for all $\xi \in \mathbb{R}^2$,

$$|\varphi(\xi)| \,\lesssim\, \int_{\mathbb{R}^2} \big|\log|\xi-\eta|\big|\,|w(\eta)|\,\mathrm{d}\eta\,, \quad |\nabla\varphi(\xi)| \,\lesssim\, \int_{\mathbb{R}^2} \frac{1}{|\xi-\eta|}\,|w(\eta)|\,\mathrm{d}\eta\,. \tag{4.33}$$

The bound (4.31) on $\nabla\varphi$ is a direct consequence of (4.33) and the Hardy-Littlewood-Sobolev inequality, see [15, Lemma 2.1]. More precisely, given $1 < p < 2 < q < \infty$ with $1/p = 1/q + 1/2$, the HLS inequality shows that $\|\nabla\varphi\|_{L^q} \lesssim \|w\|_{L^p} \lesssim \|w\|_{\mathcal{X}_\varepsilon}$, where in the last step we used the fact that $\mathcal{X}_\varepsilon \hookrightarrow L^r(\mathbb{R}^2)$ for any $r \in [1,2]$.

To conclude the proof of (4.31), we use the first inequality in (4.33) together with the crude bounds $|\log|\xi-\eta|| \le |\xi-\eta|^{-1/2}$ if $|\xi-\eta| \le 1$, and $\log|\xi-\eta| \le \log(1+|\xi|) + \log(1+|\eta|)$ if $|\xi-\eta| \ge 1$. We thus find

$$|\varphi(\xi)| \,\lesssim\, \int_{\{|\xi-\eta|\le 1\}} \frac{1}{|\xi-\eta|^{1/2}}|w(\eta)|\,\mathrm{d}\eta + \log(1+|\xi|)\int_{\{|\xi-\eta|\ge 1\}} \log(2+|\eta|)|w(\eta)|\,\mathrm{d}\eta\,,$$

and applying Hölder's inequality to the first integral we arrive at $|\varphi(\xi)| \le C\log(2+|\xi|)\|w\|_{\mathcal{X}_\varepsilon}$. This implies in particular that $(1+|\cdot|)^{-1}\varphi \in L^q(\mathbb{R}^2)$ for all $q > 2$, as asserted in (4.31).

Finally, to prove (4.32), we deduce from the second inequality in (4.33) that

$$(1+|\xi|)|\nabla\varphi(\xi)| \,\lesssim\, \int_{\mathbb{R}^2}\Big(1 + \frac{1+|\eta|}{|\xi-\eta|}\Big)\,|w(\eta)|\,\mathrm{d}\eta \,\lesssim\, \|w\|_{\mathcal{X}_\varepsilon} + \int_{\{|\xi-\eta|\le 1\}} \frac{1+|\eta|}{|\xi-\eta|}\,|w(\eta)|\,\mathrm{d}\eta\,.$$

The last integrand is equal to $|\xi-\eta|^{-1}\big((1+|\eta|)|w(\eta)|^{1/2}\big)|w(\eta)|^{1/2}$, so we can apply the trilinear Hölder inequality with exponents $8/5$, 4, 8 to obtain

$$\int_{\{|\xi-\eta|\le 1\}} \frac{1+|\eta|}{|\xi-\eta|}\,|w(\eta)|\,\mathrm{d}\eta \,\lesssim\, \big\|(1+|\cdot|)^2 w\big\|_{L^2}^{1/2}\|w\|_{L^4}^{1/2} \,\lesssim\, \|w\|_{\mathcal{X}_\varepsilon}^{1/2}\,\big(\|\nabla w\|_{\mathcal{X}_\varepsilon}^{1/2} + \|w\|_{\mathcal{X}_\varepsilon}^{1/2}\big)\,,$$

where in the last step we used the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$. This gives (4.32). □

Our second estimate focuses on the translated stream function $\mathcal{T}_\varepsilon\varphi$ near the origin.

Lemma 4.6. *Assume that $w \in \mathcal{X}_\varepsilon$ with $\mathrm{M}[w] = \mathrm{m}_1[w] = \mathrm{m}_2[w] = 0$ and let $\varphi = \Delta^{-1}w$ as in (2.6). Then, for any $q > 2$ there exists a constant $C > 0$ such that*

$$\varepsilon\big\|\mathbb{1}_{\mathrm{I}_\varepsilon}(1+|\cdot|)^{-2}\mathcal{T}_\varepsilon\varphi\big\|_{L^q} + \big\|\mathbb{1}_{\mathrm{I}_\varepsilon}(1+|\cdot|)^{-3}\nabla\mathcal{T}_\varepsilon\varphi\big\|_{L^q} \le C\varepsilon^3\|w\|_{\mathcal{X}_\varepsilon}\,. \tag{4.34}$$

Moreover $\|(1+|\cdot|)\varphi\|_{L^\infty} \le C\|w\|_{\mathcal{X}_\varepsilon}$.

The proof combines the expansion established in Lemma 3.8 with standard estimates exploiting the coercivity properties of the weight W_ε. The argument is somewhat technical, so we postpone it to Appendix A.4.

4.3. **Bound on the vertical speed.** Before setting up the nonlinear energy estimates, we apply Lemmas 4.5–4.6 to control the size of the correction ζ to the vertical speed.

Lemma 4.7. *Let ζ be defined as in (4.3). Then there exists a constant $C > 0$ such that*

$$\frac{\varepsilon}{2\pi}|\zeta(t)| \,\le\, C\big(\varepsilon^{M+1} + \delta^2\varepsilon^2 + \delta\varepsilon\,\|w\|_{\mathcal{X}_\varepsilon} + \delta^2\varepsilon^3\,\|w\|_{\mathcal{X}_\varepsilon}^2\big)\,. \tag{4.35}$$

Proof. In the definition (4.3) of ζ, we expand the vorticity Ω and the stream function Ψ as in (4.1), and we subtract the approximation ζ_{app} given by (3.45). This gives the expression

$$\frac{\varepsilon}{2\pi}\zeta \,=\, \delta\langle\partial_1\mathcal{T}_\varepsilon\Psi_{\mathrm{app}}, w\rangle_{L^2} + \delta\langle\partial_1\mathcal{T}_\varepsilon\varphi, \Omega_{\mathrm{app}}\rangle_{L^2} + \delta^2\langle\partial_1\mathcal{T}_\varepsilon\varphi, w\rangle_{L^2} + \mathcal{O}\big(\varepsilon^{M+1} + \delta^2\varepsilon^2\big)\,, \tag{4.36}$$

which is the starting point of our analysis. Since $\Omega_{\rm app} \in \mathcal{Z}$, we already know from Lemma 3.8 that $\partial_1 \mathcal{T}_\varepsilon \Psi_{\rm app} = \mathcal{O}_{\mathcal{S}_*}(\varepsilon)$. Using the fact that W_ε^{-1} vanishes rapidly at infinity (see Proposition 4.4), we deduce that

$$|\langle \partial_1 \mathcal{T}_\varepsilon \Psi_{\rm app}, w\rangle_{L^2}| \,\le\, \big\|W_\varepsilon^{-1/2}\partial_1 \mathcal{T}_\varepsilon \Psi_{\rm app}\big\|_{L^2}\|w\|_{\mathcal{X}_\varepsilon} \,\lesssim\, \varepsilon\|w\|_{\mathcal{X}_\varepsilon}\,. \tag{4.37}$$

To bound the term involving $\Omega_{\rm app}$, we split the integration domain into the region I_ε and its complement I_ε^c, and we apply Hölder's inequality with $1/p + 1/q = 1$ and $q > 2$. We thus find

$$|\langle \partial_1 \mathcal{T}_\varepsilon\varphi, \Omega_{\rm app}\rangle_{L^2}| \,\lesssim\, \|\mathbb{1}_{\mathrm{I}_\varepsilon}(1+|\cdot|)^{-3}\partial_1\mathcal{T}_\varepsilon\varphi\|_{L^q}\|(1+|\cdot|)^3\Omega_{\rm app}\|_{L^p} + \|\partial_1\varphi\|_{L^q}\|\mathbb{1}_{\mathrm{I}_\varepsilon^c}\Omega_{\rm app}\|_{L^p}\,.$$

As $\Omega_{\rm app} \in \mathcal{Z}$ and $\mathrm{I}_\varepsilon^c \subset \{\xi\in\mathbb{R}^2\,;\,|\xi| > \varepsilon^{-\sigma_1}\}$ by Proposition 4.4, it is straightforward to verify that $\|\mathbb{1}_{\mathrm{I}_\varepsilon^c}\Omega_{\rm app}\|_{L^p} \le \exp(-c_*\varepsilon^{-2\sigma_1})$ for some $c_* > 0$. Therefore using Lemma 4.6 to bound the first term in the right-hand side and Lemma 4.5 for the second one, we arrive that

$$|\langle \partial_1 \mathcal{T}_\varepsilon\varphi, \Omega_{\rm app}\rangle_{L^2}| \,\lesssim\, \varepsilon^3\|w\|_{\mathcal{X}_\varepsilon}\,. \tag{4.38}$$

Finally, the nonlinear term can be estimated in a similar way:

$$|\langle \partial_1 \mathcal{T}_\varepsilon\varphi, w\rangle_{L^2}| \,\lesssim\, \|\mathbb{1}_{\mathrm{I}_\varepsilon}(1+|\cdot|)^{-3}\partial_1\mathcal{T}_\varepsilon\varphi\|_{L^q}\|(1+|\cdot|)^3 w\|_{L^p} + \|\mathbb{1}_{\mathrm{I}_\varepsilon^c}W_\varepsilon^{-1/2}\partial_1\mathcal{T}_\varepsilon\varphi\|_{L^2}\|w\|_{\mathcal{X}_\varepsilon}\,.$$

Since $1 < p < 2$, Hölder's inequality readily implies that $\|(1+|\cdot|)^3 w\|_{L^p} \lesssim \|w\|_{\mathcal{X}_\varepsilon}$. Thus, invoking Lemmas 4.5 and 4.6, and using again Hölder's inequality with $1/q + 1/r = 1/2$, we obtain

$$|\langle \partial_1 \mathcal{T}_\varepsilon\varphi, w\rangle_{L^2}| \,\lesssim\, \varepsilon^3\|w\|_{\mathcal{X}_\varepsilon}^2 + \|\mathbb{1}_{\mathrm{I}_\varepsilon^c}W_\varepsilon^{-1/2}\|_{L^r}\|\partial_1\varphi\|_{L^q}\|w\|_{\mathcal{X}_\varepsilon} \,\lesssim\, \varepsilon^3\|w\|_{\mathcal{X}_\varepsilon}^2\,. \tag{4.39}$$

Combining (4.36) with (4.37), (4.38) and (4.39), we arrive at (4.35). ☐

4.4. **The energy functional.** We now have all the necessary ingredients to define the energy functional that will allow us to control the vorticity perturbation w in (4.1). The idea is to mimic the Arnold quadratic form that was heuristically derived in Section 4.1.

Using the function space $\mathcal{X}_\varepsilon$ introduced in (4.23), we define

$$E_\varepsilon[w] \,=\, \frac{1}{2}\Big(\|w\|_{\mathcal{X}_\varepsilon}^2 + \langle \varphi - \mathcal{T}_\varepsilon\varphi\,, w\rangle_{L^2}\Big)\,, \qquad w \in \mathcal{X}_\varepsilon\,. \tag{4.40}$$

According to (4.22), the functional $E_\varepsilon[w]$ coincides with (4.15) when w is supported in the inner region I_ε. That region expands to the whole plane $\mathbb{R}^2$ as $\varepsilon \to 0$ and, in view of Proposition 4.4, the weight W_ε in (4.23) converges to $W_0 = A$, where $A(\xi)$ is defined in (3.51). Taking formally the limit $\varepsilon \to 0$ in (4.40), we thus obtain

$$E_0[w] \,=\, \frac{1}{2}\Big(\|w\|_{\mathcal{X}_0}^2 + \langle \varphi, w\rangle_{L^2}\Big)\,, \qquad w \in \mathcal{X}_0\,, \tag{4.41}$$

where $\mathcal{X}_0 = L^2(\mathbb{R}^2, W_0\,\mathrm{d}\xi)$. This limiting functional was studied in detail in [13], in connection with the stability of the Gaussian vortex Ω_0 for the Euler and the Navier-Stokes equations. A key property is that $E_0[w]$ is coercive on the subspace of functions $w \in \mathcal{X}_0$ satisfying the moments conditions (4.6). The main goal of this section is to establish a similar result for the functional E_ε when $\varepsilon > 0$ is sufficiently small.

Proposition 4.8. *Assume that $w \in \mathcal{X}_\varepsilon$ satisfies $\mathrm{M}[w] = \mathrm{m}_1[w] = \mathrm{m}_2[w] = 0$, where $\mathrm{M}, \mathrm{m}_1, \mathrm{m}_2$ are as in (1.4). Then, if $\varepsilon, \sigma_1 > 0$ are sufficiently small, there exists a constant $\kappa_1 \in (0,1)$ independent of ε such that*

$$E_\varepsilon[w] \,\ge\, \kappa_1\,\|w\|_{\mathcal{X}_\varepsilon}^2\,. \tag{4.42}$$

Proof. Following [13, Section 4.3], the idea is to compare E_ε with E_0. First of all, using Lemma 4.6 and arguing as in the proof of (4.39), it is not difficult to verify that

$$|\langle \mathcal{T}_\varepsilon\varphi, w\rangle_{L^2}| \,\lesssim\, \varepsilon^2\|w\|_{\mathcal{X}_\varepsilon}^2\,. \tag{4.43}$$

As a consequence, there exists a constant $C_0 > 0$ such that

$$E_\varepsilon[w] \,\ge\, E_\varepsilon^1[w] - C_0\,\varepsilon^2\|w\|_{\mathcal{X}_\varepsilon}^2, \qquad \text{where} \quad E_\varepsilon^1[w] \,=\, \frac{1}{2}\Big(\|w\|_{\mathcal{X}_\varepsilon}^2 + \langle\varphi, w\rangle_{L^2}\Big)\,. \tag{4.44}$$

To compare $E_\varepsilon^1[w]$ and $E_0[w]$, we decompose

$$w \,=\, \mathbb{1}_{\mathrm{I}_\varepsilon}w + \big(1 - \mathbb{1}_{\mathrm{I}_\varepsilon}\big)w \,=:\, w_{\rm in} + w_{\rm out}\,.$$

Denoting $\varphi_{\rm in} = \Delta^{-1} w_{\rm in}$ and $\varphi_{\rm out} = \Delta^{-1} w_{\rm out}$, where Δ^{-1} is defined according to (2.6), we find

$$\begin{aligned} E^1_\varepsilon[w] \,&=\, E^1_\varepsilon[w_{\rm in}] + E^1_\varepsilon[w_{\rm out}] + \frac{1}{2}\langle \varphi_{\rm in}, w_{\rm out}\rangle_{L^2} + \frac{1}{2}\langle \varphi_{\rm out}, w_{\rm in}\rangle_{L^2} \\ &=\, E^1_\varepsilon[w_{\rm in}] + E^1_\varepsilon[w_{\rm out}] + \langle \varphi_{\rm in}, w_{\rm out}\rangle_{L^2}\,, \end{aligned} \tag{4.45}$$

where in the second line we integrate by parts using $w_{\rm in} = \Delta\varphi_{\rm in}$ and $w_{\rm out} = \Delta\varphi_{\rm out}$. To estimate the last term in (4.45), we apply Hölder's inequality with $1/q + 1/p = 1$ and $q > 2$, $1 < p < 2$. Using (4.31) to bound the stream function $\varphi_{\rm in}$ and recalling that $w_{\rm out}$ is supported outside the region I_ε, we obtain

$$|\langle \varphi_{\rm in}, w_{\rm out}\rangle_{L^2}| \,\lesssim\, \|(1+|\cdot|)^{-1}\varphi_{\rm in}\|_{L^q}\|(1+|\cdot|)w_{\rm out}\|_{L^p} \,\lesssim\, \exp(-c_*\varepsilon^{-2\sigma_1})\|w\|^2_{\mathcal{X}_\varepsilon}\,, \tag{4.46}$$

for some $c_* > 0$ sufficiently small. The same estimate holds for $\langle \varphi_{\rm out}, w_{\rm out}\rangle_{L^2}$ too, by the same argument. It follows in particular from (4.44), (4.45), (4.46) that

$$E_\varepsilon[w] \,\ge\, E^1_\varepsilon[w_{\rm in}] + E^1_\varepsilon[w_{\rm out}] - C_1\,\varepsilon^2\|w\|^2_{\mathcal{X}_\varepsilon}\,, \tag{4.47}$$

for some constant $C_1 > 0$.

It remains to estimate from below the quantities $E^1_\varepsilon[w_{\rm in}]$ and $E^1_\varepsilon[w_{\rm out}]$ in (4.47). We have just observed that

$$E^1_\varepsilon[w_{\rm out}] \,\ge\, \frac{1}{2}\|w_{\rm out}\|^2_{\mathcal{X}_\varepsilon} - |\langle \varphi_{\rm out}, w_{\rm out}\rangle_{L^2}| \,\ge\, \frac{1}{2}\|w_{\rm out}\|^2_{\mathcal{X}_\varepsilon} - \exp(-c_*\varepsilon^{-2\sigma_1})\|w\|^2_{\mathcal{X}_\varepsilon}\,. \tag{4.48}$$

To bound the other term, the strategy is to compare $E^1_\varepsilon[w_{\rm in}]$ with $E_0[w_{\rm in}]$. Using estimate (4.26) in Proposition 4.4 and the fact that $w_{\rm in}$ is supported in region I_ε, we find that, for any $\gamma_2 < 2$, there exists $C_2 > 0$ such that

$$\big|\|w_{\rm in}\|^2_{\mathcal{X}_\varepsilon} - \|w_{\rm in}\|^2_{\mathcal{X}_0}\big| \,\le\, 2C_2\,\varepsilon^{\gamma_2}\|w\|^2_{\mathcal{X}_\varepsilon}\,, \quad \text{hence} \quad E^1_\varepsilon[w_{\rm in}] \,\ge\, E_0[w_{\rm in}] - C_2\,\varepsilon^{\gamma_2}\|w\|^2_{\mathcal{X}_\varepsilon}. \tag{4.49}$$

We next invoke Proposition 4.5 in [14], which provides the lower bound

$$E_0[w_{\rm in}] \,\ge\, \kappa_0\|w_{\rm in}\|^2_{\mathcal{X}_0} - C_3\big(\mathrm{M}[w_{\rm in}]^2 + \mathrm{m}_1[w_{\rm in}]^2 + \mathrm{m}_2[w_{\rm in}]^2\big)\,, \tag{4.50}$$

for some constants $\kappa_0 \in (0, 1/2)$ and $C_3 > 0$. At this point, it is important to observe that, although the first moments of $w_{\rm in}$ do not vanish exactly, they are extremely small. Indeed, since $\mathrm{M}[w_{\rm in}] = -\mathrm{M}[w_{\rm out}]$ by assumption, we have $|\mathrm{M}[w_{\rm in}]| \le \exp(-c_*\varepsilon^{-2\sigma_1})\|w\|_{\mathcal{X}_\varepsilon}$, and a similar estimate holds for $\mathrm{m}_1[w_{\rm in}]$ and $\mathrm{m}_2[w_{\rm in}]$. It thus follows from (4.49), (4.50) that

$$E_0[w_{\rm in}] \,\ge\, \kappa_0\|w_{\rm in}\|^2_{\mathcal{X}_\varepsilon} - C_4\,\varepsilon^{\gamma_2}\|w\|^2_{\mathcal{X}_\varepsilon}\,. \tag{4.51}$$

Finally, combining (4.47), (4.49) and (4.51), we obtain

$$E_\varepsilon[w] \,\ge\, \kappa_0\|w_{\rm in}\|^2_{\mathcal{X}_\varepsilon} + \frac{1}{2}\|w_{\rm out}\|^2_{\mathcal{X}_\varepsilon} - C_5\varepsilon^{\gamma_2}\|w\|^2_{\mathcal{X}_\varepsilon} \,\ge\, \big(\kappa_0 - C_5\,\varepsilon^{\gamma_2}\big)\|w\|^2_{\mathcal{X}_\varepsilon}\,,$$

which gives the desired estimate (4.42) if $\varepsilon > 0$ is small enough. □

4.5. **The energy identity.** We next compute the time evolution of the energy $E_\varepsilon[w]$ defined in (4.40), assuming that w is the solution of (4.10)-(4.11) with zero initial data.

Lemma 4.9. *Let w be the solution of* (4.10)-(4.11) *and $E_\varepsilon[w]$ be defined as in* (4.40)*. Then*

$$t\partial_t E_\varepsilon[w] + D_\varepsilon[w] \,=\, \mathrm{A} + \mathrm{F} + \mathrm{NL}\,, \tag{4.52}$$

where we define:

- *The diffusion functional*

$$D_\varepsilon[w] \,=\, -\frac{1}{2}\,\langle (t\partial_t W_\varepsilon)w\,,w\rangle_{L^2} - \langle \mathcal{L}w\,, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\rangle_{L^2}\,; \tag{4.53}$$

- *The advection terms*

$$\begin{aligned} \mathrm{A} \,=&\, \frac{1}{\delta}\,\big\langle \big\{\Phi^E_{\rm app}\,, w\big\} + \big\{\varphi - \mathcal{T}_\varepsilon\varphi\,, \Omega^E_{\rm app}\big\}\,, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\big\rangle_{L^2} \\ &+ \big\langle \big\{\Phi^{NS}_{\rm app}\,, w\big\} + \big\{\varphi - \mathcal{T}_\varepsilon\varphi\,, \Omega^{NS}_{\rm app}\big\}\,, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\big\rangle_{L^2}\,; \end{aligned} \tag{4.54}$$

- *The forcing terms generated by the remainder and the vertical speed*

$$\mathrm{F} \,=\, \frac{1}{\delta^2}\Big\langle \mathcal{R}_M + \frac{\varepsilon\zeta}{2\pi}\left\{\xi_1,\Omega_{\mathrm{app}}\right\}, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\Big\rangle_{L^2}\,; \tag{4.55}$$

- *The nonlinear terms*

$$\mathrm{NL} \,=\, \frac{1}{4\varepsilon}\langle w, \partial_1(\mathcal{T}_\varepsilon\varphi)\rangle_{L^2} + \Big\langle \left\{\varphi - \mathcal{T}_\varepsilon\varphi, w\right\} + \frac{\varepsilon\zeta}{2\pi\delta}\left\{\xi_1, w\right\}, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\Big\rangle_{L^2}\,. \tag{4.56}$$

Proof. A direct computation gives

$$\begin{aligned} t\partial_t E_\varepsilon[w] \,&=\, \langle t\partial_t w, W_\varepsilon w\rangle_{L^2} + \frac{1}{2}\langle w, (t\partial_t W_\varepsilon)w\rangle_{L^2} \\ &\quad + \frac{1}{2}\langle t\partial_t w, \varphi - \mathcal{T}_\varepsilon\varphi\rangle_{L^2} + \frac{1}{2}\langle w, t\partial_t(\varphi - \mathcal{T}_\varepsilon\varphi)\rangle_{L^2}\,. \end{aligned}$$

The last term in the right-hand side can be handled as in Section 4.1, but we have to be more careful here because $\mathcal{T}_\varepsilon$ is now a time-dependent operator. From the definition of $\mathcal{T}_\varepsilon$ in (2.10) and the fact that $t\partial_t\varepsilon = \varepsilon/2$, we obtain

$$t\partial_t\big(\mathcal{T}_\varepsilon\varphi\big)(\xi,t) \,=\, t\partial_t\big(\varphi(-\xi_1 - \varepsilon^{-1}(t), \xi_2, t)\big) \,=\, \mathcal{T}_\varepsilon\Big(t\partial_t\varphi + \frac{1}{2\varepsilon}\,\partial_1\varphi\Big)(\xi,t)\,.$$

Observing that $\mathcal{T}_\varepsilon\partial_1\varphi = -\partial_1\mathcal{T}_\varepsilon\varphi$ and using the identities (4.17), we thus find

$$\frac{1}{2}\langle w, t\partial_t(\varphi - \mathcal{T}_\varepsilon\varphi)\rangle_{L^2} \,=\, \frac{1}{2}\langle t\partial_t w\,, \varphi - \mathcal{T}_\varepsilon\varphi\rangle_{L^2} + \frac{1}{4\varepsilon}\langle w, \partial_1(\mathcal{T}_\varepsilon\varphi)\rangle_{L^2},$$

and it follows that

$$t\partial_t E_\varepsilon[w] \,=\, \langle t\partial_t w, (W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi)\rangle_{L^2} + \frac{1}{2}\langle w, (t\partial_t W_\varepsilon)w\rangle_{L^2} + \frac{1}{4\varepsilon}\langle w, \partial_1(\mathcal{T}_\varepsilon\varphi)\rangle_{L^2}\,. \tag{4.57}$$

We now replace the time derivative $t\partial_t w$ in (4.57) by its expression (4.10)–(4.11), and this generates exactly the quantities $D_\varepsilon[w]$, A, F and NL defined in (4.53), (4.54), (4.55) and (4.56). Note that we chose to include the last term in (4.57) in the nonlinearity NL, and the previous one in the diffusion functional $D_\varepsilon[w]$. ☐

Our goal in what follows is to estimate each term in (4.53)-(4.56). In Section 4.6 we show that the diffusion functional $D_\varepsilon[w]$ controls, roughly speaking, the H^1 analogue of the weighted L^2 norm $\|w\|_{\mathcal{X}_\varepsilon}$. The remaining terms in (4.52) can be estimated using the properties of the approximate solution Ω_{app} and of the weight function W_ε. The calculations are performed in Sections 4.7–4.9, and result in the following crucial energy estimate, which is the core of the proof of Theorem 4.1 and will be established in Section 4.10.

Proposition 4.10. *Fix $\sigma \in [0,1)$, take $M \in \mathbb{N}$ such that $M > (3+\sigma)/(1-\sigma)$, and let w be the solution of (4.4)–(4.6) with initial data $w|_{t=0} = 0$. Let $\sigma_1, \sigma_2, \gamma$ be as in (4.19), with σ_1 sufficiently small and σ_2 sufficiently large. Then for any $\kappa_* > 0$ there exist positive constants $s_* \in (0,1)$ and $C_* \geq 1$ such that, if $\delta > 0$ is sufficiently small, the quantities (4.54)–(4.56) satisfy*

$$|\mathrm{A}| \,\leq\, \delta^{s_*} D_\varepsilon[w]\,, \tag{4.58}$$

$$|\mathrm{F}| \,\leq\, C_*\big(\delta^{-4}\varepsilon^{2(M+1)} + \varepsilon^4\big) + \kappa_* D_\varepsilon[w]\big(1 + \sqrt{E_\varepsilon[w]}\big)\,, \tag{4.59}$$

$$|\mathrm{NL}| \,\leq\, \delta^{s_*} D_\varepsilon[w] + C_*(\sqrt{E_\varepsilon[w]} + E_\varepsilon[w])D_\varepsilon[w]\,, \tag{4.60}$$

as long as $t < T_{\mathsf{adv}}\delta^{-\sigma}$. As a consequence, the energy (4.40) satisfies the estimate

$$E_\varepsilon[w](t) + \frac{1}{2}\int_0^t \frac{D_\varepsilon[w](\tau)}{\tau}\mathrm{d}\tau \,\leq\, C_*\big(\delta^{-4}\varepsilon^{2(M+1)} + \varepsilon^4\big)\,, \qquad t \in \big(0, T_{\mathsf{adv}}\delta^{-\sigma}\big)\,. \tag{4.61}$$

In view of Propositions 3.2 and 4.8, estimate (4.61) implies that the solution w of (4.4)–(4.6) does not become much larger than the source term $\delta^{-2}\mathcal{R}_M$, which is of size $\delta^{-2}\varepsilon^{M+1} + \varepsilon^2$.

Remark 4.11. The assumption that $t \le T_{\mathsf{adv}}\delta^{-\sigma}$ translates into an upper bound on the aspect ratio $\varepsilon = \sqrt{\nu t}/d$ in terms of the inverse Reynolds number $\delta = \nu/\Gamma$. Indeed, if we define $s_0 > 0$ such that

$$s_0 \,=\, \frac{M-3}{M+1} - \sigma, \qquad \text{namely} \qquad 1-\sigma \,=\, \frac{4}{M+1} + s_0\,, \tag{4.62}$$

we then have

$$\varepsilon \,\le\, \delta^{\frac{1-\sigma}{2}} \,=\, \delta^{\frac{2}{M+1}+\frac{s_0}{2}}\,, \qquad \text{or} \qquad \delta^{-2}\varepsilon^{M+1} \,\le\, \delta^{\frac{M+1}{2}s_0}\,. \tag{4.63}$$

The constant s_* in (4.58)–(4.60) can be taken as a small multiple of s_0.

4.6. **The diffusion functional.** As is clear from Proposition 4.10, our strategy is to control the various terms in the right-hand side of the energy identity (4.52) using the diffusion functional $D_\varepsilon[w]$ defined in (4.53). We thus need an accurate lower bound on $D_\varepsilon[w]$, which can be obtained for $\varepsilon > 0$ small enough by exploiting the coercivity properties of a limiting quadratic form that was already studied in [13], see also [14] for a similar approach. To state our result, it is convenient to introduce the continuous function $\rho_\varepsilon : \mathbb{R}^2 \to \mathbb{R}_+$ defined by

$$\rho_\varepsilon(\xi) \,=\, \begin{cases} |\xi|\,, & \text{if} \quad |\xi| < \varepsilon^{-\sigma_1}, \\ \varepsilon^{-\sigma_1}\,, & \text{if} \quad \varepsilon^{-\sigma_1} \le |\xi| \le \varepsilon^{-\sigma_2}, \\ |\xi|^\gamma\,, & \text{if} \quad |\xi| > \varepsilon^{-\sigma_2}. \end{cases} \tag{4.64}$$

In agreement with (4.64), we denote $\rho_0(\xi) = |\xi|$ in the limiting case $\varepsilon = 0$.

Proposition 4.12. *If σ_1 is sufficiently small and σ_2 sufficiently large, there exists $\kappa_D > 0$ such that the following holds for $\varepsilon > 0$ small enough. If $w \in \mathcal{X}_\varepsilon$ satisfies $\rho_\varepsilon w \in \mathcal{X}_\varepsilon$, $\nabla w \in \mathcal{X}_\varepsilon^2$, and $\mathrm{M}[w] = \mathrm{m}_1[w] = \mathrm{m}_2[w] = 0$, then*

$$D_\varepsilon[w] \,\ge\, \kappa_D\bigl(\|\nabla w\|_{\mathcal{X}_\varepsilon}^2 + \|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2 + \|w\|_{\mathcal{X}_\varepsilon}^2\bigr)\,. \tag{4.65}$$

The proof of Proposition 4.12 is rather lengthy and can be divided into four main steps.

Step 1: Preliminaries. We recall the definitions of $\mathcal{L}$ in (3.10), W_ε in (4.22) and $D_\varepsilon[w]$ in (4.53). We first handle the term involving $\mathcal{T}_\varepsilon$ in (4.53). Since $\mathcal{L}w = \Delta w + \mathrm{div}(\xi w)/2$, a simple integration by parts yields

$$\langle \mathcal{L}w, \mathcal{T}_\varepsilon\varphi\rangle_{L^2} \,=\, -\,\langle \nabla w, \nabla(\mathcal{T}_\varepsilon\varphi)\rangle_{L^2} - \frac{1}{2}\,\langle w, \xi\cdot\nabla(\mathcal{T}_\varepsilon\varphi)\rangle_{L^2}\,.$$

To estimate the right-hand side, we can argue as in the proof of Lemma 4.7, by splitting the integration domain into the region I_ε and its complement I_ε^c. Using the bound (4.34) in the inner region and the rapid decay of the weight $W_\varepsilon^{-1/2}$ in the complement, we easily obtain

$$|\langle \mathcal{L}w, \mathcal{T}_\varepsilon\varphi\rangle_{L^2}| \,\lesssim\, \varepsilon^3\|w\|_{\mathcal{X}_\varepsilon}\bigl(\|w\|_{\mathcal{X}_\varepsilon} + \|\nabla w\|_{\mathcal{X}_\varepsilon}\bigr) \,\lesssim\, \varepsilon^3\bigl(\|w\|_{\mathcal{X}_\varepsilon}^2 + \|\nabla w\|_{\mathcal{X}_\varepsilon}^2\bigr)\,.$$

We deduce that there exists a constant $C_1 > 0$ such that

$$D_\varepsilon[w] \,\ge\, \mathcal{D}_{W_\varepsilon}[w] + \mathcal{D}_{\mathcal{L},\varepsilon}[w] - C_1\varepsilon^3\bigl(\|w\|_{\mathcal{X}_\varepsilon}^2 + \|\nabla w\|_{\mathcal{X}_\varepsilon}^2\bigr)\,, \tag{4.66}$$

where we denote

$$\mathcal{D}_{W_\varepsilon}[w] \,:=\, -\frac{1}{2}\,\langle (t\partial_t W_\varepsilon)w, w\rangle_{L^2}\,, \qquad \mathcal{D}_{\mathcal{L},\varepsilon}[w] \,:=\, -\,\langle \mathcal{L}w, W_\varepsilon w + \varphi\rangle_{L^2}\,. \tag{4.67}$$

Our next task is to estimate the quadratic form $\mathcal{D}_{W_\varepsilon}$ which involves the time derivative of the weight function. From the definition of W_ε in (4.22) we get

$$\mathcal{D}_{W_\varepsilon}[w] \,=\, -\frac{1}{2}\int_{\mathrm{I}_\varepsilon} t\partial_t(F'(\Omega_{\mathrm{app}}^E))|w|^2\,\mathrm{d}\xi - \frac{1}{2}\int_{\mathrm{II}_\varepsilon}\bigl(t\partial_t(\exp(\varepsilon(t)^{-2\sigma_1}/4))\bigr)|w|^2\,\mathrm{d}\xi\,.$$

Since $t\partial_t\varepsilon = \varepsilon/2$ and $\rho_\varepsilon = \varepsilon^{-\sigma_1}$ in II_ε by (4.24) and (4.64), a direct calculation shows that

$$-\frac{1}{2}\int_{\mathrm{II}_\varepsilon}\bigl(t\partial_t(\exp(\varepsilon(t)^{-2\sigma_1}/4))\bigr)|w|^2\,\mathrm{d}\xi \,=\, \frac{\sigma_1}{8}\int_{\mathrm{II}_\varepsilon}\rho_\varepsilon^2|w|^2 W_\varepsilon\,\mathrm{d}\xi \,=\, \frac{\sigma_1}{8}\,\|\mathbb{1}_{\mathrm{II}_\varepsilon}\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2\,. \tag{4.68}$$

To bound the other term, we recall that $F = F_0 + \varepsilon^2 F_2 + \cdots + \varepsilon^M F_M$, so that

$$t\partial_t\big(F'(\Omega_{\rm app}^E)\big) \,=\, \sum_{k=2}^M \frac{k}{2}\,\varepsilon^k F_k'\big(\Omega_{\rm app}^E\big) + F''(\Omega_{\rm app}^E)\big(t\partial_t\Omega_{\rm app}^E\big)\,.$$

According to (3.1) we have $\Omega_{\rm app}^E = \Omega_0 + \varepsilon^2\Omega_2^E + \cdots + \varepsilon^M\Omega_M^E$, so that $t\partial_t\Omega_{\rm app}^E = \mathcal{O}_{\mathcal{Z}}(\varepsilon^2)$ in the sense of Definition 3.4. Moreover, since $\Omega_0/2 \le \Omega_{\rm app}^E \le 2\Omega_0$ in I_ε by (3.61), we can proceed as in (4.28) to prove that, for some $N \in \mathbb{N}$,

$$\sum_{k=2}^M \big|F_k'\big(\Omega_{\rm app}^E(\xi)\big)\big| + \big|F''\big(\Omega_{\rm app}^E(\xi)\big)\big|\,\Omega_0(\xi) \,\lesssim\, (1+|\xi|)^N\,\Omega_0(\xi)^{-1}\,, \qquad \xi \in \mathrm{I}_\varepsilon\,.$$

It follows that

$$\int_{\mathrm{I}_\varepsilon}\big|t\partial_t(F'(\Omega_{\rm app}^E))\big||w|^2\,\mathrm{d}\xi \,\lesssim\, \varepsilon^2\int_{\mathrm{I}_\varepsilon}(1+|\xi|)^N \mathrm{e}^{|\xi|^2/4}|w|^2\,\mathrm{d}\xi \,\lesssim\, \varepsilon^{2-\sigma_1(N+2)}\|w\|_{\mathcal{X}_\varepsilon}^2\,, \tag{4.69}$$

for a possibly larger integer N. In the last inequality we used Proposition 4.4 which implies that $W_\varepsilon(\xi) \approx W_0(\xi) \approx \mathrm{e}^{|\xi|^2/4}(1+|\xi|)^{-2}$ in region I_ε. Combining (4.66), (4.68), (4.69) and taking σ_1 sufficiently small, we arrive at

$$D_\varepsilon[w] \,\ge\, \frac{\sigma_1}{8}\,\|\mathbb{1}_{\mathrm{II}_\varepsilon}\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2 + \mathcal{D}_{\mathcal{L},\varepsilon}[w] - C_2\varepsilon\big(\|w\|_{\mathcal{X}_\varepsilon}^2 + \|\nabla w\|_{\mathcal{X}_\varepsilon}^2\big)\,, \tag{4.70}$$

for some constant $C_2 > 0$.

Finally we derive a more explicit expression of the diffusive quadratic form $\mathcal{D}_{\mathcal{L},\varepsilon}[w]$. Using the definition (4.67) and integrating by parts, we easily find

$$\mathcal{D}_{\mathcal{L},\varepsilon}[w] \,=\, \langle\nabla w, \nabla(W_\varepsilon w) + \nabla\varphi\rangle_{L^2} + \frac{1}{2}\,\langle w, \xi\cdot\nabla(W_\varepsilon w + \varphi)\rangle_{L^2}\,. \tag{4.71}$$

We also observe that $\langle\nabla w, \nabla\varphi\rangle_{L^2} = -\|w\|_{L^2}^2$ and $\langle w, \xi\cdot\nabla\varphi\rangle_{L^2} = 0$. The first equality is a simple integration by parts, and the second one is conveniently obtained by using polar coordinates r, θ defined by $\xi_1 = r\cos\theta$, $\xi_2 = r\sin\theta$. Indeed, since both operators $r^{-1}\partial_r$ and ∂_θ are (at least formally) skew-symmetric in the space $L^2_r := L^2((0,+\infty)\times(0,2\pi)\,, r\mathrm{d}r\mathrm{d}\theta)$, we have

$$\langle w, \xi\cdot\nabla\varphi\rangle_{L^2} \,=\, \langle\Delta\varphi, r\partial_r\varphi\rangle_{L^2_r} = \Big\langle\frac{1}{r}\partial_r(r\partial_r\varphi), r\partial_r\varphi\Big\rangle_{L^2_r} + \Big\langle\partial_{\theta\theta}\varphi, \frac{1}{r}\partial_r\varphi\Big\rangle_{L^2_r} \,=\, 0\,. \tag{4.72}$$

On the other hand, since $w(\xi\cdot\nabla w) = \mathrm{div}(\xi w^2/2) - w^2$, it is not difficult to verify that

$$\begin{aligned}\langle w, \xi\cdot\nabla(W_\varepsilon w)\rangle_{L^2} \,&=\, \langle w, (\xi\cdot\nabla W_\varepsilon)w\rangle_{L^2} + \langle w, (\xi\cdot\nabla w)W_\varepsilon\rangle_{L^2} \\ &=\, \frac{1}{2}\,\langle w, (\xi\cdot\nabla W_\varepsilon)w\rangle_{L^2} - \|w\|_{\mathcal{X}_\varepsilon}^2\,.\end{aligned} \tag{4.73}$$

Thus we can write the quantity $\mathcal{D}_{\mathcal{L},\varepsilon}[w]$ in the equivalent form

$$\mathcal{D}_{\mathcal{L},\varepsilon}[w] \,=\, \|\nabla w\|_{\mathcal{X}_\varepsilon}^2 + \langle w, \nabla w\cdot\nabla W_\varepsilon\rangle_{L^2} + \frac{1}{4}\,\langle w, w(\xi\cdot\nabla W_\varepsilon)\rangle_{L^2} - \|w\|_{L^2}^2 - \frac{1}{2}\,\|w\|_{\mathcal{X}_\varepsilon}^2\,. \tag{4.74}$$

Step 2: Decomposition of the diffusive quadratic form. The expression (4.74) is the analogue of the quadratic form $Q_\varepsilon[w]$ appearing in [14, Section 4.7]. Taking formally the limit $\varepsilon \to 0$, we obtain

$$\mathcal{D}_{\mathcal{L},0}[w] \,=\, \|\nabla w\|_{\mathcal{X}_0}^2 + \langle w, \nabla w\cdot\nabla W_0\rangle_{L^2} + \frac{1}{4}\,\langle w, w(\xi\cdot\nabla W_0)\rangle_{L^2} - \|w\|_{L^2}^2 - \frac{1}{2}\,\|w\|_{\mathcal{X}_0}^2\,, \tag{4.75}$$

which is exactly the quadratic form $Q_0[w]$ studied [13, 14]. In particular, it is established in [14, Proposition 4.14] that, for any $w \in \mathcal{X}_0$ with $\rho_0 w \in \mathcal{X}_0$ and $\nabla w \in \mathcal{X}_0^2$, the following holds

$$\mathcal{D}_{\mathcal{L},0}[w] \,\ge\, \kappa_2\big(\|\nabla w\|_{\mathcal{X}_0}^2 + \|\rho_0 w\|_{\mathcal{X}_0}^2 + \|w\|_{\mathcal{X}_0}^2\big) - C_3\big(\mathrm{M}[w]^2 + \mathrm{m}_1[w]^2 + \mathrm{m}_2[w]^2\big)\,, \tag{4.76}$$

for some constants $\kappa_2 \in (0, 1/2)$ and $C_3 > 0$.

To obtain a similar lower bound on $\mathcal{D}_{\mathcal{L},\varepsilon}[w]$ for $\varepsilon > 0$, the idea is to decompose the vorticity w so as to single out the contribution of the inner region. As opposed to what we have done in the proof of Proposition 4.8, we use here a smooth cut-off since our quadratic form involves first-order derivatives. Let $\chi_1, \chi_2 : \mathbb{R}^2 \to [0,1]$ be smooth functions satisfying $\chi_1^2 + \chi_2^2 = 1$, and such that $\chi_1(\xi) = 1$ when $|\xi| \le \varepsilon^{-\sigma_1}/2$ and $\chi_1(\xi) = 0$ when $|\xi| \ge \varepsilon^{-\sigma_1}$. In addition, we assume that $|\nabla\chi_i| \lesssim \varepsilon^{\sigma_1}$. Then, with a slight abuse in notation, we define

$$w_{\rm in} \,:=\, \chi_1 w\,, \qquad w_{\rm out} \,:=\, \chi_2 w\,, \qquad \text{so that} \quad w^2 \,=\, w_{\rm in}^2 + w_{\rm out}^2\,. \tag{4.77}$$

We obviously have $\nabla(\chi_i w) = (\nabla\chi_i)w + \chi_i\nabla w$ and $\chi_1\nabla\chi_1 + \chi_2\nabla\chi_2 = 0$, hence

$$|\nabla w|^2 \,=\, |\nabla w_{\rm in}|^2 + |\nabla w_{\rm out}|^2 - (|\nabla\chi_1|^2 + |\nabla\chi_2|^2)w^2\,. \tag{4.78}$$

Since

$$\langle \nabla w\cdot\nabla W_\varepsilon, w\rangle_{L^2} \,=\, \frac{1}{2}\int_{\mathbb{R}^2}\nabla(w^2)\cdot\nabla W_\varepsilon\,{\rm d}\xi\,,$$

and since the remaining terms in $\mathcal{D}_{\mathcal{L},\varepsilon}$ do not involve derivatives of w, it follows from (4.77), (4.78) that

$$\begin{aligned}\mathcal{D}_{\mathcal{L},\varepsilon}[w] \,&=\, \mathcal{D}_{\mathcal{L},\varepsilon}[w_{\rm in}] + \mathcal{D}_{\mathcal{L},\varepsilon}[w_{\rm out}] - \Big\|\sqrt{|\nabla\chi_1|^2 + |\nabla\chi_2|^2}\,w\Big\|_{\mathcal{X}_\varepsilon}^2\,,\\ &\ge\, \mathcal{D}_{\mathcal{L},\varepsilon}[w_{\rm in}] + \mathcal{D}_{\mathcal{L},\varepsilon}[w_{\rm out}] - C_4\varepsilon^{2\sigma_1}\|w\|_{\mathcal{X}_\varepsilon}^2\,,\end{aligned} \tag{4.79}$$

where in the last inequality we use the assumption that $|\nabla\chi_i| \lesssim \varepsilon^{\sigma_1}$.

Step 3: Contribution of the inner region. We now decompose

$$\mathcal{D}_{\mathcal{L},\varepsilon}[w_{\rm in}] \,=\, \mathcal{D}_{\mathcal{L},0}[w_{\rm in}] + \mathcal{D}_{\rm err}[w_{\rm in}]\,,$$

where

$$\begin{aligned}\mathcal{D}_{\rm err}[w_{\rm in}] \,:=\, &\big(\|\nabla w_{\rm in}\|_{\mathcal{X}_\varepsilon}^2 - \|\nabla w_{\rm in}\|_{\mathcal{X}_0}^2\big) + \langle w_{\rm in}, \nabla w_{\rm in}\cdot\nabla(W_\varepsilon - W_0)\rangle_{L^2}\\ &+ \frac{1}{4}\,\langle w_{\rm in}, w_{\rm in}(\xi\cdot\nabla(W_\varepsilon - W_0)\rangle_{L^2} - \frac{1}{2}\big(\|w_{\rm in}\|_{\mathcal{X}_\varepsilon}^2 - \|w_{\rm in}\|_{\mathcal{X}_0}^2\big)\,.\end{aligned}$$

Since $w_{\rm in}$ is supported in the region ${\rm I}_\varepsilon$, the bounds (4.26) in Proposition 4.4 readily imply that

$$\big|\mathcal{D}_{\rm err}[w_{\rm in}]\big| \,\lesssim\, \varepsilon^{\gamma_2}\big(\|\nabla w_{\rm in}\|_{\mathcal{X}_0}^2 + \|\rho_0 w_{\rm in}\|_{\mathcal{X}_0}^2 + \|w_{\rm in}\|_{\mathcal{X}_0}^2\big)\,,$$

for some $\gamma_2 \in (1,2)$. To estimate $\mathcal{D}_{\mathcal{L},0}[w_{\rm in}]$, we use the lower bound (4.76). As was observed in the proof of Proposition 4.8, our assumptions on w imply that the moments of $w_{\rm in}$ are extremely small, namely

$${\rm M}[w_{\rm in}]^2 + {\rm m}_1[w_{\rm in}]^2 + {\rm m}_2[w_{\rm in}]^2 \,\le\, \exp(-c_*\varepsilon^{-2\sigma_1})\|w\|_{\mathcal{X}_\varepsilon}^2 \,\le\, \varepsilon\|w\|_{\mathcal{X}_\varepsilon}^2\,.$$

Altogether, if $\varepsilon > 0$ small enough, we obtain

$$\begin{aligned}\mathcal{D}_{\mathcal{L},\varepsilon}[w_{\rm in}] \,&\ge\, \kappa_2\big(\|\nabla w_{\rm in}\|_{\mathcal{X}_0}^2 + \|\rho_0 w_{\rm in}\|_{\mathcal{X}_0}^2 + \|w_{\rm in}\|_{\mathcal{X}_0}^2\big) - C_3\varepsilon\|w\|_{\mathcal{X}_\varepsilon}^2 - \big|\mathcal{D}_{\rm err}[w_{\rm in}]\big|\\ &\ge\, \frac{\kappa_2}{2}\big(\|\nabla w_{\rm in}\|_{\mathcal{X}_\varepsilon}^2 + \|\rho_\varepsilon w_{\rm in}\|_{\mathcal{X}_\varepsilon}^2 + \|w_{\rm in}\|_{\mathcal{X}_\varepsilon}^2\big) - C_3\varepsilon\|w\|_{\mathcal{X}_\varepsilon}^2\,.\end{aligned} \tag{4.80}$$

In the second line, we used Proposition 4.4 again to compare the norms of $\mathcal{X}_0$ and $\mathcal{X}_\varepsilon$.

Step 4: Contribution of the outer region. It remains to estimate the term $\mathcal{D}_{\mathcal{L},\varepsilon}[w_{\rm out}]$ in (4.79), which is more complicated because $w_{\rm out}$ is nonzero in all three regions ${\rm I}_\varepsilon$, ${\rm II}_\varepsilon$, and ${\rm III}_\varepsilon$. We recall, however, that $w_{\rm out}$ is supported in the domain where $|\xi| \ge \varepsilon^{-\sigma_1}/2$, which implies

$$\|w_{\rm out}\|_{L^2} \,\le\, \|w_{\rm out}\|_{\mathcal{X}_\varepsilon} \,\lesssim\, \varepsilon^{\sigma_1}\,\|\rho_\varepsilon w_{\rm out}\|_{\mathcal{X}_\varepsilon} \,\le\, \varepsilon^{\sigma_1}\,\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}\,. \tag{4.81}$$

On the other hand, using Young's inequality, we find

$$\big|\langle w_{\rm out}, \nabla w_{\rm out}\cdot\nabla W_\varepsilon\rangle_{L^2}\big| \,\le\, \frac{3}{8}\,\big\|(W_\varepsilon^{-1}\nabla W_\varepsilon)w_{\rm out}\big\|_{\mathcal{X}_\varepsilon}^2 + \frac{2}{3}\,\|\nabla w_{\rm out}\|_{\mathcal{X}_\varepsilon}^2\,.$$

In view of (4.74) we thus have, for some constant $C_5 > 0$,

$$\mathcal{D}_{\mathcal{L},\varepsilon}[w_{\rm out}] \,\ge\, \frac{1}{3}\,\|\nabla w_{\rm out}\|_{\mathcal{X}_\varepsilon}^2 + \mathcal{I}_\varepsilon[w_{\rm out}] - C_5\varepsilon^{2\sigma_1}\,\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2\,, \tag{4.82}$$

where

$$\mathcal{I}_\varepsilon[w_{\rm out}] \,:=\, \frac{1}{4}\,\big\langle w_{\rm out}^2, \xi\cdot\nabla W_\varepsilon\big\rangle_{L^2} - \frac{3}{8}\,\big\|(W_\varepsilon^{-1}\nabla W_\varepsilon)w_{\rm out}\big\|_{\mathcal{X}_\varepsilon}^2\,.$$

Our goal is to find a lower bound on $\mathcal{I}_\varepsilon[w_{\rm out}]$ in regions I_ε and III_ε, keeping in mind that $\nabla W_\varepsilon = 0$ in II_ε. In the outer region III_ε we have $W_\varepsilon = \mathrm{e}^{|\xi|^{2\gamma}/4}$ with $\gamma = \sigma_1/\sigma_2 < 1/2$, so that

$$\xi\cdot\nabla W_\varepsilon \,=\, \frac{\gamma}{2}\,|\xi|^{2\gamma}\,W_\varepsilon \,=\, \frac{\gamma}{2}\,\rho_\varepsilon^2\,W_\varepsilon\,, \qquad |\nabla W_\varepsilon| \,\le\, \frac{\gamma}{2}\,|\xi|^{\gamma-1}\rho_\varepsilon W_\varepsilon \,\le\, \frac{\gamma}{2}\,\varepsilon^{\sigma_2/2}\rho_\varepsilon W_\varepsilon\,. \tag{4.83}$$

It follows that

$$\mathcal{I}_\varepsilon\bigl[\mathbb{1}_{\mathrm{III}_\varepsilon} w_{\rm out}\bigr] \,=\, \frac{\gamma}{8}\,\|\mathbb{1}_{\mathrm{III}_\varepsilon}\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2 + \mathcal{O}\bigl(\varepsilon^{\sigma_2}\,\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2\bigr)\,. \tag{4.84}$$

In the inner region I_ε, we can use Proposition 4.4 to compare the weight W_ε with the function $W_0 = A$ defined in (3.51). This gives the estimates

$$\bigl|\langle \mathbb{1}_{\mathrm{I}_\varepsilon} w_{\rm out}^2, \xi\cdot\nabla W_\varepsilon\rangle_{L^2} - \langle \mathbb{1}_{\mathrm{I}_\varepsilon} w_{\rm out}^2, \xi\cdot\nabla W_0\rangle_{L^2}\bigr| \,\lesssim\, \varepsilon^{\gamma_2+\sigma_1}\,\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2\,, \tag{4.85}$$

and

$$\Bigl|\bigl\|\mathbb{1}_{\mathrm{I}_\varepsilon}\bigl(W_\varepsilon^{-1}\nabla W_\varepsilon\bigr)w_{\rm out}\bigr\|_{\mathcal{X}_\varepsilon}^2 - \bigl\|\mathbb{1}_{\mathrm{I}_\varepsilon}\bigl(W_0^{-1}\nabla W_0\bigr)w_{\rm out}\bigr\|_{\mathcal{X}_0}^2\Bigr| \,\lesssim\, \varepsilon^{\gamma_2+2\sigma_1}\,\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2\,. \tag{4.86}$$

Moreover, using the explicit expression (3.51), we easily find

$$\xi\cdot\nabla W_0(\xi) \,=\, \frac{|\xi|^2}{2}\,W_0(\xi) - 2W_0(\xi) + 2\,, \qquad \xi\in\mathbb{R}^2\,.$$

In view of (4.81), we deduce that

$$\begin{aligned} \frac{1}{4}\langle \mathbb{1}_{\mathrm{I}_\varepsilon} w_{\rm out}^2, \xi\cdot\nabla W_0\rangle_{L^2} \,&=\, \frac{1}{8}\,\bigl\|\mathbb{1}_{\mathrm{I}_\varepsilon}\rho_0 w_{\rm out}\bigr\|_{\mathcal{X}_0}^2 + \mathcal{O}\bigl(\varepsilon^{2\sigma_1}\,\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2\bigr)\,,\\ \frac{3}{8}\,\bigl\|\mathbb{1}_{\mathrm{I}_\varepsilon} w_{\rm out}\bigl(W_0^{-1}\nabla W_0\bigr)\bigr\|_{\mathcal{X}_0}^2 \,&=\, \frac{3}{32}\,\bigl\|\mathbb{1}_{\mathrm{I}_\varepsilon}\rho_0 w_{\rm out}\bigr\|_{\mathcal{X}_0}^2 + \mathcal{O}\bigl(\varepsilon^{2\sigma_1}\,\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2\bigr)\,. \end{aligned} \tag{4.87}$$

Combining (4.85), (4.86), (4.87), and using (4.26) again, we obtain

$$\begin{aligned} \mathcal{I}_\varepsilon\bigl[\mathbb{1}_{\mathrm{I}_\varepsilon} w_{\rm out}\bigr] \,&\ge\, \frac{1}{32}\,\bigl\|\mathbb{1}_{\mathrm{I}_\varepsilon}\rho_0 w_{\rm out}\bigr\|_{\mathcal{X}_0}^2 - C\bigl(\varepsilon^{2\sigma_1} + \varepsilon^{\gamma_2+\sigma_1}\bigr)\,\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2\\ &\ge\, \frac{1}{32}\,\bigl\|\mathbb{1}_{\mathrm{I}_\varepsilon}\rho_\varepsilon w_{\rm out}\bigr\|_{\mathcal{X}_\varepsilon}^2 - C_6\bigl(\varepsilon^{2\sigma_1} + \varepsilon^{\gamma_2}\bigr)\,\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2\,, \end{aligned} \tag{4.88}$$

for some $C_6 > 0$. Finally, in view of (4.82), (4.84), (4.88), we arrive at

$$\mathcal{D}_{\mathcal{L},\varepsilon}[w_{\rm out}] \ge \frac{1}{3}\,\|\nabla w_{\rm out}\|_{\mathcal{X}_\varepsilon}^2 + \frac{1}{32}\,\bigl\|\mathbb{1}_{\mathrm{I}_\varepsilon}\rho_\varepsilon w_{\rm out}\bigr\|_{\mathcal{X}_\varepsilon}^2 + \frac{\gamma}{8}\,\|\mathbb{1}_{\mathrm{III}_\varepsilon}\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2 - C_7\varepsilon^{2\sigma_1}\,\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2\,, \tag{4.89}$$

for some $C_7 > 0$. Here we used the fact that $2\sigma_1 < 1 < \min(\gamma_2,\sigma_2)$.

It is now a simple task to conclude the proof of Proposition 4.12. If we combine (4.70), (4.79), (4.80), (4.89), we see that there exist $\kappa_3 > 0$ and $C_8 > 0$ such that

$$\begin{aligned} \mathcal{D}_{\mathcal{L},\varepsilon}[w] \,\ge\, &\kappa_3\bigl(\|\nabla w_{\rm in}\|_{\mathcal{X}_\varepsilon}^2 + \|\nabla w_{\rm out}\|_{\mathcal{X}_\varepsilon}^2 + \|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2 + \|w\|_{\mathcal{X}_\varepsilon}^2\bigr)\\ &- C_8\,\varepsilon^{2\sigma_1}\bigl(\|\nabla w\|_{\mathcal{X}_\varepsilon}^2 + \|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}^2 + \|w\|_{\mathcal{X}_\varepsilon}^2\bigr)\,. \end{aligned}$$

Since $|\nabla w_{\rm in}|^2 + |\nabla w_{\rm out}|^2 \ge |\nabla w|^2$ by (4.78), we arrive at (4.65) by taking $\varepsilon > 0$ small enough. □

4.7. **The advection terms.** In this section we estimate the advection terms defined in (4.54), which are potentially dangerous because of the large prefactor δ^{-1}. In the inner region I_ε, we exploit crucial cancellations related to the structure of the energy functional (4.40), which were explained in an informal way in Section 4.1. In the other regions the dominant term in the energy is the weighted enstrophy $\frac{1}{2}\|w\|_{\mathcal{X}_\varepsilon}^2$, and the influence of the large advection terms can be controlled by an appropriate choice of the weight function W_ε.

Starting from (4.54) we decompose $\mathrm{A} = \delta^{-1}\bigl(\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3\bigr) + \mathrm{A}_{NS}$, where

$$\mathrm{A}_1 \,=\, \bigl\langle \bigl\{\Phi_{\rm app}^E, w\bigr\}\,, W_\varepsilon w\bigr\rangle_{L^2}\,, \tag{4.90}$$

$$\mathrm{A}_2 \,=\, \bigl\langle \bigl\{\varphi - \mathcal{T}_\varepsilon\varphi, \Omega_{\rm app}^E\bigr\}\,, W_\varepsilon w\bigr\rangle_{L^2} + \bigl\langle \mathbb{1}_{\mathrm{I}_\varepsilon}\,\bigl\{\Phi_{\rm app}^E, w\bigr\}\,, \varphi - \mathcal{T}_\varepsilon\varphi\bigr\rangle_{L^2} \tag{4.91}$$

$$\mathrm{A}_3 \,=\, \bigl\langle \mathbb{1}_{\mathrm{II}_\varepsilon\cup\mathrm{III}_\varepsilon}\,\bigl\{\Phi_{\rm app}^E, w\bigr\}\,, \varphi - \mathcal{T}_\varepsilon\varphi\bigr\rangle_{L^2}\,, \tag{4.92}$$

$$\mathrm{A}_{NS} \,=\, \bigl\langle \bigl\{\Phi_{\rm app}^{NS}, w\bigr\} + \bigl\{\varphi - \mathcal{T}_\varepsilon\varphi, \Omega_{\rm app}^{NS}\bigr\}\,, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\bigr\rangle_{L^2}\,. \tag{4.93}$$

Here we use the fact that $\big\langle\{\varphi-\mathcal{T}_\varepsilon\varphi,\Omega^E_{\rm app}\},\varphi-\mathcal{T}_\varepsilon\varphi\big\rangle_{L^2}=0$. This is a consequence of standard identities such as

$$\langle\{f,g\},h\rangle_{L^2} \,=\, -\langle\{f,h\},g\rangle_{L^2}\,, \qquad \langle\{f,g\},hg\rangle \,=\, -\frac{1}{2}\,\langle\{f,h\}g,g\rangle\,, \tag{4.94}$$

which are repeatedly used in the sequel. Our goal is to prove the following set of estimates.

Lemma 4.13. *There exists an integer $N>0$ depending only on M such that the quantities (4.90)–(4.93) satisfy*

$$\delta^{-1}|\mathrm{A}_1| \,\lesssim\, \delta^{-1}\varepsilon^{M+1-\sigma_1 N}\|w\|^2_{\mathcal{X}_\varepsilon}+\delta^{-1}\varepsilon^{1+\sigma_2}\|\rho_\varepsilon w\|^2_{\mathcal{X}_\varepsilon}\,, \tag{4.95}$$

$$\delta^{-1}|\mathrm{A}_2| \,\lesssim\, \delta^{-1}\varepsilon^{M+1}\big(\|w_\varepsilon\|^2_{\mathcal{X}_\varepsilon}+\|\nabla w\|^2_{\mathcal{X}_\varepsilon}\big)\,, \tag{4.96}$$

$$\delta^{-1}|\mathrm{A}_3| \,\lesssim\, \delta^{-1}\exp(-c_*\varepsilon^{-2\sigma_1})\big(\|w\|^2_{\mathcal{X}_\varepsilon}+\|\nabla w\|^2_{\mathcal{X}_\varepsilon}\big)\,, \tag{4.97}$$

$$|\mathrm{A}_{NS}| \,\lesssim\, \varepsilon\big(\|w\|^2_{\mathcal{X}_\varepsilon}+\|\nabla w\|^2_{\mathcal{X}_\varepsilon}\big)\,, \tag{4.98}$$

for some $c_*>0$ sufficiently small. Consequently, under the assumptions of Proposition 4.10, there exists $s_*\in(0,1)$ such that estimate (4.58) holds.

Proof. We start with the bound for A_1. Since $W_\varepsilon=F'(\Omega^E_{\rm app})$ in region I_ε and $\nabla W_\varepsilon=0$ in region II_ε, we find using (4.94)

$$\begin{aligned}\mathrm{A}_1 \,&=\, \big\langle\big\{\Phi^E_{\rm app},w\big\}\,,W_\varepsilon w\big\rangle_{L^2} \,=\, -\frac{1}{2}\,\big\langle\big\{\Phi^E_{\rm app},W_\varepsilon\big\}\,w,w\big\rangle_{L^2}\\ &=\, -\frac{1}{2}\,\big\langle\mathbb{1}_{\mathrm{I}_\varepsilon}\big\{\Phi^E_{\rm app},F'(\Omega^E_{\rm app})\big\}\,w,w\big\rangle_{L^2}-\frac{1}{2}\,\big\langle\mathbb{1}_{\mathrm{III}_\varepsilon}\big\{\Phi^E_{\rm app},W_\varepsilon\big\}\,w,w\big\rangle_{L^2}\,.\end{aligned}$$

The first term in the right-hand side is clearly unchanged if we replace $\Phi^E_{\rm app}$ by the quantity $\Theta=\Phi^E_{\rm app}+F(\Omega^E_{\rm app})$ which is introduced in (4.20). We thus obtain

$$\mathrm{A}_1 \,=\, -\frac{1}{2}\,\langle\mathbb{1}_{\mathrm{I}_\varepsilon}\{\Theta,W_\varepsilon\}\,w,w\rangle_{L^2}-\frac{1}{2}\,\big\langle\mathbb{1}_{\mathrm{III}_\varepsilon}\big\{\Phi^E_{\rm app},W_\varepsilon\big\}\,w,w\big\rangle_{L^2}\,. \tag{4.99}$$

To control the first term in (4.99), we use the bound (4.21) on $\nabla\Theta$ and the estimate (4.26) on ∇W_ε in region I_ε, which give $|\{\Theta,W_\varepsilon\}|\lesssim\varepsilon^{M+1}(1+|\xi|)^N W_\varepsilon$ for some $N\in\mathbb{N}$. Since $|\xi|\le2\varepsilon^{-\sigma_1}$ in that region, we infer that

$$\big|\langle\mathbb{1}_{\mathrm{I}_\varepsilon}\{\Theta,W_\varepsilon\}\,w,w\rangle_{L^2}\big| \,\lesssim\, \varepsilon^{M+1-\sigma_1 N}\|w\|^2_{\mathcal{X}_\varepsilon}\,. \tag{4.100}$$

To bound the second term in (4.99), we recall that $W_\varepsilon=\exp(|\xi|^{2\gamma}/4)$ in region III_ε, so that

$$\big|\big\langle\mathbb{1}_{\mathrm{III}_\varepsilon}\big\{\Phi^E_{\rm app},W_\varepsilon\big\}\,w,w\big\rangle_{L^2}\big| \,\lesssim\, \int_{\{|\xi|>\varepsilon^{-\sigma_2}\}}|\nabla\Phi^E_{\rm app}|\,|\xi|^{2\gamma-1}\,W_\varepsilon|w|^2\,\mathrm{d}\xi\,.$$

In view of (3.49) we have $|\nabla\Phi^E_{\rm app}|\le|\nabla\Psi^E_{\rm app}|+|\nabla\mathcal{T}_\varepsilon\Psi^E_{\rm app}|+C\varepsilon$, and applying estimate (4.32) with $w=\Omega^E_{\rm app}$ we obtain $|\nabla\Psi^E_{\rm app}(\xi)|\le C(1+|\xi|)^{-1}$ and $|\nabla\mathcal{T}_\varepsilon\Psi^E_{\rm app}(\xi)|\le C(1+|\xi+\varepsilon^{-1}e_1|)^{-1}$. It follows that $\mathbb{1}_{\{|\xi|>\varepsilon^{-\sigma_2}\}}|\nabla\Phi^E_{\rm app}(\xi)|\lesssim\varepsilon+\varepsilon^{\sigma_2}\lesssim\varepsilon$. Since $\rho_\varepsilon(\xi)=|\xi|^\gamma$ in region III_ε, we conclude that

$$\big|\big\langle\mathbb{1}_{\mathrm{III}_\varepsilon}\big\{\Phi^E_{\rm app},W_\varepsilon\big\}\,w,w\big\rangle_{L^2}\big| \,\lesssim\, \varepsilon^{1+\sigma_2}\|\rho_\varepsilon w\|^2_{\mathcal{X}_\varepsilon}\,, \tag{4.101}$$

and estimate (4.95) follows directly from (4.100), (4.101).

We next consider the term A_2 defined by (4.91). Using (4.94) and the Leibniz rule, we find

$$\big\langle\big\{\varphi-\mathcal{T}_\varepsilon\varphi,\Omega^E_{\rm app}\big\}\,,W_\varepsilon w\big\rangle_{L^2} \,=\, \big\langle\big\{\Omega^E_{\rm app},w\big\}\,W_\varepsilon,\varphi-\mathcal{T}_\varepsilon\varphi\big\rangle_{L^2}+\big\langle\big\{\Omega^E_{\rm app},W_\varepsilon\big\}\,w,\varphi-\mathcal{T}_\varepsilon\varphi\big\rangle_{L^2}\,.$$

Only the region III_ε contributes to the last term, since $W_\varepsilon=F'(\Omega^E_{\rm app})$ in I_ε and $\nabla W_\varepsilon=0$ in II_ε. As for the first term, we observe that

$$\mathbb{1}_{\mathrm{I}_\varepsilon}\big\{\Omega^E_{\rm app},w\big\}W_\varepsilon \,=\, \mathbb{1}_{\mathrm{I}_\varepsilon}\big\{F(\Omega^E_{\rm app}),w\big\} \,=\, \mathbb{1}_{\mathrm{I}_\varepsilon}\big\{\Theta,w\big\}-\mathbb{1}_{\mathrm{I}_\varepsilon}\big\{\Phi^E_{\rm app},w\big\}\,,$$

where $\Theta=\Phi^E_{\rm app}+F(\Omega^E_{\rm app})$. We thus obtain the following alternative expression

$$\mathrm{A}_2 \,=\, \big\langle\mathbb{1}_{\mathrm{I}_\varepsilon}\{\Theta,w\}+\mathbb{1}_{\mathrm{II}_\varepsilon\cup\mathrm{III}_\varepsilon}\{\Omega^E_{\rm app},w\}W_\varepsilon+\mathbb{1}_{\mathrm{III}_\varepsilon}\big\{\Omega^E_{\rm app},W_\varepsilon\big\}\,w,\varphi-\mathcal{T}_\varepsilon\varphi\big\rangle_{L^2}\,. \tag{4.102}$$

By Hölder's inequality we have

$$|\langle \mathbb{1}_{\mathrm{I}_\varepsilon}\{\Theta,\, w\}, \varphi - \mathcal{T}_\varepsilon\varphi\rangle_{L^2}| \;\le\; \|\mathbb{1}_{\mathrm{I}_\varepsilon} W_\varepsilon^{-1/2}\nabla\Theta\|_{L^2}\, \|\varphi - \mathcal{T}_\varepsilon\varphi\|_{L^\infty}\, \|\nabla w\|_{\mathcal{X}_\varepsilon}\,.$$

Using the bound (4.21) on $\nabla\Theta$ and the rapid decay of the function $W_\varepsilon^{-1/2}$, we see that the first factor in the right-hand side is of order ε^{M+1}. Moreover, we know from Lemma 4.6 that $\|\varphi - \mathcal{T}_\varepsilon\varphi\|_{L^\infty} \le 2\|\varphi\|_{L^\infty} \lesssim \|w\|_{\mathcal{X}_\varepsilon}$. This gives

$$|\,\langle \mathbb{1}_{\mathrm{I}_\varepsilon}\,\{\Theta, w\}\,, \varphi - \mathcal{T}_\varepsilon\varphi\rangle_{L^2}\,| \;\lesssim\; \varepsilon^{M+1}\|w_\varepsilon\|_{\mathcal{X}_\varepsilon}\|\nabla w\|_{\mathcal{X}_\varepsilon}\,. \tag{4.103}$$

To treat the remaining terms in (4.102), we use the estimate $|\nabla\Omega^E_{\mathrm{app}}|(W_\varepsilon + |\nabla W_\varepsilon|) \lesssim 1$, which follows from the definition (4.22) of W_ε and from the properties of the approximate solution $\Omega^E_{\mathrm{app}} \in \mathcal{Z}$. Proceeding as above we find

$$\begin{aligned}|\,\big\langle \mathbb{1}_{\mathrm{II}_\varepsilon\cup\mathrm{III}_\varepsilon}\{w, \Omega^E_{\mathrm{app}}\}W_\varepsilon + \mathbb{1}_{\mathrm{III}_\varepsilon}\,\{W_\varepsilon, \Omega^E_{\mathrm{app}}\}\,w, \varphi - \mathcal{T}_\varepsilon\varphi\big\rangle_{L^2}\,| \\ \lesssim\; \exp(-c_*\varepsilon^{-2\sigma_1})\|w\|_{\mathcal{X}_\varepsilon}(\|w\|_{\mathcal{X}_\varepsilon} + \|\nabla w\|_{\mathcal{X}_\varepsilon})\,,\end{aligned} \tag{4.104}$$

for some $c_* > 0$ sufficiently small, and (4.96) is a direct consequence of (4.103), (4.104). We estimate A_3 in a similar way:

$$|\mathrm{A}_3| \;\le\; \|\mathbb{1}_{\mathrm{II}_\varepsilon\cup\mathrm{III}_\varepsilon} W_\varepsilon^{-1/2}\nabla\Phi^E_{\mathrm{app}}\|_{L^2}\, \|w\|_{\mathcal{X}_\varepsilon}\|\nabla w\|_{\mathcal{X}_\varepsilon} \;\lesssim\; \exp(-c_*\varepsilon^{-2\sigma_1})\|w\|_{\mathcal{X}_\varepsilon}\|\nabla w\|_{\mathcal{X}_\varepsilon}\,,$$

where in the last inequality we used the fact that $\Phi^E_{\mathrm{app}} \in \mathcal{S}_*$. This proves (4.97).

We now consider the viscous term A_{NS}. Since $\Omega^{NS}_{\mathrm{app}} = \mathcal{O}_{\mathcal{Z}}(\varepsilon^2)$ we find using (4.31)

$$|\,\big\langle\{\varphi - \mathcal{T}_\varepsilon\varphi, \Omega^{NS}_{\mathrm{app}}\}\,, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\big\rangle_{L^2}\,| \;=\; |\,\big\langle\{\varphi - \mathcal{T}_\varepsilon\varphi, \Omega^{NS}_{\mathrm{app}}\}\,, W_\varepsilon w\big\rangle_{L^2}\,| \;\lesssim\; \varepsilon^2\|w\|^2_{\mathcal{X}_\varepsilon}\,. \tag{4.105}$$

Similarly, using the definition (4.9) and the bound (4.32), one can verify that $\|\nabla\Phi^{NS}_{\mathrm{app}}\|_{L^\infty} \lesssim \varepsilon$, which gives

$$\big|\big\langle\{\Phi^{NS}_{\mathrm{app}}, w\}\,, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\big\rangle_{L^2}\big| \;\lesssim\; \varepsilon\,\|\nabla w\|_{\mathcal{X}_\varepsilon}\|w\|_{\mathcal{X}_\varepsilon}\,, \tag{4.106}$$

and (4.98) results from the combination of (4.105), (4.106).

Finally, if we assume that $\sigma_1 > 0$ is small enough so that $\sigma_1 N \le (M+1)/2$, and $\sigma_2 > 1$ large enough so that $1 + \sigma_2 \ge (M+1)/2$, it follows from (4.63) that

$$\delta^{-1}\varepsilon^{M+1-\sigma_1 N} + \delta^{-1}\varepsilon^{1+\sigma_2} \;\lesssim\; \delta^{-1}\varepsilon^{\frac{M+1}{2}} \;\le\; \delta^{s_*}\,,$$

provided $0 < s_* \le s_0(M+1)/4$. Under this assumption the bound (4.58) is a straightforward consequence of (4.95)–(4.98) and Proposition 4.12. □

4.8. **The forcing term.** To control the forcing term F defined in (4.55), we exploit the estimate of the remainder $\mathcal{R}_M$ given in Proposition 3.2, and we use the argument sketched at the end of Section 4.1 to handle the term involving the correction ζ to the vertical speed. For the latter, we can rely on the bound established in Lemma 4.7.

Lemma 4.14. *Let* F *be the forcing term* (4.55)*. Under the assumptions of Proposition 4.10, for any $\kappa_* > 0$, there exists a constant $C_* > 0$ such that*

$$|\mathrm{F}| \;\le\; C_*\big(\delta^{-4}\varepsilon^{2(M+1)} + \varepsilon^4\big) + \kappa_* D_\varepsilon[w]\big(1 + \sqrt{E_\varepsilon[w]}\big)\,. \tag{4.107}$$

Proof. Using the bound (3.5) on $\mathcal{R}_M$ and applying Lemma 4.6 to estimate the stream function $\varphi - \mathcal{T}_\varepsilon\varphi$, we easily obtain

$$|\,\langle\mathcal{R}_M, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\rangle_{L^2}\,| \;\lesssim\; \big(\varepsilon^{M+1} + \delta^2\varepsilon^2\big)\|w\|_{\mathcal{X}_\varepsilon}\,. \tag{4.108}$$

To handle the term in F involving ζ, we proceed as in Section 4.1. We first observe that

$$\big\langle\{\xi_1, \Omega^E_{\mathrm{app}}\}, \varphi - \mathcal{T}_\varepsilon\varphi\big\rangle_{L^2} = \big\langle\partial_2\Delta\Psi^E_{\mathrm{app}}, \varphi - \mathcal{T}_\varepsilon\varphi\big\rangle_{L^2} = \big\langle\partial_2\Psi^E_{\mathrm{app}}, w - \mathcal{T}_\varepsilon w\big\rangle_{L^2} = \big\langle\partial_2\Phi^E_{\mathrm{app}}, w\big\rangle_{L^2}\,,$$

because $\partial_2\Phi^E_{\mathrm{app}} = \partial_2(\Psi^E_{\mathrm{app}} - \mathcal{T}_\varepsilon\Psi^E_{\mathrm{app}})$ by (3.49). It follows that

$$\begin{aligned}\langle\{\xi_1, \Omega_{\mathrm{app}}\}, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\rangle_{L^2} \;=\; &\big\langle(W_\varepsilon\partial_2\Omega^E_{\mathrm{app}} + \partial_2\Phi^E_{\mathrm{app}}), w\big\rangle_{L^2} \\ &+ \delta\,\big\langle\partial_2\Omega^{NS}_{\mathrm{app}}, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\big\rangle_{L^2}\,.\end{aligned} \tag{4.109}$$

To estimate the first term in the right-hand side, we split the domain of integration. In the inner region I_ε, we have $W_\varepsilon \partial_2 \Omega^E_{\mathrm{app}} = \partial_2 F(\Omega^E_{\mathrm{app}})$, so using the definition (4.20) and the bound (4.21) we obtain

$$\left|\left\langle \mathbb{1}_{\mathrm{I}_\varepsilon} \partial_2\big(F(\Omega^E_{\mathrm{app}}) + \Phi^E_{\mathrm{app}}\big), w\right\rangle_{L^2}\right| = \left|\langle \mathbb{1}_{\mathrm{I}_\varepsilon}\partial_2\Theta, w\rangle_{L^2}\right| \lesssim \varepsilon^{M+1}\|w\|_{\mathcal{X}_\varepsilon}\,. \tag{4.110}$$

Outside I_ε we rely on the bound $W_\varepsilon|\partial_2\Omega^E_{\mathrm{app}}| + |\partial_2\Phi^E_{\mathrm{app}}| \lesssim 1$, which gives

$$\left|\left\langle \mathbb{1}_{\{\mathrm{II}_\varepsilon\cup\mathrm{III}_\varepsilon\}}(W_\varepsilon\partial_2\Omega^E_{\mathrm{app}} + \partial_2\Phi^E_{\mathrm{app}}), w\right\rangle_{L^2}\right| \lesssim \exp(-c_*\varepsilon^{-2\sigma_1})\|w\|_{\mathcal{X}_\varepsilon}\,, \tag{4.111}$$

for some $c_* > 0$ sufficiently small. To treat the last term in the right-hand side of (4.109), we remark that $\Omega^{NS}_{\mathrm{app}} = \mathcal{O}_{\mathcal{Z}}(\varepsilon^2)$ and we apply Lemma 4.6 again to arrive at

$$\left|\left\langle \partial_2\Omega^{NS}_{\mathrm{app}}, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\right\rangle_{L^2}\right| \lesssim \varepsilon^2\|w\|_{\mathcal{X}_\varepsilon}\,. \tag{4.112}$$

Summarizing, in view of (4.108)–(4.112), the forcing term (4.55) satisfies

$$|\mathrm{F}| \lesssim \Big(\delta^{-2}\varepsilon^{M+1} + \varepsilon^2 + \frac{\varepsilon|\zeta|}{\delta^2}\left(\varepsilon^{M+1} + \delta\varepsilon^2\right)\Big)\|w\|_{\mathcal{X}_\varepsilon}\,.$$

Using in addition the bound (4.35) on $\varepsilon\zeta$, we thus find

$$|\mathrm{F}| \lesssim \big(\delta^{-2}\varepsilon^{M+1} + \varepsilon^2\big)\|w\|_{\mathcal{X}_\varepsilon} + \big(\delta^{-1}\varepsilon^{M+2} + \varepsilon^3\big)\|w\|^2_{\mathcal{X}_\varepsilon} + \big(\varepsilon^{M+4} + \delta\varepsilon^5\big)\|w\|^3_{\mathcal{X}_\varepsilon}\,. \tag{4.113}$$

Under the assumptions of Proposition 4.10, we know that $\delta^{-2}\varepsilon^{M+1} \le \delta^{s_*} \ll 1$, and the other coefficients in (4.113) are small too. Therefore, applying Young's inequality to the first term in the right-hand side, and using the lower bound (4.65) on the dissipation functional D_ε, we obtain the desired inequality (4.107). □

4.9. **The nonlinear term.** Finally we estimate the nonlinear term in (4.56).

Lemma 4.15. *Let* NL *be the nonlinearity defined in* (4.56). *Under the assumptions of Proposition 4.10, there exist constants $C_* > 0$ and $0 < s_* < 1$ such that*

$$|\mathrm{NL}| \le \delta^{s_*} D_\varepsilon[w] + C_*\big(\sqrt{E_\varepsilon[w]} + E_\varepsilon[w]\big)D_\varepsilon[w]\,. \tag{4.114}$$

Proof. First of all, applying Cauchy-Schwarz's inequality, we find

$$\left|\langle w, \partial_1(\mathcal{T}_\varepsilon\varphi)\rangle_{L^2}\right| \le \|W_\varepsilon^{-\frac12}\nabla\mathcal{T}_\varepsilon\varphi\|_{L^2}\|w\|_{\mathcal{X}_\varepsilon}\,.$$

To estimate the right-hand side, we consider separately the regions I_ε and I^c_ε. Applying Hölder's inequality with $1/q + 1/p = 1/2$ and $q > 2$, and using Lemmas 4.5–4.6 to bound the stream function $\mathcal{T}_\varepsilon\varphi$, we obtain

$$\begin{aligned}
\|\mathbb{1}_{\mathrm{I}_\varepsilon}W_\varepsilon^{-\frac12}\nabla\mathcal{T}_\varepsilon\varphi\|_{L^2} &\lesssim \|\mathbb{1}_{\mathrm{I}_\varepsilon}(1+|\cdot|)^3W_\varepsilon^{-\frac12}\|_{L^p}\|\mathbb{1}_{\mathrm{I}_\varepsilon}(1+|\cdot|)^{-3}\nabla\mathcal{T}_\varepsilon\varphi\|_{L^q} \lesssim \varepsilon^3\|w\|_{\mathcal{X}_\varepsilon}\,,\\
\|\mathbb{1}_{\mathrm{I}^c_\varepsilon}W_\varepsilon^{-\frac12}\nabla\mathcal{T}_\varepsilon\varphi\|_{L^2} &\lesssim \|\mathbb{1}_{\mathbb{1}_{\mathrm{I}^c_\varepsilon}}W_\varepsilon^{-\frac12}\|_{L^p}\|\nabla\varphi\|_{L^q} \lesssim \exp(-c_*\varepsilon^{-2\sigma_1})\|w\|_{\mathcal{X}_\varepsilon}\,,
\end{aligned}$$

for some $c_* > 0$ sufficiently small. Altogether this gives

$$\frac{1}{4\varepsilon}\left|\langle w, \partial_1(\mathcal{T}_\varepsilon\varphi)\rangle_{L^2}\right| \lesssim \varepsilon^2\|w\|^2_{\mathcal{X}_\varepsilon} \lesssim \varepsilon^2 D_\varepsilon[w]\,, \tag{4.115}$$

where in the last step we used the lower bound (4.65) in Proposition 4.12.

Next, in view of the Poisson bracket identities (4.94), we have

$$\langle\{\varphi - \mathcal{T}_\varepsilon\varphi, w\}, W_\varepsilon w + \varphi - \mathcal{T}_\varepsilon\varphi\rangle_{L^2} = -\frac12\langle\{\varphi - \mathcal{T}_\varepsilon\varphi, W_\varepsilon\}\, w, w\rangle_{L^2}\,.$$

Here we use the bound (4.32) in Lemma 4.5 to control the streamfunction, and we make the following observations concerning ∇W_ε. In region I_ε, thanks to Proposition 4.4, we can approximate ∇W_ε by ∇W_0, which gives $|\nabla W_\varepsilon(\xi)| \lesssim |\nabla W_0(\xi)| + W_0(\xi) \lesssim \rho_\varepsilon(\xi)W_\varepsilon(\xi)$, for $|\xi| \le 2\varepsilon^{-\sigma_1}$. In region II_ε we have $\nabla W_\varepsilon = 0$ whereas in region III_ε we know that $|\nabla W_\varepsilon| \lesssim |\xi|^{\gamma-1}\rho_\varepsilon W_\varepsilon$. Overall, we obtain

$$\begin{aligned}
\left|\langle\{\varphi - \mathcal{T}_\varepsilon\varphi, W_\varepsilon\}\, w, w\rangle_{L^2}\right| &\lesssim \|\nabla\varphi\|_{L^\infty}\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}\|w\|_{\mathcal{X}_\varepsilon}\\
&\lesssim \big(\|\nabla w\|^{\frac12}_{\mathcal{X}_\varepsilon} + \|w\|^{\frac12}_{\mathcal{X}_\varepsilon}\big)\|\rho_\varepsilon w\|_{\mathcal{X}_\varepsilon}\|w\|^{\frac32}_{\mathcal{X}_\varepsilon} \lesssim \sqrt{E_\varepsilon[w]}D_\varepsilon[w]\,,
\end{aligned} \tag{4.116}$$

where in the last inequality we used Proposition 4.12.

Finally, for the remaining term in NL, we use Lemma 4.6 again together with the estimate on ζ in Lemma 4.7. We thus obtain

$$\begin{aligned}\frac{\varepsilon|\zeta|}{2\pi\delta}\,\big|\langle\{\xi_1,w\},W_\varepsilon w+\varphi-\mathcal{T}_\varepsilon\varphi\rangle_{L^2}\big| &\lesssim \frac{\varepsilon}{\delta}\,|\zeta|\,\|\nabla w\|_{\mathcal{X}_\varepsilon}\|w\|_{\mathcal{X}_\varepsilon}\\ &\lesssim \big(\delta^{-1}\varepsilon^{M+1}+\delta\varepsilon^2+\varepsilon\|w\|_{\mathcal{X}_\varepsilon}+\delta\varepsilon^3\|w\|_{\mathcal{X}_\varepsilon}^2\big)\|\nabla w\|_{\mathcal{X}_\varepsilon}\|w\|_{\mathcal{X}_\varepsilon}\\ &\lesssim \big(\delta^{-1}\varepsilon^{M+1}+\delta\varepsilon^2+\varepsilon\sqrt{E_\varepsilon[w]}+\delta\varepsilon^3E_\varepsilon[w]\big)D_\varepsilon[w]\,.\end{aligned}\tag{4.117}$$

Under the assumptions of Proposition 4.10, we have $\delta^{-1}\varepsilon^{M+1}+\delta\varepsilon^2\le\delta^{s_*}$ for some $s_*>0$, see Remark 4.11. Therefore, combining the the bounds (4.115), (4.116) and (4.117), we see that the nonlinearity NL defined by (4.56) satisfies estimate (4.114) for some constant $C_*>0$. ☐

4.10. **Conclusion of the proof.** We are now in position to conclude the proof Proposition 4.10, hence also of Theorem 4.1. Under the assumptions of Proposition 4.10, we consider the solution w of (4.4)–(4.6) with initial data $w|_{t=0}=0$. Thanks to Proposition 4.8, we can use the energy $E_\varepsilon[w]$ defined in (4.40) to control the size of w in the function space $\mathcal{X}_\varepsilon$. The energy evolves in time according to (4.52), where the quantities A, F, NL in the right-hand side satisfy the estimates (4.58), (4.59), (4.60) for some constants $s_*>0$, $\kappa_*>0$, and $C_*\ge 1$. Without loss of generality, we assume that $\kappa_*\le 1/8$ and we take $\delta>0$ small enough so that $\delta^{s_*}\le 1/16$.

As long as the energy satisfies $E_\varepsilon[w]\le\mathcal{E}_0:=\min\{1,(16C_*)^{-2}\}$, we have

$$\begin{aligned}t\partial_tE_\varepsilon[w]+D_\varepsilon[w] &\le 2\delta^{s_*}D_\varepsilon[w]+2\kappa_*D_\varepsilon[w]+2C_*\sqrt{E_\varepsilon[w]}D_\varepsilon[w]+C_*\big(\delta^{-4}\varepsilon^{2(M+1)}+\varepsilon^4\big)\\ &\le \frac18\,D_\varepsilon[w]+\frac14\,D_\varepsilon[w]+\frac18\,D_\varepsilon[w]+C_*\big(\delta^{-4}\varepsilon^{2(M+1)}+\varepsilon^4\big)\,,\end{aligned}$$

so that $t\partial_tE_\varepsilon[w]+D_\varepsilon[w]/2\le C_*\big(\delta^{-4}\varepsilon^{2(M+1)}+\varepsilon^4\big)$. Recalling that $\varepsilon=\varepsilon(t)=\sqrt{\nu t}/d$, we can integrating this differential inequality on the time interval $(0,t)$ to obtain the bound

$$E_\varepsilon[w(t)]+\frac12\int_0^t\frac{D_\varepsilon[w(\tau)]}{\tau}\,\mathrm{d}\tau \le \frac{C_*}{M+1}\,\delta^{-4}\varepsilon^{2(M+1)}+\frac{C_*}{2}\,\varepsilon^4 \le C_*\big(\delta^{-4}\varepsilon^{2(M+1)}+\varepsilon^4\big)\,.$$

According to (4.63), the right-hand side is smaller than a fractional power of δ as long as $t\in(0,T_{\mathsf{adv}}\delta^{-\sigma})$, so we can make it smaller than the fixed number $\mathcal{E}_0$ by taking $\delta>0$ small enough. In that case, the argument above holds for all $t\in(0,T_{\mathsf{adv}}\delta^{-\sigma})$, which concludes the proof of (4.61).

With Proposition 4.10 at hand, it is straightforward to conclude the proof of Theorem 4.1. Indeed, as already noticed, the estimate (4.7) follows directly from (4.61) in view of (4.42), and the formula (4.8) for the vertical speed is a consequence of the decomposition (4.2), the expression (3.45) of ζ_{app}, and the bound on the correction ζ in Lemma 4.7. ☐

Appendix A. Appendix

A.1. **Homogeneous polynomials.** For any integer $n\in\mathbb{N}$ we denote by $Q_n^c(x)$ and $Q_n^s(x)$ the n-homogeneous polynomials on $\mathbb{R}^2$ defined by

$$Q_n^c(x) = \mathrm{Re}\big(x_1+ix_2\big)^n\,,\qquad Q_n^s(x) = \mathrm{Im}\big(x_1+ix_2\big)^n\,.\tag{A.1}$$

Note that $Q_n^c(\cos\theta,\sin\theta)=\cos(n\theta)$ and $Q_n^s(\cos\theta,\sin\theta)=\sin(n\theta)$ by De Moivre's formula. For the first values of n we have

$$\begin{aligned}&Q_0^c(x) = 1\,,\quad Q_1^c(x) = x_1\,,\quad Q_2^c(x) = x_1^2-x_2^2\,,\quad Q_3^c(x) = x_1^3-3x_1x_2^2\,,\\ &Q_0^s(x) = 0\,,\quad Q_1^s(x) = x_2\,,\quad Q_2^s(x) = 2x_1x_2\,,\quad Q_3^s(x) = 3x_1^2x_2-x_2^3\,.\end{aligned}$$

Assume that $x\in\mathbb{R}^2$ satisfies $|x|^2=x_1^2+x_2^2<1$. Denoting $z=x_1+ix_2$ we find

$$\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n}\,Q_n^c(x) = \mathrm{Re}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n}\,z^n = \mathrm{Re}\log(1+z) = \frac12\log(1+2x_1+|x|^2)\,,\tag{A.2}$$

which is (3.19). Since $\partial_1 Q_n^c(x) = n\, Q_{n-1}^c(x)$ and $\partial_2 Q_n^c(x) = -n\, Q_{n-1}^s(x)$, we can differentiate both sides of (A.2) to arrive at the formulas

$$\sum_{n=0}^{\infty}(-1)^n Q_n^c(x) \,=\, \frac{1+x_1}{1+2x_1+|x|^2}\,, \qquad \sum_{n=1}^{\infty}(-1)^{n-1} Q_n^s(x) \,=\, \frac{x_2}{1+2x_1+|x|^2}\,, \tag{A.3}$$

which were used several times in the previous sections.

A.2. **Inverting the diffusion operator.** This section is devoted to the proof of Lemma 3.5. Given $\kappa > 0$ and $f \in \mathcal{Y}$, we have the Laplace formula

$$(\kappa - \mathcal{L})^{-1} f \,=\, \int_0^\infty e^{-\kappa\tau} S(\tau) f \,\mathrm{d}\tau\,, \tag{A.4}$$

where $S(\tau) = \exp(\tau\mathcal{L})$ is the analytic semigroup in $\mathcal{Y}$ generated by the selfadjoint operator $\mathcal{L}$, see [15, Appendix A]. It is well known that

$$\big(S(\tau)f\big)(\xi) \,=\, \frac{1}{4\pi a(\tau)}\int_{\mathbb{R}^2} \mathrm{e}^{-|\xi-\eta\,\mathrm{e}^{-\tau/2}|^2/4a(\tau)} f(\eta)\,\mathrm{d}\eta\,, \qquad \tau > 0\,, \quad \xi \in \mathbb{R}^2\,, \tag{A.5}$$

where $a(\tau) = 1 - \mathrm{e}^{-\tau}$. Multiplying both sides by $\mathrm{e}^{|\xi|^2/4}$ and using the identity

$$\frac{1}{4}\big(|\xi|^2 - |\eta|^2\big) - \frac{1}{4a(\tau)}\,|\xi - \eta\,\mathrm{e}^{-\tau/2}|^2 \,=\, -\frac{1}{4a(\tau)}\,|\eta - \xi\,\mathrm{e}^{-\tau/2}|^2\,,$$

we obtain the equivalent formula

$$G(\xi,\tau) \,:=\, \mathrm{e}^{|\xi|^2/4}\big(S(\tau)f\big)(\xi) \,=\, \frac{1}{4\pi a(\tau)}\int_{\mathbb{R}^2} \mathrm{e}^{-|\eta-\xi\,\mathrm{e}^{-\tau/2}|^2/4a(\tau)} g(\eta)\,\mathrm{d}\eta\,,$$

where $g(\eta) = \mathrm{e}^{|\eta|^2/4} f(\eta)$.

We assume henceforth that $f \in \mathcal{Z}$, hence $g \in \mathcal{S}_*(\mathbb{R}^2)$. In particular $|g(\eta)| \le C(1+|\eta|)^N$ for some $C > 0$ and some $N \in \mathbb{N}$, so that

$$|G(\xi,\tau)| \,\le\, \frac{C}{a(\tau)}\int_{\mathbb{R}^2}(1+|\eta|)^N \mathrm{e}^{-|\eta-\xi\,\mathrm{e}^{-\tau/2}|^2/4a(\tau)}\,\mathrm{d}\eta \,\le\, C(1+|\xi|)^N\,.$$

Returning to (A.4) we thus obtain

$$\mathrm{e}^{|\xi|^2/4}\big|\big((\kappa-\mathcal{L})^{-1}f\big)(\xi)\big| \,\le\, \int_0^\infty e^{-\kappa\tau}|G(\xi,\tau)|\,\mathrm{d}\tau \,\le\, C\kappa^{-1}(1+|\xi|)^N\,. \tag{A.6}$$

The derivative $\partial^\alpha(\kappa-\mathcal{L})^{-1}f$ can be estimated in the same way, for any multi-index $\alpha \in \mathbb{N}^2$. If $0 < \tau < 1$ we differentiate (A.5) and integrate by parts to obtain

$$\partial_\xi^\alpha\big(S(\tau)f\big)(\xi) \,=\, \frac{\mathrm{e}^{|\alpha|\tau/2}}{4\pi a(\tau)}\int_{\mathbb{R}^2} \mathrm{e}^{-|\xi-\eta\,\mathrm{e}^{-\tau/2}|^2/4a(\tau)}\partial_\eta^\alpha f(\eta)\,\mathrm{d}\eta\,, \qquad 0 < \tau < 1\,.$$

Since $f \in \mathcal{Z}$ we have $e^{|\eta|^2/4}|\partial_\eta^\alpha f(\eta)| \le C(1+|\eta|)^{N'}$ for some $N' \in \mathbb{N}$ (depending on α), and proceeding as before we deduce that $e^{|\xi|^2/4}|\partial_\xi^\alpha\big(S(\tau)f\big)(\xi)| \le C(1+|\xi|)^{N'}$. If $\tau \ge 1$, we observe that $1 - \mathrm{e}^{-1} \le a(\tau) \le 1$ and we differentiate (A.5) to obtain

$$\big|\partial_\xi^\alpha\big(S(\tau)f\big)(\xi)\big| \,\le\, C\int_{\mathbb{R}^2}\big(1+|\xi-\eta\,\mathrm{e}^{-\tau/2}|\big)^{|\alpha|}\,\mathrm{e}^{-|\xi-\eta\,\mathrm{e}^{-\tau/2}|^2/4a(\tau)}|f(\eta)|\,\mathrm{d}\eta\,,$$

which gives $e^{|\xi|^2/4}|\partial_\xi^\alpha\big(S(\tau)f\big)(\xi)| \le C(1+|\xi|)^{N+|\alpha|}$. Thus using (A.4) we find, as in (A.6),

$$\mathrm{e}^{|\xi|^2/4}\big|\partial^\alpha\big((\kappa-\mathcal{L})^{-1}f\big)(\xi)\big| \,\le\, \int_0^\infty e^{-\kappa\tau}\mathrm{e}^{|\xi|^2/4}\big|\partial_\xi^\alpha\big(S(\tau)f\big)(\xi)\big|\,\mathrm{d}\tau \,\le\, C\kappa^{-1}(1+|\xi|)^{N''}\,, \tag{A.7}$$

for some $N'' \in \mathbb{N}$. This shows that $(\kappa-\mathcal{L})^{-1}f \in \mathcal{Z}$. □

A.3. **Inverting the advection operator.** In this section, for the reader's convenience, we recall some known results about the (partial) inverse of the operator Λ introduced in (3.12). In particular, we prove the second half of Proposition 3.6. We work in the function space $\mathcal{Y}$ defined by (3.6), and we recall that Λ leaves invariant the direct sum decomposition (3.8), so that it is sufficient to consider the restriction of Λ to each subspace $\mathcal{Y}_n$. To do that, we use polar coordinates $\xi = (r\cos\theta, r\sin\theta)$ in $\mathbb{R}^2$, and we define the radially symmetric functions

$$v_0(r) \,=\, \frac{1}{2\pi r^2}\bigl(1 - \mathrm{e}^{-r^2/4}\bigr)\,, \qquad g(r) \,=\, \frac{1}{8\pi}\,\mathrm{e}^{-r^2/4}\,, \qquad h(r) \,=\, \frac{g(r)}{v_0(r)} \,=\, \frac{r^2/4}{\mathrm{e}^{r^2/4}-1}\,. \tag{A.8}$$

We observe that $\partial_r\Omega_0 = -rg$ and $\partial_r\Psi_0 = rv_0$, where Ω_0, Ψ_0 are given by (2.15), (2.16).

Since Λ vanishes on the subspace $\mathcal{Y}_0$ of radially symmetric functions, we assume henceforth that $n \ge 1$. If $\Omega \in \mathcal{Y}_n$ takes the form $\Omega = w(r)\cos(n\theta)$ for some function $w : \mathbb{R}_+ \to \mathbb{R}$, the associated stream function is $\Psi = \varphi(r)\cos(n\theta)$, where φ denotes the unique solution of the ordinary differential equation

$$\varphi''(r) + \frac{1}{r}\,\varphi'(r) - \frac{n^2}{r^2}\varphi(r) \,=\, w(r)\,, \qquad r > 0\,, \tag{A.9}$$

satisfying the boundary conditions $\varphi(r) = \mathcal{O}(r^n)$ as $r \to 0$ and $\varphi(r) = \mathcal{O}(r^{-n})$ as $r \to +\infty$. A direct calculation shows that

$$\Lambda\Omega \,=\, \bigl\{\Psi_0\,,\,\Omega\bigr\} + \bigl\{\Psi\,,\,\Omega_0\bigr\} \,=\, -n\bigl(v_0 w + \varphi g\bigr)\sin(n\theta)\,. \tag{A.10}$$

Similarly, if $\Omega = w(r)\sin(n\theta)$, then $\Psi = \varphi(r)\sin(n\theta)$ and $\Lambda\Omega = n\bigl(v_0 w + \varphi g\bigr)\cos(n\theta)$.

Suppose now that $f = b(r)\sin(n\theta) \in \mathcal{Y}_n$. If the inhomogeneous differential equation

$$-\,\varphi''(r) - \frac{1}{r}\,\varphi'(r) + \Bigl(\frac{n^2}{r^2} - h(r)\Bigr)\varphi(r) \,=\, \frac{b(r)}{nv_0(r)}\,, \qquad r > 0\,, \tag{A.11}$$

has a solution satisfying the boundary conditions, we can define $\Omega = w(r)\cos(n\theta)$ with

$$w(r) \,=\, -\varphi(r)h(r) - \frac{b(r)}{nv_0(r)}\,. \tag{A.12}$$

Then (A.9) is satisfied, and it follows from (A.10) that $\Lambda\Omega = f$. The same conclusion holds if $f = b(r)\cos(n\theta)$ and $\Omega = -w(r)\sin(n\theta)$.

So the invertibility of the operator Λ in the subspace $\mathcal{Y}_n$ is reduced to the solvability of the ODE (A.11). If $n \ge 2$, the coefficient $n^2/r^2 - h(r)$ is positive, which ensures that equation (A.11) always has a unique solution satisfying the boundary conditions. This leads to the following statement, where $\mathcal{Z}$ denotes the function space introduced in (3.9).

Lemma A.1. [12] *If $n \ge 2$ and $f \in \mathcal{Y}_n \cap \mathcal{Z}$, there exists a unique $\Omega \in \mathcal{Y}_n \cap \mathcal{Z}$ such that $\Lambda\Omega = f$. Moreover, if $f = b(r)\sin(n\theta)$ (respectively, $f = b(r)\cos(n\theta)$) then $\Omega = w(r)\cos(n\theta)$ (respectively, $\Omega = -w(r)\sin(n\theta)$) where w is defined by* (A.12) *with φ given by* (A.11).

If $n = 1$, the homogeneous ODE (A.11) with $b = 0$ has a nontrivial solution $\varphi = rv_0$ satisfying the boundary conditions. As a consequence, the inhomogeneous equation can be solved only if the source term satisfies $\int_0^\infty b(r)r^2\,\mathrm{d}r = 0$, and the solution is not unique. This solvability condition ensures that the function $f = b(r)\sin\theta$ (or $f = b(r)\cos\theta$) belongs to the subspace $\mathcal{Y}_1'$ defined by

$$\mathcal{Y}_1' \,=\, \mathcal{Y}_1 \cap \mathrm{Ker}(\Lambda)^\perp \,=\, \Bigl\{f \in \mathcal{Y}_1 \,:\, \int_{\mathbb{R}^2} \xi_1 f(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^2} \xi_2 f(\xi)\mathrm{d}\xi = 0\Bigr\}\,, \tag{A.13}$$

see also Remark 3.7. This leads to the following result, which complements Lemma A.1.

Lemma A.2. [14] *If $n = 1$ and $f \in \mathcal{Y}_1' \cap \mathcal{Z}$, there exists a unique $\Omega \in \mathcal{Y}_1' \cap \mathcal{Z}$ such that $\Lambda\Omega = f$. Moreover, if $f = b(r)\sin\theta$ (respectively, $f = b(r)\cos\theta$) then $\Omega = w(r)\cos\theta$ (respectively, $\Omega = -w(r)\sin\theta$) where w is defined by* (A.12) *with φ given by* (A.11).

Remark A.3. In the lemmas above, the assumption that $f \in \mathcal{Y}_n$ (respectively, $f \in \mathcal{Y}_1'$) implies that $f \in \mathrm{Ker}(\Lambda)^\perp$, but does not ensure that $f \in \mathrm{Ran}(\Lambda)$ because $f(r)$ may not decay to zero sufficiently rapidly as $r \to +\infty$. This problem disappears if we assume in addition that $f \in \mathcal{Z}$, in which case $f \in \mathrm{Ran}(\Lambda)$ and the unique preimage $\Omega = \Lambda^{-1} f$ in $\mathcal{Y}_n$ (respectively, in $\mathcal{Y}_1'$) still belongs to $\mathcal{Z}$. Note also that, if f is an odd (respectively, even) function of ξ_2, then Ω is an even (respectively, odd) function of ξ_2.

A.4. **Proof of Lemma 4.6.** Since $\mathrm{M}[w] = 0$, using (2.6) and the definition of $\mathcal{T}_\varepsilon$ in (2.10) we find as in Lemma 3.8

$$\begin{aligned}(\mathcal{T}_\varepsilon \varphi)(\xi) &= \frac{1}{4\pi}\int_{\mathbb{R}^2} \log\big(|1+\varepsilon(\xi_1+\eta_1)|^2 + \varepsilon^2|\xi_2-\eta_2|^2\big)\, w(\eta)\,\mathrm{d}\eta \\ &=: \int_{\mathbb{R}^2} K_\varepsilon(\xi_1+\eta_1, \xi_2-\eta_2) w(\eta)\,\mathrm{d}\eta\,, \qquad \xi \in \mathbb{R}^2\,.\end{aligned}$$

We first prove the bound on $\nabla \mathcal{T}_\varepsilon \varphi$ in (4.34). To evaluate the contribution of the vorticity w in the outer region, we define

$$I_{\mathrm{out}}(\xi) := \int_{\{|\eta|>4\varepsilon^{-\sigma_1}\}} \nabla_\xi K_\varepsilon(\xi_1+\eta_1, \xi_2-\eta_2) w(\eta)\,\mathrm{d}\eta\,, \qquad \xi \in \mathbb{R}^2\,.$$

Given any $q > 2$, we claim that

$$\|I_{\mathrm{out}}\|_{L^q} \lesssim \exp(-c_*\varepsilon^{-2\sigma_1})\|w\|_{\mathcal{X}_\varepsilon}\,, \tag{A.14}$$

for some sufficiently small constant $c_* > 0$. To this end, we observe that

$$|\nabla_\xi K_\varepsilon(\xi_1+\eta_1, \xi_2-\eta_2)| \lesssim \frac{1}{|\varepsilon^{-1} + (\xi_1+\eta_1)| + |\xi_2-\eta_2|}\,.$$

Using the change of variables $\eta_1 = -\varsigma_1 - \varepsilon^{-1}$, $\eta_2 = \varsigma_2$, we thus get

$$\int_{\{|\eta|>4\varepsilon^{-\sigma_1}\}} |\nabla_\xi K_\varepsilon(\xi_1+\eta_1, \xi_2-\eta_2) w(\eta)|\,\mathrm{d}\eta \lesssim \int_{\mathbb{R}^2} \frac{1}{|\xi-\varsigma|}\, \mathcal{T}_\varepsilon\big(\mathbb{1}_{\{|\cdot|>4\varepsilon^{-\sigma_1}\}}|w|\big)(\varsigma)\mathrm{d}\varsigma\,.$$

Taking $p \in (1,2)$ such that $1/p = 1/q + 1/2$ and applying the Hardy–Littlewood–Sobolev inequality, we deduce that

$$\|I_{\mathrm{out}}\|_{L^q} \lesssim \bigg(\int_{\{|\eta|>4\varepsilon^{-\sigma_1}\}} W_\varepsilon^{-\frac{p}{2}}(\eta)\big(\sqrt{W_\varepsilon}|w|\big)^p(\eta)\,\mathrm{d}\eta\bigg)^{\frac{1}{p}} \lesssim \|\mathbb{1}_{\{|\cdot|>4\varepsilon^{-\sigma_1}\}} W_\varepsilon^{-\frac{1}{2}}\|_{L^{\frac{2p}{2-p}}} \|w\|_{\mathcal{X}_\varepsilon}\,,$$

where the last step follows from Hölder's inequality with conjugate exponents $2/(2-p)$ and $2/p$. Finally, using the lower bound on W_ε given by the definition (4.22), we arrive at the bound (A.14) for some sufficiently small $c_* > 0$.

We next consider the contribution of the vorticity $w(\eta)$ in the complementary region where $|\eta| \le 4\varepsilon^{-\sigma_1}$. We assume henceforth that $\xi \in \mathrm{I}_\varepsilon$, so that $|\xi| \le 2\varepsilon^{-\sigma_1}$. This of course implies that $\varepsilon(|\xi| + |\eta|) \ll 1$, so that we can expand the kernel $K_\varepsilon(\xi_1+\eta_1, \xi_2-\eta_2)$ into a convergent power series as in (3.19). In particular, if we define

$$B_{\mathrm{err}}(\xi,\eta) := \nabla_\xi K_\varepsilon(\xi_1+\eta_1, \xi_2-\eta_2) - \sum_{n=1}^{2} \frac{(-1)^{n-1}}{2\pi n}\,\varepsilon^n \nabla_\xi Q_n^c(\xi_1+\eta_1, \xi_2-\eta_2)\,,$$

we have the accurate bound $|B_{\mathrm{err}}(\xi,\eta)| \lesssim \varepsilon^3(1+|\xi|+|\eta|)^2$. Moreover, by construction,

$$\begin{aligned}(\nabla \mathcal{T}_\varepsilon \varphi)(\xi) &= \sum_{n=1}^{2} \frac{(-1)^{n-1}}{2\pi n}\,\varepsilon^n \nabla_\xi \int_{\{|\eta|\le 4\varepsilon^{-\sigma_1}\}} Q_n^c(\xi_1+\eta_1, \xi_2-\eta_2) w(\eta)\,\mathrm{d}\eta \\ &\quad + \int_{\{|\eta|\le 4\varepsilon^{-\sigma_1}\}} B_{\mathrm{err}}(\xi,\eta) w(\eta)\,\mathrm{d}\eta + I_{\mathrm{out}}(\xi)\,, \quad \text{for } |\xi| \le 2\varepsilon^{-\sigma_1}\,.\end{aligned} \tag{A.15}$$

In view of the definition of the polynomials $P_n(\xi)$ in (3.17) (with $\Omega = w$), the first term in the right-hand side of (A.15) is equal to

$$\nabla P_1(\xi) + \nabla P_2(\xi) - \sum_{n=1}^{2} \frac{(-1)^{n-1}}{2\pi n} \varepsilon^n \nabla_\xi \int_{\{|\eta|>4\varepsilon^{-\sigma_1}\}} Q_n^c(\xi_1+\eta_1, \xi_2-\eta_2) w(\eta)\,\mathrm{d}\eta\,.$$

As observed in Remark 3.9, we know that $\nabla P_1(\xi) = \nabla P_2(\xi) = 0$ if $\mathrm{M}[w] = \mathrm{m}_1[w] = \mathrm{m}_0[w] = 0$. Therefore, since $|\nabla Q_n^c(x)| \le n|x|^{n-1}$, we get

$$\begin{aligned} \big|(\nabla \mathcal{T}_\varepsilon \varphi)(\xi)\big| &\lesssim \varepsilon \int_{\{|\eta|>4\varepsilon^{-\sigma_1}\}} (1+|\xi|+|\eta|)\,|w(\eta)|\,\mathrm{d}\eta + \varepsilon^3 \int_{\mathbb{R}^2} (1+|\xi|+|\eta|)^2 |w(\eta)|\,\mathrm{d}\eta + |I_{\mathrm{out}}(\xi)| \\ &\lesssim \varepsilon^3 (1+|\xi|)^2 \|w\|_{\mathcal{X}_\varepsilon} + (1+|\xi|)\exp(-c_*\varepsilon^{-2\sigma_1})\|w\|_{\mathcal{X}_\varepsilon} + |I_{\mathrm{out}}(\xi)|\,, \end{aligned}$$

whenever $|\xi| \le 2\varepsilon^{-\sigma_1}$. Dividing both sides by $(1+|\xi|)^3$ and taking the L^q norm with $q > 2$, we obtain the bound involving $\nabla \mathcal{T}_\varepsilon \varphi$ in (4.34), in view of (A.14).

The estimate involving $\mathcal{T}_\varepsilon \varphi$ in (4.34) is established following exactly the same lines, and the details can thus be omitted. When $\varepsilon(|\xi| + |\eta|) \ll 1$, we now expand the kernel K_ε itself, and not its derivative ∇K_ε, which makes a difference because we only know that $P_1(\xi) = 0$ under the moment assumptions in Lemma 4.6. Therefore our expansion stops at $n = 1$ (instead of $n = 2$ in the previous case), which explains why we only gain a factor ε^2 (instead of ε^3).

It remains to prove the estimate $\|(1+|\cdot|)\varphi\|_{L^\infty} \le C\|w\|_{\mathcal{X}_\varepsilon}$. We already know from the proof of Lemma 4.5 that $|\varphi(\xi)| \le C\log(2+|\xi|)\|w\|_{\mathcal{X}_\varepsilon}$, so we can assume henceforth that $|\xi|$ is large. As $\mathrm{M}[w] = 0$ by assumption, we can decompose $\varphi(\xi) = \varphi_1(\xi) + \varphi_2(\xi)$ where

$$\varphi_1(\xi) \;=\; \frac{1}{2\pi} \int_{\{|\eta|\le|\xi|/2\}} \log\frac{|\xi-\eta|}{|\xi|}\, w(\eta)\,\mathrm{d}\eta\,, \quad \varphi_2(\xi) \;=\; \frac{1}{2\pi} \int_{\{|\eta|>|\xi|/2\}} \log\frac{|\xi-\eta|}{|\xi|}\, w(\eta)\,\mathrm{d}\eta\,.$$

Since $|\log(|\xi-\eta|/|\xi|)| \le C|\eta|/|\xi|$ when $|\eta| \le |\xi|/2$, we have $|\xi||\varphi_1(\xi)| \le C\|w\|_{\mathcal{X}_\varepsilon}$. To bound φ_2, we use Hölder's inequality and the fact that the L^2 norm of $(1+|\eta|)^2 w(\eta)$ in the exterior domain $|\eta| > |\xi|/2$ decays rapidly as $|\xi| \to +\infty$. This gives the desired result. □

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