Orbital stability in the cubic defocusing NLS equation: II. The black soliton

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Abstract

Combining the usual energy functional with a higher-order conserved quantity originating from integrability theory, we show that the black soliton is a local minimizer of a quantity that is conserved along the flow of the cubic defocusing NLS equation in one space dimension. This unconstrained variational characterization gives an elementary proof of the orbital stability of the black soliton with respect to perturbations in $H^2(\mathbb{R})$.

1 Introduction

In this work we show how the techniques developed in the companion paper [5] to investigate the stability properties of the cnoidal periodic waves of the cubic defocusing nonlinear Schrödinger equation in one space dimension can be extended to provide a new and rather elementary proof of orbital stability in the limiting case of the black soliton. We thus consider the cubic defocusing NLS equation

$$i\psi_t(x,t) + \psi_{xx}(x,t) - |\psi(x,t)|^2\psi(x,t) \,=\, 0, \tag{1.1}$$

where ψ is a complex-valued function of $(x,t) \in \mathbb{R}\times\mathbb{R}$. The black soliton is the particular solution of (1.1) given by $\psi(x,t) = e^{-it}u_0(x)$, where

$$u_0(x) \,=\, \tanh\Big(\frac{x}{\sqrt{2}}\Big), \qquad x \in \mathbb{R}. \tag{1.2}$$

For later use, we note that the soliton profile $u_0 : \mathbb{R} \to \mathbb{R}$ satisfies the differential equations

$$u_0' \,=\, \frac{1}{\sqrt{2}}\Big(1 - u_0^2\Big), \qquad \text{hence} \qquad u_0'' + u_0 - u_0^3 \,=\, 0. \tag{1.3}$$

The NLS equation (1.1) has many symmetries and conserved quantities, which play a crucial role in the dynamics of the system. In particular, the gauge invariance $\psi \mapsto e^{i\theta}\psi$ and the translation invariance $\psi \mapsto \psi(\cdot - \xi)$ give rise to the conservation of the charge Q and the momentum M, respectively, where

$$Q(\psi) \,=\, \int_{\mathbb{R}} \Big(|\psi|^2 - 1\Big)\,\mathrm{d}x, \qquad M(\psi) \,=\, \frac{i}{2}\int_{\mathbb{R}} \Big(\bar\psi\psi_x - \psi\bar\psi_x\Big)\,\mathrm{d}x. \tag{1.4}$$

Since the NLS equation (1.1) is an autonomous Hamiltonian system, we also have the conservation of the energy

$$E(\psi) = \int_{\mathbb{R}} \left(|\psi_x|^2 + \frac{1}{2}(1-|\psi|^2)^2 \right) \mathrm{d}x. \qquad (1.5)$$

In what follows, our goal is to study the stability of the black soliton (1.2), and we shall therefore restrict ourselves to solutions of (1.1) for which $|\psi| \to 1$ as $|x| \to \infty$. This is why we defined the conserved quantities (1.4), (1.5) in such a way that the integrands vanish when $|\psi| = 1$ and $\psi_x = 0$.

The nonlinear stability of the black soliton (1.2) has been studied in several recent works. In [1] the authors apply the variational method of Cazenave and Lions [3], which relies on the fact that the black soliton (1.2) is a global minimizer of the energy E for a fixed value of the momentum M. The difficulty with this approach is that the momentum is not defined for all finite-energy solutions, so that the integral defining M in (1.4) has to be renormalized and properly interpreted. A slightly different proof was subsequently given in [8], in the spirit of the work by Weinstein [12] and Grillakis, Shatah, and Strauss [9]. The main idea is to show that the energy functional (1.5) becomes coercive in a neighborhood of the black soliton (1.2) if the conservation of the momentum is used to get rid of one unstable direction. Both results in [1, 8] are variational in nature and establish orbital stability of the black soliton in the energy space. Note that asymptotic stability of the black soliton is also proved in [8], using ideas and techniques developed by Martel and Merle for the generalized Korteweg-de Vries equation [10]. In a different direction, a more precise orbital stability result was obtained in [7] for sufficiently smooth and localized perturbations, using the inverse scattering transform method which relies on the integrability of the cubic defocusing NLS equation (1.1). Similarly, asymptotic stability of the black soliton and several dark solitons was recently proved in [4].

As as consequence of integrability, the NLS equation (1.1) has many conserved quantities in addition to the charge, the momentum, and the energy. In the present work, we introduce a new variational approach based on the higher-order functional

$$S(\psi) = \int_{\mathbb{R}} \left(|\psi_{xx}|^2 + 3|\psi|^2|\psi_x|^2 + \frac{1}{2}(\bar{\psi}\psi_x + \psi\bar{\psi}_x)^2 + (1-|\psi|^2)^2\Big(1 + \frac{1}{2}|\psi|^2\Big) \right) \mathrm{d}x, \qquad (1.6)$$

which is also conserved under the evolution defined by (1.1). The latter claim can be proved by a straightforward but cumbersome calculation, or by more educated techniques as described, e.g., in [11, Section 2.3]. The natural domain of definition for the functional (1.6) is the H^2 energy space defined by

$$X = \Big\{ \psi \in H^2_{\mathrm{loc}}(\mathbb{R}) : \quad \psi_x \in H^1(\mathbb{R}),\ 1 - |\psi|^2 \in L^2(\mathbb{R}) \Big\}. \qquad (1.7)$$

Indeed, if $\psi \in X$, then $\zeta := 1 - |\psi|$ belongs to $H^1(\mathbb{R})$, because $|\zeta| \le |1 - |\psi|^2| \in L^2(\mathbb{R})$ and $\zeta_x = -|\psi|_x \in L^2(\mathbb{R})$. By Sobolev's embedding of $H^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$, we thus have $|\psi| = 1 - \zeta \in L^\infty(\mathbb{R})$, and from the definitions (1.6) and (1.7) it follows easily that $S(\psi) < \infty$. Since u_0', u_0'', and $1 - u_0^2$ decay exponentially to zero as $|x| \to \infty$, it is clear that $u_0 + H^2(\mathbb{R}) \subset X$, so that the functional (1.6) is well defined for H^2 perturbations of the soliton profile u_0. This allows us to define the differential of S at u_0, and a direct calculation using the differential equations (1.3) reveals that u_0 is a *critical point* of S, in the sense that $S'(u_0) = 0$.

Unfortunately, the second variation $S''(u_0)$ has no definite sign [5], hence it is not possible to prove orbital stability of the black soliton using the functional S alone. As is explained in the companion paper [5], which is devoted to the stability of periodic waves for the NLS equation (1.1), it is possible to cure that problem by subtracting from S an appropriate multiple of the energy E, which is well defined on X and also satisfies $E'(u_0) = 0$. The optimal choice is

$$\Lambda(\psi) \;=\; S(\psi) - 2E(\psi), \qquad \psi \in X. \tag{1.8}$$

We then have $\Lambda'(u_0) = 0$, and the starting point of our approach is the following result, which asserts that the second variation $\Lambda''(u_0)$ is nonnegative.

Proposition 1.1. *The second variation of the functional* (1.8) *at the black soliton* (1.2) *is nonnegative for perturbations in* $H^2(\mathbb{R})$*.*

It is important to realize that Proposition 1.1 gives an *unconstrained* variational characterization of the black soliton u_0, which is our main motivation for introducing the higher-order conserved quantity (1.6). In contrast, the approach in [1, 8] relies on the fact that u_0 is a minimum of the energy $E(\psi)$ subject to the constraint $\mathcal{M}(\psi) = \mathcal{M}(u_0)$, where $\mathcal{M}$ is a suitably renormalized version of the momentum M defined in (1.4).

The proof of Proposition 1.1 developed in Section 2 actually shows that the second variation $\Lambda''(u_0)$ is positive except for degeneracies due to symmetries: the nonnegative self-adjoint operator associated with $\Lambda''(u_0)$ has a simple zero eigenvalue which is due to translation invariance, and the essential spectrum extends all the way to the origin due to gauge invariance. As a consequence, perturbations in $H^2(\mathbb{R})$ can include slow modulations of the phase of the black soliton far away from the origin, which hardly increase the functional Λ. This means that the second variation $\Lambda''(u_0)$ is not coercive in $H^2(\mathbb{R})$, even if modulation parameters are used to remove the zero modes due to the symmetries. For that reason, we are not able to control the perturbations of the black soliton in the topology of $H^2(\mathbb{R})$, but only in a weaker sense that allows for a slow drift of the phase at infinity, see Section 3 below for a more detailed discussion.

To formulate our main result, we equip the space X with the distance

$$d_R(\psi_1, \psi_2) \;=\; \|(\psi_1 - \psi_2)_x\|_{H^1(\mathbb{R})} + \||\psi_1|^2 - |\psi_2|^2\|_{L^2(\mathbb{R})} + \|\psi_1 - \psi_2\|_{L^2(-R,R)}, \tag{1.9}$$

where $R \geq 1$ is a parameter. Note that d_R is the exact analogue, at the H^2 level, of the distance that is used in previous variational studies of the black soliton, including [1, 6, 8]. As is easily verified, a function $\psi \in H^2_{\text{loc}}(\mathbb{R})$ belongs to X if and only if $d_R(\psi, u_0) < \infty$; moreover, different choices of R give equivalent distances on X. To prove orbital stability of the black soliton with profile u_0, the idea is to consider solutions ψ of the NLS equation (1.1) for which $d_R(\psi, u_0)$ is small. This is certainly the case if $\|\psi - u_0\|_{H^2}$ is small, but the converse is not true because $d_R(\psi, u_0)$ does not control the L^2 norm of the difference $\psi - u_0$ on the whole real line. We shall prove in Section 4 that the distance d_R is well adapted to the functional Λ near u_0, in the sense that

$$\Lambda(\psi) - \Lambda(u_0) \;\geq\; C d_R(\psi, u_0)^2 \qquad \text{when} \quad d_R(\psi, u_0) \ll 1, \tag{1.10}$$

provided the perturbation $\psi - u_0$ satisfies a pair of orthogonality conditions. As is usual in orbital stability theory, these orthogonality conditions can be fulfilled if we replace ψ by $e^{i\theta}\psi(\cdot + \xi)$ for some appropriate modulation parameters $\theta, \xi \in \mathbb{R}$, see Section 3 below. It is then easy to deduce from (1.10) that solutions of the NLS equation (1.1) with initial data ψ satisfying $d_R(\psi_0, u_0) \ll 1$ will stay close for all times to the orbit of the black soliton under the group of translations and phase rotations. The precise statement is:

Theorem 1.2. *Fix $R \geq 1$ and let $u_0 \in X$ be the black soliton* (1.2)*. Given any $\epsilon > 0$, there exists $\delta > 0$ such that, for any $\psi_0 \in X$ satisfying*

$$d_R(\psi_0, u_0) \leq \delta, \tag{1.11}$$

the global solution $\psi(\cdot, t)$ of the NLS equation (1.1) *with initial data u_0 has the following property. For any $t \in \mathbb{R}$, there exist $\xi(t) \in \mathbb{R}$ and $\theta(t) \in \mathbb{R}/(2\pi\mathbb{Z})$ such that*

$$d_R\Big(e^{i(t+\theta(t))}\psi(\cdot + \xi(t), t)\,, u_0\Big) \leq \epsilon. \tag{1.12}$$

Moreover ξ and θ are continuously differentiable functions of t which satisfy

$$|\dot{\xi}(t)| + |\dot{\theta}(t)| \leq C\epsilon, \quad t \in \mathbb{R}, \tag{1.13}$$

for some positive constant C.

Remark 1.3. It is known from the work of Zhidkov [13] that the Cauchy problem for the NLS equation (1.1) is globally well-posed in X. This is the functional framework that is used to define solutions of (1.1) in Theorem 1.2.

Remark 1.4. Except for the use of a different distance d_R, which controls the perturbations in the topology of $H^2_{\mathrm{loc}}(\mathbb{R})$, Theorem 1.2 is the exact analogue of the orbital stability results obtained in [1, 8]. However the proof is quite different, and in some sense simpler, because the profile u_0 of the black soliton is an unconstrained local minimizer of the higher-order functional Λ.

Remark 1.5. It is also possible to prove asymptotic stability results for the black soliton of the cubic NLS equation (1.1). In that perspective, it is useful to consider the black soliton as a member of the one-parameter family of traveling dark solitons, given by the exact expression

$$e^{it}\psi_\nu(x + \nu t, t) = \sqrt{1 - \tfrac{1}{2}\nu^2}\, \tanh\left(\sqrt{\tfrac{1}{2} - \tfrac{1}{4}\nu^2}\, x\right) + \frac{i\nu}{\sqrt{2}}, \tag{1.14}$$

where $\nu \in (-\sqrt{2}, \sqrt{2})$. Asymptotic stability of the family of dark solitons with nonzero speed ν was proved in [2], using the Madelung transformation and the hydrodynamic formulation of the NLS equation. This approach applies to solutions whose modulus is strictly positive, and therefore excludes the case of the black soliton. Very recently, the asymptotic stability of the black soliton (within the one-parameter family of all dark solitons) has been established in [4, 8].

The rest of this article is organized as follows. In Section 2 we establish positivity and coercivity properties for the quadratic form associated with the second variation of the functional (1.8) at u_0. In Section 3, we introduce modulation parameters in a neighborhood of the soliton profile to eliminate the zero modes of the second variation $\Lambda''(u_0)$. Combining these results and using a new variable borrowed from [8], we prove in Section 4 the orbital stability of the black soliton (1.3) in the space X.

2 Positivity and coercivity of the second variation

Let u_0 be the soliton profile (1.2) and $\Lambda = S - 2E$ be the functional defined by (1.5), (1.6), and (1.8). In this section, we prove that the second variation $\Lambda''(u_0)$ is nonnegative, as stated in Proposition 1.1, and we deduce some coercivity properties that will be used in the proof of

Theorem 1.2. We consider perturbations of u_0 of the form $\psi = u_0 + u + iv$, where $u, v \in H^2(\mathbb{R})$ are real-valued. As in [5], the second variations at u_0 of the functionals E and S satisfy

$$\begin{aligned} \tfrac{1}{2}\langle E''(u_0)[u,v],[u,v]\rangle \;&=\; \langle L_+u,u\rangle_{L^2} + \langle L_-v,v\rangle_{L^2}\,, \\ \tfrac{1}{2}\langle S''(u_0)[u,v],[u,v]\rangle \;&=\; \langle M_+u,u\rangle_{L^2} + \langle M_-v,v\rangle_{L^2}\,, \end{aligned}$$

where $\langle\cdot\,,\cdot\rangle_{L^2}$ denotes the usual scalar product in $L^2(\mathbb{R})$. The self-adjoint operators $L_\pm$ and $M_\pm$ have the following expressions:

$$\begin{aligned} L_+ \;&=\; -\partial_x^2 + 3u_0^2 - 1\,, \qquad & M_+ \;&=\; \partial_x^4 - 5\partial_x u_0^2\partial_x - 5u_0^4 + 15u_0^2 - 4\,, \\ L_- \;&=\; -\partial_x^2 + u_0^2 - 1\,, \qquad & M_- \;&=\; \partial_x^4 - 3\partial_x u_0^2\partial_x + u_0^2 - 1\,. \end{aligned} \tag{2.1}$$

In view of (1.8), it follows that

$$\tfrac{1}{2}\langle \Lambda''(u_0)[u,v],[u,v]\rangle \;=\; \langle K_+u,u\rangle_{L^2} + \langle K_-v,v\rangle_{L^2}\,, \tag{2.2}$$

where $K_\pm = M_\pm - 2L_\pm$. More explicitly, the quadratic forms associated with $K_\pm$ are given by

$$\langle K_+u,u\rangle_{L^2} \;=\; \int_{\mathbb{R}}\Bigl(u_{xx}^2 + (5u_0^2-2)u_x^2 + (9u_0^2 - 5u_0^4 - 2)u^2\Bigr)\,\mathrm{d}x\,, \tag{2.3}$$

$$\langle K_-v,v\rangle_{L^2} \;=\; \int_{\mathbb{R}}\Bigl(v_{xx}^2 + (3u_0^2-2)v_x^2 + (1-u_0^2)v^2\Bigr)\,\mathrm{d}x\,. \tag{2.4}$$

Our first task is to show that the quadratic forms (2.3), (2.4) are nonnegative on $H^2(\mathbb{R})$. Due to translation invariance of the NLS equation (1.1), we have $L_+u_0' = M_+u_0' = 0$, hence also $K_+u_0' = 0$. As $u_0' \in H^2(\mathbb{R})$, this shows that the quadratic form associated with K_+ has a neutral direction, hence is not strictly positive, see Lemma 2.1 below. The situation is slightly different for K_-: due to gauge invariance, we have $L_-u_0 = M_-u_0 = 0$, hence $K_-u_0 = 0$, but of course $u_0 \notin H^2(\mathbb{R})$. In fact, the result of Lemma 2.3 below shows that the quadratic form associated with K_- is strictly positive on $H^2(\mathbb{R})$.

We first prove that the quadratic form (2.3) is nonnegative, see also [5, Corollary 4.5].

Lemma 2.1. *For any $u \in H^2(\mathbb{R})$, we have*

$$\langle K_+u,u\rangle_{L^2} \;=\; \|w_x\|_{L^2}^2 + \|w\|_{L^2}^2 \;\ge\; 0\,, \tag{2.5}$$

where $w = u_x + \sqrt{2}u_0u$.

Proof. Integrating by parts and using the differential equations (1.3) satisfied by u_0, we easily obtain

$$\int_{\mathbb{R}} w^2\,\mathrm{d}x \;=\; \int_{\mathbb{R}}\Bigl(u_x^2 + 2\sqrt{2}u_0uu_x + 2u_0^2u^2\Bigr)\,\mathrm{d}x \;=\; \int_{\mathbb{R}}\Bigl(u_x^2 + (3u_0^2-1)u^2\Bigr)\,\mathrm{d}x\,. \tag{2.6}$$

Similarly, as $w_x = u_{xx} + \sqrt{2}u_0u_x + \sqrt{2}u_0'u$, we find

$$\begin{aligned} \int_{\mathbb{R}} w_x^2\,\mathrm{d}x \;&=\; \int_{\mathbb{R}}\Bigl(u_{xx}^2 + 2\sqrt{2}u_0u_xu_{xx} + 2u_0^2u_x^2 + 2\sqrt{2}u_0'uu_{xx} + 4u_0u_0'uu_x + 2u_0'^2u^2\Bigr)\,\mathrm{d}x \\ &=\; \int_{\mathbb{R}}\Bigl(u_{xx}^2 + (5u_0^2-3)u_x^2 + 8u_0u_0'uu_x + 2u_0'^2u^2\Bigr)\,\mathrm{d}x \\ &=\; \int_{\mathbb{R}}\Bigl(u_{xx}^2 + (5u_0^2-3)u_x^2 + (1-u_0^2)(5u_0^2-1)u^2\Bigr)\,\mathrm{d}x\,, \end{aligned} \tag{2.7}$$

because $2u_0'^2 - 4(u_0u_0')' = (1-u_0^2)(5u_0^2-1)$. If we now combine (2.6) and (2.7), we see that $\|w_x\|_{L^2}^2 + \|w\|_{L^2}^2$ is equal to the right-hand side of (2.3), which is the desired conclusion. ◻

Remark 2.2. The right-hand side of (2.5) vanishes if and only if $w = 0$, which is equivalent to $u = Cu_0'$ for some constant C. Thus zero is a simple eigenvalue of K_+ in $L^2(\mathbb{R})$. Moreover, since $u_0(x) \to \pm 1$ as $x \to \pm\infty$, it is clear from (2.3) that the essential spectrum of K_+ is the interval $[2, \infty)$. Thus if we restrict ourselves to the orthogonal complement of u_0' with respect to the scalar product $\langle \cdot\,, \cdot \rangle_{L^2}$, the spectrum of K_+ is bounded from below by a strictly positive constant, and the corresponding quadratic form is thus coercive in the topology of $H^2(\mathbb{R})$, see Remark 2.7 below.

We next prove the positivity of the quadratic form (2.4), see also [5, Lemma 4.1].

Lemma 2.3. *For any $v \in H^2(\mathbb{R})$, we have*

$$\langle K_- v, v\rangle_{L^2} \;=\; \|L_- v\|_{L^2}^2 + \|u_0 v_x - u_0' v\|_{L^2}^2 \;\geq\; 0, \tag{2.8}$$

where $L_- = -\partial_x^2 + u_0^2 - 1$.

Proof. Integrating by parts we obtain

$$\begin{aligned}\int_{\mathbb{R}} (L_- v)^2 \,\mathrm{d}x \;&=\; \int_{\mathbb{R}} \Big(v_{xx}^2 + 2(1-u_0^2) v v_{xx} + (1-u_0^2)^2 v^2 \Big)\,\mathrm{d}x \\ &=\; \int_{\mathbb{R}} \Big(v_{xx}^2 + 2(u_0^2 - 1) v_x^2 - 2(u_0 u_0')' v^2 + (1-u_0^2)^2 v^2 \Big)\,\mathrm{d}x.\end{aligned}$$

Similarly, we have

$$\int_{\mathbb{R}} \Big(u_0 v_x - u_0' v \Big)^2 \,\mathrm{d}x \;=\; \int_{\mathbb{R}} \Big(u_0^2 v_x^2 + (u_0 u_0')' v^2 + u_0'^2 v^2 \Big)\,\mathrm{d}x.$$

It follows that

$$\|L_- v\|_{L^2}^2 + \|u_0 v_x - u_0' v\|_{L^2}^2 \;=\; \int_{\mathbb{R}} \Big(v_{xx}^2 + (3u_0^2 - 2) v_x^2 + [(1-u_0^2)^2 - u_0 u_0''] v^2 \Big)\,\mathrm{d}x,$$

and that expression coincides with the right-hand side of (2.4) since $(1-u_0^2)^2 - u_0 u_0'' = 1 - u_0^2$ by (1.3). This proves (2.8). ∎

Remark 2.4. The right-hand side of (2.8) vanishes if and only if $L_- v = 0$ and $u_0 v_x - u_0' v = 0$, namely if $v = Cu_0$ for some constant C. As $u_0 \notin H^2(\mathbb{R})$, this shows that $\langle K_- v, v\rangle_{L^2} > 0$ for any nonzero $v \in H^2(\mathbb{R})$. However, since $|u_0(x)| \to 1$ as $|x| \to \infty$, it is clear from the representation (2.4) that zero belongs to the essential spectrum of the operator K_-, hence the associated quadratic form is not coercive in the topology of $H^2(\mathbb{R})$. Some weaker coercivity property will nevertheless be established below, see Remark 2.9.

Remark 2.5. In view of the decomposition (2.2), Proposition 1.1 is an immediate consequence of Lemmas 2.1 and 2.3.

In the rest of this section, we show that the quadratic forms (2.3), (2.4) are not only positive, but also coercive in some appropriate sense.

Lemma 2.6. *Let u_0 be the black soliton* (1.2). *There exists a positive constant C such that, for any $u \in H^2(\mathbb{R})$ satisfying $\langle u_0', u\rangle_{L^2} = 0$, we have the estimate*

$$\|u\|_{H^2} \;\leq\; C \|w\|_{H^1}, \tag{2.9}$$

where $w = u_x + \sqrt{2} u_0 u$.

Proof. Solving the linear differential equation $u_x + \sqrt{2}u_0 u = w$ by Duhamel's formula, we find $u = Au_0' + W$ for some $A \in \mathbb{R}$, where

$$W(x) \;=\; \int_0^x K(x,y)w(y)\,\mathrm{d}y, \qquad K(x,y) \;=\; \frac{\cosh^2(y/\sqrt{2})}{\cosh^2(x/\sqrt{2})}. \tag{2.10}$$

The constant A is uniquely determined by the orthogonality condition $\langle u_0', u\rangle_{L^2} = 0$, which implies that $A\|u_0'\|_{L^2}^2 + \langle u_0', W\rangle_{L^2} = 0$. Using (2.10), we easily obtain

$$\begin{aligned}\langle u_0', W\rangle_{L^2} \;&=\; \int_{-\infty}^{\infty}\Big\{\int_0^x K(x,y)w(y)\,\mathrm{d}y\Big\}u_0'(x)\,\mathrm{d}x \\ &=\; \int_0^{\infty}\Big\{\int_y^{\infty} K(x,y)u_0'(x)\,\mathrm{d}x\Big\}\Big(w(y) - w(-y)\Big)\,\mathrm{d}y \\ &=\; \frac{1}{3}\int_0^{\infty} e^{-\sqrt{2}y}\,\frac{3+e^{-\sqrt{2}y}}{1+e^{-\sqrt{2}y}}\Big(w(y) - w(-y)\Big)\,\mathrm{d}y, \end{aligned} \tag{2.11}$$

hence $|\langle u_0', W\rangle_{L^2}| \le 2^{-1/4}\|w\|_{L^2}$. It follows that $|A| \le C\|w\|_{L^2}$ for some $C > 0$.

On the other hand, if we introduce the operator notation $W = \hat{K}(w)$ for the representation (2.10), we note that $\hat{K}$ is a bounded operator from $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$ with norm

$$K_\infty \;=\; \sup_{x\in\mathbb{R}}\int_0^{|x|} K(x,y)\,\mathrm{d}y \;=\; \frac{1}{\sqrt{2}}\sup_{x\in\mathbb{R}}\frac{1+2\sqrt{2}|x|e^{-\sqrt{2}|x|} - e^{-2\sqrt{2}|x|}}{1+2e^{-\sqrt{2}|x|}+e^{-2\sqrt{2}|x|}} \;<\; \infty,$$

as well as a bounded operator from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$ with norm

$$K_1 \;=\; \sup_{y\in\mathbb{R}}\int_{|y|}^{\infty} K(x,y)\,\mathrm{d}x \;=\; \frac{1}{\sqrt{2}}\sup_{y\in\mathbb{R}}\Big(1+e^{-\sqrt{2}|y|}\Big) \;=\; \sqrt{2}.$$

By the Riesz-Thorin interpolation theorem, it follows that $\hat{K}$ is a bounded operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, and we have the estimate $\|W\|_{L^2} = \|\hat{K}(w)\|_{L^2} \le (K_1K_\infty)^{1/2}\|w\|_{L^2}$.

Summarizing, we have shown that $\|u\|_{L^2} \le |A|\|u_0'\|_{L^2} + \|W\|_{L^2} \le C\|w\|_{L^2}$ for some $C > 0$. Since $w = u_x + \sqrt{2}u_0u$, we also have $\|u_x\|_{L^2} \le \|w\|_{L^2} + \sqrt{2}\|u\|_{L^2}$ and (after differentiating) $\|u_{xx}\|_{L^2} \le \|w_x\|_{L^2} + \sqrt{2}\|u_x\|_{L^2} + \|u\|_{L^2}$. This proves the bound (2.9). □

Remark 2.7. Combining (2.5) and (2.9), we conclude that there exists a constant $C_+ > 0$ such that

$$\langle K_+u, u\rangle_{L^2} \;\ge\; C_+\|u\|_{H^2}^2, \tag{2.12}$$

for all $u \in H^2(\mathbb{R})$ satisfying $\langle u_0', u\rangle_{L^2} = 0$.

Lemma 2.8. *Let u_0 be the black soliton* (1.2). *There exists a positive constant C such that, for any $v \in H^2_{\mathrm{loc}}(\mathbb{R})$ satisfying $v_x \in H^1(\mathbb{R})$ and $\langle u_0'', v\rangle_{L^2} = 0$, we have the estimate*

$$\|v_{xx}\|_{L^2} + \|v_x\|_{L^2} + |v(0)| \;\le\; C(\|p\|_{L^2} + \|q\|_{L^2}), \tag{2.13}$$

where $p = u_0v_x - u_0'v$ and $q = -L_-v = v_{xx} + (1-u_0^2)v$.

Proof. Any solution of the linear differential equation $u_0 v_x - u_0' v = p$ has the form $v = Bu_0 + Z$ for some $B \in \mathbb{R}$, where

$$Z(x) \;=\; u_0(x) \int_0^x \Big(p(y) + \sqrt{2}q(y)\Big)\,\mathrm{d}y - \sqrt{2}p(x). \tag{2.14}$$

Indeed, we observe that $p_x = u_0 v_{xx} - u_0'' v = u_0(v_{xx} + (1-u_0^2)v) = u_0 q$. Thus, if $v = Bu_0 + Z$, we have

$$v_x(x) \;=\; u_0'(x)\left(B + \int_0^x \Big(p(y) + \sqrt{2}q(y)\Big)\,\mathrm{d}y\right) + u_0(x)p(x), \tag{2.15}$$

hence $u_0 v_x - u_0' v = (u_0^2 + \sqrt{2}u_0')p = p$. The constant B is uniquely determined by the orthogonality condition $\langle u_0'', v\rangle_{L^2} = 0$, which implies that $B\|u_0'\|_{L^2}^2 = \langle u_0'', Z\rangle$.

Since $p \in L^2(\mathbb{R})$ and $p_x = u_0 q \in L^2(\mathbb{R})$, we have $p \in L^\infty(\mathbb{R})$ by Sobolev's embedding, which implies the bound $\|p\|_{L^\infty}^2 \le \|p\|_{L^2}\|p_x\|_{L^2} \le \|p\|_{L^2}\|q\|_{L^2}$. Thus, using (2.14) and Hölder's inequality, we deduce that

$$|Z(x)| \;\le\; \sqrt{2}(|x|^{1/2} + 1)(\|p\|_{L^2} + \|q\|_{L^2}), \qquad x \in \mathbb{R}.$$

This moderate growth of Z is compensated for by the exponential decay of u_0'' to zero at infinity, and we obtain $|\langle u_0'', Z\rangle| \le C(\|p\|_{L^2} + \|q\|_{L^2})$ for some $C > 0$, hence also $|B| \le C(\|p\|_{L^2} + \|q\|_{L^2})$. In the same way, it follows from (2.15) that $\|v_x\|_{L^2} \le C(\|p\|_{L^2} + \|q\|_{L^2})$. A similar estimate holds for $\|v_{xx}\|_{L^2}$ because $v_{xx} = q - (1-u_0^2)v$ and $1 - u_0^2$ has the exponential decay to zero at infinity. Finally, since $v(0) = -\sqrt{2}p(0)$, we also have $|v(0)| \le C(\|p\|_{L^2} + \|q\|_{L^2})$. This proves the bound (2.13). $\square$

Remark 2.9. Combining (2.8) and (2.13), we conclude that there exists a constant $C_- > 0$ such that

$$\langle K_- v, v\rangle_{L^2} \;\ge\; C_-\Big(\|v_x\|_{H^1}^2 + |v(0)|^2\Big), \tag{2.16}$$

for all $v \in H^2_{\mathrm{loc}}(\mathbb{R})$ satisfying $v_x \in H^1(\mathbb{R})$ and $\langle u_0'', v\rangle_{L^2} = 0$. As is clear from the proof of Lemma 2.8, we need some orthogonality condition on v to prove estimate (2.16), and since $u_0 \notin L^2(\mathbb{R})$ we cannot impose $\langle u_0, v\rangle_{L^2} = 0$. Thus we use $u_0'' = u_0(u_0^2 - 1)$ instead of u_0. Although u_0'' is only an approximate eigenfunction of K_-, the orthogonality condition $\langle u_0'', v\rangle_{L^2} = 0$ is good enough for our purposes, as we shall see in Section 3.

3 Modulation parameters near the black soliton

This section contains some important preliminary steps in the proof of Theorem 1.2. To establish the orbital stability of the black soliton with profile u_0, our general strategy is to consider solutions $\psi(x,t)$ of the cubic NLS equation (1.1) of the form

$$e^{i(t+\theta(t))}\psi(x + \xi(t), t) \;=\; u_0(x) + u(x,t) + iv(x,t), \qquad (x,t) \in \mathbb{R}\times\mathbb{R}, \tag{3.1}$$

where the perturbations u, v are real-valued and satisfy the orthogonality conditions

$$\langle u_0', u(\cdot,t)\rangle_{L^2} \;=\; 0, \qquad \langle u_0'', v(\cdot,t)\rangle_{L^2} \;=\; 0, \qquad t \in \mathbb{R}. \tag{3.2}$$

As was discussed in Remarks 2.7 and 2.9, these conditions are needed to exploit the coercivity properties of the second variation $\Lambda''(u_0)$, where Λ is the conserved quantity (1.8). They also allow us to determine uniquely the "modulation parameters", namely the translation $\xi(t)$ and the phase $\theta(t)$, at least for solutions $\psi(x,t)$ in a small neighborhood of the black soliton. To make these considerations rigorous, we first need to specify in which topology that neighborhood is understood; in other words, we need to choose an appropriate perturbation space. Next we have to verify that the modulation parameters exist and depend smoothly on the solution $\psi(x,t)$ in the vicinity of the black soliton.

Concerning the first point, we observe that the functional (1.8) which serves as a basis for our analysis is invariant under translations and gauge transformations, and we recall that $\Lambda'(u_0) = 0$. Thus, if $\psi(x,t)$ is a solution of the NLS equation (1.1) of the form (3.1) with $u, v \in H^2(\mathbb{R})$, we have for each fixed $t \in \mathbb{R}$ the following expansion

$$\Lambda(\psi) - \Lambda(u_0) \;=\; \langle K_+ u, u\rangle_{L^2} + \langle K_- v, v\rangle_{L^2} + N(u,v), \tag{3.3}$$

where $N(u,v)$ collects all terms that are at least cubic in u and v. However, unlike in the periodic case considered in the companion paper [5], the decomposition (3.3) is not sufficient to prove the orbital stability of the black soliton. Indeed, the quadratic terms in (3.3) are nonnegative, but they are degenerate in the sense that they do not control the $L^2(\mathbb{R})$ norm of v, as can be seen from the lower bound (2.16). This is due to the fact that the operator K_- has essential spectrum touching the origin, with generalized eigenfunctions corresponding to slow modulations of the phase of the black soliton. As is clear from the proof of Lemma 2.8, one cannot even prove that $v \in L^\infty(\mathbb{R})$ if we only know that $\langle K_- v, v\rangle_{L^2} < \infty$. This in turn makes it impossible to control the nonlinearity $N(u,v)$ in (3.3) in terms of the quadratic part $\langle K_+ u, u\rangle_{L^2} + \langle K_- v, v\rangle_{L^2}$.

There are good reasons to believe that the above problem is not just a technical one, and that the H^2 topology for the perturbations u, v is not appropriate to prove orbital stability of the black soliton. Indeed, as is well known, the cubic NLS equation (1.1) has a family of travelling dark solitons $\psi_\nu(x,t)$ given by (1.14). Rigorous results [8] and numerical simulations indicate that a small, localized perturbation of the black soliton ψ_0 can lead to the formation of a dark soliton ψ_ν with a small nonzero speed ν. If this happens, the functions u, v defined in (3.1) cannot stay bounded in $L^2(\mathbb{R})$ for all times, because $\psi_\nu - \psi_0 \notin L^2(\mathbb{R})$ if $\nu \neq 0$. Note, however, that the quantity $|\psi_\nu| - |\psi_0|$ does belong to $L^2(\mathbb{R})$ and decays exponentially at infinity. This suggests that a particular combination of u, v may be controlled in $L^2(\mathbb{R})$ for all times.

Following [8], we introduce the auxiliary variable

$$\eta \;=\; |u_0 + u + iv|^2 - |u_0|^2 \;=\; 2u_0 u + u^2 + v^2, \tag{3.4}$$

which allows us to control the perturbations of the modulus of the black soliton u_0. The idea is now to consider perturbations u, v for which $u_x, v_x \in H^1(\mathbb{R})$, $\eta \in L^2(\mathbb{R})$, and $u, v \in L^2(-R,R)$ for some fixed $R \geq 1$. If $\psi = u_0 + u + iv$, this is equivalent to requiring that $\psi \in X$, where X is the function space (1.7), or that $d_R(\psi, u_0) < \infty$, where d_R is the distance (1.9). Indeed, we have by definition

$$d_R(\psi, u_0) \;=\; \|u_x + iv_x\|_{H^1(\mathbb{R})} + \|\eta\|_{L^2(\mathbb{R})} + \|u + iv\|_{L^2(-R,R)}. \tag{3.5}$$

Note, however, that we do not assume any longer that u, v are square integrable at infinity. In particular, the perturbed solutions we consider include dark solitons ψ_ν with nonzero speed ν.

Now that we have defined a precise perturbation space, we can state our first result showing the existence and the continuity of the modulation parameters ξ and θ in a neighborhood of the orbit of the soliton profile u_0. The following statement is very close in spirit to Proposition 2 in [8] or Lemma 6.1 in [5].

Lemma 3.1. *Fix any $R \geq 1$. There exists $\epsilon_0 > 0$ such that, for any $\psi \in X$ satisfying*

$$\inf_{\xi,\theta\in\mathbb{R}} d_R\Big(e^{i\theta}\psi(\cdot+\xi), u_0\Big) \leq \epsilon_0, \tag{3.6}$$

there exist $\xi \in \mathbb{R}$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ such that

$$e^{i\theta}\psi(x+\xi) = u_0(x) + u(x) + iv(x), \quad x \in \mathbb{R}, \tag{3.7}$$

where the real-valued functions u and v satisfy the orthogonality conditions (3.2)*. Moreover, the modulation parameters $\xi \in \mathbb{R}$ and $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ depend continuously on ψ in the topology defined by the distance* (1.9)*.*

Proof. It is sufficient to prove (3.7) for all $\psi \in X$ such that $\epsilon := d_R(\psi, u_0)$ is sufficiently small. Given such a $\psi \in X$, we consider the smooth function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mathbf{f}(\xi,\theta) = \begin{pmatrix} \langle u_0'(\cdot-\xi), \mathrm{Re}(e^{i\theta}\psi)\rangle_{L^2} \\ \langle u_0''(\cdot-\xi), \mathrm{Im}(e^{i\theta}\psi)\rangle_{L^2} \end{pmatrix}, \qquad (\xi,\theta) \in \mathbb{R}^2.$$

By construction, we have $\mathbf{f}(\xi,\theta) = \mathbf{0}$ if and only if ψ can be represented as in (3.7) for some real-valued functions u, v satisfying the orthogonality conditions (3.2).

If we decompose $\psi = u_0 + u + iv$ where u, v are real-valued, we have $\langle u_0', \mathrm{Re}(\psi)\rangle_{L^2} = \langle u_0', u\rangle_{L^2}$ because $\langle u_0', u_0\rangle_{L^2} = 0$. As in the proof of Lemma 2.8, we observe that

$$|u(x)| \leq C\Big(\|u\|_{L^2(-1,1)} + (1+|x|^{1/2})\|u_x\|_{L^2(\mathbb{R})}\Big) \leq C(1+|x|^{1/2})d_R(\psi, u_0),$$

where in the last inequality we have used (3.5). Thus $|\langle u_0', \mathrm{Re}(\psi)\rangle_{L^2}| \leq C d_R(\psi, u_0)$, and a similar argument gives $|\langle u_0'', \mathrm{Im}(\psi)\rangle_{L^2}| \leq C d_R(\psi, u_0)$. This shows that $\|\mathbf{f}(0,0)\| \leq C\epsilon$ for some positive constant C independent of ϵ.

On the other hand, the Jacobian matrix of the function $\mathbf{f}$ at the origin $(0,0)$ is given by

$$D\mathbf{f}(0,0) = \begin{pmatrix} \|u_0'\|_{L^2}^2 & 0 \\ 0 & -\|u_0'\|_{L^2}^2 \end{pmatrix} + \begin{pmatrix} -\langle u_0'', \mathrm{Re}(\psi-u_0)\rangle_{L^2} & -\langle u_0', \mathrm{Im}(\psi-u_0)\rangle_{L^2} \\ -\langle u_0''', \mathrm{Im}(\psi-u_0)\rangle_{L^2} & \langle u_0'', \mathrm{Re}(\psi-u_0)\rangle_{L^2} \end{pmatrix}.$$

The first term in the right-hand side is a fixed invertible matrix and the second term is bounded in norm by $C\epsilon$, hence $D\mathbf{f}(0,0)$ is invertible if ϵ is small enough. In addition, the norm of the inverse of $D\mathbf{f}(0,0)$ is bounded by a constant independent of ϵ. Finally, it is straightforward to verify that the second-order derivatives of $\mathbf{f}$ are uniformly bounded when $\epsilon \leq 1$. These observations together imply that there exists a unique pair (ξ,θ), in a neighborhood of size $\mathcal{O}(\epsilon)$ of the origin, such that $\mathbf{f}(\xi,\theta) = \mathbf{0}$. Thus the decomposition (3.1) holds for these values of (ξ,θ). In addition, the above argument shows that the modulation parameters ξ, θ depend continuously on $\psi \in X$ in the topology defined by the distance (1.9). This concludes the proof. ∎

As was already mentioned, the Cauchy problem for the NLS equation (1.1) is globally well-posed in the space X [13]. If $\psi(\cdot,t)$ is a solution of (1.1) in X which stays for all times in a neighborhood of the orbit of the black soliton, the modulation parameters $\xi(t)$, $\theta(t)$ given by the decomposition (3.1) subject to the orthogonality conditions (3.2) are continuous functions of time. In fact, as in [5, Lemma 6.3], we have the following stronger conclusion:

Lemma 3.2. *If $\epsilon > 0$ is sufficiently small and if $\psi(\cdot,t)$ is any solution of the NLS equation* (1.1) *in X satisfying, for all $t \in \mathbb{R}$,*

$$\inf_{\xi,\theta\in\mathbb{R}} d_R\Big(e^{i\theta}\psi(\cdot+\xi,t),u_0\Big) \le \epsilon, \tag{3.8}$$

then the modulation parameters $\xi(t),\theta(t)$ in the decomposition (3.1), (3.2) *are continuously differentiable functions of t satisfying* (1.13).

Proof. If $\psi(\cdot,t)$ is any solution of the NLS equation (1.1) in X, it is easy to verify that the map $t \mapsto e^{it}\psi(\cdot,t) - \psi(\cdot,0)$ is continuously differentiable in the topology of $L^2(\mathbb{R})$, with

$$\partial_t\Big(e^{it}\psi(\cdot,t)-\psi(\cdot,0)\Big) = ie^{it}\Big(\psi_{xx}+(1-|\psi|^2)\psi\Big) \in L^2(\mathbb{R}).$$

In particular, for all $(\xi,\theta) \in \mathbb{R}^2$, the scalar products

$$\langle u_0'(\cdot-\xi),\mathrm{Re}(e^{i\theta}\psi(\cdot,t))\rangle_{L^2}\,, \quad \langle u_0''(\cdot-\xi),\mathrm{Im}(e^{i\theta}\psi(\cdot,t))\rangle_{L^2}\,,$$

are continuously differentiable functions of time. Thus, if assumption (3.8) holds for all times, the proof of Lemma 3.1 shows that the modulations parameters $\xi(t),\theta(t)$ in the decomposition (3.1), (3.2) are C^1 functions of time.

Differentiating both sides of (3.1) and using (1.1), we obtain the evolution system

$$\begin{cases} u_t = L_-v+\dot\xi(u_0'+u_x)-\dot\theta v+(2u_0u+u^2+v^2)v, \\ -v_t = L_+u-\dot\xi v_x-\dot\theta(u_0+u)+(3u_0u+u^2+v^2)u+u_0v^2, \end{cases}$$

where the operators $L_\pm$ are defined in (2.1). Using the orthogonality conditions (3.2), we eliminate the time derivatives u_t, v_t by taking the scalar product of the first line with u_0' and of the second line with u_0''. This gives the following linear system for the derivatives $\dot\xi$ and $\dot\theta$:

$$B\begin{pmatrix}\dot\xi\\ \dot\theta\end{pmatrix} = \begin{pmatrix}\langle L_-u_0',v\rangle_{L^2}\\ \langle L_+u_0'',u\rangle_{L^2}\end{pmatrix} + \begin{pmatrix}\langle u_0',(2u_0u+u^2+v^2)v\rangle_{L^2}\\ \langle u_0'',(3u_0u+u^2+v^2)u+u_0v^2\rangle_{L^2}\end{pmatrix}, \tag{3.9}$$

where

$$B = \begin{pmatrix}-\|u_0'\|_{L^2}^2 & 0\\ 0 & -\|u_0'\|_{L^2}^2\end{pmatrix} + \begin{pmatrix}-\langle u_0',u_x\rangle_{L^2} & \langle u_0',v\rangle_{L^2}\\ \langle u_0'',v_x\rangle_{L^2} & \langle u_0'',u\rangle_{L^2}\end{pmatrix}. \tag{3.10}$$

As in the proof of Lemma 3.1, it is easy to verify using (3.8) that the second term in the right-hand side of (3.10) is bounded by $C\epsilon$ for some positive constant C, hence the matrix B is invertible if ϵ is small enough, with uniformly bounded inverse. On the other hand, the first term in the right-hand side of (3.9) is of size $\mathcal{O}(\epsilon)$, whereas the second term is $\mathcal{O}(\epsilon^2)$, hence $|\dot\xi(t)|+|\dot\theta(t)| \le C\epsilon$ for all $t \in \mathbb{R}$, where the positive constant C is independent of t. This concludes the proof. □

4 Proof of orbital stability of the black soliton

This final section is entirely devoted to the proof of Theorem 1.2. As in the previous section, we consider solutions of the NLS equation (1.1) of the form (3.1), where the real-valued perturbations u, v satisfy the orthogonality conditions (3.2). Our main task is a detailed analysis of the functional (1.8) in a neighborhood of the orbit of the soliton profile u_0. Instead of using the straightforward decomposition (3.3), the main idea is to express the difference $\Lambda(\psi)-\Lambda(u_0)$ in terms of the variables u, v, and η, where η is defined in (3.4).

Lemma 4.1. *If $\psi = u_0 + u + iv$ satisfies $d_R(\psi, u_0) < \infty$, then*

$$\begin{aligned}\Lambda(\psi) - \Lambda(u_0) \;=\; \int_{\mathbb{R}} \Big(& u_{xx}^2 + v_{xx}^2 + (3u_0^2 - 2)(u_x^2 + v_x^2) + (1 - u_0^2)(u^2 + v^2) \\ & - 3(1 - u_0^2)(1 - 3u_0^2)u^2 + \frac{1}{2}\eta_x^2 + \frac{1}{2}(3u_0^2 - 2)\eta^2 \\ & + \frac{1}{2}\eta^3 + 3\eta(u_x^2 + v_x^2) + 6u_0'(u^2 + v^2)u_x \Big)\,\mathrm{d}x\ . \end{aligned} \tag{4.1}$$

Proof. We observe that $|\psi|^2 = u_0^2 + \eta$ and $\bar\psi\psi_x + \psi\bar\psi_x = 2u_0u_0' + \eta_x$. Thus, if

$$A(\psi) \;=\; |\psi_{xx}|^2 + |\psi_x|^2(3|\psi|^2 - 2) + \frac{1}{2}(\bar\psi\psi_x + \psi\bar\psi_x)^2 + \frac{1}{2}|\psi|^2(1 - |\psi|^2)^2$$

denotes the integrand in the functional $\Lambda = S - 2E$, a direct calculation shows that

$$\begin{aligned}A(\psi) - A(u_0) \;=\; & \mathcal{L}(u, \eta) + 6\eta u_0' u_x + u_{xx}^2 + v_{xx}^2 + (3u_0^2 - 2)(u_x^2 + v_x^2) \\ & + \frac{1}{2}\eta_x^2 + \frac{1}{2}(3u_0^2 - 2)\eta^2 + \frac{1}{2}\eta^3 + 3\eta(u_x^2 + v_x^2), \end{aligned} \tag{4.2}$$

where $\mathcal{L}(u, \eta) = 2u_0''u_{xx} + 2(3u_0^2 - 2)u_0'u_x + 2u_0u_0'\eta_x + \eta(1 - u_0^2)(2 - 3u_0^2)$. We now integrate the right-hand side of (4.2) over $x \in \mathbb{R}$, starting with the terms $\mathcal{L}(u, \eta)$ which are linear in u and η. Using the identities $u_0'' + u_0 - u_0^3 = 0$ and $u_0'''' + (1 - 3u_0^2)u_0'' - 6u_0u_0'^2 = 0$, we find

$$\begin{aligned} 2\int_{\mathbb{R}} \Big(u_0'' u_{xx} + (3u_0^2 - 2)u_0' u_x \Big)\,\mathrm{d}x \;&=\; 2\int_{\mathbb{R}} \Big(u_0'''' - (3u_0^2 - 2)u_0'' - 6u_0u_0'^2 \Big) u\,\mathrm{d}x \\ &=\; 2\int_{\mathbb{R}} u_0'' u\,\mathrm{d}x \;=\; -2\int_{\mathbb{R}} (1 - u_0^2)u_0 u\,\mathrm{d}x. \end{aligned}$$

Similarly, as $2(u_0u_0')' = (1 - u_0^2)(1 - 3u_0^2)$, we have

$$2\int_{\mathbb{R}} u_0u_0'\eta_x\,\mathrm{d}x = -2\int_R (u_0u_0')'\eta\,\mathrm{d}x = -\int_{\mathbb{R}} (1 - u_0^2)(1 - 3u_0^2)\eta dx.$$

We conclude that

$$\int_{\mathbb{R}} \mathcal{L}(u, \eta)\,\mathrm{d}x \;=\; \int_{\mathbb{R}} (1 - u_0^2)(\eta - 2u_0u)\,\mathrm{d}x \;=\; \int_{\mathbb{R}} (1 - u_0^2)(u^2 + v^2)\,\mathrm{d}x. \tag{4.3}$$

Note that (4.3) is now quadratic in u and v, which could be expected since u_0 is a critical point of the functional Λ. We next consider the quadratic term $6\eta u_0'u_x$ in (4.2), which has no definite sign. Using the definition (3.4), we find $6\eta u_0'u_x = 12u_0u_0'uu_x + 6u_0'(u^2 + v^2)u_x$, and integrating by parts, we obtain

$$6\int_{\mathbb{R}} \eta u_0' u_x\,\mathrm{d}x \;=\; -3\int_{\mathbb{R}} (1 - u_0^2)(1 - 3u_0^2)u^2\,\mathrm{d}x + 6\int_{\mathbb{R}} u_0'(u^2 + v^2)u_x\,\mathrm{d}x. \tag{4.4}$$

Now, combining (4.2), (4.3), and (4.4), we arrive at (4.1). □

To simplify the notations, we define

$$\begin{aligned}
B_0(u) &= u_{xx}^2 + (5u_0^2 - 2)u_x^2 + (9u_0^2 - 5u_0^4 - 2)u^2 \\
B_1(u) &= u_{xx}^2 + (3u_0^2 - 2)u_x^2 + (1 - u_0^2)u^2 - 3(1 - u_0^2)(1 - 3u_0^2)u^2 \\
B_2(v) &= v_{xx}^2 + (3u_0^2 - 2)v_x^2 + (1 - u_0^2)v^2 \\
B_3(\eta) &= \tfrac{1}{2}\eta_x^2 + \tfrac{1}{2}(3u_0^2 - 2)\eta^2.
\end{aligned} \tag{4.5}$$

The quadratic terms in the right-hand side of (4.1) can be written in the compact form

$$Q(u, v, \eta) = \int_{\mathbb{R}} \Big(B_1(u) + B_2(v) + B_3(\eta)\Big)\,\mathrm{d}x. \tag{4.6}$$

We see that $Q(u, v, \eta)$ contains $\langle K_- v, v\rangle \equiv \int_{\mathbb{R}} B_2(v)\,\mathrm{d}x$, but not $\langle K_+ u, u\rangle \equiv \int_{\mathbb{R}} B_0(u)\,\mathrm{d}x$. Instead, it only contains $\int_{\mathbb{R}} B_1(u)\,\mathrm{d}x$ and $\int_{\mathbb{R}} B_3(\eta)\,\mathrm{d}x$. This discrepancy is due to that fact that the variables u and η are not independent. As $\eta = 2u_0 u + u^2 + v^2$, the quantity $\int_{\mathbb{R}} B_3(\eta)\,\mathrm{d}x$ also contains quadratic terms in u and u_x, which should be added to $\int_{\mathbb{R}} B_1(u)\,\mathrm{d}x$ to obtain $\int_{\mathbb{R}} B_0(u)\,\mathrm{d}x$.

Due to the relation between u and η, it is not obvious that each quadratic term in (4.6) is positive independently of the others. To avoid that difficulty, we fix some $R \geq 1$ (which will be chosen large enough below) and we split the integration domain into two regions. When $|x| \leq R$, we replace η by $2u_0 u + u^2 + v^2$, and we use extensions of Lemmas 2.6 and 2.8 to prove positivity of the quadratic terms in (4.6). In the outer region $|x| > R$, the analysis is much simpler, because the expressions $B_1(u)$, $B_2(v)$, and $B_3(\eta)$ are obviously positive if R is large enough.

Since η is a nonlinear function of u and v, the analysis of the quadratic expression (4.6) will produce higher-order terms, which will be controlled using a smallness assumption on the distance $d_R(\psi, u_0)$. To that purpose, we find it convenient to introduce the quantity

$$\rho^2(u, v, \eta) \;=\; \int_{\mathbb{R}} \Big(u_{xx}^2 + v_{xx}^2 + u_x^2 + v_x^2\Big)\,\mathrm{d}x + \int_{|x|\leq R} \Big(u^2 + R^{-2}v^2\Big)\,\mathrm{d}x + \int_{|x|\geq R} \Big(\eta_x^2 + \eta^2\Big)\,\mathrm{d}x, \tag{4.7}$$

which is equivalent to the squared distance (3.5) in a neighborhood of u_0. Indeed, we have the following elementary result:

Lemma 4.2. *Fix $R \geq 1$, and assume that $\psi = u_0 + u + iv$, where $u, v \in H^2_{\mathrm{loc}}(\mathbb{R})$ are real-valued. Let $d_R(\psi, u_0)$ be given by (3.5) and $\rho(u, v, \eta)$ by (4.7).*

a) *One has $d_R(\psi, u_0) < \infty$ if and only if $\rho(u, v, \eta) < \infty$.*

b) *There exists a constant $C_0 \geq 1$ (independent of R) such that, if $d_R(\psi, u_0) \leq 1$ or if $R^{1/2}\rho(u, v, \eta) \leq 1$, then*

$$C_0^{-1}\rho(u, v, \eta) \leq d_R(\psi, u_0) \leq C_0 R\rho(u, v, \eta). \tag{4.8}$$

Proof. Throughout the proof, we denote $d_R(\psi, u_0)$ by d_R and $\rho(u, v, \eta)$ simply by ρ. We proceed in three steps.

Step 1: Assume first that $d_R < \infty$, so that $u_x, v_x \in H^1(\mathbb{R})$, $u, v \in L^2(-R, R)$, and $\eta \in L^2(\mathbb{R})$, where $\eta = |\psi|^2 - |u_0|^2 = 2u_0 u + u^2 + v^2$. We claim that $u, v \in L^\infty(\mathbb{R})$ and that

$$K \;:=\; \|u\|_{L^\infty(\mathbb{R})} + \|v\|_{L^\infty(\mathbb{R})} \;\leq\; C(1 + d_R), \tag{4.9}$$

for some universal constant $C > 0$. Indeed, if $f = |\psi| - |u_0|$, we observe that

$$d_R^2 \ge \int_{\mathbb{R}} \eta^2 \, \mathrm{d}x \ge \int_{|x|\ge 1} (|\psi| - |u_0|)^2 (|\psi| + |u_0|)^2 \, \mathrm{d}x \ge C \int_{|x|\ge 1} f^2 \, \mathrm{d}x,$$

hence $f \in L^2(I)$, where $I = \{x \in \mathbb{R} : |x| \ge 1\}$, and $\|f\|_{L^2(I)} \le C d_R$. Moreover, we have $|f_x| \le 2u_0' + |u_x| + |v_x|$ almost everywhere, hence $f_x \in L^2(\mathbb{R})$ and $\|f_x\|_{L^2(\mathbb{R})} \le C(1 + d_R)$. By Sobolev embedding, this implies that $f \in L^\infty(I)$, hence also $u, v \in L^\infty(I)$, and we have the bound $\|u\|_{L^\infty(I)} + \|v\|_{L^\infty(I)} \le C(1 + d_R)$. Finally, since $\|u_x\|_{L^2(\mathbb{R})} + \|v_x\|_{L^2(\mathbb{R})} \le C d_R$, we conclude that $u, v \in L^\infty(\mathbb{R})$ and that (4.9) holds.

Step 2: Next, we assume that $\rho < \infty$, so that $u_x, v_x \in H^1(\mathbb{R})$, $u, v \in L^2(-R, R)$, and $\eta \in H^1(I_R)$, where $I_R = \{x \in \mathbb{R} : |x| \ge R\}$. We claim that $u, v \in L^\infty(\mathbb{R})$ and that

$$K := \|u\|_{L^\infty(\mathbb{R})} + \|v\|_{L^\infty(\mathbb{R})} \le C(1 + R^{1/2}\rho), \tag{4.10}$$

for some universal constant $C > 0$. Indeed, we know that $\eta \in L^\infty(I_R)$ with $\|\eta\|_{L^\infty(I_R)} \le C\rho$. This implies that $\psi \in L^\infty(I_R)$, hence also $u, v \in L^\infty(I_R)$, and that $\|u\|_{L^\infty(I_R)} + \|v\|_{L^\infty(I_R)} \le C(1+\rho)^{1/2}$. On the other hand, we know that $\|u\|_{L^\infty(-R,R)} \le C\|u\|_{H^1(-R,R)} \le C\rho$ and that

$$\|v\|_{L^\infty(-R,R)} \le C\biggl(\frac{\|v\|_{L^2(-R,R)}}{R^{1/2}} + \|v\|_{L^2(-R,R)}^{1/2} \|v_x\|_{L^2(-R,R)}^{1/2}\biggr) \le C R^{1/2} \rho,$$

because $\|v\|_{L^2(-R,R)} \le R\rho$ and $\|v_x\|_{L^2(-R,R)} \le \rho$. Thus we conclude that $u, v \in L^\infty(\mathbb{R})$ and that (4.10) holds.

Step 3: Finally we assume that $K = \|u\|_{L^\infty(\mathbb{R})} + \|v\|_{L^\infty(\mathbb{R})} < \infty$, which is the case if $d_R < \infty$ or if $\rho < \infty$. As $\eta = 2u_0 u + u^2 + v^2$, we find

$$\|\eta\|_{L^2(-R,R)} \le C(1 + K)\Bigl(\|u\|_{L^2(-R,R)} + \|v\|_{L^2(-R,R)}\Bigr) \le C(1 + K) R\rho,$$

because $\|u\|_{L^2(-R,R)} \le \rho$ and $\|v\|_{L^2(-R,R)} \le R\rho$. This shows that, if $\rho < \infty$, then $\eta \in L^2(\mathbb{R})$, so that $d_R < \infty$, and we have the bound $d_R \le C(1 + K)R\rho$. Conversely, since $\eta_x = 2(u_0' u + u_0 u_x + u u_x + v v_x)$, we obtain

$$\|\eta_x\|_{L^2(\mathbb{R})} \le C(1 + K)\Bigl(\|u\|_{L^2(-1,1)} + \|u_x\|_{L^2(\mathbb{R})} + \|v_x\|_{L^2(\mathbb{R})}\Bigr) \le C(1 + K) d_R,$$

where to estimate $u_0' u$ we used the fact that $|u(x)| \le C(\|u\|_{L^2(-1,1)} + (1 + |x|)^{1/2} \|u_x\|_{L^2(\mathbb{R})})$. This shows that, if $d_R < \infty$, then $\eta_x \in L^2(\mathbb{R})$, so that $\rho < \infty$, and we have the bound $\rho \le C(1+K) d_R$. This concludes the proof. $\square$

In the calculations below, to avoid boundary terms when integrating by parts in expressions such as (4.6), it is technically convenient to split the integration domain using a smooth partition of unity. Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth cut-off function such that

$$\chi(x) = 1 \quad \text{for} \quad |x| \le \frac{1}{2}, \qquad \text{and} \qquad \chi(x) = 0 \quad \text{for} \quad |x| \ge \frac{3}{2}.$$

We further assume that χ is even, that $\chi'(x) \le 0$ for $x \ge 0$, and that $\chi(1) = \frac{1}{2}$. Given $R \ge 1$, we denote $\chi_R(x) = \chi(x/R)$. The following estimates will be useful to control the functions u, v on the support of χ_R'.

Lemma 4.3. *Fix $R \ge 1$, and assume that $\psi = u_0 + u + iv$ satisfies $d_R(\psi, u_0) < \infty$. Then there exists a constant $C_1 > 0$ (independent of R) such that*

$$\|u\|_{L^2(-2R,2R)} \;\le\; C_1(\rho(u,v,\eta) + R^{3/2}\rho(u,v,\eta)^2), \tag{4.11}$$

$$\|u\|_{L^\infty(-2R,2R)} + \|v\|_{L^\infty(-2R,2R)} \;\le\; C_1 R^{1/2}\rho(u,v,\eta), \tag{4.12}$$

where $\rho(u,v,\eta)$ is given by (4.7).

Proof. If f is either u or v, then $|f(x)| \le C(R^{-1/2}\|f\|_{L^2(-R,R)} + (|x|+R)^{1/2}\|f_x\|_{L^2(\mathbb{R})})$, and this gives the bound (4.12). To prove estimate (4.11), we recall that $\|u\|_{L^2(-R,R)} \le \rho(u,v,\eta)$, so we only need to control $u(x)$ for $R \le |x| \le 2R$. In that region we have $|u| \le C(|\eta| + u^2 + v^2)$, hence using the bound (4.12) and the fact that $\|\eta\|_{L^2(|x|\ge R)} \le \rho(u,v,\eta)$ we obtain the desired result. ☐

We now analyze the quadratic terms in the representation (4.6).

Lemma 4.4. *Under the assumptions of Lemma 4.2, if $d_R(\psi,u_0) \le 1$, we have*

$$\int_{\mathbb{R}} \Big(B_1(u) + B_3(\eta)\Big)\chi_R(x)\,\mathrm{d}x \;=\; \int_{\mathbb{R}} B_0(u)\chi_R(x)\,\mathrm{d}x + \mathcal{O}(R^3\rho(u,v,\eta)^3 + e^{-R}\rho(u,v,\eta)^2), \tag{4.13}$$

where the estimate in the big O term holds uniformly for $R \ge 1$.

Proof. Since $\eta = 2u_0u + u^2 + v^2$, we find by a direct calculation

$$B_3(\eta) \;=\; 2u_0'^2u^2 + 2u_0^2u_x^2 + 4u_0u_0'uu_x + 2(3u_0^2-2)u_0^2u^2 + \tilde{N}(u,v),$$

where

$$\begin{aligned}\tilde{N}(u,v) \;=\;& 4(uu_x+vv_x)(u_0'u+u_0u_x) + 2(uu_x+vv_x)^2\\ &+ 2(3u_0^2-2)u_0u(u^2+v^2) + \frac{1}{2}(3u_0^2-2)(u^2+v^2)^2.\end{aligned}$$

In view of the definitions (4.5), this implies that

$$B_1(u) + B_3(\eta) \;=\; B_0(u) + (2u_0u_0'u^2)_x + \tilde{N}(u,v).$$

If we now multiply both sides by $\chi_R(x)$ and integrate over $x \in \mathbb{R}$, we arrive at (4.13), because it is straightforward to verify using (4.7), (4.9) and (4.12) that

$$-2\int_{\mathbb{R}} u_0u_0'u^2\chi_R'(x)\,\mathrm{d}x \;=\; \mathcal{O}(e^{-R}\rho(u,v,\eta)^2), \quad \text{and} \quad \int_{\mathbb{R}} \tilde{N}(u,v)\chi_R(x)\,\mathrm{d}x \;=\; \mathcal{O}(R^3\rho(u,v,\eta)^3).$$

This concludes the proof of the lemma. ☐

Using Lemma 4.4, we are able to derive the desired lower bound on the difference $\Lambda(\psi)-\Lambda(u_0)$ in terms of the quantity $\rho(u,v,\eta)$.

Proposition 4.5. *If $R \ge 1$ is sufficiently large, there exists a constant $C_2 > 0$ such that, if $\psi = u_0 + u + iv$ satisfies $d_R(\psi,u_0) \le 1$ and if $\langle u_0', u\rangle_{L^2} = \langle u_0'', v\rangle_{L^2} = 0$, then*

$$\Lambda(\psi) - \Lambda(u_0) \;\ge\; C_2\rho(u,v,\eta)^2 + \mathcal{O}(R^3\rho(u,v,\eta)^3), \tag{4.14}$$

where the estimate in the big O term is uniform in R.

Proof. Proceeding as in the proof of Lemma 4.2, it is easy to estimate the cubic terms in (4.1) in terms of $\rho(u,v,\eta)$ using, in particular, the uniform bound (4.9) and the estimate (4.12). We thus find

$$\Lambda(\psi) - \Lambda(u_0) \;=\; Q(u,v,\eta) + \mathcal{O}(R^3\rho(u,v,\eta)^3), \tag{4.15}$$

where $Q(u,v,\eta)$ is given by (4.5) and (4.6). Then, in the definition (4.6), we split the integral using the partition of unity $1 = \chi_R + (1-\chi_R)$ and we use Lemma 4.4. This gives

$$\begin{aligned} Q(u,v,\eta) \;&=\; \int_{\mathbb{R}} B_2(v)\,\mathrm{d}x + \int_{\mathbb{R}} B_0(u)\chi_R(x)\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} \Big(B_1(u) + B_3(\eta)\Big)(1-\chi_R(x))\,\mathrm{d}x + \mathcal{O}(R^3\rho(u,v,\eta)^3 + e^{-R}\rho(u,v,\eta)^2). \end{aligned} \tag{4.16}$$

As $\langle u_0'', v\rangle = 0$, we know from Lemmas 2.3 and 2.8 that

$$\int_{\mathbb{R}} B_2(v)\,\mathrm{d}x \;\ge\; C\int_{\mathbb{R}} (v_{xx}^2 + v_x^2)\,\mathrm{d}x + \frac{C}{R^2}\int_{|x|\le R} v^2\,\mathrm{d}x, \tag{4.17}$$

where the last term in the right-hand side follows from the bound $|v(x)| \le |v(0)| + |x|^{1/2}\|v_x\|_{L^2}$, which implies

$$\int_{|x|\le R} v^2\,\mathrm{d}x \;\le\; 4R|v(0)|^2 + 2R^2\int_{\mathbb{R}} v_x^2\,\mathrm{d}x \;\le\; CR^2\int_{\mathbb{R}} B_2(v)\,\mathrm{d}x.$$

On the other hand, if $R \ge 1$ is large enough so that $3u_0^2 - 2 \ge \frac{1}{2}$ for $|x| \ge R$, it is clear from (4.5) that

$$\int_{\mathbb{R}} \Big(B_1(u) + B_3(\eta)\Big)(1-\chi_R(x))\,\mathrm{d}x \;\ge\; C\int_{|x|\ge R} (u_{xx}^2 + u_x^2 + \eta_x^2 + \eta^2)\,\mathrm{d}x. \tag{4.18}$$

Finally, we estimate from below the term $\int_{\mathbb{R}} B_0(u)\chi_R(x)\,\mathrm{d}x$ under the orthogonality assumption $\langle u_0', u\rangle_{L^2} = 0$. Arguing as in Lemma 2.1 and Corollary 2.3, we introduce the auxiliary variable $w = u_x + \sqrt{2}u_0 u$. After integrating by parts, we obtain the identity

$$\int_{\mathbb{R}} B_0(u)\chi_R(x)\,\mathrm{d}x = \int_{\mathbb{R}} \Big(w_x^2 + w^2\Big)\chi_R(x)\,\mathrm{d}x + J_R,$$

where

$$J_R \;=\; \int_{\mathbb{R}} \Big(\sqrt{2}u_0 u_x^2 + 2\sqrt{2}u_0' u u_x + (2u_0u_0' - \sqrt{2}u_0'')u^2 + \sqrt{2}u_0^2 u^2\Big)\chi_R'(x)\,\mathrm{d}x.$$

Since $\chi_R'(x) = R^{-1}\chi'(x/R)$, we have using the estimate (4.11)

$$|J_R| \;\le\; \frac{C}{R}\int_{|x|\le 3R/2} \Big(u_x^2 + u^2\Big)\,\mathrm{d}x \;\le\; \frac{C_3\rho(u,v,\eta)^2}{R} + \mathcal{O}(R^2\rho(u,v,\eta)^4),$$

where $C_3 > 0$ is independent of R. Moreover, proceeding as in the proof of Lemma 2.6, we find

$$\int_{|x|\le R} \Big(u_{xx}^2 + u_x^2 + u^2\Big)\,\mathrm{d}x \;\le\; C\int_{|x|\le R} \Big(w_x^2 + w^2\Big)\,\mathrm{d}x + \mathcal{O}(e^{-R}\rho(u,v,\eta)^2). \tag{4.19}$$

Indeed, we have the representation $u = Au_0' + W$, where the function W is defined in (2.10) and the constant A is fixed by the orthogonality condition $\langle u_0', u\rangle_{L^2} = 0$. The proof of Lemma 2.6 shows that $\|W\|_{L^2(|x|\le R)} \le C\|w\|_{L^2(|x|\le R)}$. From the orthogonality relation

$$0 \;=\; \int_{|x|\le R} u_0'(x)\Big(Au_0'(x) + W(x)\Big)\,\mathrm{d}x + \int_{|x|\ge R} u_0'(x)u(x)\,\mathrm{d}x,$$

we easily obtain the bound $|A| \le C\|W\|_{L^2(|x|\le R)} + \mathcal{O}(e^{-R}\rho(u,v,\eta))$. This shows that

$$\|u\|_{L^2(|x|\le R)} \;\le\; C\|w\|_{L^2(|x|\le R)} + \mathcal{O}(e^{-R}\rho(u,v,\eta)),$$

and since $u_x = w - \sqrt{2}u_0 u$ we obtain similar estimates for the derivatives u_x and u_{xx}, which altogether give (4.19). Summarizing, we have shown

$$\begin{aligned}\int_{\mathbb{R}} B_0(u)\chi_R(x)\,\mathrm{d}x \;\ge\; C\int_{|x|\le R}\Big(u_{xx}^2 + u_x^2 + u^2\Big)\,\mathrm{d}x - \frac{C_3\rho(u,v,\eta)^2}{R}\\ + \mathcal{O}(R^2\rho(u,v,\eta)^3 + e^{-R}\rho(u,v,\eta)^2),\end{aligned} \tag{4.20}$$

where in the big O term we replaced $R^2\rho(u,v,\eta)^4$ with $R^2\rho(u,v,\eta)^3$ using the fact that $\rho(u,v,\eta) \le C_0 d_R(\psi,u_0) \le C_0$ by (4.8). Now, combining (4.15), (4.16), (4.17), (4.18), (4.20), and taking $R \ge 1$ sufficiently large, we arrive at (4.14). □

Corollary 4.6. *Fix any $R \ge 1$. There exist $\epsilon_1 \in (0,1)$ and $C_4 \ge 1$ such that, if $\psi = u_0 + u + iv$ satisfies $d_R(\psi,u_0) \le \epsilon_1$ and if $\langle u_0', u\rangle_{L^2} = \langle u_0'', v\rangle_{L^2} = 0$, then*

$$C_4^{-1}d_R(\psi,u_0)^2 \le \Lambda(\psi) - \Lambda(u_0) \;\le\; C_4 d_R(\psi,u_0)^2. \tag{4.21}$$

Proof. Choose $R \ge 1$ large enough so that the conclusion of Proposition 4.5 holds, and $\rho_0 > 0$ small enough so that $R^3\rho_0 \ll C_2$, where C_2 is as in (4.14). Take $\epsilon_1 \le 1$ such that $C_0\epsilon_1 \le \rho_0$, where C_0 is as in (4.8). If $\psi = u_0 + u + iv$ satisfies $d_R(\psi,u_0) \le \epsilon_1$ and $\langle u_0', u\rangle_{L^2} = \langle u_0'', v\rangle_{L^2} = 0$, it follows from (4.8) that the quantity $\rho(u,v,\eta)$ defined in (4.7) satisfies $\rho(u,v,\eta) \le \rho_0$. By Proposition 4.5, we thus have

$$\frac{1}{2}C_2\rho(u,v,\eta)^2 \;\le\; \Lambda(\psi) - \Lambda(u_0) \le C_2'\rho(u,v,\eta)^2,$$

where the lower bound follows from (4.14), and the upper bound can be established by a much simpler argument (which does not use any orthogonality condition). Since $\rho(u,v,\eta)$ is equivalent to $d_R(\psi,u_0)$ by Lemma 4.2, we obtain (4.21). Finally, Corollary 4.6 holds for any $R \ge 1$ because different values of R give equivalent distances d_R on X. □

It is now easy to conclude the proof of Theorem 1.2. Fix any $R \ge 1$. Given any $\epsilon > 0$, we take

$$\delta \;=\; \frac{1}{2C_4}\,\min(2\epsilon, \epsilon_0, \epsilon_1),$$

where $C_4 \ge 1$ and $\epsilon_1 > 0$ are as in Corollary 4.6 and $\epsilon_0 > 0$ is as in Lemma 3.1. If $\psi_0 \in X$ satisfies $d_R(\psi_0,u_0) \le \delta$, then $\Lambda(\psi_0) - \Lambda(u_0) \le C_4\delta^2$ by the upper bound in (4.21), which does not require any orthogonality condition. Since Λ is a conserved quantity, we deduce that the solution

$\psi(\cdot,t)$ of the cubic NLS equation (1.1) with initial data ψ_0 satisfies $\Lambda(\psi(\cdot,t)) - \Lambda(u_0) \le C_4\delta^2$ for all $t \in \mathbb{R}$. We claim that, for all $t \in \mathbb{R}$, we have

$$\inf_{\xi,\theta\in\mathbb{R}} d_R\Big(e^{i\theta}\psi(\cdot+\xi,t), u_0\Big) \;\le\; 2C_4\delta \le \epsilon_0. \tag{4.22}$$

Indeed, the bound (4.22) holds for $t=0$ by assumption. Let $\mathcal{J} \subset \mathbb{R}$ be the largest time interval containing the origin such that the bound (4.22) holds for all $t \in \mathcal{J}$. As is well-known [6, 13], the solutions of the cubic NLS equation (1.1) with initial data in X depend continuously on time with respect to the distance $d_R(\psi, u_0)$. This implies that the left-hand side of the bound (4.22) is a continuous function of t, so that $\mathcal{J}$ is closed. On the other hand, if $t \in \mathcal{J}$, then by Lemma 3.1 we can find $\xi, \theta \in \mathbb{R}$ such that the function $\tilde\psi(x) = e^{i(\theta+t)}\psi(x+\xi,t)$ can be decomposed as in (3.7) with u, v satisfying the orthogonality conditions (3.2). Applying Corollary 4.6 to $\tilde\psi$, we deduce that

$$C_4^{-1} d_R(\tilde\psi, u_0)^2 \le \Lambda(\tilde\psi) - \Lambda(u_0) \;=\; \Lambda(\psi_0) - \Lambda(u_0) \le C_4\delta^2,$$

so that $d_R(\tilde\psi, u_0) \le C_4\delta$. Using again a continuity argument, we conclude that $\mathcal{J}$ contains a neighborhood of t. Thus $\mathcal{J}$ is open, hence finally $\mathcal{J} = \mathbb{R}$, so that the bound (4.22) holds for all $t \in \mathbb{R}$. Using Lemma 3.1, we thus obtain modulations parameters $\xi(t)$, $\theta(t)$ such that

$$d_R\Big(e^{i(\theta(t)+t)}\psi(\cdot+\xi(t),t)\,,u_0\Big) \;\le\; C_4\delta \le \epsilon, \qquad t \in \mathbb{R}.$$

Finally, Lemma 3.2 shows that the functions $\xi : \mathbb{R} \to \mathbb{R}$ and $\theta : \mathbb{R} \to \mathbb{R}/(2\pi\mathbb{Z})$ are continuously differentiable and satisfy the bounds (1.13). The proof of Theorem 1.2 is now complete.

Remark 4.7. Instead of introducing the auxiliary variable η to cure the imperfect decomposition (3.1), it would be advantageous to find a parametrization of the perturbations that fully takes into account the geometry of the functional Λ, and in particular the degeneracy of $\Lambda''(u_0)$. Near the constant solution $u_1 \equiv 1$, it is most natural to write $\psi(x,t) = (1 + r(x,t))e^{i\varphi(x,t)}$, where r and φ are real-valued functions. In that case, the usual energy function (1.5) allows us to control r in $H^1(\mathbb{R})$ and φ_x in $L^2(\mathbb{R})$. In the same spirit, it is tempting to consider perturbations of the black soliton of the form

$$\psi(x,t) \;=\; (u_0(x) + r(x,t))e^{i\varphi(x,t)}, \quad x \in \mathbb{R}, \tag{4.23}$$

where r, φ are again real-valued functions. With this representation, we find

$$\Lambda(\psi) - \Lambda(u_0) \;=\; \langle K_+ r, r\rangle + \int_{\mathbb{R}} \Big(u_0^2\varphi_{xx}^2 + \varphi_x^2)\Big)\,\mathrm{d}x + \tilde N(r,\varphi_x), \tag{4.24}$$

where $\tilde N(r,\varphi_x)$ collects the higher order terms. This formula is interesting, because it is not difficult to verify that $\tilde N(r,\varphi_x)$ can be controlled by the quadratic terms in (4.24) if r is small in $H^2(\mathbb{R})$ and φ_x small in $H^1(\mathbb{R})$. However, not all perturbations of the black soliton can be written in the form (4.23) with r, φ satisfying such smallness conditions, because u_0 vanishes at $x = 0$ in (4.23).

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