Infinite energy solutions of the two-dimensional Navier-Stokes equations

Thierry Gallay
Université de Grenoble I et CNRS
Institut Fourier, UMR 5582
100 rue des maths, B.P. 74
38402 Saint-Martin-d'Hères, France
`Thierry.Gallay@ujf-grenoble.fr`

Abstract

These notes are based on a series of lectures delivered by the author at the University of Toulouse in February 2014. They are entirely devoted to the initial value problem and the long-time behavior of solutions for the two-dimensional incompressible Navier-Stokes equations, in the particular case where the domain occupied by the fluid is the whole plane $\mathbb{R}^2$ and the velocity field is only assumed to be bounded. In this context, local well-posedness is not difficult to establish [19], and a priori estimates on the vorticity distribution imply that all solutions are global and grow at most exponentially in time [20, 40]. Moreover, as was recently shown by S. Zelik, localized energy estimates can be used to obtain a much better control on the uniformly local energy norm of the velocity field [46]. The aim of these notes is to present, in an explanatory and self-contained way, a simplified and optimized version of Zelik's argument which, in combination with a new formulation of the Biot-Savart law for bounded vorticities, allows one to show that the L^∞ norm of the velocity field grows at most linearly in time. The results do not rely on the viscous dissipation, and remain therefore valid for the so-called "Serfati solutions" of the two-dimensional Euler equations [2]. In the viscous case, a recent work by S. Slijepčević and the author shows that all solutions stay uniformly bounded if the velocity field and the pressure are periodic in a given space direction [16, 17].

Résumé

Ces notes s'appuient sur un cours donné par l'auteur à l'université de Toulouse en février 2014. Elles sont entièrement consacrées au problème de Cauchy et au comportement en temps grands des solutions des équations de Navier-Stokes incompressibles à deux dimensions, dans le cas particulier où le domaine occupé par le fluide est le plan $\mathbb{R}^2$ tout entier et où le champ de vitesse est seulement supposé borné. Dans ce contexte, il n'est pas difficile de montrer que le problème est localement bien posé [19], et des estimations a priori sur le tourbillon impliquent que toutes les solutions sont globales et que leur croissance temporelle est au plus exponentielle [20, 40]. En outre, comme l'a récemment montré S. Zelik, on peut utiliser des estimations d'énergie localisées pour obtenir un contrôle beaucoup plus précis sur le champ de vitesse dans l'espace d'énergie uniformément local [46]. Le but de ces notes est de présenter, de façon pédagogique et indépendante, une version simplifiée et optimisée de l'argument de Zelik qui, combinée avec une nouvelle formulation de la loi de Biot-Savart pour des tourbillons bornés, permet de montrer que la croissance temporelle du champ de vitesse est au plus linéaire. Ces résultats sont établis sans utiliser la dissipation d'énergie due à la viscosité, et restent donc valables pour les solutions dites "de Serfati" des équations d'Euler en dimension deux [2]. Dans le cas visqueux, un travail récent de S. Slijepčević et de l'auteur montre que toutes les solutions demeurent uniformément bornées si le champ de vitesse et la pression sont périodiques dans une direction donnée du plan [16, 17].

Contents

1 Introduction

The aim of these notes is to present in a unified and rather self-contained way a set of recent results by various authors which give some valuable insight into the dynamics of the incompressible Navier-Stokes equations in large or unbounded two-dimensional domains. We must immediately point out that the restriction to the two-dimensional case is more a technical necessity than a deliberate choice: all questions that are discussed below would be equally important and considerably more challenging for three-dimensional fluids, but in the present state of affairs we simply do not know how to address them mathematically. It should be mentioned, however, that there exist situations where a two-dimensional approximation is undoubtedly relevant for real fluids. This is the case, for instance, when the aspect ratio of the domain containing the fluid is very large, so that the motion in one space direction can be neglected under certain conditions. Large-scale oceanic motion is a typical example that is good to keep in mind, although in that particular case a realistic model should take into account additional effects such as the Coriolis force, the wind forcing at the free surface, the topography of the bottom, or the energy dissipation in boundary layers.

If the Navier-Stokes equations are considered in a smooth two-dimensional domain $\Omega \subset \mathbb{R}^2$, with no-slip boundary conditions, it is well known that there exists a unique global solution in the energy space $L^2(\Omega)$ if the initial data have finite kinetic energy. This fundamental result was first established by J. Leray in the particular situation where Ω is the whole plane $\mathbb{R}^2$ [27]. Bounded domains were also considered by Leray, who proved local well-posedness in that case as well as global well-posedness for small initial data [28]. The restriction on the size of the data was completely removed later [26], and the existence proof was subsequently written in a nice functional-analytic setting [13] which is applicable to essentially arbitrary domains with smooth boundary, including for example exterior domains [25]. If no exterior force is exerted on the fluid, the kinetic energy is a nonincreasing function of time that converges to zero as $t \to \infty$, see [32]. The rate of convergence is exponential if Ω is bounded, due to the boundary conditions, and in the unbounded case it depends on the localization properties of the initial data [41, 45]. To conclude this brief survey, we also mention that infinite-energy solutions can be considered in unbounded two-dimensional domains, and may exhibit nontrivial long-time asymptotics. For instance, in the whole plane $\mathbb{R}^2$, solutions with integrable vorticity distribution but nonzero total circulation have infinite kinetic energy, and converge toward nontrivial self-similar solutions as $t \to \infty$ [18].

The results mentioned above, and many others that were omitted, may sometimes lead to the hasty conclusion that "everything is known" about the dynamics of the two-dimensional Navier-Stokes equations. This is of course deeply incorrect, and a more careful thinking reveals that even simple and natural questions still lack a satisfactory answer. Here is a typical example, which motivates some of the questions investigated in the present notes. Consider the free evolution of a viscous incompressible fluid in a bounded two-dimensional domain $\Omega \subset \mathbb{R}^2$, with no-slip boundary conditions. We are interested in the situation where the domain is very large compared to the length scale given by the kinematic viscosity and the typical size of the velocity; in other words, the *Reynolds number* of the flow is very high. If $D \subset \Omega$ is a small subdomain located far from the boundary $\partial\Omega$, we are interested in estimating the kinetic energy of the fluid in the observation domain D at a given time t. That energy is certainly smaller than the total kinetic energy of the fluid at time t, which in turn is less than the same quantity at initial time, but such an estimate is ridiculously non optimal. When sailing the ocean, nobody expects that the total energy of the sea, or a substantial fraction of it, could suddenly get concentrated in a small neighborhood of the boat, and we certainly do not suggest this mechanism as a possible explanation for the formation of rogue waves! It is intuitively clear that the energy in the subdomain D at time t should be essentially independent of the size of the domain Ω and of the total kinetic energy of the fluid; instead it should be possible to estimate that quantity in terms of the size of D and the initial energy *density* only, but to the author's knowledge no such result has been established so far. In a more mathematical language, we are lacking *uniformly local energy estimates* for the fluid velocity that would hold uniformly in time and depend only on the initial energy density. Such estimates would tell us how the energy can be redistributed in the system, due to advection and diffusion, until it is dissipated by the viscosity.

Since the questions we have just mentioned are independent of the size of the fluid domain and of the exact nature of the boundary conditions, it seems reasonable to attack them first in the idealized situation where the fluid fills the whole plane $\mathbb{R}^2$ and the velocity field is merely bounded. We thus consider the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla)u \,=\, \nu\Delta u - \frac{1}{\rho}\nabla\pi \;, \qquad \operatorname{div} u \,=\, 0 \;, \tag{1.1}$$

where the vector field $u(x,t) \in \mathbb{R}^2$ is the velocity of the fluid at point $x \in \mathbb{R}^2$ and time $t \in \mathbb{R}_+$, and the scalar field $\pi(x,t) \in \mathbb{R}$ is the pressure in the fluid at the same point. The physical parameters in (1.1) are the kinematic viscosity $\nu > 0$ and the fluid density $\rho > 0$, which are both assumed to be constant. To eliminate the fluid density from (1.1), we introduce the new function $p = \pi/\rho$, which we still call (somewhat incorrectly) the "pressure" in the fluid. Many authors also eliminate the kinematic viscosity by an appropriate rescaling, but dimensionality checks then become more cumbersome, so we prefer keeping the parameter ν.

The first equation in (1.1) corresponds to Newton's equation for a fluid particle moving under the action of the pressure gradient $-\nabla p$ and the internal friction $\nu\Delta u$, whereas the relation $\operatorname{div} u = 0$ is the mathematical formulation of the incompressibility of the fluid. The nonlinear advection term in (1.1) is due to the definition of the velocity field in the Eulerian representation, which implies that the acceleration of a fluid particle located at point $x \in \mathbb{R}^2$ is not $\partial_t u(x,t)$ but $\partial_t u(x,t) + (u(x,t)\cdot\nabla)u(x,t)$. No evolution equation for the pressure is needed, because p can be expressed as a nonlinear and nonlocal function of the velocity field u by solving the elliptic equation

$$-\Delta p \,=\, \mathrm{div}((u\cdot\nabla)u)\;, \tag{1.2}$$

which is obtained by taking the divergence with respect to x of the first equation in (1.1). Note that (1.2) only determines the pressure up to a harmonic function in $\mathbb{R}^2$, but if the velocity field is bounded and divergence free one can show that (1.2) has a solution $p \in \mathrm{BMO}(\mathbb{R}^2)$ which is unique up to an irrelevant additive constant. This is the canonical choice of the pressure, which will always be made, albeit tacitly, in what follows. Here $\mathrm{BMO}(\mathbb{R}^2)$ denotes the space of functions of bounded mean oscillation in $\mathbb{R}^2$, see Section 2.2 below for a brief presentation. The interested reader should consult the monographs [8, 30, 31, 44] for a careful derivation and a detailed discussion of the model (1.1).

In most mathematical studies of the Navier-Stokes equations (1.1), it is assumed that the total (kinetic) energy of the fluid is finite:

$$E(t) \,:=\, \frac{1}{2}\int_{\mathbb{R}^2}|u(x,t)|^2\,\mathrm{d}x \,<\, \infty\;. \tag{1.3}$$

Strictly speaking the physical energy is $\rho E(t)$, but we use definition (1.3) in agreement with our previous choice of eliminating the density parameter ρ. It is important to realize that $E(t)$ is a *Lyapunov function* for the flow of (1.1), because a formal calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \,=\, -\nu\int_{\mathbb{R}^2}|\nabla u(x,t)|^2\,\mathrm{d}x \,\le\, 0\;. \tag{1.4}$$

As a consequence, we have

$$E(t) + \nu\int_0^t\int_{\mathbb{R}^2}|\nabla u(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s \,=\, E(0)\;, \qquad t \ge 0\;. \tag{1.5}$$

The energy equality (1.5) plays a crucial role in Leray's construction of global solutions to the Navier-Stokes equations in $\mathbb{R}^2$ [27].

As was already mentioned, we consider in these notes the more general situation where the velocity field is only assumed to be bounded. To avoid inessential technical problems related to continuity at initial time, we assume that u belongs to the Banach space

$$X \,=\, C_{\mathrm{bu}}(\mathbb{R}^2)^2 \,=\, \Big\{u : \mathbb{R}^2 \to \mathbb{R}^2 \,\Big|\, u \text{ is bounded and uniformly continuous}\Big\}\;, \tag{1.6}$$

equipped with the uniform norm. If $u \in X$, the energy (1.3) is infinite in general, but we can still consider the energy density $e = \frac{1}{2}|u|^2$, which satisfies the following local version of (1.4):

$$\partial_t e + \mathrm{div}\Big((p+e)u\Big) \,=\, \nu\Delta e - \nu|\nabla u|^2\;, \qquad x \in \mathbb{R}^2\;, \quad t \ge 0\;. \tag{1.7}$$

Another important quantity is the vorticity $\omega = \mathrm{curl}\, u = \partial_1 u_2 - \partial_2 u_1$, which evolves according to the simple advection-diffusion equation

$$\partial_t\omega + u\cdot\nabla\omega \,=\, \nu\Delta\omega\;. \tag{1.8}$$

If the velocity field is bounded, one can apply the parabolic maximum principle to (1.8) and prove that all L^p norms of ω are Lyapunov functions for the flow of (1.8). The case $p = \infty$ is most relevant for us, because if we assume that the initial velocity u_0 belongs to X, standard parabolic smoothing estimates imply that, for any positive time, the derivative ∇u is a bounded function on $\mathbb{R}^2$, see (1.10) below. The quantity $\|\omega(\cdot,t)\|_{L^\infty}$ is therefore finite and nonincreasing with time for all $t > 0$. We also mention that, since $\operatorname{div} u = 0$ and $\operatorname{curl} u = \omega$, it is possible to reconstruct the velocity field u from the vorticity ω, up to an additive constant, by the Biot-Savart formula, see Section 4.1 for a detailed discussion. However, a uniform bound on the vorticity does not allow to control the L^∞ norm of the velocity field, hence a priori estimates are not sufficient to prove that solutions of (1.1) stay uniformly bounded in time.

A global existence result for solutions of the Navier-Stokes equations (1.1) in the space X was first published by Giga, Matsui, and Sawada [19, 20], although a similar result was previously obtained in [6]. The proof in [20] shows that the L^∞ norm of the velocity field cannot grow faster than a double exponential as $t \to \infty$, but that pessimistic estimate was subsequently improved by Sawada and Taniuchi [40] who obtained a single exponential bound. These early results are summarized in our first statement :

Theorem 1.1. [20, 40] *For any $u_0 \in X$ with $\operatorname{div} u_0 = 0$, the Navier-Stokes equations (1.1) have a unique global (mild) solution $u \in C^0([0,+\infty),X)$ with initial data u_0. Moreover, if the initial vorticity $\omega_0 = \operatorname{curl} u_0$ is bounded, we have the estimate*

$$\|u(\cdot,t)\|_{L^\infty} \;\le\; K_0\|u_0\|_{L^\infty}\, \exp\Bigl(K_0\,\|\omega_0\|_{L^\infty}t\Bigr)\;, \qquad t \ge 0\;, \tag{1.9}$$

where $K_0 \ge 1$ is a universal constant.

In Theorem 1.1, a *mild* solution refers to a solution of the integral equation associated with (1.1), see Section 2.4 below for details. The assumption that the initial vorticity be bounded is only needed to derive the nice estimate (1.9), which does not depend on the viscosity parameter. If we only suppose that $u_0 \in X$, $\operatorname{div} u_0 = 0$, and $u_0 \not\equiv 0$, the local existence theory shows that

$$\sup_{0\le t\le T}\|u(\cdot,t)\|_{L^\infty} + \sup_{0<t\le T}(\nu t)^{1/2}\|\nabla u(\cdot,t)\|_{L^\infty} \;\le\; K_1\|u_0\|_{L^\infty}\,, \quad \text{for } T \;=\; \frac{\nu}{K_1^2\|u_0\|_{L^\infty}^2}\;, \tag{1.10}$$

where $K_1 \ge 1$ is a universal constant, see Section 2.4. It follows in particular from (1.10) that $\|\omega(\cdot,T)\|_{L^\infty} \le 2\|\nabla u(\cdot,T)\|_{L^\infty} \le 2K_1^2\nu^{-1}\|u_0\|_{L^\infty}^2$, so if we use (1.10) for $t \in [0,T]$ and (1.9) for $t \ge T$ we obtain a bound of the form

$$\|u(\cdot,t)\|_{L^\infty} \;\le\; K_2\|u_0\|_{L^\infty}\, \exp\Bigl(K_2\,\nu^{-1}\|u_0\|_{L^\infty}^2 t\Bigr)\;, \qquad t \ge 0\;, \tag{1.11}$$

for some universal constant $K_2 \ge 1$. Estimate (1.11) holds for all $u_0 \in X$ with $\operatorname{div} u_0 = 0$, but the right-hand side depends explicitly on the viscosity parameter ν.

There are reasons to believe that the exponential upper bound (1.9) is the best one can obtain using only the a priori estimates given by the vorticity equation. However, as was shown recently by S. Zelik [46], the above results can be improved in a spectacular way if one also exploits the local dissipation law (1.7), which asserts that no energy is created inside the system. The work of Zelik is devoted to a more general Navier-Stokes system, which includes an additional linear damping term and an external force, but in the particular case of equation (1.1) a slight extension of the results of [46] gives the following statement :

Theorem 1.2. [46, revisited] *If $u_0 \in X$, $\operatorname{div} u_0 = 0$, and $\omega_0 = \operatorname{curl} u_0 \in L^\infty(\mathbb{R}^2)$, the solution of the Navier-Stokes equations (1.1) given by Theorem 1.1 satisfies*

$$\|u(\cdot,t)\|_{L^\infty} \;\le\; K_3\|u_0\|_{L^\infty}\Bigl(1 + \|\omega_0\|_{L^\infty}t\Bigr)\;, \qquad t \ge 0\;, \tag{1.12}$$

where $K_3 \ge 1$ is a universal constant.

Estimate (1.12) is clearly superior to (1.9), because it shows that the L^∞ norm of the velocity field grows at most linearly as $t \to \infty$. As before, if we do not assume that $\omega_0 \in L^\infty(\mathbb{R}^2)$, we can use (1.10) for short times to prove that the bound (1.12) remains valid if $\|\omega_0\|_{L^\infty}$ is replaced by $2K_1^2\nu^{-1}\|u_0\|_{L^\infty}^2$ in the right-hand side. We thus find

$$\|u(\cdot,t)\|_{L^\infty} \,\le\, K_4\|u_0\|_{L^\infty}\biggl(1 + \frac{\|u_0\|_{L^\infty}^2 t}{\nu}\biggr)~, \qquad t \ge 0~, \tag{1.13}$$

for some universal constant $K_4 \ge 1$.

The strategy of the proof of Theorem 1.2 in [46] can be roughly explained as follows. Suppose that we want to control the solution of (1.1) given by Theorem 1.1 on some large time interval $[0,T]$. A natural idea is to compute, for $t \in [0,T]$, the amount of energy contained in the ball of radius $R > 0$ centered at $x \in \mathbb{R}^2$:

$$E_R(x,t) \,=\, \frac{1}{2}\int_{B_x^R} |u(y,t)|^2 \,\mathrm{d}y~, \qquad \text{where} \quad B_x^R \,=\, \Bigl\{y \in \mathbb{R}^2 \,\Big|\, |y-x| \le R\Bigr\}~. \tag{1.14}$$

Although the Navier-Stokes equations are dissipative, it is clear that $E_R(x,t)$ is not necessarily a decreasing function of time, because energy may enter the ball B_x^R through the boundary due to the advection term $\mathrm{div}((p+e)u)$ and the diffusion term $\nu\Delta e$ in (1.7). However, the key observation is that these energy fluxes become relatively negligible when the radius R is taken sufficiently large. Indeed, since the velocity field $u(\cdot,t)$ is bounded on $\mathbb{R}^2$ for any $t \in [0,T]$, we expect that for large R the energy $E_R(x,t)$ will be proportional to the area of the ball B_x^R, which is πR^2, whereas the flux terms will be proportional to the length of the boundary ∂B_x^R, which is $2\pi R$. This suggests that taking R sufficiently large, depending on T, may help controlling the relative contribution of the energy entering the ball B_x^R through the boundary. As a matter of fact, S. Zelik proved in [46] that there exists a universal constant $K_5 \ge 1$ such that

$$\sup_{t\in[0,T]}\,\sup_{x\in\mathbb{R}^2}\frac{1}{\pi R^2}\int_{B_x^R}|u(y,t)|^2\,\mathrm{d}y \,\le\, K_5\sup_{x\in\mathbb{R}^2}\frac{1}{\pi R^2}\int_{B_x^R}|u_0(y)|^2\,\mathrm{d}y \,\le\, K_5\|u_0\|_{L^\infty}^2~, \tag{1.15}$$

provided R is taken sufficiently large, depending on T. More precisely, we shall see in Section 3.2 below that one can take $R = \max\{R_0\,,\,C\sqrt{\nu T}\,,\,C\|u_0\|_{L^\infty}\|\omega_0\|_{L^\infty}T^2\}$, where $C > 0$ is a universal constant and $R_0 = \|u_0\|_{L^\infty}/\|\omega_0\|_{L^\infty}$. Estimate (1.15) is an example of *uniformly local energy estimate* for the Navier-Stokes equations, because the quantity it involves is equivalent to the square of the norm of u in the uniformly local Lebesgue space $L^2_{\mathrm{ul}}(\mathbb{R}^2)$, see Section 4.3 for an introduction to these spaces. It is clear that (1.15) is optimal in the sense that, if the initial velocity u_0 is a nonzero constant, then $u(\cdot,t) = u_0$ for all $t \ge 0$ and (1.15) becomes an equality if $K_5 = 1$. What may not be optimal is the dependence of the radius R upon the observation time T, namely $R = \mathcal{O}(T^2)$ as $T \to \infty$. If we had (1.15) for a smaller value of R, this would improve inequality (1.12), because as we shall see in Section 3.3 the right-hand side of (1.12) behaves like $\|u_0\|_{L^\infty}^{1/2}\|\omega_0\|_{L^\infty}^{1/2}R(t)^{1/2}$ as $t \to \infty$.

It is worth emphasizing that estimates (1.9) and (1.12) do not involve the viscosity parameter ν, and thus do not rely on energy dissipation in the system. Passing to the limit as $\nu \to 0$, they remain valid for global solutions of the Euler equations in $\mathbb{R}^2$ with bounded velocity and vorticity. Such solutions were recently studied by Ambrose, Kelliher, Lopes Filho, and Nussenzveig Lopes in [2, 24], following an earlier work by Ph. Serfati [42], and the vanishing viscosity limit (over a short time interval) was investigated in [7]. Existence of solutions in the ideal case $\nu = 0$ can be proved by an approximation argument, which is quite different from the simple proof presented in Section 2 for the Navier-Stokes equations, but once global solutions have been constructed the bounds (1.9), (1.12) can be established just as in the viscous case, see [5].

On the other hand, if one does use energy dissipation when $\nu > 0$, it is possible to obtain the following uniformly local enstrophy estimate:

$$\sup_{x\in\mathbb{R}^2} \frac{1}{\pi R^2}\int_{B_x^R} |\omega(y,t)|^2 \,\mathrm{d}y \;\le\; K_6\,\frac{\|u_0\|_{L^\infty}^2}{\nu t}\;, \qquad 0 < t \le T\;, \tag{1.16}$$

where $R = R(T)$ is as in (1.15) and $K_6 > 0$ is a universal constant. Of course, if the initial vorticity is bounded, the left-hand side of (1.16) is also smaller than $\|\omega_0\|_{L^\infty}^2$ by the maximum principle. Estimate (1.16) shows that a suitable average of the vorticity distribution converges to zero like $t^{-1/2}$ as $t \to \infty$. This strongly suggests that the long-time behavior of solutions to (1.1) should be governed by irrotational flows, although no precise statement is available so far.

Theorem 1.2 is the best we can do without further assumptions on the initial data. Since the right-hand side of (1.12) still depends on time, although in a rather mild way, we do not have a satisfactory answer yet to the original question of estimating the energy of a solution of (1.1) in an observation domain $D \subset \mathbb{R}^2$ in terms of the initial energy density only. There is no reason to believe that the linear time dependence in (1.12) is sharp, and to the author's knowledge there is no example of a solution to the Navier-Stokes equations (1.1) for which the L^∞ norm of the velocity field grows unboundedly in time. However, we believe that genuinely new ideas are needed to improve estimate (1.12).

To conclude this introduction, we briefly present an interesting particular case where the conclusion of Theorem 1.2 can be substantially strengthened. Following [1, 16, 17], we consider the Navier-Stokes equations (1.1) in the infinite strip $\Omega_L = \mathbb{R}\times[0,L]$, with periodic boundary conditions. Equivalently, we restrict ourselves to solutions of (1.1) in $\mathbb{R}^2$ for which the velocity field $u(x,t)$ and the pressure $p(x,t)$ are periodic of period $L > 0$ in one space direction, which is chosen to be the second coordinate axis. We denote by X_L the set of all $u \in X$ such that $u(x_1,x_2) = u(x_1,x_2+L)$ for all $x = (x_1,x_2) \in \mathbb{R}^2$. If $u \in X_L$ is divergence free, one can show that the elliptic equation (1.2) has a bounded solution which is L-periodic with respect to the second coordinate x_2, and that this solution is unique up to an additive constant. This is the canonical definition of the pressure in the present context, which agrees with the choice made in Theorems 1.1 and 1.2. We are now in position to state our last result:

Theorem 1.3. **[17]** *For any $u_0 \in X_L$ with $\operatorname{div} u_0 = 0$, the Navier-Stokes equations (1.1) have a unique global (mild) solution $u \in C^0([0,+\infty), X_L)$ with initial data u_0. Moreover, we have the estimate*

$$\|u(\cdot,t)\|_{L^\infty} + (\nu t)^{1/2}\|\omega(\cdot,t)\|_{L^\infty} \;\le\; K_7\|u_0\|_{L^\infty}(1+R_u^5)\;, \qquad t > 0\;, \tag{1.17}$$

where $R_u = \nu^{-1}L\|u_0\|_{L^\infty}$ is the initial Reynolds number and $K_7 \ge 1$ is a universal constant.

The conclusion of Theorem 1.3 is obviously much stronger than that of Theorem 1.2. First, the right-hand side of (1.17) does not depend on time, so that the velocity field $u(\cdot,t)$ is uniformly bounded for all times. This is the result that we were not able to prove in the general case. Next, the vorticity distribution $\omega(\cdot,t)$ converges uniformly to zero as $t \to \infty$, at the optimal rate $\mathcal{O}(t^{-1/2})$ which is the same as for the linear heat equation. This is clearly compatible with (1.16), but the estimate is now much more precise. In addition, the proof of Theorem 1.3 given in [17] provides detailed informations on the long-time behavior of the solutions, which are shown to converge exponentially in time to a shear flow governed by a linear advection-diffusion equation on the real line $\mathbb{R}$. These very strong conclusions are obtained using, in particular, the crucial observation made in [1] that the Biot-Savart law is more powerful when periodicity is assumed in one space direction. Indeed, a uniform bound on the vorticity ω allows to control the L^∞ norm of the velocity field, except for the quantity $m(x_1,t) = L^{-1}\int_0^L u_2(x_1,x_2,t)\,\mathrm{d}x_2$, which represents the average of the second component of the velocity over one period. We observe, however, that the right-hand side of (1.17) depends on the viscosity parameter ν, through the initial Reynolds number R_u, and does not have a finite limit as $\nu \to 0$. As a matter of fact, energy dissipation is an essential ingredient in the proof of Theorem 1.3.

The rest of these notes is organized as follows. Section 2 is entirely devoted to the proof of Theorem 1.1. For the reader's convenience, we first recall well known properties of the heat semigroup in the space (1.6), we study the elliptic equation (1.2), and we define the Leray-Hopf projection which allows us to eliminate the pressure from (1.1). We then establish local existence of solutions in X by applying a fixed point argument to the integral equation associated with (1.1). Finally, we prove global existence and obtain the exponential bound (1.9) using a nice Fourier-splitting argument borrowed from [40]. In Section 3, we develop the uniformly local energy estimates which are the key ingredient in the proof of Theorem 1.2. As a warm-up, we first apply uniformly local L^2 and L^p estimates to solutions of the linear heat equation. The core of the proof is Section 3.2 where, following the approach of Zelik [46], we use uniformly local energy estimates to control the solutions of the Navier-Stokes equations in $\mathbb{R}^2$. This gives estimate (1.15), and it is then relatively simple to deduce the upper bound (1.12) as well as the enstrophy estimate (1.16). The final section is an appendix, where important auxiliary results are collected. We first discuss the Biot-Savart formula, which allows us to reconstruct the velocity field from the vorticity up to an additive constant. We also establish a new representation formula for the pressure, which can be expressed as an absolutely convergent integral involving the velocity field and the vorticity. In Section 4.3 we define the uniformly local Lebesgue spaces, and we specify a class of weight functions that can be used to construct equivalent norms. Finally miscellaneous notations and results are collected in the last subsection, for easy reference.

Disclaimer. The present text is a set of lecture notes, not an original research article. Most of the results presented here have already been published elsewhere, and are not due to the author. In particular, the proof of Theorem 1.1 in Section 2 is entirely taken from [19, 20, 40], and the preliminary material collected in Sections 2.1–2.3 can be found in many textbooks. Section 3 is a little bit more original, although the statement and the proof of Theorem 1.2 are taken from the work of Zelik [46] with relatively minor modifications, see also [5] for recent improvements in the same direction. In particular, the way we treat the pressure in Section 3.2 differs notably from [46] and simplifies somewhat the argument by avoiding the use of delicate interpolation inequalities established in the appendices of [46] and [5]. The linear bound (1.12) does not appear explicitly in [46], but follows quite easily from the uniformly local energy estimate (1.15) and the a priori bound on the vorticity, see [5]. The uniformly local enstrophy estimate (1.16) is also new. Somewhat paradoxically, the most original material is perhaps in the Appendix. The Biot-Savart formula and the representation of the pressure given in Sections 4.1 and 4.2 are apparently new, although the recent work [24] contains several interesting results in the same spirit. In Section 4.3, Lemma 4.7 provides a characterization of admissible weights which is substantially more general than what can be found in the literature, see e.g. [3, Definition 4.1].

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2 The Cauchy problem with bounded initial data

In this section we study the Cauchy problem for the Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla) u \;=\; \nu \Delta u - \nabla p \;, \qquad \operatorname{div} u \;=\; 0 \;, \tag{2.1}$$

in the whole plane $\mathbb{R}^2$, with bounded initial data. We thus assume that the velocity field $u = (u_1, u_2)$ belongs to the Banach space X defined in (1.6), which is equipped with the uniform norm

$$\|u\|_{L^\infty} \;=\; \sup_{x\in\mathbb{R}^2} |u(x)| \;, \qquad \text{where} \quad |u| \;=\; (u_1^2 + u_2^2)^{1/2} \;.$$

Our first goal is to reformulate the Navier-Stokes equations (2.1) as an integral equation in X. This requires three preliminary steps, which are performed in Sections 2.1-2.3.

2.1 The heat semigroup on $C_{\mathrm{bu}}(\mathbb{R}^2)$

Let $\mathcal{L}(X)$ be the space of all bounded linear operators on X. For any $t > 0$, we denote by $S(t) \in \mathcal{L}(X)$ the linear operator defined, for all $u_0 \in X$, by the formula

$$\Bigl(S(t)u_0\Bigr)(x) \;=\; \frac{1}{4\pi t}\int_{\mathbb{R}^2} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\, u_0(y)\,\mathrm{d}y \;, \qquad x \in \mathbb{R}^2 \;. \tag{2.2}$$

We also set $S(0) = \mathbf{1}$ (the identity map). The family $\{S(t)\}_{t\ge 0}$ has the following properties, which are well known and easy to verify [11, Section 2.3].

1. For any $u_0 \in X$, one has $S(t)u_0 \in X$ for any $t \ge 0$ and $\|S(t)u_0\|_{L^\infty} \le \|u_0\|_{L^\infty}$. That bound holds because the heat kernel in (2.2) is positive and normalized so that
$$\frac{1}{4\pi t}\int_{\mathbb{R}^2} \mathrm{e}^{-\frac{|x|^2}{4t}}\,\mathrm{d}x \;=\; 1 \;, \qquad \text{for any } t > 0 \;.$$

2. One has $S(t_1 + t_2) = S(t_1)S(t_2)$ for all $t_1, t_2 \ge 0$. If both t_1, t_2 are positive, this follows from the identity
$$\frac{1}{4\pi}\int_{\mathbb{R}^2} \mathrm{e}^{-\frac{|x-y|^2}{4t_1}}\,\mathrm{e}^{-\frac{|y|^2}{4t_2}}\,\mathrm{d}y \;=\; \frac{t_1 t_2}{t_1+t_2}\,\mathrm{e}^{-\frac{|x|^2}{4(t_1+t_2)}} \;, \qquad x \in \mathbb{R}^2 \;,$$
which can be established by a direct calculation, or by using the Fourier transform to compute the convolution product in the left-hand side.

3. For any $u_0 \in X$, the map $t \mapsto S(t)u_0$ is continuous from $[0,\infty)$ into X. More generally, for any $u_0 \in L^\infty(\mathbb{R}^2)^2$, one can verify that $t \mapsto S(t)u_0$ is continuous from $(0,\infty)$ into X, but right-continuity at $t = 0$ holds only if $u_0 \in X$.

4. If $u_0 \in X$ and if we set $u(x,t) = (S(t)u_0)(x)$ for $x \in \mathbb{R}^2$ and $t \ge 0$, then u is smooth for $t > 0$ and satisfies the heat equation
$$\left\{\begin{array}{rcll} \partial_t u(x,t) &=& \Delta u(x,t) \;, & x \in \mathbb{R}^2 \,, \quad t > 0 \,, \\ u(x,0) &=& u_0(x) \;, & x \in \mathbb{R}^2 \,. \end{array}\right. \tag{2.3}$$
In fact u is the unique bounded solution of (2.3).

5. For any $u_0 \in X$ and any multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, there exists a constant $C > 0$ such that
$$\|\partial^\alpha S(t) u_0\|_{L^\infty} \le \frac{C}{t^{|\alpha|/2}} \|u_0\|_{L^\infty} , \qquad \text{for all } t > 0 , \tag{2.4}$$
where $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2}$ and $|\alpha| = \alpha_1 + \alpha_2$. In particular $\|\nabla S(t) u_0\|_{L^\infty} \le C t^{-1/2} \|u_0\|_{L^\infty}$.

Properties 1–3 above can be summarized by saying that the family $\{S(t)\}_{t \ge 0}$ is a *strongly continuous semigroup of contractions in* X, see [10, 36]. Property 4 implies that the Laplacian operator is the *generator* of the heat semigroup. Finally, the smoothing estimates (2.4) are related to the *analyticity* of the semigroup $\{S(t)\}_{t \ge 0}$ in X.

2.2 Determination of the pressure

Applying the divergence operator to the first equation in (2.1), we obtain the elliptic equation
$$-\Delta p(x) = \operatorname{div}\Big((u(x) \cdot \nabla) u(x)\Big) , \qquad x \in \mathbb{R}^2 , \tag{2.5}$$
which determines the pressure p up to a harmonic function on $\mathbb{R}^2$. To construct a particular solution we observe that, since $\operatorname{div} u = 0$, we can write (2.5) in the equivalent form
$$-\Delta p(x) = \sum_{k,\ell=1}^{2} \partial_k \partial_\ell \Big(u_k(x) u_\ell(x)\Big) , \qquad x \in \mathbb{R}^2 .$$
If we take the Fourier transform of both sides and use the conventions specified in Section 4.4, we thus find
$$|\xi|^2 \hat{p}(\xi) = \sum_{k,\ell=1}^{2} (i\xi_k)(i\xi_\ell) \widehat{u_k u_\ell}(\xi) , \qquad \text{hence} \quad \hat{p}(\xi) = \sum_{k,\ell=1}^{2} \frac{i\xi_k}{|\xi|} \frac{i\xi_\ell}{|\xi|} \widehat{u_k u_\ell}(\xi) ,$$
where equality holds in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^2)$. This gives (at least formally) the following solution to (2.5)
$$p = \sum_{k,\ell=1}^{2} R_k R_\ell (u_k u_\ell) , \tag{2.6}$$
where R_1, R_2 are the *Riesz transforms* on $\mathbb{R}^2$, namely the linear operators defined as Fourier multipliers through the formulas
$$(\widehat{R_k f})(\xi) = \frac{i\xi_k}{|\xi|} \hat{f}(\xi) , \qquad k = 1, 2 , \qquad \xi \in \mathbb{R}^2 \setminus \{0\} . \tag{2.7}$$
Here $f \in \mathcal{S}(\mathbb{R}^2)$ is an arbitrary test function. In ordinary space, the Riesz transforms are *singular integral operators* of the form
$$(R_k f)(x) = -\frac{1}{2\pi} \lim_{\epsilon \to 0} \int_{|y| \ge \epsilon} f(x-y) \frac{y_k}{|y|^3} \,\mathrm{d}y , \qquad k = 1, 2 , \qquad x \in \mathbb{R}^2 .$$
Using the Calderón-Zygmund theory [43, Chapter I], one can prove that the Riesz transforms define bounded linear operators on $L^p(\mathbb{R}^2)$ for $p \in (1, \infty)$:
$$\|R_k f\|_{L^p} \le C_p \|f\|_{L^p} , \qquad k = 1, 2 , \qquad 1 < p < \infty . \tag{2.8}$$
Unfortunately, estimate (2.8) fails both for $p = 1$ and $p = \infty$. In particular, if $f \in L^\infty(\mathbb{R}^2)$, the Riesz transform $R_k f$ is not a bounded function in general, but a function of *bounded mean oscillation* in the sense of the following definition.

Definition 2.1. *A locally integrable function f on $\mathbb{R}^2$ belongs to* $\mathrm{BMO}(\mathbb{R}^2)$ *if there exists $A \ge 0$ such that, for any ball $B \subset \mathbb{R}^2$ with nonzero Lebesgue measure $|B|$, one has*

$$\frac{1}{|B|}\int_B |f(x) - f_B|\,\mathrm{d}x \,\le\, A\ , \qquad \textit{where} \quad f_B \,=\, \frac{1}{|B|}\int_B f(x)\,\mathrm{d}x\ . \tag{2.9}$$

If $f \in \mathrm{BMO}(\mathbb{R}^2)$, the smallest bound A in (2.9) *is denoted by $\|f\|_{\mathrm{BMO}}$.*

If $f \in L^\infty(\mathbb{R}^2)$, it is clear that $f \in \mathrm{BMO}(\mathbb{R}^2)$ and $\|f\|_{\mathrm{BMO}} \le 2\|f\|_{L^\infty}$. However, the space $\mathrm{BMO}(\mathbb{R}^2)$ is strictly larger than $L^\infty(\mathbb{R}^2)$. For instance, if $f(x) = \log|x|$, then f has bounded mean oscillation [43, §IV.1.1], but f is obviously unbounded. It is also clear that adding a constant to f does not alter the quantity $\|f\|_{\mathrm{BMO}}$ which, therefore, is not a norm. However, one can show that $\|\cdot\|_{\mathrm{BMO}}$ defines a norm on the quotient space of $\mathrm{BMO}(\mathbb{R}^2)$ modulo the space of constant functions, which becomes in this way a Banach space. We refer to [43, Chapter IV] for a comprehensive study of functions of bounded mean oscillation.

Returning to Riesz transforms, we mention the important fact that R_1, R_2 can be extended to bounded linear operators from $L^\infty(\mathbb{R}^2)$ into $\mathrm{BMO}(\mathbb{R}^2)$, and even from $\mathrm{BMO}(\mathbb{R}^2)$ into itself, see e.g. [35, Section VII.4] for more general results implying that particular one. We point out that these extensions have the property that R_1, R_2 vanish on constant functions. As a consequence, if $u \in X$ is divergence free, the formula (2.6) makes sense and defines a function $p \in \mathrm{BMO}(\mathbb{R}^2)$, which satisfies the elliptic equation (2.5) in the sense of distributions. Thus we have proved:

Lemma 2.2. *If $u \in X$ and $\mathrm{div}\, u = 0$, the elliptic equation* (2.5) *has a solution $p \in \mathrm{BMO}(\mathbb{R}^2)$ given by* (2.6)*, which is unique up to an additive constant.*

The uniqueness claim in Lemma 2.2 is easy to prove. Indeed, any harmonic function on $\mathbb{R}^2$ that belongs to $\mathrm{BMO}(\mathbb{R}^2)$ is identically constant (this can be seen as a modest generalization of Liouville's theorem). More generally, we could consider other solutions of (2.5), but since we want to solve the Navier-Stokes equations (2.1) in the space X it is natural to assume that the pressure gradient is bounded. So the most general admissible solution of (2.5) is $p + \alpha + \beta_1 x_1 + \beta_2 x_2$, where p is given by (2.6) and $\alpha, \beta_1, \beta_2 \in \mathbb{R}$. The constant α is irrelevant, but nonzero values of β_1, β_2 would correspond to driving the fluid by a pressure gradient (like, for instance, in the classical Poiseuille flow). In these notes, we are interested in the intrinsic dynamics of the Navier-Stokes equations (2.1) in the absence of exterior forcing, so we always use the canonical choice of the pressure given by Lemma 2.2.

2.3 The Leray-Hopf projection

With the canonical choice of the pressure (2.6), the Navier-Stokes equations (2.1) can be written in the equivalent form

$$\partial_t u + \mathbb{P}(u\cdot\nabla)u \,=\, \nu\Delta u\ , \qquad \mathrm{div}\, u \,=\, 0\ , \tag{2.10}$$

where the Leray-Hopf projection $\mathbb{P}$ is the matrix-valued operator defined by

$$(\mathbb{P}u)_j \,=\, \sum_{k=1}^2 \mathbb{P}_{jk} u_k\ , \qquad \text{with} \quad \mathbb{P}_{jk} = \delta_{jk} + R_j R_k\ .$$

Indeed, using Einstein's summation convention over repeated indices, we have from (2.6):

$$\partial_j p \,=\, \partial_j R_k R_\ell(u_k u_\ell) \,=\, R_k R_j \partial_\ell(u_k u_\ell) \,=\, R_j R_k(u_\ell \partial_\ell u_k)\ ,$$

hence

$$\partial_j p + u_\ell \partial_\ell u_j \;=\; \Big(\delta_{jk} + R_j R_k\Big)(u_\ell \partial_\ell u_k)\;.$$

This shows that $\nabla p + (u\cdot\nabla)u = \mathbb{P}(u\cdot\nabla)u$. In the calculations above, we have used the commutations relations $R_1R_2 = R_2R_1$, $\partial_j R_k = R_k\partial_j$, $\partial_j R_\ell = R_j\partial_\ell$, as well as the incompressibility condition $\operatorname{div} u = 0$. Symbolically, we may also write

$$\mathbb{P}(u\cdot\nabla)u \;=\; \nabla\cdot\mathbb{P}(u\otimes u)\;. \tag{2.11}$$

It is clear that the Leray-Hopf projection $\mathbb{P}$ is a bounded linear operator on $L^p(\mathbb{R}^2)^2$ for $1 < p < \infty$, and from $L^\infty(\mathbb{R}^2)^2$ into $\mathrm{BMO}(\mathbb{R}^2)^2$. For later use, we also mention that $\nabla S(t)\mathbb{P}$ defines a bounded operator on $X = C_{\mathrm{bu}}(\mathbb{R}^2)^2$ for any $t > 0$, where $S(t)$ is the heat semigroup defined by (2.2).

Lemma 2.3. *There exists a constant $C_0 > 0$ such that*

$$\|\nabla S(t)\mathbb{P}f\|_{L^\infty} \;\le\; \frac{C_0}{\sqrt{t}}\,\|f\|_{L^\infty}\;, \qquad t > 0\;, \tag{2.12}$$

for all $f \in X$.

Proof. For any $t > 0$ and any choice of $j,k,\ell \in \{1,2\}$, the operator $\partial_j S(t)\mathbb{P}_{k\ell}$ is the Fourier multiplier with symbol

$$i\xi_j\Big(\delta_{k\ell} - \frac{\xi_k\xi_\ell}{|\xi|^2}\Big)\,\mathrm{e}^{-t|\xi|^2} \;=\; i\xi_j\delta_{k\ell}\,\mathrm{e}^{-t|\xi|^2} - i\xi_j\xi_k\xi_\ell\int_t^\infty \mathrm{e}^{-\tau|\xi|^2}\,\mathrm{d}\tau\;, \qquad \xi\in\mathbb{R}^2\setminus\{0\}\;.$$

We thus have the following identity

$$\partial_j S(t)\mathbb{P}_{k\ell} f \;=\; \delta_{k\ell}\partial_j S(t) f + \int_t^\infty \partial_j\partial_k\partial_\ell S(\tau) f\,\mathrm{d}\tau\;, \tag{2.13}$$

which holds in particular for any $f \in C_{\mathrm{bu}}(\mathbb{R}^2)$. Both terms in the right-hand side of (2.13) belong to $C_{\mathrm{bu}}(\mathbb{R}^2)$ and can be easily estimated using (2.4) :

$$\begin{aligned}
\|\partial_j S(t) f\|_{L^\infty} \;&\le\; \frac{C}{\sqrt{t}}\,\|f\|_{L^\infty}\;,\\
\Big\|\int_t^\infty \partial_j\partial_k\partial_\ell S(\tau) f\,\mathrm{d}\tau\Big\|_{L^\infty} \;&\le\; C\int_t^\infty \frac{1}{\tau^{3/2}}\,\|f\|_{L^\infty}\,\mathrm{d}\tau \;\le\; \frac{C}{\sqrt{t}}\,\|f\|_{L^\infty}\;,
\end{aligned}$$

This immediately yields estimate (2.12). ◻

2.4 Local existence of solutions

Let $u_0 \in X$ be such that $\operatorname{div} u_0 = 0$. We consider the integral equation associated with the Navier-Stokes equations (2.10) :

$$u(t) \;=\; S(\nu t)u_0 - \int_0^t \nabla\cdot S(\nu(t-s))\,\mathbb{P}(u(s)\otimes u(s))\,\mathrm{d}s\;, \qquad t > 0\;, \tag{2.14}$$

where $S(t)$ is the heat semigroup (2.2), and the notation $\mathbb{P}(u\otimes u)$ is explained in (2.11). Here and in the sequel, the map $x \mapsto u(x,t)$ is simply denoted by $u(t)$ instead of $u(\cdot,t)$. The goal of this section is to prove that the integral equation (2.14) has a unique local solution that it continuous in time with values in the space X defined by (1.6). Such a solution of (2.14) is usually called a *mild solution* of the Navier-Stokes equations (2.1) in X.

Proposition 2.4. *Fix $\nu > 0$. For any $M > 0$, there exists a time $T = T(M, \nu) > 0$ such that, for all initial data $u_0 \in X$ with $\operatorname{div} u_0 = 0$ and $\|u_0\|_{L^\infty} \le M$, the integral equation (2.14) has a unique local solution $u \in C^0([0,T], X)$, which moreover satisfies*

$$\sup_{0\le t\le T} \|u(t)\|_{L^\infty} \le 2M , \qquad \textit{and} \qquad \sup_{0<t\le T} (\nu t)^{1/2} \|\nabla u(t)\|_{L^\infty} \le CM , \tag{2.15}$$

where $C > 0$ is a universal constant. In addition, the solution $u \in C^0([0,T], X)$ depends continuously on the initial data $u_0 \in X$.

Proof. Take $T > 0$ small enough so that

$$\kappa := 8C_0 M \frac{T^{1/2}}{\nu^{1/2}} < 1 , \tag{2.16}$$

where $C_0 > 0$ is as in Lemma 2.3. We introduce the Banach space $Y = C^0([0,T], X)$ equipped with the norm

$$\|u\|_Y = \sup_{0\le t\le T} \|u(t)\|_{L^\infty} .$$

Using Lemma 2.3 it is not difficult to verify that, if $u \in Y$, the integral in the right-hand side of (2.14) is well defined and depends continuously on time in the topology of X. Moreover, if $u_0 \in X$, the results of Section 2.1 show that the map $t \mapsto S(\nu t)u_0$ also belongs to Y. Thus, given any $u_0 \in X$ with $\operatorname{div} u_0 = 0$ and $\|u_0\|_{L^\infty} \le M$, we can consider the map $F : Y \to Y$ defined by

$$(Fu)(t) = S(\nu t)u_0 - \int_0^t \nabla \cdot S(\nu(t-s))\, \mathbb{P}(u(s) \otimes u(s))\, \mathrm{d}s , \qquad t \in [0,T] . \tag{2.17}$$

Denoting $B = \{u \in Y \mid \|u\|_Y \le 2M\} \subset Y$, we claim that

i) *F maps B into itself.* Indeed, if $u \in B$, we find using (2.12) and (2.16)

$$\begin{aligned}
\|(Fu)(t)\|_{L^\infty} &\le \|u_0\|_{L^\infty} + \int_0^t \frac{C_0}{\sqrt{\nu(t-s)}} \|u(s)\|_{L^\infty}^2 \,\mathrm{d}s \\
&\le M + \|u\|_Y^2 \int_0^t \frac{C_0}{\sqrt{\nu\tau}} \,\mathrm{d}\tau \le M + 8M^2 C_0 (T/\nu)^{1/2} = (1+\kappa)M ,
\end{aligned}$$

for any $t \in [0,T]$. As $\kappa < 1$, we deduce that $\|Fu\|_Y < 2M$, hence $Fu \in B$.

ii) *F is a strict contraction in B.* Indeed, if $u, v \in B$, then

$$(Fu)(t) - (Fv)(t) = \int_0^t \nabla \cdot S(\nu(t-s))\, \mathbb{P}\Big((v(s) \otimes v(s)) - (u(s) \otimes u(s))\Big)\, \mathrm{d}s ,$$

hence decomposing $v \otimes v - u \otimes u = v \otimes (v-u) + (v-u) \otimes u$ and proceeding as above, we find

$$\begin{aligned}
\|(Fu)(t) - (Fv)(t)\|_{L^\infty} &\le \int_0^t \frac{C_0}{\sqrt{\nu(t-s)}} \Big(\|v(s)\|_{L^\infty} + \|u(s)\|_{L^\infty}\Big) \|u(s) - v(s)\|_{L^\infty} \,\mathrm{d}s \\
&\le 4M\|u-v\|_Y \int_0^t \frac{C_0}{\sqrt{\nu\tau}} \,\mathrm{d}\tau \le 8MC_0 (T/\nu)^{1/2} \|u-v\|_Y ,
\end{aligned}$$

for all $t \in [0,T]$. Thus $\|Fu - Fv\|_Y \le \kappa \|u-v\|_Y$ where $\kappa < 1$ is as in (2.16).

By the Banach fixed point theorem, the map $F : Y \to Y$ has a unique fixed point u in B, which satisfies by construction the integral equation (2.14) as well as the first bound in (2.15).

If $\tilde{u} \in Y$ is another solution of (2.14), then applying Gronwall's lemma to the integral equation satisfied by the difference $\tilde{u} - u$ it is easy to verify that $\tilde{u} = u$, see [22] and Section 4.4. Thus the solution u of (2.14) constructed by the fixed point argument above is unique not only in the ball B, but also in the whole space Y. A similar argument shows that the solution $u \in Y$ is a locally Lipschitz function of the initial data $u_0 \in X$. Finally, the simplest way to prove the second bound in (2.15) is to repeat the existence proof using the smaller function space

$$Z = \left\{ u \in C^0([0,T], X) \,\Big|\, t^{1/2}\nabla u \in C_b^0((0,T], X^2) \right\} ,$$

equipped with the norm

$$\|u\|_Z = \sup_{0 \le t \le T} \|u(t)\|_{L^\infty} + \sup_{0 < t \le T} (\nu t)^{1/2} \|\nabla u(t)\|_{L^\infty} .$$

Proceeding as above one obtains the existence of a local solution $u \in Z$ of (2.14) for a slightly smaller value of T, which is determined by a condition of the form (2.16) where C_0 is replaced by a larger constant. $\square$

Remark 2.5. If $u_0 \neq 0$, the local existence time T given by the proof of Proposition 2.4 satisfies

$$T = \frac{C_1 \nu}{\|u_0\|_{L^\infty}^2} , \tag{2.18}$$

where $C_1 > 0$ is a universal constant. This implies in particular that, if we consider the maximal solution $u \in C^0([0,T_*), X)$ of (2.14) in X, then either $T_* = \infty$, which means that the solution is global, or $\|u(t)\|_{L^\infty} \to \infty$ as $t \to T_*$. More precisely, we must have $\|u(t)\|_{L^\infty}^2 > C_1 \nu (T_* - t)^{-1}$ for all $t \in [0, T_*)$. Note also that estimate (1.10) follows from (2.15) and (2.18).

Remark 2.6. Using standard parabolic smoothing estimates, it is not difficult to show that, if $u \in C^0([0,T], X)$ is the mild solution of (2.1) constructed in Proposition 2.4, then $u(x,t)$ is a smooth function for $(x,t) \in \mathbb{R}^2 \times (0,T]$ which satisfies the Navier-Stokes equations (2.1) in the classical sense, with the pressure $p(x,t)$ given by (2.6), see [19, 20].

2.5 Global existence and a priori estimates

Take $u_0 \in X$ such that $\operatorname{div} u_0 = 0$, and let $u \in C^0([0,T_*), X)$ be the maximal solution of (2.1) with initial data u_0, the existence of which follows from Proposition 2.4. In view of Remark 2.5, to prove that this solution is global (namely, $T_* = \infty$), it is sufficient to show that the norm $\|u(t)\|_{L^\infty}$ cannot blow up in finite time. The easiest way to do that is to consider the vorticity distribution $\omega = \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$, which satisfies the advection-diffusion equation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega . \tag{2.19}$$

We know from Proposition 2.4 that $\|\omega(t)\|_{L^\infty} \le 2\|\nabla u(t)\|_{L^\infty} < \infty$ for any $t \in (0, T_*)$, hence shifting the origin of time we can assume without loss of generality that $\omega_0 = \omega(\cdot, 0) \in L^\infty(\mathbb{R}^2)$. Now, the parabolic maximum principle [37] asserts that $\|\omega(t)\|_{L^\infty}$ is a *nonincreasing function of time*, which gives the a priori estimate

$$\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty} , \qquad \text{for all } t \ge 0 . \tag{2.20}$$

Unfortunately, the bound (2.20) does not provide any direct control on $\|u(t)\|_{L^\infty}$, because we work in an unbounded domain where the velocity field can have arbitrarily low frequencies. In Fourier space, the relation between $\hat{u} = \mathcal{F}u$ and $\hat{\omega} = \mathcal{F}\omega$ takes the simple form

$$\hat{u}(\xi) = \frac{-i\xi^\perp}{|\xi|^2} \hat{\omega}(\xi) , \qquad \xi \in \mathbb{R}^2 \setminus \{0\} , \tag{2.21}$$

where $\xi^\perp = (-\xi_2, \xi_1)$ if $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. This shows that the first-order derivatives of the velocity field u satisfy

$$\partial_1 u_1 = -\partial_2 u_2 = R_1 R_2 \,\omega\ , \qquad \partial_1 u_2 = -R_1^2\,\omega\ , \qquad \partial_2 u_1 = R_2^2\,\omega\ ,$$

where R_1, R_2 are the Riesz transforms (2.7). In particular, we deduce the a priori estimate

$$\|\nabla u(t)\|_{\mathrm{BMO}} \le C\|\omega(t)\|_{L^\infty} \le C\|\omega_0\|_{L^\infty}\ , \qquad \text{for all } t \ge 0\ . \tag{2.22}$$

We refer to Section 4.1 for a more detailed discussion of the Biot-Savart law in $\mathbb{R}^2$.

To go further we observe that, since $\mathrm{div}\, u = 0$, we have the identity

$$(u\cdot\nabla)u = \tfrac{1}{2}\nabla|u|^2 + u^\perp\omega\ , \tag{2.23}$$

where $u^\perp = (-u_2, u_1)$ if $u = (u_1, u_2)$. We can thus write the Navier-Stokes equations (2.1) in the equivalent form

$$\partial_t u + u^\perp\omega = \nu\Delta u - \nabla q\ , \qquad \mathrm{div}\, u = 0\ , \tag{2.24}$$

where $q = p + \frac{1}{2}|u|^2$. Applying the Leray-Hopf projection, we obtain the analog of (2.10)

$$\partial_t u + \mathbb{P}(u^\perp\omega) = \nu\Delta u\ , \qquad \mathrm{div}\, u = 0\ . \tag{2.25}$$

Since ω is under control, the nonlinear term $\mathbb{P}(u^\perp\omega)$ can be considered as a linear expression in the velocity field u, and this strongly suggests that the solutions of (2.25) should not grow faster than $\exp(C\|\omega_0\|_{L^\infty}t)$ as $t \to \infty$. The problem with this naive argument is that we cannot estimate $\|\mathbb{P}(u^\perp\omega)\|_{L^\infty}$ in terms of $\|u\|_{L^\infty}\|\omega\|_{L^\infty}$, because the Leray-Hopf projection $\mathbb{P}$ is not continuous on $L^\infty(\mathbb{R}^2)^2$. This difficulty was solved in an elegant way by O. Sawada and Y. Taniuchi, who obtained the following result.

Proposition 2.7. **[40]** *Assume that $u_0 \in X$, $\mathrm{div}\, u_0 = 0$, and $\omega_0 = \mathrm{curl}\, u_0 \in L^\infty(\mathbb{R}^2)$. Then the Navier-Stokes equations (2.1) have a unique global (mild) solution $u \in C^0([0,\infty), X)$ with initial data u_0. Moreover, we have the estimate*

$$\|u(t)\|_{L^\infty} \le C\|u_0\|_{L^\infty}\exp\Bigl(C\|\omega\|_{L^\infty}t\Bigr)\ , \qquad t \ge 0\ , \tag{2.26}$$

for some universal constant $C > 0$.

The proof of Proposition 2.7 relies on a clever Fourier-splitting argument which we now describe. Let $\hat\chi : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function such that

$$\hat\chi(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \le 1\,, \\ 0 & \text{if} \quad |\xi| \ge 2\,. \end{cases}$$

We further assume that χ is radially symmetric and nonincreasing along rays. Let $\chi = \mathcal{F}^{-1}\hat\chi$ be the inverse Fourier transform of $\hat\chi$, so that $\chi \in \mathcal{S}(\mathbb{R}^2)$. Given any $\delta > 0$, we denote by Q_δ the Fourier multiplier with symbol $\hat\chi(\xi/\delta)$:

$$(\widehat{Q_\delta f})(\xi) = \hat\chi(\xi/\delta)\hat f(\xi)\ , \qquad \xi \in \mathbb{R}^2\ . \tag{2.27}$$

It is clear that Q_δ is a bounded linear operator on $\mathcal{S}'(\mathbb{R}^2)$.

Lemma 2.8. *There exists a constant $C_2 > 0$ such that the following bounds hold for any $\delta > 0$.*

1. $\|Q_\delta f\|_{L^\infty} \le C_2\|f\|_{L^\infty}$, *for any $f \in C_{\mathrm{bu}}(\mathbb{R}^2)$;*
2. $\|Q_\delta \nabla\,\mathbb{P} f\|_{L^\infty} \le C_2\delta\|f\|_{L^\infty}$, *for any $f \in C_{\mathrm{bu}}(\mathbb{R}^2)^2$;*
3. $\|(\mathbf{1} - Q_\delta)u\|_{L^\infty} \le C_2\delta^{-1}\|\omega\|_{L^\infty}$, *for any $u \in X$ with $\mathrm{div}\, u = 0$ and $\mathrm{curl}\, u = \omega$.*

Proof. The first estimate follows immediately from Young's inequality (see Section 4.4), because Q_δ is the convolution operator with the integrable function $x \mapsto \delta^2\chi(\delta x)$, the L^1 norm of which does not depend on δ. To prove the second estimate we have to show that, for any $j,k,\ell \in \{1,2\}$, the Fourier multiplier M with symbol

$$m(\xi) \;=\; \frac{i\xi_j\xi_k\xi_\ell}{|\xi|^2}\,\hat\chi(\xi/\delta)~, \qquad \xi \in \mathbb{R}^2 \setminus \{0\}~,$$

is continuous on $L^\infty(\mathbb{R}^2)$ with operator norm bounded by $C\delta$. We observe that

$$m(\xi) \;=\; \delta\hat\psi(\xi/\delta)~, \qquad \text{where} \quad \hat\psi(\xi) \;=\; \frac{i\xi_j\xi_k\xi_\ell}{|\xi|^2}\,\hat\chi(\xi)~.$$

It follows that $Mf = \psi_\delta * f$, where $\psi_\delta(x) = \delta^3\psi(\delta x)$ and $\psi = \mathcal{F}^{-1}\hat\psi$. It is clear that $\psi \in C^\infty(\mathbb{R}^2)$ (because $\hat\psi$ has compact support), and from the explicit formula

$$\psi(x) \;=\; \frac{1}{2\pi}\,\partial_j\partial_k\partial_\ell \int_{\mathbb{R}^2} \log(|x-y|)\chi(y)\,\mathrm{d}y~, \qquad x \in \mathbb{R}^2~,$$

it is straightforward to verify that $|\psi(x)| \le C|x|^{-3}$ for $|x| \ge 1$. Thus $\psi \in L^1(\mathbb{R}^2)$, and using Young's inequality we conclude that $\|Mf\|_{L^\infty} \le \|\psi_\delta\|_{L^1}\|f\|_{L^\infty} = \delta\|\psi\|_{L^1}\|f\|_{L^\infty}$, which is the desired result.

Finally, to prove the third estimate in Lemma 2.8, we use formula (2.21) to derive the relation

$$\hat u(\xi) - \widehat{Q_\delta u}(\xi) \;=\; \Bigl(1-\hat\chi(\xi/\delta)\Bigr)\frac{-i\xi^\perp}{|\xi|^2}\,\hat\omega(\xi) \;=\; \frac{1}{\delta}\,\hat\phi(\xi/\delta)\hat\omega(\xi)~,$$

where

$$\hat\phi(\xi) \;=\; \Bigl(1-\hat\chi(\xi)\Bigr)\frac{-i\xi^\perp}{|\xi|^2}~, \qquad \xi \in \mathbb{R}^2 \setminus \{0\}~. \tag{2.28}$$

As before, if $\phi = \mathcal{F}^{-1}\hat\phi$, this implies that $\|(\mathbf{1}-Q_\delta)u\|_{L^\infty} \le \delta^{-1}\|\phi\|_{L^1}\|\omega\|_{L^\infty}$, so we only need to verify that $\phi \in L^1(\mathbb{R}^2)$. Since $\hat\chi$ has compact support in $\mathbb{R}^2$, it follows from (2.28) that

$$\phi(x) \;=\; \frac{1}{2\pi}\frac{x^\perp}{|x|^2} + \Phi(x)~, \qquad x \in \mathbb{R}^2 \setminus \{0\}~,$$

where $\Phi : \mathbb{R}^2 \to \mathbb{R}$ is smooth, hence ϕ is integrable on any bounded neighborhood of the origin. On the other hand, if we apply the Laplacian Δ_ξ to both sides of (2.28), the resulting expression belongs to $L^2(\mathbb{R}^2, \mathrm{d}\xi)$. This shows that $|x|^2\phi \in L^2(\mathbb{R}^2, \mathrm{d}x)$, hence ϕ is integrable on the complement of any neighborhood of the origin. Thus altogether $\phi \in L^1(\mathbb{R}^2)$, which is the desired result. $\square$

Proof of Proposition 2.7. Let $u \in C^0([0,T_*), X)$ be the maximal solution of (2.1) with initial data u_0. Without loss of generality, we assume that $u_0 \not\equiv 0$, and we fix $t \in (0,T_*)$. The idea is to control the low frequencies $|\xi| \le 2\delta$ in the solution $u(t)$ using the integral equation (2.14), and the high frequencies $|\xi| \ge \delta$ using the third estimate in Lemma 2.8 together with the a priori bound on the vorticity. The threshold frequency δ will depend on time and on the solution itself.

Given any $\delta > 0$, we apply the Fourier multiplier Q_δ defined in (2.27) to the integral equation (2.14) and obtain

$$Q_\delta u(t) \;=\; S(\nu t)Q_\delta u_0 - \int_0^t S(\nu(t-s))Q_\delta\nabla\cdot\mathbb{P}(u(s)\otimes u(s))\,\mathrm{d}s~,$$

where we have used the fact that Q_δ commutes with the heat semigroup $S(t)$. Using the first two estimates in Lemma 2.8, we thus find

$$\|Q_\delta u(t)\|_{L^\infty} \;\le\; C_2\|u_0\|_{L^\infty} + C_2\delta\int_0^t \|u(s)\|_{L^\infty}^2\,\mathrm{d}s\;.$$

On the other hand, the third estimate in Lemma 2.8 implies that

$$\|(\mathbf{1}-Q_\delta)u(t)\|_{L^\infty} \;\le\; C_2\delta^{-1}\|\omega(t)\|_{L^\infty} \;\le\; C_2\delta^{-1}\|\omega_0\|_{L^\infty}\;.$$

This bound shows how the high frequencies in the velocity field $u(t)$ can be controlled in terms of the vorticity. Combining both results, we find

$$\|u(t)\|_{L^\infty} \;\le\; C_2\|u_0\|_{L^\infty} + C_2\delta\int_0^t \|u(s)\|_{L^\infty}^2\,\mathrm{d}s + C_2\delta^{-1}\|\omega_0\|_{L^\infty}\;.$$

If we now choose

$$\delta \;=\; \|\omega_0\|_{L^\infty}^{1/2}\left(\int_0^t \|u(s)\|_{L^\infty}^2\,\mathrm{d}s\right)^{-1/2}\;,$$

we obtain the bound

$$\|u(t)\|_{L^\infty} \;\le\; C_2\|u_0\|_{L^\infty} + 2C_2\|\omega_0\|_{L^\infty}^{1/2}\left(\int_0^t \|u(s)\|_{L^\infty}^2\,\mathrm{d}s\right)^{1/2}\;, \tag{2.29}$$

which holds for any $t \in (0,T_*)$. Finally, squaring both sides of (2.29) and applying Gronwall's lemma (see Section 4.4), we arrive at the inequality

$$\|u(t)\|_{L^\infty}^2 \;\le\; 2C_2^2\|u_0\|_{L^\infty}^2 \exp\Bigl(8C_2^2\|\omega_0\|_{L^\infty}t\Bigr)\;, \qquad t\in(0,T_*)\;, \tag{2.30}$$

which shows that the norm $\|u(t)\|_{L^\infty}$ cannot blow up in finite time. Thus $T_* = \infty$, and estimate (2.30) holds for all $t > 0$. ☐

Remark 2.9. Theorem 1.1 is an immediate consequence of Propositions 2.4 and 2.7.

3 Uniformly local energy estimates

In the study of nonlinear partial differential equations on unbounded spatial domains, if one considers solutions that do not decay to zero at infinity, it is not always convenient to use function spaces based on the uniform norm $\|\cdot\|_\infty$, because those spaces may not take into account some essential properties of the system, such as locally conserved or locally dissipated quantities. From this point of view, the larger family of *uniformly local Lebesgue spaces* offers an interesting compromise between simplicity and flexibility. In the analysis of evolution PDE's, uniformly local spaces were introduced by T. Kato in 1975 [23], and subsequently used by many authors, see [3, 9, 12, 15, 21, 33, 34, 46] for a few examples. For the reader's convenience, the definition and the main properties of those spaces are given in the Appendix, see Section 4.3. In short, if $1 \le p < \infty$, the uniformly local Lebesgue space $L^p_{\mathrm{ul}}(\mathbb{R}^d)$ is the completion of the space of all bounded and uniformly continuous functions on $\mathbb{R}^d$ with respect to the norm

$$\|f\|_{L^p_{\mathrm{ul}}} \;=\; \sup_{x\in\mathbb{R}^d}\left(\int_{|y-x|\le 1}|f(y)|^p\,\mathrm{d}y\right)^{1/p}\;. \tag{3.1}$$

Our goal in this section is to control the solutions of the two-dimensional Navier-Stokes equations in the uniformly local energy space $L^2_{\mathrm{ul}}(\mathbb{R}^2)$. This can be done using uniformly local energy estimates, a flexible and powerful technique that we first explain on a simple example in Section 3.1 before applying it to the original problem in Section 3.2.

3.1 Uniformly local energy estimates for the heat equation

In this section we show on a simple example how uniformly local energy estimates can be used to obtain information on the solutions of partial differential equations on unbounded domains. We concentrate on the linear heat equation on $\mathbb{R}^d$, with nondecaying initial data u_0. In that particular example, the solution can be written in explicit form, but we shall not use the heat kernel (4.22) because we want to develop robust methods that can be applied to more complicated situations, such as the two-dimensional Navier-Stokes equations which will considered later.

Let $u_0 \in L^2_{\mathrm{ul}}(\mathbb{R}^d)$, and let $u(x,t)$ be the solution of the heat equation

$$\partial_t u(x,t) \;=\; \Delta u(x,t)\ , \qquad x \in \mathbb{R}^d\ , \quad t > 0\ , \tag{3.2}$$

with initial data $u(\cdot,0) = u_0$. It follows from Proposition 4.6 below that $u \in C^0(\mathbb{R}_+, L^2_{\mathrm{ul}}(\mathbb{R}^d))$, and our goal is to derive accurate bounds on u using localized energy estimates. Let $\rho : \mathbb{R}^d \to (0,+\infty)$ be a Lipschitz continuous function satisfying the assumptions of Proposition 4.7 and such that $|\nabla\rho(x)| \le \rho(x)$ for almost every $x \in \mathbb{R}^d$. Typical examples are

$$\rho(x) \;=\; e^{-|x|}\ , \qquad \text{or} \qquad \rho(x) \;=\; \frac{1}{(m+|x|)^m} \quad \text{where } m > d\ . \tag{3.3}$$

Note that ρ cannot decay to zero faster than an exponential as $|x| \to \infty$, because of the assumption $|\nabla\rho| \le \rho$. For any $R > 0$, we also define $\rho_R(x) = \rho(x/R)$.

Since the solution u of (3.2) is smooth and bounded for $t > 0$, we can compute

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \rho_R u^2 \,\mathrm{d}x \;&=\; \int_{\mathbb{R}^d} \rho_R u u_t \,\mathrm{d}x \;=\; \int_{\mathbb{R}^d} \rho_R u \Delta u \,\mathrm{d}x \\
&=\; -\int_{\mathbb{R}^d} \rho_R |\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^d} (\nabla\rho_R \cdot \nabla u) u \,\mathrm{d}x \\
&\le\; -\int_{\mathbb{R}^d} \rho_R |\nabla u|^2 \,\mathrm{d}x + \frac{1}{R}\int_{\mathbb{R}^d} \rho_R |\nabla u||u| \,\mathrm{d}x \\
&\le\; -\frac{1}{2}\int_{\mathbb{R}^d} \rho_R |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2R^2}\int_{\mathbb{R}^d} \rho_R u^2 \,\mathrm{d}x\ .
\end{aligned}$$

Using Gronwall's lemma (see Section 4.4), we deduce that

$$\int_{\mathbb{R}^d} \rho_R(x) u(x,t)^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^d} \rho_R(x) |\nabla u(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s \;\le\; \Bigl(\int_{\mathbb{R}^d} \rho_R(x) u_0(x)^2 \,\mathrm{d}x\Bigr)\, \mathrm{e}^{t/R^2}\ , \tag{3.4}$$

for all $t > 0$. This estimate looks rather pessimistic, because it predicts an exponential growth of the solution as $t \to \infty$, but one should keep in mind that $\int \rho_R u_0^2 \,\mathrm{d}x < \infty$ is the only assumption on the initial data that was really used in the derivation of (3.4). If $\rho(x) = \mathrm{e}^{-|x|}$, this means that u_0 is allowed to grow exponentially as $|x| \to \infty$, in which case the solution of (3.2) indeed grows exponentially in time.

To improve (3.4) for large times, we must use the assumption that $u_0 \in L^2_{\mathrm{ul}}(\mathbb{R}^d)$. If $R \ge 1$, an easy calculation shows that

$$\int_{\mathbb{R}^d} \rho_R(x) u_0(x)^2 \,\mathrm{d}x \;\le\; C_d R^d \|u_0\|^2_{L^2_{\mathrm{ul}}}\ , \tag{3.5}$$

for some constant $C_d > 0$ depending on the dimension d. So if we take $R = R(t) = (1+t)^{1/2}$ we obtain from (3.4) and (3.5)

$$\int_{\mathbb{R}^d} \rho_{R(t)}(x) u(x,t)^2 \,\mathrm{d}x + \int_0^t \int_{\mathbb{R}^d} \rho_{R(t)}(x) |\nabla u(x,s)|^2 \,\mathrm{d}x\,\mathrm{d}s \;\le\; C\|u_0\|^2_{L^2_{\mathrm{ul}}} (1+t)^{d/2}\ , \tag{3.6}$$

for all $t > 0$. This estimate is clearly superior to (3.4), because the right-hand side grows only polynomially as $t \to \infty$. Another elementary but important observation is that the same estimate holds if we replace $\rho_R(x)$ with $\rho_R(x-y)$ for any fixed $y \in \mathbb{R}^d$. Taking the supremum over all translations and using the fact that $\rho_R(x) \ge c > 0$ whenever $|x| \le R$, we obtain

$$\sup_{x\in\mathbb{R}^d} \frac{1}{R(t)^d}\int_{|y-x|\le R(t)} u(y,t)^2 \,\mathrm{d}y \;\le\; C\|u_0\|_{L^2_{\mathrm{ul}}}^2 \;, \qquad t \ge 0\;, \tag{3.7}$$

where $R(t) = (1+t)^{1/2}$ and $C > 0$ is independent of t. In particular, if $u_0 \equiv 1$, then $u(\cdot,t) \equiv 1$ for all positive times, and (3.7) is sharp in that particular case, as far as the time dependence is concerned.

Unfortunately, the approach developed so far does not allow to bound the norm $\|u(t)\|_{L^2_{\mathrm{ul}}}$ in an optimal way. Indeed, the best we can deduce directly from (3.6) is

$$\|u(t)\|_{L^2_{\mathrm{ul}}} \;\le\; C\|u_0\|_{L^2_{\mathrm{ul}}}(1+t)^{d/4} \;, \qquad t \ge 0\;,$$

which is not sharp in view of Proposition 4.6. To improve that result, a possibility is to use uniformly local L^p estimates for higher values of p. Indeed, if $p \in \mathbb{N}^*$ we have as before

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\rho_R u^{2p}\,\mathrm{d}x \;&=\; p\int_{\mathbb{R}^d}\rho_R u^{2p-1}u_t\,\mathrm{d}x \;=\; p\int_{\mathbb{R}^d}\rho_R u^{2p-1}\Delta u\,\mathrm{d}x\\
&=\; -p(2p-1)\int_{\mathbb{R}^d}\rho_R u^{2p-2}|\nabla u|^2\,\mathrm{d}x - p\int_{\mathbb{R}^d}(\nabla\rho_R\cdot\nabla u)u^{2p-1}\,\mathrm{d}x\\
&=\; -\frac{2p-1}{p}\int_{\mathbb{R}^d}\rho_R|\nabla u^p|^2\,\mathrm{d}x - \int_{\mathbb{R}^d}(\nabla\rho_R\cdot\nabla u^p)u^p\,\mathrm{d}x\\
&\le\; -\int_{\mathbb{R}^d}\rho_R|\nabla u^p|^2\,\mathrm{d}x + \frac{1}{R}\int_{\mathbb{R}^d}\rho_R|\nabla u^p||u|^p\,\mathrm{d}x\\
&\le\; -\frac{1}{2}\int_{\mathbb{R}^d}\rho_R|\nabla u^p|^2\,\mathrm{d}x + \frac{1}{2R^2}\int_{\mathbb{R}^d}\rho_R u^{2p}\,\mathrm{d}x\;.
\end{aligned}$$

Proceeding as above, we find if $u_0 \in L^{2p}_{\mathrm{ul}}(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d}\rho_{R(t)}(x)u(x,t)^{2p}\,\mathrm{d}x \;\le\; C\int_{\mathbb{R}^d}\rho_{R(t)}(x)u_0(x)^{2p}\,\mathrm{d}x \;\le\; C\|u_0\|_{L^{2p}_{\mathrm{ul}}}^{2p}(1+t)^{d/2}\;,$$

where $R(t) = (1+t)^{1/2}$, and this implies

$$\|u(t)\|_{L^{2p}_{\mathrm{ul}}} \;\le\; C^{\frac{1}{2p}}\|u_0\|_{L^{2p}_{\mathrm{ul}}}(1+t)^{\frac{d}{4p}}\;, \qquad t \ge 0\;.$$

Here the constant C does not depend on p. Now, if we assume that $u_0 \in C_{\mathrm{bu}}(\mathbb{R}^d)$, we can take the limit $p \to \infty$ in the above inequality, and we obtain the bound $\|u(t)\|_{L^\infty} \le \|u_0\|_{L^\infty}$ which is clearly optimal.

3.2 Uniformly local energy estimates for the $2D$ Navier-Stokes equations

After these preliminaries, we return to the two-dimensional Navier-Stokes equations in $\mathbb{R}^2$. We assume that the initial data $u_0 \in X$ satisfy $\operatorname{div} u_0 = 0$ and $\omega_0 = \operatorname{curl} u_0 \in L^\infty(\mathbb{R}^2)$. Let $u \in C^0([0,+\infty),X)$ be the unique mild solution of (2.1) given by Proposition 2.7. Our goal is to control the velocity field $u(x,t)$ for large times using uniformly local L^2 estimates.

For technical reasons, related to the control of the pressure term in (2.1), it is convenient here to used compactly supported weight functions, see [46]. Let $\psi : \mathbb{R}^2 \to \mathbb{R}_+$ be a smooth function satisfying

$$\psi(x) \;=\; \begin{cases} 1 & \text{if} \quad |x| \le 1\,, \\ 0 & \text{if} \quad |x| \ge 2\,. \end{cases}$$

We also assume that ψ is radially symmetric and nonincreasing along rays, and we define $\phi = \psi^2$. Then $\phi(x)$ is also equal to 1 if $|x| \le 1$ and to 0 if $|x| \ge 2$. In addition, we have the estimate

$$|\nabla\phi(x)| \;\le\; C_3\phi(x)^{1/2}\;, \qquad x \in \mathbb{R}^2\;, \tag{3.8}$$

where $C_3 = 2\|\nabla\psi\|_{L^\infty}$. Note that a compactly supported function ϕ cannot satisfy $|\nabla\phi| \le C\phi$, unless $\phi \equiv 0$, and this is why we shall only use the weaker property (3.8). Given $x_0 \in \mathbb{R}^2$ and $R > 0$, we also consider the translated and rescaled localization function ϕ_{R,x_0} defined by

$$\phi_{R,x_0}(x) \;=\; \phi\Big(\frac{x-x_0}{R}\Big)\;, \qquad x \in \mathbb{R}^2. \tag{3.9}$$

By construction we have $\phi_{R,x_0} = 1$ on $B^R_{x_0}$ and $\phi_{R,x_0} = 0$ on the complement of $B^{2R}_{x_0}$, where $B^R_{x_0}$ denotes the closed ball with radius R centered at x_0 :

$$B^R_{x_0} \;=\; \Big\{x \in \mathbb{R}^2 \,\Big|\, |x-x_0| \le R\Big\}\;.$$

Our starting point is the following localized energy estimate :

Lemma 3.1. *For any $x_0 \in \mathbb{R}^2$ and any $R > 0$, the solution of* (2.1) *satisfies*

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\phi_{R,x_0}|u|^2\,\mathrm{d}x + \nu\int_{\mathbb{R}^2}|\nabla(\phi_{R,x_0}^{1/2}u)|^2\,\mathrm{d}x \\ = \nu\int_{\mathbb{R}^2}|\nabla\phi_{R,x_0}^{1/2}|^2|u|^2\,\mathrm{d}x + \int_{\mathbb{R}^2}q(u\cdot\nabla\phi_{R,x_0})\,\mathrm{d}x\;,\end{aligned} \tag{3.10}$$

where $q = p + \frac{1}{2}|u|^2$.

Proof. For simplicity we write ϕ instead of ϕ_{R,x_0}, and we denote $\psi = \phi^{1/2}$. Using the equivalent form (2.24) of the Navier-Stokes equations, we easily obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\phi|u|^2\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2}\phi u\cdot u_t\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2}\phi u\cdot(\nu\Delta u - \nabla q)\,\mathrm{d}x\;. \tag{3.11}$$

Now, we have for $k = 1, 2$:

$$\int_{\mathbb{R}^2}|\nabla(\psi u_k)|^2 \;=\; \int_{\mathbb{R}^2}|u_k\nabla\psi + \psi\nabla u_k|^2 \;=\; \int_{\mathbb{R}^2}(u_k^2|\nabla\psi|^2 + \psi^2|\nabla u_k|^2 + 2\psi u_k\nabla\psi\cdot\nabla u_k)\,\mathrm{d}x\;,$$

hence

$$\begin{aligned}\int_{\mathbb{R}^2}\psi^2 u_k\Delta u_k\,\mathrm{d}x \;&=\; -\int_{\mathbb{R}^2}\psi^2|\nabla u_k|^2\,\mathrm{d}x - 2\int_{\mathbb{R}^2}\psi u_k\nabla\psi\cdot\nabla u_k\,\mathrm{d}x \\ &=\; -\int_{\mathbb{R}^2}|\nabla(\psi u_k)|^2\,\mathrm{d}x + \int_{\mathbb{R}^2}|\nabla\psi|^2 u_k^2\,\mathrm{d}x\;.\end{aligned}$$

Thus summing over k and multiplying by ν, we obtain

$$\nu\int_{\mathbb{R}^2}\phi u\cdot(\Delta u)\,\mathrm{d}x \;=\; -\nu\int_{\mathbb{R}^2}|\nabla(\psi u)|^2\,\mathrm{d}x + \nu\int_{\mathbb{R}^2}|\nabla\psi|^2|u|^2\,\mathrm{d}x\;. \tag{3.12}$$

On the other hand, since $\operatorname{div} u = 0$, we easily find

$$\int_{\mathbb{R}^2} \phi u \cdot \nabla q \,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} \phi \operatorname{div}(uq) \,\mathrm{d}x \;=\; -\int_{\mathbb{R}^2} q(u \cdot \nabla \phi) \,\mathrm{d}x \;. \tag{3.13}$$

Combining (3.11)–(3.13) we arrive at (3.10). $\square$

Our next task is to transform the identity (3.10) into a differential inequality for the uniformly local energy norm of the velocity field. The terms proportional to the viscosity parameter ν originate from the linear part of the equation and can be easily estimated, as in Section 3.1. The difficulty is concentrated in the last term, which contains the modified pressure $q = p + \frac{1}{2}|u|^2$. That term is nonlocal in space and cubic in the velocity field u. Using the results of Section 4.2, we obtain the following important estimate:

Lemma 3.2. *There exists a constant $C_4 > 0$ such that, if $x_0 \in \mathbb{R}^2$ and $0 < r \le R$, one has*

$$\begin{aligned}\left|\int_{\mathbb{R}^2} q(u \cdot \nabla \phi_{R,x_0}) \,\mathrm{d}x\right| \;\le\; C_4 \biggl(&\frac{r}{R} \|\omega\|_{L^\infty} \|u\|^2_{L^2(B^{3R}_{x_0})} \\ &+ \frac{1}{rR} \|u\|^3_{L^2(B^{3R}_{x_0})} + \frac{1}{R^2} \sup_{z \in \mathbb{R}^2} \|u\|^3_{L^2(B^{2R}_z)}\biggr) \,.\end{aligned} \tag{3.14}$$

Remark 3.3. As often in the sequel, we completely omit here the time dependence of the quantities u, ω, and q, for notational simplicity. This is harmless since time plays no role in an "elliptic" estimate such as (3.14). But in Lemma 3.4 below that estimate will be applied to the solution of the Navier-Stokes equations which, of course, depends on time.

Proof. For simplicity we assume that $x_0 = 0$, and we write ϕ_R instead of ϕ_{R,x_0}. To use the results of Section 4.2 we choose a smooth function $\chi : \mathbb{R}^2 \to [0,1]$ satisfying

$$\chi(x) \;=\; \begin{cases} 1 & \text{if} \quad |x| \le 1/2\,, \\ 0 & \text{if} \quad |x| \ge 1\,, \end{cases}$$

and we denote $\chi_r(x) = \chi(x/r)$, where $0 < r \le R$. To estimate the left-hand side of (3.14), it is sufficient to control the modified pressure q in the ball B^{2R}_0, because $\nabla \phi_R$ vanishes outside that ball. Also, as is clear from (3.13), adding a constant to q does not change the quantity we want to bound. Thus, using Lemma 4.5, we can assume that

$$q(x) \;=\; q_1(x) + q_2(x) + q_3(x) + q_4(x) \;, \qquad x \in B^{2R}_0 \;, \tag{3.15}$$

where

$$\begin{aligned}
q_1(x) \;&=\; \frac{1}{2\pi} \int_{\mathbb{R}^2} \chi_r(x-y) \frac{(x-y)^\perp}{|x-y|^2} \cdot u(y)\omega(y) \,\mathrm{d}y \;, \\
q_2(x) \;&=\; \frac{1}{4\pi} \sum_{k,\ell=1}^{2} \int_{\mathbb{R}^2} M^{(r)}_{k\ell}(x-y) u_k(y) u_\ell(y) \,\mathrm{d}y \;, \\
q_3(x) \;&=\; \frac{1}{2\pi} \sum_{k,\ell=1}^{2} \int_{|y| \le 3R} \chi^c_r(x-y) K_{k\ell}(x-y) u_k(y) u_\ell(y) \,\mathrm{d}y \;, \\
q_4(x) \;&=\; \frac{1}{2\pi} \sum_{k,\ell=1}^{2} \int_{|y| \ge 3R} \Bigl\{K_{k\ell}(x-y) - K_{k\ell}(-y)\Bigr\} u_k(y) u_\ell(y) \,\mathrm{d}y \;.
\end{aligned}$$

The expressions above agree with (4.15) up to the following inessential differences. First we use everywhere the rescaled cut-off χ_r instead of χ. In particular we denote by $M_{k\ell}^{(r)}$ the functions defined by (4.16) with χ replaced by χ_r, and we write $\chi_r^c = 1-\chi_r$. Next, in the definition (4.15) of $q_3(x,x_0)$, we take $x_0 = 0$ and we decompose the domain of integration as $\mathbb{R}^2 = B_0^{3R} \cup (B_0^{3R})^c$. The integral over $y \in B_0^{3R}$ coincides with the function q_3 above, up to an irrelevant additive constant, and the integral over $y \notin B_0^{3R}$ gives exactly q_4, because $\chi_r^c(x-y) = \chi_r^c(-y) = 1$ when $|x| \le 2R$ and $|y| \ge 3R$.

We now estimate the various terms in (3.15). As χ_r is supported in the ball $B_0^r \subset B_0^R$, we have

$$|q_1(x)| \,\le\, \frac{1}{2\pi}\int_{|y|\le 3R} \frac{\chi_r(x-y)}{|x-y|}\,|u(y)||\omega(y)|\,\mathrm{d}y\;, \qquad x \in B_0^{2R}\;.$$

Since

$$\int_{\mathbb{R}^2} \frac{\chi_r(z)}{|z|}\,\mathrm{d}z \,=\, r\int_{\mathbb{R}^2}\frac{\chi(z)}{|z|}\,\mathrm{d}z \,=\, Cr\;,$$

it follows from Young's inequality (see Section 4.4) that

$$\|q_1\|_{L^2(B_0^{2R})} \,\le\, Cr\|u\,\omega\|_{L^2(B_0^{3R})} \,\le\, Cr\|\omega\|_{L^\infty}\|u\|_{L^2(B_0^{3R})}\;. \tag{3.16}$$

Similarly,

$$|q_2(x)| \,\le\, \frac{1}{4\pi}\sum_{k,\ell=1}^{2}\int_{|y|\le 3R} |M_{k\ell}^{(r)}(x-y)||u_k(y)||u_\ell(y)|\,\mathrm{d}y\;, \qquad x\in B_0^{2R}\;.$$

As $|M_{k\ell}^{(r)}(z)| \le C|z|^{-1}|\nabla\chi_r(z)| = Cr^{-1}|z|^{-1}|\nabla\chi(z/r)|$, we have

$$\sum_{k,\ell=1}^{2}\int_{\mathbb{R}^2}|M_{k\ell}^{(r)}(z)|^2\,\mathrm{d}z \,\le\, \frac{C}{r^2}\int_{r/2\le|z|\le r}\frac{1}{|z|^2}\,\mathrm{d}z \,=\, \frac{C}{r^2}\;,$$

and using Young's inequality again we find

$$\|q_2\|_{L^2(B_0^{2R})} \,\le\, \frac{C}{r}\,\||u|^2\|_{L^1(B_0^{3R})} \,=\, \frac{C}{r}\,\|u\|^2_{L^2(B_0^{3R})}\;. \tag{3.17}$$

Exactly the same estimate holds for q_3 too, because in view of (4.16)

$$\sum_{k,\ell=1}^{2}\int_{\mathbb{R}^2}\chi_r^c(z)^2 K_{k\ell}(z)^2\,\mathrm{d}z \,\le\, C\int_{|z|\ge r/2}\frac{1}{|z|^4}\,\mathrm{d}z \,=\, \frac{C}{r^2}\;.$$

Finally, to bound q_4, we use estimate (4.19) which shows that, for any $x \in B_0^{2R}$ and $y \notin B_0^{3R}$,

$$\sum_{k,\ell=1}^{2}|K_{k\ell}(x-y) - K_{k\ell}(-y)| \,\le\, \frac{CR}{2|y|^3} \,\le\, \frac{CR}{R^3+|y|^3}\;.$$

Thus

$$|q_4(x)| \,\le\, C\int_{\mathbb{R}^2}\frac{R}{R^3+|y|^3}\,|u(y)|^2\,\mathrm{d}y\;, \qquad x\in B_0^{2R}\;.$$

To evaluate the integral we decompose $\mathbb{R}^2 = \cup_{k\in\mathbb{Z}^2} Q_k^R$, where $Q_k^R \subset B_{kR}^R$ denotes the square of measure R^2 centered at $kR \in \mathbb{R}^2$. For $k \in \mathbb{Z}^2$ we also define

$$S_k \,=\, \sup_{y\in Q_k^R}\frac{R}{R^3+|y|^3} \,\le\, \frac{C}{R^2}\,\frac{1}{1+|k|^3}\;, \qquad \text{so that} \quad \sum_{k\in\mathbb{Z}^2} S_k \,\le\, \frac{C}{R^2}\;.$$

With these notations we find for $x \in B_0^{2R}$:

$$|q_4(x)| \,\le\, C \sum_{k\in\mathbb{Z}^2} \int_{Q_k^R} \frac{R}{R^3+|y|^3}\,|u(y)|^2 \,\mathrm{d}y \,\le\, C\Bigl(\sum_{k\in\mathbb{Z}^2} S_k\Bigr) \sup_{k\in\mathbb{Z}^2} \|u\|_{L^2(Q_k^R)}^2 \,,$$

hence

$$\|q_4\|_{L^2(B_0^{2R})} \,\le\, 2\pi^{1/2} R \|q_4\|_{L^\infty(B_0^{2R})} \,\le\, \frac{C}{R} \sup_{z\in\mathbb{R}^2} \|u\|_{L^2(B_z^R)}^2 \,. \tag{3.18}$$

Summarizing, estimates (3.16)–(3.18) show that

$$\|q\|_{L^2(B_0^{2R})} \,\le\, Cr\|\omega\|_{L^\infty}\|u\|_{L^2(B_0^{3R})} + \frac{C}{r}\,\|u\|_{L^2(B_0^{3R})}^2 + \frac{C}{R} \sup_{z\in\mathbb{R}^2} \|u\|_{L^2(B_z^R)}^2 \,. \tag{3.19}$$

On the other hand, as $\nabla\phi_R(x) = R^{-1}\nabla\phi(x/R)$, we have by Hölder's inequality

$$\biggl|\int_{\mathbb{R}^2} q(u\cdot\nabla\phi_R)\,\mathrm{d}x\biggr| \,\le\, \frac{C}{R}\,\|u\|_{L^2(B_0^{2R})}\|q\|_{L^2(B_0^{2R})} \,,$$

thus using (3.19) we easily obtain estimate (3.14). □

Combining Lemmas 3.1 and 3.2, we now derive an integral inequality for the quantity

$$Z_R(t) \,=\, \sup_{x\in\mathbb{R}^2} \|u(t)\|_{L^2(B_x^R)} \,, \tag{3.20}$$

which is equivalent (up to R-dependent constants) to the norm $\|u(t)\|_{L^2_{\mathrm{ul}}}$.

Lemma 3.4. *There exists a constant $C_5 \ge 1$ such that, for any $t > 0$ and any $R \ge r > 0$,*

$$\begin{aligned} Z_R(t)^2 + 2\nu \sup_{x\in\mathbb{R}^2} \int_0^t \|\nabla u(s)\|_{L^2(B_x^R)}^2 \,\mathrm{d}s \,&\le\, 7Z_R(0)^2 \\ &+ C_5 \int_0^t \Bigl\{\frac{\nu}{R^2}\,Z_R(s)^2 + \frac{r}{R}\|\omega(s)\|_{L^\infty} Z_R(s)^2 + \frac{1}{rR} Z_R(s)^3\Bigr\}\,\mathrm{d}s \,. \end{aligned} \tag{3.21}$$

Proof. Fix $R > 0$. Integrating (3.10) with respect to time, we obtain for any $t > 0$:

$$\begin{aligned} \int_{\mathbb{R}^2} \phi_{R,x_0}|u(t)|^2\,\mathrm{d}x + 2\nu\int_0^t\!\!\int_{\mathbb{R}^2} |\nabla(\phi_{R,x_0}^{1/2}u(s))|^2\,\mathrm{d}x\,\mathrm{d}s \,&=\, \int_{\mathbb{R}^2}\phi_{R,x_0}|u_0|^2\,\mathrm{d}x \\ + 2\nu\int_0^t\!\!\int_{\mathbb{R}^2} |\nabla\phi_{R,x_0}^{1/2}|^2|u(s)|^2\,\mathrm{d}x\,\mathrm{d}s &+ 2\int_0^t\!\!\int_{\mathbb{R}^2} q(s)(u(s)\cdot\nabla\phi_{R,x_0})\,\mathrm{d}x\,\mathrm{d}s \,, \end{aligned} \tag{3.22}$$

where $x_0 \in \mathbb{R}^2$ is arbitrary. We now take the supremum over $x_0 \in \mathbb{R}^2$ in both sides. As $\phi_{R,x_0} = 1$ on $B_{x_0}^R$, the supremum over $x_0 \in \mathbb{R}^2$ of the left-hand side of (3.22) is bounded from below by the left-hand side of (3.21). To bound the right-hand side of (3.22), we observe that

$$\sup_{x_0\in\mathbb{R}^2}\int_{\mathbb{R}^2}\phi_{R,x_0}|u_0|^2\,\mathrm{d}x \,\le\, \sup_{x_0\in\mathbb{R}^2}\|u_0\|_{L^2(B_{x_0}^{2R})}^2 \,\le\, 7Z_R(0)^2 \,,$$

because each ball $B_{x_0}^{2R}$ can be covered by 7 balls of radius R, centered at appropriate points. Similarly, as $|\nabla\phi_{R,x_0}^{1/2}|$ is bounded by C/R and vanishes outside $B_{x_0}^{2R}$, we have

$$\sup_{x_0\in\mathbb{R}^2}\int_0^t\!\!\int_{\mathbb{R}^2}|\nabla\phi_{R,x_0}^{1/2}|^2|u(s)|^2\,\mathrm{d}s \,\le\, \frac{C}{R^2}\sup_{x_0\in\mathbb{R}^2}\int_0^t\|u(s)\|_{L^2(B_{x_0}^{2R})}^2\,\mathrm{d}s \,\le\, \frac{7C}{R^2}\int_0^t Z_R(s)^2\,\mathrm{d}s \,.$$

Finally, for the last term in (3.22), we choose $r \in (0, R]$ and we use inequality (3.14). Collecting all estimates, we see that the supremum over $x_0 \in \mathbb{R}^2$ of the right-hand side of (3.22) is bounded by the right-hand side of (3.21), provided $C_5 > 0$ is large enough. □

Using Lemma 3.4 and Gronwall's lemma, we arrive at the main result of this section.

Proposition 3.5. *There exist positive constants C_6 and C_7 such that the following holds for any $\nu > 0$. Let $u \in C^0([0,+\infty), X)$ be the mild solution of the Navier-Stokes equations* (2.1) *with initial data $u_0 \in X$ satisfying* $\operatorname{div} u_0 = 0$*, $\omega_0 \in L^\infty(\mathbb{R}^2)$, and $\omega_0 \not\equiv 0$. For any $t > 0$, if $R > 0$ is large enough so that*

$$R \;\ge\; \max\Bigl\{R_0\,,\, C_7\sqrt{\nu t}\,,\, C_7\|u_0\|_{L^\infty}\|\omega_0\|_{L^\infty}t^2\Bigr\}\;, \qquad \textit{where} \quad R_0 \;=\; \frac{\|u_0\|_{L^\infty}}{\|\omega_0\|_{L^\infty}}\;, \tag{3.23}$$

we have

$$Z_R(t) \;\equiv\; \sup_{x\in\mathbb{R}^2} \|u(t)\|_{L^2(B_x^R)} \;\le\; C_6 R\|u_0\|_{L^\infty}\;. \tag{3.24}$$

Proof. We observe that $u_0 \not\equiv 0$, since by assumption $\omega_0 \not\equiv 0$. Take $C_6 > 0$ and $C_7 > 0$ such that

$$C_6^2 \;>\; 7\,\mathrm{e}\,\pi\;, \qquad C_7^2 \;\ge\; 2C_5\;, \qquad C_7^{1/2} \;\ge\; 2C_5(1+C_6)\;, \tag{3.25}$$

where C_5 is as in Lemma 3.4. Given any $t > 0$, choose R as in (3.23). From the definition (3.20), we see that $Z_R(0) \le \pi^{1/2}R\|u_0\|_{L^\infty}$, hence by continuity we necessarily have $Z_R(s) \le C_6R\|u_0\|_{L^\infty}$ for sufficiently small $s > 0$. Define

$$t_* \;=\; \sup\Bigl\{\tau\in[0,t]\;\Big|\; Z_R(s) \le C_6R\|u_0\|_{L^\infty} \text{ for all } s\in[0,\tau]\Bigr\} \;\in\; (0,t]\;. \tag{3.26}$$

We shall prove that $t_* = t$, hence $Z_R(t) \le C_6R\|u_0\|_{L^\infty}$, which is (3.24).

According to Lemma 3.4, we have for all $\tau \in [0,t_*]$:

$$Z_R(\tau)^2 \;\le\; 7Z_R(0)^2 + C_5\int_0^\tau A(s)Z_R(s)^2\,\mathrm{d}s\;, \tag{3.27}$$

where

$$A(s) \;=\; \frac{\nu}{R^2} + \frac{r}{R}\|\omega(s)\|_{L^\infty} + \frac{1}{rR}\,Z_R(s) \;\le\; \frac{\nu}{R^2} + \frac{r}{R}\|\omega_0\|_{L^\infty} + \frac{C_6}{r}\|u_0\|_{L^\infty}\;.$$

Here $r \le R$ is arbitrary, but we can optimize the right-hand side by choosing $r = (RR_0)^{1/2}$, which gives

$$A(s) \;\le\; \frac{\nu}{R^2} + (1+C_6)\frac{\|u_0\|_{L^\infty}^{1/2}\|\omega_0\|_{L^\infty}^{1/2}}{R^{1/2}}\;, \qquad s\in[0,t_*]\;.$$

In particular, using (3.23), (3.25) and the fact that $t_* \le t$, we find

$$C_5\int_0^{t_*} A(s)\,\mathrm{d}s \;\le\; C_5\frac{\nu t}{R^2} + C_5(1+C_6)\frac{\|u_0\|_{L^\infty}^{1/2}\|\omega_0\|_{L^\infty}^{1/2}t}{R^{1/2}} \;\le\; \frac12 + \frac12 \;=\; 1\;.$$

If we now apply Gronwall's lemma (see Section 4.4) to (3.27), we obtain

$$Z_R(t_*)^2 \;\le\; 7Z_R(0)^2\exp\Bigl(C_5\int_0^{t_*}A(s)\,\mathrm{d}s\Bigr) \;\le\; 7\,\mathrm{e}\,Z_R(0)^2 \;\le\; 7\,\mathrm{e}\,\pi R^2\|u_0\|_{L^\infty}^2\;. \tag{3.28}$$

By (3.25) we thus have $Z_R(t_*) < C_6R\|u_0\|_{L^\infty}$, which contradicts (3.26) if $t_* < t$. Thus we must have $t_* = t$, which proves (3.24). $\square$

Remark 3.6. Estimate (1.15) follows immediately from (3.28) with $t_* = t$.

3.3 Velocity bounds and uniformly local enstrophy estimates

In this final section, we derive a few important consequences of the previous results. First, combining Proposition 3.5 with Corollary 4.3, we derive an upper bound on the L^∞ norm of the velocity field which greatly improves (2.26).

Proposition 3.7. *There exist a positive constant C_8 such that, for any $u_0 \in X$ satisfying $\operatorname{div} u_0 = 0$ and $\omega_0 = \operatorname{curl} u_0 \in L^\infty(\mathbb{R}^2)$, the solution of the Navier-Stokes equations (2.1) with initial data u_0 satisfies*

$$\|u(t)\|_{L^\infty} \,\le\, C_8 \|u_0\|_{L^\infty}\biggl\{1 + \|\omega_0\|_{L^\infty} t + \Bigl(\frac{\sqrt{\nu t}}{R_0}\Bigr)^{1/2}\biggr\}~, \qquad t \ge 0~, \tag{3.29}$$

where $R_0 = \|u_0\|_{L^\infty}/\|\omega_0\|_{L^\infty}$.

Proof. If $\omega_0 \equiv 0$, then u_0 is a constant and $u(t) = u_0$ for all $t \ge 0$, hence we can assume without loss of generality that $\omega_0 \not\equiv 0$. Fix $t > 0$, and let $R = \max\{R_0\,,\, C_7\sqrt{\nu t}\,,\, C_7\|u_0\|_{L^\infty}\|\omega_0\|_{L^\infty} t^2\}$, as in (3.23). If $M = \|u(t)\|_{L^\infty} > 0$, there exists $\bar x \in \mathbb{R}^2$ such that $|u(\bar x, t)| \ge M/2$. For simplicity, we assume without loss of generality that $\bar x = 0$. We know from Proposition 3.5 that

$$I := \int_{|x|\le R} |u(x,t)|^2 \,\mathrm{d}x \,\le\, C_6^2 R^2 \|u_0\|_{L^\infty}^2~. \tag{3.30}$$

On the other hand, applying Corollary 4.3 to $u(x,t)$, we deduce from (4.10) that

$$|u(x,t) + u(-x,t)| \,\ge\, 2|u(0,t)| - C_*|x|\|\omega(t)\|_{L^\infty} \,\ge\, M - C_*|x|\|\omega_0\|_{L^\infty}~, \tag{3.31}$$

for all $x \in \mathbb{R}^2$, where $C_* > 0$ is a universal constant. The idea is to use estimate (3.31) to obtain a lower bound on the quantity I defined in (3.30), in terms of M. Let

$$R_* \,=\, \frac{M}{C_*\|\omega_0\|_{L^\infty}}~.$$

We consider separately the following two cases:

Case 1: $R_* \le R$. In that case, denoting $D_+ = \{x \in \mathbb{R}^2 \,|\, |x| \le R_*\,,\ x_1 \ge 0\}$, we compute

$$\begin{aligned} I \,&\ge\, \int_{|x|\le R_*} |u(x,t)|^2 \,\mathrm{d}x \,=\, \int_{D_+} \Bigl(|u(x,t)|^2 + |u(-x,t)|^2\Bigr)\,\mathrm{d}x \\ &\ge\, \frac{1}{2}\int_{D_+} |u(x,t) + u(-x,t)|^2 \,\mathrm{d}x \,\ge\, \frac{1}{4}\int_{|x|\le R_*} \Bigl(M - C_*|x|\|\omega_0\|_{L^\infty}\Bigr)^2 \mathrm{d}x \,=\, \frac{\pi}{24}\, M^2 R_*^2~, \end{aligned}$$

where in the last inequality we used (3.31). Comparing with (3.30), we deduce that

$$M^2 R_*^2 \,\le\, C R^2 \|u_0\|_{L^\infty}^2~, \qquad \text{or} \quad M^4 \,\le\, C R^2 \|u_0\|_{L^\infty}^2 \|\omega_0\|_{L^\infty}^2~.$$

Since $M = \|u(t)\|_{L^\infty}$ and $R = \max\{R_0\,,\, C_7\sqrt{\nu t}\,,\, C_7\|u_0\|_{L^\infty}\|\omega_0\|_{L^\infty} t^2\}$, this gives (3.29).

Case 2: $R_* \ge R$. A similar calculation gives

$$I \,\ge\, \frac{1}{4}\int_{|x|\le R} \Bigl(M - C_*|x|\|\omega_0\|_{L^\infty}\Bigr)^2 \mathrm{d}x \,\ge\, \frac{\pi}{24}\, M^2 R^2~,$$

hence $M^2 R^2 \le C R^2 \|u_0\|_{L^\infty}^2$, namely $M^2 \le C\|u_0\|_{L^\infty}^2$. Thus we obtain (3.29) in both cases. $\square$

Proof of Theorem 1.2. We can assume without loss of generality that $u_0 \not\equiv 0$. Since global existence of solutions is already asserted by Theorem 1.1, we only need to prove estimate (1.12). In view of the local existence theory, it is sufficient to prove (1.12) for $t \geq T$, where $T > 0$ is as in (1.10). In that case we have $\sqrt{\nu t} \leq K_1 \|u_0\|_{L^\infty} t$, or equivalently

$$\frac{\sqrt{\nu t}}{R_0} = \frac{\|\omega_0\|_{L^\infty}}{\|u_0\|_{L^\infty}} \sqrt{\nu t} \leq K_1 \|\omega_0\|_{L^\infty} t ,$$

hence (1.12) follows directly from (3.29). $\square$

The results obtained so far do not rely on the viscous dissipation term in (2.1), and remain therefore valid in the vanishing viscosity limit. Now, assuming that $\nu > 0$, we can also derive uniformly local enstrophy estimates.

Proposition 3.8. *Under the assumptions of Proposition 3.5, there exists a constant $C_9 > 0$ such that, for all $t > 0$,*

$$\sup_{x\in\mathbb{R}^2} \|\omega(t)\|_{L^2(B_x^R)}^2 \,\mathrm{d}y \leq C_9 \|u_0\|_{L^2}^2 \Big(1 + \frac{R^2}{\nu t} + \frac{Rt}{\sqrt{\nu t}} \|\omega_0\|_{L^\infty}\Big) , \tag{3.32}$$

where $R = R(t)$ is as in (3.23).

Proof. Fix $t > 0$ and let R be as in (3.23). If one does not neglect the second term in the left-hand side of (3.21), the proof of Proposition 3.5 shows that

$$Z_R(t)^2 + 2\nu \sup_{x\in\mathbb{R}^2} \int_0^t \|\nabla u(s)\|_{L^2(B_x^R)}^2 \,\mathrm{d}s \leq C_6^2 R^2 \|u_0\|_{L^\infty}^2 . \tag{3.33}$$

Unfortunately, we cannot extract from (3.33) a pointwise estimate in time on the uniformly local L^2 norm of ∇u, because we cannot exchange the supremum and the integral in the left-hand side of (3.33).

To avoid that difficulty, we use localized energy estimates for the vorticity equation (1.8). Let $x_0 \in \mathbb{R}^2$. As in Lemma 3.1 we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \phi_{R,x_0} |\omega|^2 \,\mathrm{d}x + \nu \int_{\mathbb{R}^2} |\nabla(\phi_{R,x_0}^{1/2}\omega)|^2 \,\mathrm{d}x \\ = \nu \int_{\mathbb{R}^2} |\nabla \phi_{R,x_0}^{1/2}|^2 |\omega|^2 \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^2} \omega^2 (u \cdot \nabla \phi_{R,x_0}) \,\mathrm{d}x ,\end{aligned} \tag{3.34}$$

where ϕ_{R,x_0} is the localization function defined in (3.9). In view of (3.33), there exists a time $t_0 \in [0, t/2]$ (depending on x_0) such that

$$\nu \|\omega(t_0)\|_{L^2(B_{x_0}^{2R})}^2 \leq 2\nu \|\nabla u(t_0)\|_{L^2(B_{x_0}^{2R})}^2 \leq \frac{CR^2}{t} \|u_0\|_{L^\infty}^2 . \tag{3.35}$$

Integrating (3.34) over the time interval $[t_0, t]$, we find

$$\begin{aligned}\int_{\mathbb{R}^2} \phi_{R,x_0} |\omega(t)|^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^2} \phi_{R,x_0} |\omega(t_0)|^2 \,\mathrm{d}x + 2\nu \int_{t_0}^t \int_{\mathbb{R}^2} |\nabla \phi_{R,x_0}^{1/2}|^2 |\omega(s)|^2 \,\mathrm{d}x \,\mathrm{d}s \\ + \int_{t_0}^t \int_{\mathbb{R}^2} \omega(s)^2 (u(s) \cdot \nabla \phi_{R,x_0}) \,\mathrm{d}x \,\mathrm{d}s .\end{aligned} \tag{3.36}$$

By (3.35) we have

$$\int_{\mathbb{R}^2} \phi_{R,x_0} |\omega(t_0)|^2 \,\mathrm{d}x \;\le\; \|\omega(t_0)\|^2_{L^2(B^{2R}_{x_0})} \;\le\; \frac{CR^2}{\nu t}\,\|u_0\|^2_{L^\infty}\;. \tag{3.37}$$

It follows also from (3.33) that

$$2\nu \int_{t_0}^t \int_{\mathbb{R}^2} |\nabla \phi^{1/2}_{R,x_0}|^2 |\omega(s)|^2 \,\mathrm{d}x\,\mathrm{d}s \;\le\; \frac{C\nu}{R^2}\int_0^t \|\omega(s)\|^2_{L^2(B^{2R}_{x_0})}\,\mathrm{d}s \;\le\; C\|u_0\|^2_{L^\infty}\;. \tag{3.38}$$

Finally, the cubic term in (3.36) can be estimated as follows:

$$\begin{aligned}
\Big|\int_{t_0}^t \int_{\mathbb{R}^2} & \omega(s)^2 (u(s)\cdot\nabla\phi_{R,x_0})\,\mathrm{d}x\,\mathrm{d}s\Big| \\
&\le\; \frac{C}{R}\int_0^t \|\omega(s)\|_{L^\infty}\|\omega(s)\|_{L^2(B^{2R}_{x_0})}\|u(s)\|_{L^2(B^{2R}_{x_0})}\,\mathrm{d}s \\
&\le\; \frac{C}{R}\,\|\omega_0\|_{L^\infty}\Big(\int_0^t \|\omega(s)\|^2_{L^2(B^{2R}_{x_0})}\,\mathrm{d}s\Big)^{1/2} t^{1/2} \sup_{0\le s\le t}\|u(s)\|_{L^2(B^{2R}_{x_0})} \\
&\le\; \frac{C}{R}\,\|\omega_0\|_{L^\infty}\cdot\frac{CR\|u_0\|_{L^\infty}}{\nu^{1/2}}\cdot t^{1/2}\cdot CR\|u_0\|_{L^\infty} \;\le\; \frac{CRt}{\sqrt{\nu t}}\,\|\omega_0\|_{L^\infty}\|u_0\|^2_{L^\infty}\;.
\end{aligned} \tag{3.39}$$

If we now insert (3.37)–(3.39) into (3.36) we obtain for all $x_0\in\mathbb{R}^2$:

$$\int_{\mathbb{R}^2}\phi_{R,x_0}|\omega(t)|^2\,\mathrm{d}x \;\le\; C\|u_0\|^2_{L^2}\Big(1+\frac{R^2}{\nu t}+\frac{Rt}{\sqrt{\nu t}}\,\|\omega_0\|_{L^\infty}\Big)\;.$$

Taking the supremum over $x_0\in\mathbb{R}^2$, we arrive at (3.32). □

Remark 3.9. Estimate (1.16) easily follows from (3.32). Indeed, in view of (1.10), it is clearly sufficient to prove (1.16) for $t\ge T$. In that case we have $\sqrt{\nu t}\le K_1\|u_0\|_{L^\infty}t$, hence

$$\frac{Rt}{\sqrt{\nu t}}\,\|\omega_0\|_{L^\infty} \;=\; \frac{R}{\nu}\,\|\omega_0\|_{L^\infty}\sqrt{\nu t} \;\le\; \frac{K_1 R}{\nu t}\,\|u_0\|_{L^\infty}\|\omega_0\|_{L^\infty}t^2 \;\le\; \frac{K_1}{C_7}\frac{R^2}{\nu t}\;,$$

where in the last inequality we used definition (3.23), which also implies that $R^2\ge C_7^2\nu t$. Thus (1.16) follows from (3.32) when $t\ge T$.

4 Appendix

4.1 The Biot-Savart law for bounded velocities and vorticities

Assume that $u\in X$ satisfies $\operatorname{div} u = 0$, and let $\omega = \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$. If ω is strongly localized, for instance if $\omega\in L^p(\mathbb{R}^2)$ for some $p\in(1,2)$, then u can be reconstructed from ω, up to an additive constant u_∞, by the classical Biot-Savart formula (see Section 4.4):

$$u(x) \;=\; u_\infty + \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\,\omega(y)\,\mathrm{d}y\;, \qquad x\in\mathbb{R}^2\;. \tag{4.1}$$

Moreover, the Hardy-Littlewood-Sobolev inequality [29] implies that $u-u_\infty\in L^q(\mathbb{R}^2)^2$ for $q=2p/(2-p)$, hence $u(x)$ converges in some sense to u_∞ as $|x|\to\infty$.

If the vorticity distribution ω is only weakly localized, the integral in (4.1) does not converge any more, and the Biot-Savart formula has to be modified. Here is a reasonable possibility:

Proposition 4.1. *If $u \in X$ is divergence free and if $\omega = \operatorname{curl} u \in L^p(\mathbb{R}^2)$ for some $p \in (2,\infty)$, then*

$$u(x) \;=\; u(0) + \frac{1}{2\pi}\int_{\mathbb{R}^2}\Big\{\frac{(x-y)^\perp}{|x-y|^2} + \frac{y^\perp}{|y|^2}\Big\}\omega(y)\,\mathrm{d}y\;, \qquad x \in \mathbb{R}^2\;. \tag{4.2}$$

Proof. Let $q = p/(p-1)$, so that $q \in (1,2)$ and $\frac{1}{p}+\frac{1}{q} = 1$. For all $x, y \in \mathbb{R}^2$ with $x \neq y$ and $y \neq 0$, we denote

$$F(x,y) \;=\; \frac{(x-y)^\perp}{|x-y|^2} + \frac{y^\perp}{|y|^2} \;=\; \frac{x^\perp(y\cdot(y{-}x)) + y^\perp(x\cdot(x{-}y)) + (x^\perp{-}y^\perp)(x\cdot y)}{|x-y|^2|y|^2}\;. \tag{4.3}$$

We claim that, for any $x \in \mathbb{R}^2$, the map $y \mapsto F(x,y)$ belongs to $L^q(\mathbb{R}^2)$ and

$$\Big(\int_{\mathbb{R}^2}|F(x,y)|^q\,\mathrm{d}y\Big)^{1/q} \;\le\; C|x|^{\frac{2}{q}-1}\;, \tag{4.4}$$

where $C > 0$ is a universal constant. Indeed, as $|F(x,y)| \le |x-y|^{-1} + |y|^{-1}$, we obtain using Minkowski's inequality

$$\Big(\int_{|y|\le 2|x|}|F(x,y)|^q\,\mathrm{d}y\Big)^{1/q} \;\le\; 2\Big(\int_{|y|\le 3|x|}\frac{1}{|y|^q}\,\mathrm{d}y\Big)^{1/q} \;\le\; C|x|^{\frac{2}{q}-1}\;.$$

On the other hand, since $|F(x,y)| \le 3|x||x-y|^{-1}|y|^{-1}$ in view of the last expression in (4.3), we have $|F(x,y)| \le 6|x||y|^{-2}$ when $|y| \ge 2|x|$, hence

$$\Big(\int_{|y|\ge 2|x|}|F(x,y)|^q\,\mathrm{d}y\Big)^{1/q} \;\le\; 6|x|\Big(\int_{|y|\ge 2|x|}\frac{1}{|y|^{2q}}\,\mathrm{d}y\Big)^{1/q} \;\le\; C|x|^{\frac{2}{q}-1}\;.$$

This proves (4.4). Now, let

$$v(x) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}F(x,y)\omega(y)\,\mathrm{d}y \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}\Big\{\frac{(x-y)^\perp}{|x-y|^2} + \frac{y^\perp}{|y|^2}\Big\}\omega(y)\,\mathrm{d}y\;, \qquad x \in \mathbb{R}^2\;.$$

Since $\frac{2}{q} - 1 = 1 - \frac{2}{p}$, we deduce from (4.4) that $|v(x)| \le C|x|^{1-\frac{2}{p}}\|\omega\|_{L^p}$ for all $x \in \mathbb{R}^2$. Moreover, a standard calculation in distribution theory shows that $\operatorname{div} v = 0$ and $\operatorname{curl} v = \omega$. If $w = u - v$, we thus have $\operatorname{div} w = 0$ and $\operatorname{curl} w = 0$, so that w is a harmonic vector field on $\mathbb{R}^2$. As $w(x)$ has a sublinear growth as $|x| \to \infty$, we conclude that w is identically constant, and since $v(0) = 0$ by definition we must have $w = u(0)$, which proves (4.2). □

In these notes, we mainly deal with the situation where the vorticity ω is not localized at all, namely $\omega \in L^\infty(\mathbb{R}^2)$. In that case, the integral in (4.2) is logarithmically divergent, and has to be interpreted in an appropriate way. The main result of this section is:

Proposition 4.2. *Assume that $u \in X$, $\operatorname{div} u = 0$, and $\omega = \operatorname{curl} u \in L^\infty(\mathbb{R}^2)$. Then*

$$u(x) \;=\; u(0) + \lim_{R\to\infty}\frac{1}{2\pi}\int_{|y|\le R}\Big\{\frac{(x-y)^\perp}{|x-y|^2} + \frac{y^\perp}{|y|^2}\Big\}\omega(y)\,\mathrm{d}y\;, \qquad x \in \mathbb{R}^2\;. \tag{4.5}$$

Proof. For any $R > 0$ we denote

$$u_R(x) \;=\; \frac{1}{2\pi}\int_{|y|\le R}\Big\{\frac{(x-y)^\perp}{|x-y|^2} + \frac{y^\perp}{|y|^2}\Big\}\omega(y)\,\mathrm{d}y\;, \qquad x \in \mathbb{R}^2\;.$$

Clearly $\operatorname{div} u_R = 0$ and $\operatorname{curl} u_R = \omega\, \mathbf{1}_{\{|x|\le R\}}$ in the sense of distributions on $\mathbb{R}^2$. Moreover u_R is Hölder continuous and satisfies $u_R(0) = 0$. For $R > 2|x|+3$, we decompose $u_R(x) = v(x)+w_R(x)$, where

$$v(x) \;=\; \frac{1}{2\pi}\int_{D_x} F(x,y)\omega(y)\,\mathrm{d}y\;, \qquad w_R(x) \;=\; \frac{1}{2\pi}\int_{B_0^R\setminus D_x} F(x,y)\omega(y)\,\mathrm{d}y\;. \tag{4.6}$$

Here the following notations have been used. As in Section 3, we denote by B_x^r the closed ball of radius $r \ge 0$ centered at $x \in \mathbb{R}^2$. For any $x \in \mathbb{R}^2$, we define

$$D_x \;=\; \begin{cases} B_0^3 & \text{if} \quad |x| \le 2\,, \\ B_0^1 \cup B_x^1 & \text{if} \quad |x| > 2\,, \end{cases}$$

so that $D_x \subset B_0^R$ and D_x has smooth boundary. Finally $F(x,y)$ is as in (4.3).

We now estimate both terms in (4.6). As $|F(x,y)| \le |x-y|^{-1} + |y|^{-1}$, we have

$$|v(x)| \;\le\; \frac{1}{\pi}\int_{|y|\le 5}\frac{1}{|y|}|\omega(y)|\,\mathrm{d}y \;\le\; 10\|\omega\|_{L^\infty}\;, \qquad \text{if } |x| \le 2\;,$$

and

$$|v(x)| \;\le\; \frac{2}{\pi}\int_{|y|\le 1}\frac{1}{|y|}|\omega(y)|\,\mathrm{d}y \;\le\; 4\|\omega\|_{L^\infty}\;, \qquad \text{if } |x| > 2\;.$$

Moreover v is continuous and $v(0) = 0$. To bound w_R, we use the fact that $\omega = \operatorname{curl} u = -\operatorname{div} u^\perp$, and we integrate by parts using Green's formula. We obtain

$$\begin{aligned} w_R(x) \;=\; &\frac{1}{2\pi}\int_{\partial D_x} F(x,y)u^\perp(y)\cdot\nu(y)\,\mathrm{d}\ell_y - \frac{1}{2\pi}\int_{\partial B_0^R} F(x,y)u^\perp(y)\cdot\nu(y)\,\mathrm{d}\ell_y \\ &+ \frac{1}{2\pi}\int_{B_0^R\setminus D_x} u^\perp(y)\cdot\nabla_y F(x,y)\,\mathrm{d}y \;=\; w^{(1)}(x) - w_R^{(2)}(x) + w_R^{(3)}(x)\;, \end{aligned}$$

where on the circles ∂D_x or ∂B_0^R we denote by ν the exterior unit normal and $\mathrm{d}\ell$ the elementary arc length. Proceeding as in the proof of Proposition 4.1, it is straightforward to verify that $|w^{(1)}(x)| \le 4\|u\|_{L^\infty}$ and $|w_R^{(2)}(x)| \le 6|x|\|u\|_{L^\infty}/R$. In particular $w_R^{(2)}(x)$ converges to zero as $R \to \infty$, for any $x \in \mathbb{R}^2$. Finally, using the estimate

$$|\nabla_y F(x,y)| \;\le\; C\biggl(\frac{|x|}{|x-y|^2|y|} + \frac{|x|}{|x-y|\,|y|^2}\biggr)\;, \qquad x \ne y\;, \quad y \ne 0\;, \tag{4.7}$$

which can be obtained by a direct calculation, it is not difficult to show that

$$\frac{1}{2\pi}\int_{\mathbb{R}^2\setminus D_x} |u(y)||\nabla_y F(x,y)|\,\mathrm{d}y \;\le\; C\|u\|_{L^\infty}\log(1+|x|)\;, \tag{4.8}$$

for some universal constant $C > 0$. When evaluating that integral for large $|x|$, it is convenient to consider separately the regions where $1 \le |y| \le |x|/2$, where $1 \le |y-x| \le |x|/2$, where $|y| \ge 2|x|$, and the region where $|x|/2 \le |y| \le 2|x|$ with $|y-x| \ge |x|/2$. Estimate (4.8) implies in particular that $w_R^{(3)}(x)$ has a limit as $R \to \infty$, so that $w_R(x) \to w_\infty(x)$ for some continuous vector field w_∞.

Summarizing, we have shown that $u_R(x)$ converges as $R \to \infty$ to some continuous vector field $\bar{u}(x) = v(x) + w_\infty(x)$ which satisfies

$$|\bar{u}(x)| \;\le\; C\Bigl(\|\omega\|_{L^\infty} + \|u\|_{L^\infty}\Bigr)\log(2+|x|)\;, \qquad x \in \mathbb{R}^2\;.$$

By construction, we have $\operatorname{div}\bar{u} = 0$, $\operatorname{curl}\bar{u} = \omega$, and $\bar{u}(0) = 0$. As in the proof of Proposition 4.1, we conclude that $u - \bar{u}$ is identically constant, and this gives (4.5). $\square$

Proposition 4.2 shows that the divergence free velocity field $u \in X$ is entirely determined, up to an additive constant, by its vorticity distribution ω even in the case where ω is merely bounded. However this result does not provide a good reconstruction formula, because the integral in (4.5) is not absolutely convergent. In particular, we cannot use (4.5) to derive an estimate on $\|u\|_{L^\infty}$ in terms of $\|\omega\|_{L^\infty}$, but the following consequence of (4.5) will be useful:

Corollary 4.3. *Under the assumptions of Proposition 4.2, we have for all $x \in \mathbb{R}^2$:*

$$u(x) + u(-x) - 2u(0) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}\Bigl\{\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x+y)^\perp}{|x+y|^2} + 2\frac{y^\perp}{|y|^2}\Bigr\}\omega(y)\,\mathrm{d}z\;. \tag{4.9}$$

In particular, the following estimate holds

$$|u(x) + u(-x) - 2u(0)| \;\le\; C_*|x|\,\|\omega\|_{L^\infty}\;, \qquad x \in \mathbb{R}^2\;, \tag{4.10}$$

where $C_* > 0$ is a universal constant.

Proof. Using (4.5) we easily obtain

$$u(x) + u(-x) - 2u(0) \;=\; \lim_{R\to\infty}\frac{1}{2\pi}\int_{|y|\le R} G(x,y)\omega(y)\,\mathrm{d}y\;, \tag{4.11}$$

where

$$G(x,y) \;=\; \frac{(x-y)^\perp}{|x-y|^2} - \frac{(x+y)^\perp}{|x+y|^2} + 2\frac{y^\perp}{|y|^2}\;, \qquad x \neq \pm y\;, \quad y \neq 0\;.$$

A direct calculation yields the bound

$$|G(x,y)| \;\le\; C\,\frac{|x|^2}{|x-y|\,|x+y|\,|y|}\;,$$

which implies that

$$\int_{\mathbb{R}^2} |G(x,y)|\,\mathrm{d}y \;\le\; C|x|\;, \qquad x \in \mathbb{R}^2\;, \tag{4.12}$$

for some universal constant $C > 0$. When evaluating that integral, it is convenient to consider separately the regions where $|y| \le |x|/2$, where $|x|/2 \le |y| \le 2|x|$, and where $|y| \ge 2|x|$. Thus we have shown that the integral in (4.11) is absolutely convergent, so that (4.9) holds, and (4.10) follows from (4.12). $\square$

Remark 4.4. The origin plays a distinguished role in formulas (4.5) and (4.9), but this is by no mean essential, and more general expressions can easily be obtained using translation invariance.

4.2 A representation formula for the pressure

Assume that $u \in X$ is such that $\operatorname{div} u = 0$ and $\omega = \partial_1 u_2 - \partial_2 u_1 \in L^\infty(\mathbb{R}^2)$. As was discussed in Section 2.2, the elliptic equation (2.5) determines a unique pressure field $p \in \mathrm{BMO}(\mathbb{R}^2)$, up to an irrelevant additive constant. Setting $q = p + \frac{1}{2}|u|^2$ and using identity (2.23), we obtain for q the equation

$$-\Delta q \;=\; \operatorname{div}(u^\perp\omega)\;, \qquad x \in \mathbb{R}^2\;. \tag{4.13}$$

The goal of this section is to obtain a representation formula for the solution of (4.13) involving absolutely convergent integrals only, and not singular integrals as in (2.6).

Lemma 4.5. *Assume that $u \in X$, $\operatorname{div} u = 0$ and $\omega = \partial_1 u_2 - \partial_2 u_1 \in L^\infty(\mathbb{R}^2)$. If $q \in \mathrm{BMO}(\mathbb{R}^2)$ is a solution to (4.13), we have for any $x_0 \in \mathbb{R}^2$*

$$q(x) \;=\; q_0 + q_1(x) + q_2(x) + q_3(x, x_0)\;, \qquad x \in \mathbb{R}^2\;, \tag{4.14}$$

where $q_0 \in \mathbb{R}$ and

$$\begin{aligned} q_1(x) \;&=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \chi(x-y)\frac{(x-y)^\perp}{|x-y|^2}\cdot u(y)\omega(y)\,\mathrm{d}y\;,\\ q_2(x) \;&=\; \frac{1}{4\pi}\sum_{k,\ell=1}^2 \int_{\mathbb{R}^2} M_{k\ell}(x-y)u_k(y)u_\ell(y)\,\mathrm{d}y\;,\\ q_3(x,x_0) \;&=\; \frac{1}{2\pi}\sum_{k,\ell=1}^2 \int_{\mathbb{R}^2}\Big\{\chi^c(x-y)K_{k\ell}(x-y) - \chi^c(x_0-y)K_{k\ell}(x_0-y)\Big\}u_k(y)u_\ell(y)\,\mathrm{d}y\;. \end{aligned} \tag{4.15}$$

Here the following notations have been used: $\chi \in C_0^\infty(\mathbb{R}^2)$ is a cut-off function which is equal to 1 on a neighborhood of the origin, $\chi^c = 1 - \chi$, and

$$M_{k\ell}(z) \;=\; \frac{2z_k\partial_\ell\chi(z) - \delta_{k\ell}(z_1\partial_1\chi(z) + z_2\partial_2\chi(z))}{|z|^2}\;, \qquad K_{k\ell}(z) \;=\; \frac{2z_kz_\ell - |z|^2\delta_{k\ell}}{|z|^4}\;. \tag{4.16}$$

Proof. We first explain how the formulas (4.15) are obtained. Assume for simplicity that ω has compact support in $\mathbb{R}^2$. In that case, using the fundamental solution of the Laplace operator in $\mathbb{R}^2$ (see Section 4.4) and integrating by parts, we obtain the unique solution of (4.13) which decays to zero at infinity:

$$q(x) \;=\; -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\,\mathrm{div}(u^\perp(y)\omega(y))\,\mathrm{d}y \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}\cdot u(y)\omega(y)\,\mathrm{d}y\;. \tag{4.17}$$

Our goal is to transform the integral in the right-hand side of (4.17) into an expression that makes sense even if ω is not localized. To do that, we use the partition of unity $1 = \chi + \chi^c$ and we decompose $q(x) = q_1(x) + \tilde{q}(x)$, where q_1 is given by (4.15) and

$$\tilde{q}(x) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}\chi^c(x-y)\frac{(x-y)^\perp}{|x-y|^2}\cdot u(y)\omega(y)\,\mathrm{d}y\;. \tag{4.18}$$

We next invoke the identities

$$u_1\omega \;=\; \partial_1(u_1u_2) + \tfrac{1}{2}\partial_2(u_2^2 - u_1^2)\;, \qquad u_2\omega \;=\; -\partial_2(u_1u_2) + \tfrac{1}{2}\partial_1(u_2^2 - u_1^2)\;,$$

which allow us to integrate by parts in (4.18). This gives two different terms, according to whether the derivative acts on the cut-off or on the Biot-Savart kernel. After careful calculations, we obtain decomposition $\tilde{q}(x) = q_2(x) + q_3^*(x)$, where q_2 is as in (4.15) and

$$q_3^*(x) \;=\; \frac{1}{2\pi}\sum_{k,\ell=1}^2\int_{\mathbb{R}^2}\chi^c(x-y)K_{k\ell}(x-y)u_k(y)u_\ell(y)\,\mathrm{d}y\;.$$

Summarizing, we have $q(x) = q_1(x) + q_2(x) + q_3^*(x)$ when ω is localized, which gives (4.14) with $q_0 = q_3^*(x_0)$ since $q_3(x,x_0) = q_3^*(x) - q_3^*(x_0)$.

We now consider the general case where u and ω are only supposed to be bounded. Under these assumptions the functions q_1, q_2 defined by (4.15) are clearly continuous and bounded. On the other hand, using the estimate

$$|K_{k\ell}(x-y) - K_{k\ell}(x_0-y)| \;\le\; C\bigg(\frac{|x-x_0|}{|x-y|^2|x_0-y|} + \frac{|x-x_0|}{|x-y|\,|x_0-y|^2}\bigg)\;, \tag{4.19}$$

which is obtained as in (4.7), it is straightforward to verify that the integral defining $q_3(x,x_0)$ in (4.15) is absolutely convergent for any pair of points $x,x_0 \in \mathbb{R}^2$, because the integrand is bounded and decays to zero like $|y|^{-3}$ as $|y| \to \infty$. In fact, proceeding as in the proof of Proposition 4.2, one can show that $q_3(x,x_0)$ is a continuous function of x which grows at most logarithmically as $|x| \to \infty$. Now, if we define $\bar{q}(x) = q_1(x) + q_2(x) + q_3(x,x_0)$, then $\bar{q}$ satisfies the elliptic equation (4.13). A convenient way to verify that is to approximate u by an sequence of compactly supported divergence free vector fields u_n (this can be done using Bogovskii's operator, see [14]). The corresponding pressure $\bar{q}_n$ satisfies $-\Delta\bar{q}_n = \operatorname{div}(u_n^\perp \omega_n)$ by construction, where $\omega_n = \operatorname{curl} u_n$, and taking the limit $n \to \infty$ we obtain the desired property for the limit $\bar{q}$. Finally, if $q \in \mathrm{BMO}(\mathbb{R}^2)$ is any other solution of (4.13), it follows that $q - \bar{q}$ is a harmonic function on $\mathbb{R}^2$ with sublinear growth at infinity, hence $q - \bar{q} = q_0$ for some $q_0 \in \mathbb{R}$. $\square$

4.3 Uniformly local Lebesgue spaces

Let $d \in \mathbb{N}^*$ and $1 \le p < \infty$. We introduce the space $\mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ defined by

$$\mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d) \;=\; \Big\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^d) \,\Big|\, \|f\|_{L^p_{\mathrm{ul}}} < \infty \Big\}\,, \tag{4.20}$$

where $\|f\|_{L^p_{\mathrm{ul}}}$ is given by (3.1). In other words, a function f belongs to $\mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ if and only if $f \in L^p(B_x)$ for any $x \in \mathbb{R}^d$, where $B_x \subset \mathbb{R}^d$ denotes the ball of unit radius centered at x, and if moreover the norm $\|f\|_{L^p(B_x)}$ is uniformly bounded for all $x \in \mathbb{R}^2$. Roughly speaking, a function $f \in \mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ is locally in L^p but behaves at large scales like a bounded function.

The *uniformly local L^p space* $L^p_{\mathrm{ul}}(\mathbb{R}^d)$ is the subspace of $\mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ defined by

$$L^p_{\mathrm{ul}}(\mathbb{R}^d) \;=\; \Big\{ f \in \mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d) \,\Big|\, \|\tau_y f - f\|_{L^p_{\mathrm{ul}}} \xrightarrow[y\to 0]{} 0 \Big\}\,, \tag{4.21}$$

where τ_y denotes the translation operator: $(\tau_y f)(x) = f(x-y)$ for $x,y \in \mathbb{R}^d$. The following properties are well known [3]:

1. The space $\mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ equipped with the norm (3.1) is a Banach space, which contains $L^p_{\mathrm{ul}}(\mathbb{R}^d)$ as a closed subspace. In fact $L^p_{\mathrm{ul}}(\mathbb{R}^d)$ is the closure of $C_{\mathrm{bu}}(\mathbb{R}^d)$ in $\mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$, so that $C_{\mathrm{bu}}(\mathbb{R}^d)$ and even $C^\infty_{\mathrm{bu}}(\mathbb{R}^d)$ are dense in $L^p_{\mathrm{ul}}(\mathbb{R}^d)$.

2. If $p = \infty$ the norm (3.1) should be understood as the uniform norm over $\mathbb{R}^d$. We thus have $\mathcal{L}^\infty_{\mathrm{ul}}(\mathbb{R}^d) = L^\infty(\mathbb{R}^d)$ and the definition (4.21) shows that $L^\infty_{\mathrm{ul}}(\mathbb{R}^d) = C_{\mathrm{bu}}(\mathbb{R}^d)$.

3. For any $p \in [1,\infty]$ one has $L^p_{\mathrm{ul}}(\mathbb{R}^d) \neq \mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$. For instance, if $f(x) = \sin(|x|^2)$, it is easy to verify that $f \in \mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d) \setminus L^p_{\mathrm{ul}}(\mathbb{R}^d)$. Such a function cannot be approximated by uniformly continuous functions in the topology defined by the norm (3.1).

4. As a Banach space, $L^p_{\mathrm{ul}}(\mathbb{R}^d)$ is neither reflexive nor separable.

5. If $1 \le p \le q \le \infty$ one has the embeddings $C_{\mathrm{bu}}(\mathbb{R}^d) \hookrightarrow L^q_{\mathrm{ul}}(\mathbb{R}^d) \hookrightarrow L^p_{\mathrm{ul}}(\mathbb{R}^d) \hookrightarrow L^1_{\mathrm{ul}}(\mathbb{R}^d)$.

Uniformly local Sobolev spaces can be constructed in a similar way. For instance one can define $W^{1,p}_{\mathrm{ul}}(\mathbb{R}^d)$ as the space of all $f \in L^p_{\mathrm{ul}}(\mathbb{R}^d)$ such that the distributional derivatives $\partial_i f$ belong to $L^p_{\mathrm{ul}}(\mathbb{R}^d)$ for $i = 1,\dots,d$.

Uniformly local Lebesgue spaces provide a convenient framework for solving evolution PDE's. As a simple example, we consider the linear heat equation $\partial_t u = \Delta u$ in $\mathbb{R}^d$. The solution with initial data $u_0 \in L^p_{\mathrm{ul}}(\mathbb{R}^d)$ is $u(t) = S(t)u_0$, where $S(0) = \mathbf{1}$ and

$$(S(t)u_0)(x) \;=\; \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{|x-y|^2}{4t}}\, u_0(y)\,\mathrm{d}y \;, \qquad x \in \mathbb{R}^d \;, \quad t > 0 \;. \tag{4.22}$$

The following result is not difficult to establish [3]:

Proposition 4.6. *The family $\{S(t)\}_{t\ge 0}$ given by (4.22) defines a strongly continuous semigroup on $L^p_{\mathrm{ul}}(\mathbb{R}^d)$ for $1 \le p \le \infty$. Moreover, if $1 \le p \le q \le \infty$ one has the estimate*

$$\|S(t)u_0\|_{L^q_{\mathrm{ul}}} \;\le\; C\Bigl(1 + t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\Bigr)\|u_0\|_{L^p_{\mathrm{ul}}} \;, \qquad t > 0 \;. \tag{4.23}$$

In Proposition 4.6, it is important to use $L^p_{\mathrm{ul}}(\mathbb{R}^d)$ instead of the larger space $\mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$, because if $u_0 \in \mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d) \setminus L^p_{\mathrm{ul}}(\mathbb{R}^d)$ the solution $u(t) = S(t)u_0$ is not right continuous at $t = 0$. For short times ($t \le 1$), the bound (4.23) reduces to the usual $L^p - L^q$ estimate for the heat semigroup in $\mathbb{R}^d$, whereas for large times ($t \ge 1$) we recover the $L^\infty - L^\infty$ estimate. This is not surprising if one remembers that elements of $L^p_{\mathrm{ul}}(\mathbb{R}^d)$ behave locally like L^p functions, but look like bounded functions when considered at a sufficiently large scale. It is also possible to obtain smoothing estimates for the heat semigroup in uniformly local Lebesgue spaces. For instance, if $1 \le p \le q \le \infty$, we have

$$\|\nabla S(t)u_0\|_{L^q_{\mathrm{ul}}} \;\le\; Ct^{-1/2}\Bigl(1 + t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\Bigr)\|u_0\|_{L^p_{\mathrm{ul}}} \;, \qquad t > 0 \;. \tag{4.24}$$

In the applications to partial differential equations, it is often convenient to use slightly different norms on $L^p_{\mathrm{ul}}(\mathbb{R}^d)$ which turn out to be equivalent to (3.1). Let $\rho : \mathbb{R}^d \to \mathbb{R}_+$ be a measurable function with the following two properties:

a) ρ is positive on a set of nonzero measure;

b) $\tilde\rho \in L^1(\mathbb{R}^d)$, where $\tilde\rho(x) = \sup\{\rho(y)\,|\,|y-x| \le 1\}$.

The second assumption implies that $\rho \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Indeed, we clearly have $\rho \le \tilde\rho$, hence $\|\rho\|_{L^1} \le \|\tilde\rho\|_{L^1}$. Moreover, if $B \subset \mathbb{R}^d$ is any ball of unit diameter, the definition of $\tilde\rho$ implies that $\rho(x) \le \tilde\rho(y)$ for all $x, y \in B$, hence

$$\sup_{x\in B} \rho(x) \;\le\; \inf_{y\in B} \tilde\rho(y) \;\le\; \frac{1}{|B|}\int_B \tilde\rho(y)\,\mathrm{d}y \;. \tag{4.25}$$

This shows that $\|\rho\|_{L^\infty} \le |B|^{-1}\|\tilde\rho\|_{L^1}$. Assumption a) ensures that ρ is not zero almost everywhere, so that $\int_{\mathbb{R}^d} \rho\,\mathrm{d}x > 0$.

The following result provides a plethora of equivalent norms on $L^p_{\mathrm{ul}}(\mathbb{R}^d)$.

Proposition 4.7. *If $\rho : \mathbb{R}^d \to \mathbb{R}_+$ satisfies assumptions a) and b) above, then for $1 \le p < \infty$ the quantity*

$$\|f\|_{p,\rho} \;=\; \sup_{x\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \rho(x-y)|f(y)|^p\,\mathrm{d}y\right)^{1/p} \tag{4.26}$$

is equivalent to the norm $\|\cdot\|_{L^p_{\mathrm{ul}}}$ on $L^p_{\mathrm{ul}}(\mathbb{R}^d)$.

Remark 4.8. Obviously, if ρ is the characteristic function of the unit ball $B_0 \subset \mathbb{R}^d$, the definition (4.26) reduces to (3.1). So (4.26) is clearly a generalization of (3.1).

Proof. For any $z \in \mathbb{R}^d$, we denote by $Q_z \subset \mathbb{R}^d$ the cube of unit diameter centered at z, the edges of which are parallel to the coordinate axes. The Lebesgue measure of Q_z is λ^d, where $\lambda = d^{-1/2}$. Given any $f \in \mathcal{L}^p_{\mathrm{ul}}(\mathbb{R}^d)$ and any $x \in \mathbb{R}^d$, we estimate

$$\int_{\mathbb{R}^d} \rho(x-y)|f(y)|^p \,\mathrm{d}y \;=\; \sum_{k\in\lambda\mathbb{Z}^d} \int_{Q_k} \rho(x-y)|f(y)|^p \,\mathrm{d}y \;\le\; \sum_{k\in\lambda\mathbb{Z}^d} r(x-k) \int_{Q_k} |f(y)|^p \,\mathrm{d}y \;, \tag{4.27}$$

where $r(x-k) = \sup\{\rho(x-y) \,|\, y \in Q_k\} = \sup\{\rho(z) \,|\, z \in Q_{x-k}\}$. We observe that

$$\int_{Q_k} |f(y)|^p \,\mathrm{d}y \;\le\; \int_{B_k} |f(y)|^p \,\mathrm{d}y \;\le\; \|f\|^p_{L^p_{\mathrm{ul}}} \;,$$

because $Q_k \subset B_k$ (the ball of unit radius centered at k). On the other hand, we have as in (4.25) :

$$r(x-k) \;=\; \sup_{z\in Q_{x-k}} \rho(z) \;\le\; \inf_{y\in Q_{x-k}} \tilde\rho(y) \;\le\; \frac{1}{\lambda^d}\int_{Q_{x-k}} \tilde\rho(y)\,\mathrm{d}y \;,$$

hence

$$\sum_{k\in\lambda\mathbb{Z}^d} r(x-k) \;\le\; \frac{1}{\lambda^d} \sum_{k\in\lambda\mathbb{Z}^d} \int_{Q_{x-k}} \tilde\rho(y)\,\mathrm{d}y \;=\; d^{d/2}\|\tilde\rho\|_{L^1} \;.$$

Taking the supremum over $x \in \mathbb{R}^d$ in (4.27), we conclude that $\|f\|^p_{p,\rho} \le d^{d/2}\|\tilde\rho\|_{L^1}\|f\|^p_{L^p_{\mathrm{ul}}}$.

Conversely, as ρ is positive on a set of nonzero measure, we can assume without loss of generality that $\int_{B_0} \rho \,\mathrm{d}x = \epsilon > 0$, where $B_0 \subset \mathbb{R}^d$ is the unit ball centered at the origin. If $\|f\|_{L^p_{\mathrm{ul}}} > 0$, we choose $x \in \mathbb{R}^d$ such that

$$\int_{|y-x|\le 1} |f(y)|^d \,\mathrm{d}y \;\ge\; \frac{1}{2}\|f\|^p_{L^p_{\mathrm{ul}}} \;.$$

We then have

$$\int_{|z|\le 2}\Big\{\int_{\mathbb{R}^d} \rho(x-y-z)|f(y)|^p\,\mathrm{d}y\Big\}\,\mathrm{d}z \;=\; \int_{\mathbb{R}^d}\Big\{\int_{|z|\le 2} \rho(x-y-z)\,\mathrm{d}z\Big\}|f(y)|^p\,\mathrm{d}y \;\ge\; \frac{\epsilon}{2}\|f\|^p_{L^p_{\mathrm{ul}}} \;,$$

because by assumption $\int_{|z|\le 2} \rho(x-y-z)\,\mathrm{d}z \ge \epsilon$ whenever $|y-x| \le 1$. Thus there exists $z \in \mathbb{R}^d$ with $|z| \le 2$ such that

$$\|f\|^p_{p,\rho} \;\ge\; \int_{\mathbb{R}^d} \rho(x-y-z)|f(y)|^p\,\mathrm{d}y \;\ge\; \frac{\epsilon}{2}\,\mathrm{meas}\{z\in\mathbb{R}^d \,|\, |z|\le 2\}^{-1}\|f\|^p_{L^p_{\mathrm{ul}}} \;.$$

This proves the desired equivalence. ☐

Remark 4.9. Proposition 4.7 provides sufficient conditions on the weight function ρ so that (4.26) is equivalent to (3.1). These conditions are weaker than what can be found in the existing literature (see e.g. [3, Definition 4.1]), but it is not clear that assumptions a) and b) are optimal. It is easy to verify that any weight ρ for which (4.26) is equivalent to (3.1) should satisfy $\rho \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\int \rho \,\mathrm{d}x > 0$, but these properties alone are not sufficient, as can be seen from the following example. Assume that $d = 1$ and take

$$\rho \;=\; \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}\,\mathbf{1}_{[-k-k^{-1},-k]} \;, \qquad f \;=\; \sum_{k=1}^{\infty} k\,\mathbf{1}_{[k,k+k^{-1}]} \;,$$

where $\mathbf{1}_I$ denotes the characteristic function of an interval $I \subset \mathbb{R}$. Then $\rho \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and using definition (3.1) we see that $f \in \mathcal{L}^1_{\mathrm{ul}}(\mathbb{R})$. But

$$\int_{\mathbb{R}} \rho(-y) f(y)\,\mathrm{d}x \;=\; \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \;=\; +\infty\;,$$

so that $\|f\|_{1,\rho} = +\infty$.

4.4 A few elementary tools

We collect here, for easy reference, a few elementary definitions and results that are used several times in these notes.

I. Fourier transforms. We use the following conventions for Fourier transforms on $\mathbb{R}^2$. Let $\mathcal{S}(\mathbb{R}^2)$ denote the (Schwartz) space of all smooth and rapidly decreasing functions $f : \mathbb{R}^2 \to \mathbb{C}$, see [38]. If $f \in \mathcal{S}(\mathbb{R}^2)$ we set

$$\hat{f}(\xi) \;=\; \int_{\mathbb{R}^2} f(x)\,\mathrm{e}^{-i\xi\cdot x}\,\mathrm{d}x\;, \qquad f(x) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{f}(\xi)\,\mathrm{e}^{i\xi\cdot x}\,\mathrm{d}\xi\;, \tag{4.28}$$

for all $x \in \mathbb{R}^2$ and $\xi \in \mathbb{R}^2$. We also denote $\hat{f} = \mathcal{F}f$ and $f = \mathcal{F}^{-1}\hat{f}$. According to (4.28), we have for any $f \in \mathcal{S}(\mathbb{R}^2)$:

$$\mathcal{F}(\nabla_x f) \;=\; i\xi(\mathcal{F}f)\;, \qquad \text{and} \quad \mathcal{F}(xf) \;=\; i\nabla_\xi(\mathcal{F}f)\;.$$

The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are linear isomorphisms on $\mathcal{S}(\mathbb{R}^2)$, and can be extended to linear isomorphisms on the dual space $\mathcal{S}'(\mathbb{R}^2)$, which is the space of tempered distributions on $\mathbb{R}^2$ [38, 39]. For instance, if δ_0 denotes the Dirac measure located at the origin, we have $(\mathcal{F}\delta_0)(\xi) = 1$ for all $\xi \in \mathbb{R}^2$.

II. Young's inequality. If $f, g \in \mathcal{S}(\mathbb{R}^2)$, we define the convolution product $h = f * g \in \mathcal{S}(\mathbb{R}^2)$ by the formula

$$h(x) \;=\; \int_{\mathbb{R}^2} f(x-y)g(y)\,\mathrm{d}y \;=\; \int_{\mathbb{R}^2} g(x-y)f(y)\,\mathrm{d}y\;, \qquad x \in \mathbb{R}^2\;. \tag{4.29}$$

In Fourier space, we then have $\hat{h}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$ for all $\xi \in \mathbb{R}^2$, so that $\mathcal{F}(f*g) = (\mathcal{F}f)(\mathcal{F}g)$. Moreover, for all exponents $p, q, r \in [1,\infty]$ satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, we have Young's inequality

$$\|h\|_{L^r(\mathbb{R}^2)} \;=\; \|f*g\|_{L^r(\mathbb{R}^2)} \;\le\; \|f\|_{L^p(\mathbb{R}^2)}\|g\|_{L^q(\mathbb{R}^2)}\;. \tag{4.30}$$

More generally, if $f \in L^p(\mathbb{R}^2)$ and $g \in L^q(\mathbb{R}^2)$, one can show that the integral in (4.29) converges for almost every $x \in \mathbb{R}^2$ and defines a function $h \in L^r(\mathbb{R}^2)$ satisfying (4.30).

III. Fundamental solutions. The Fourier transform can be used to compute fundamental solutions of partial differential operators with constant coefficients. Two particular examples play an important role in these notes. First, the fundamental solution of the Poisson equation $\Delta\Phi = \delta_0$ in $\mathbb{R}^2$ is

$$\Phi(x) \;=\; \frac{1}{2\pi}\log|x|\;, \qquad x \in \mathbb{R}^2 \setminus \{0\}\;,$$

see [11, 29]. It follows that $u = \Phi * \rho$ is the solution of the Poisson equation $\Delta u = \rho$ for any $\rho \in \mathcal{S}(\mathbb{R}^2)$. Similarly, the vector field

$$V(x) \;=\; \frac{1}{2\pi}\frac{x^\perp}{|x|^2}\;, \qquad x \in \mathbb{R}^2 \setminus \{0\}\;,$$

satisfies $\operatorname{div} V = 0$ and $\operatorname{curl} V \equiv \partial_1 V_2 - \partial_2 V_1 = \delta_0$. Thus, if $u = V * \omega$ for some $\omega \in \mathcal{S}(\mathbb{R}^2)$, we have $\operatorname{div} u = 0$ and $\operatorname{curl} u = \omega$. The vector field $V = \nabla^\perp \Phi$ is therefore the fundamental solution associated to the Biot-Savart law.

IV. Gronwall's lemma. There exist many versions of Gronwall's lemma, but the following one is sufficient for our purposes.

Lemma 4.10. *Let $T > 0$, $a \ge 0$, and assume that $f, g, b : [0,T] \to \mathbb{R}_+$ are continuous functions satisfying*

$$f(t) + \int_0^t g(s)\,\mathrm{d}s \;\le\; a + \int_0^t b(s) f(s)\,\mathrm{d}s\;, \qquad 0 \le t \le T\;. \tag{4.31}$$

Then

$$f(t) + \int_0^t g(s)\,\mathrm{d}s \;\le\; a \exp\Bigl(\int_0^t b(s)\,\mathrm{d}s\Bigr)\;, \qquad 0 \le t \le T\;. \tag{4.32}$$

Proof. Let $F(t) = \int_0^t b(s) f(s)\,\mathrm{d}s$. Then F is continuously differentiable on $[0,T]$ and satisfies, in view of (4.31),

$$F'(t) = b(t) f(t) \;\le\; a b(t) + b(t) F(t)\;, \qquad 0 \le t \le T\;.$$

Integrating that differential inequality and observing that $F(0) = 0$, we obtain the bound

$$F(t) \;\le\; a \exp\Bigl(\int_0^t b(s)\,\mathrm{d}s\Bigr) - a\;, \qquad 0 \le t \le T\;,$$

which can be inserted in the right-hand side of (4.31) to give (4.32). ∎

References

[1] A. Afendikov and A. Mielke, Dynamical properties of spatially non-decaying 2D Navier-Stokes flows with Kolmogorov forcing in an infinite strip, J. Math. Fluid. Mech. **7** (2005), suppl. 1, S51–S67.

[2] D. Ambrose, J. Kelliher, M. Lopes Filho, and H. Nussenzveig Lopes, Serfati solutions to the 2D Euler equations on exterior domains, J. Differential Equations **259** (2015), 4509–4560.

[3] J. Arrieta, A. Rodriguez-Bernal, J. Cholewa, and T. Dlotko, Linear parabolic equations in locally uniform spaces, Math. Models Methods Appl. Sci. **14** (2004), 253–293.

[4] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Grundlehren der Mathematischen Wissenschaften **223**, Springer, 1976.

[5] V. Chepyzhov and S. Zelik, Infinite-energy solutions for dissipative Euler equations in $\mathbb{R}^2$, J. Math. Fluid Mech. **17** (2015), 513–532.

[6] P. Collet, A global existence result for the Navier-Stokes equation in the plane, unpublished manuscript, June 1994.

[7] E. Cozzi, Vanishing viscosity in the plane for nondecaying velocity and vorticity, SIAM J. Math. Anal. **41** (2009), 495–510.

[8] P. Constantin and C. Foias, *Navier-Stokes equations*, Chicago Lectures in Mathematics, University of Chicago Press, 1988.

[9] M. Efendiev and S. Zelik, The attractor for a nonlinear reaction-diffusion system in an unbounded domain, Comm. Pure Appl. Math. **54** (2001), 625–688.

[10] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics **194**, Springer, 2000.

[11] L. Evans, *Partial differential equations*, Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, RI, 1998.

[12] E. Feireisl, Bounded, locally compact global attractors for semilinear damped wave equations on $\mathbb{R}^n$, J. Diff. Integral Eqns. **9** (1996), 1147–1156.

[13] H. Fujita and T. Kato, On the Navier-Stokes initial value problem. I, Arch. Rational Mech. Anal. **16** (1964), 269–315.

[14] G. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems*, Springer Monographs in Mathematics, Springer, 2011.

[15] Th. Gallay and S. Slijepčević, Energy flow in formally gradient partial differential equations on unbounded domains. J. Dynam. Differential Equations **13** (2001), 757–789.

[16] Th. Gallay and S. Slijepčević, Energy bounds for the two-dimensional Navier-Stokes equations in an infinite cylinder, Comm. Partial Differential Equations **39** (2014), 1741–1769.

[17] Th. Gallay and S. Slijepčević, Uniform boundedness and long-time asymptotics for the two-dimensional Navier-Stokes equations in an infinite cylinder, J. Math. Fluid Mech. **17** (2015), 23–46.

[18] Th. Gallay and C.E. Wayne, Global stability of vortex solutions of the two-dimensional Navier-Stokes equation, Comm. Math. Phys. **255** (2005), 97–129.

[19] Y. Giga, Yoshikazu, K. Inui, and S. Matsui, On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data, in *Advances in fluid dynamics*, Quaderni di Mathematica **4** (1999), 27–68.

[20] Y. Giga, S. Matsui, and O. Sawada, Global existence of two dimensional Navier-Stokes flow with non-decaying initial velocity, J. Math. Fluid. Mech. **3** (2001), 302–315.

[21] J. Ginibre and G. Velo, The Cauchy problem in local spaces for the complex Ginzburg-Landau equation, I. Compactness methods, Physica D **95** (1996), 191–228. II. Contraction methods, Comm. Math. Phys. **187** (1997), 45–79.

[22] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics **840**, Springer, 1981.

[23] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, Arch. Rational Mech. Anal. **58** (1975), 181–205.

[24] J. Kelliher, A characterization at infinity of bounded vorticity, bounded velocity solutions to the 2D Euler equations, preprint `arXiv:1404.3404`.

[25] H. Kozono and T. Ogawa, Two-dimensional Navier-Stokes flow in unbounded domains, Math. Ann. **297** (1993), 1–31.

[26] O. A. Ladyženskaja, Solution "in the large" of the nonstationary boundary value problem for the Navier-Stokes system with two space variables, Comm. Pure Appl. Math. **12** (1959), 427–433.

[27] J. Leray, Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique, J. Math. Pures Appliquées 9ème série **12** (1933), 1–82.

[28] J. Leray, Essai sur les mouvements plans d'un fluide visqueux que limitent des parois, J. Math. Pures Appliquées 9ème série **13** (1934), 331–418.

[29] E. Lieb and M. Loss, *Analysis*, Graduate Studies in Mathematics **14**, American Mathematical Society, Providence, RI, 1997.

[30] P.-L. Lions, *Mathematical topics in fluid mechanics. Vol. 1. Incompressible models*, Oxford Lecture Series in Mathematics and its Applications **3**, Oxford University Press, 1996.

[31] A. Majda and A. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics **27**, Cambridge University Press, Cambridge, 2002.

[32] K. Masuda, Weak solutions of Navier-Stokes equations, Tohoku Math. J. (2) **36** (1984), 623–646.

[33] Y. Maekawa and Y. Terasawa, The Navier-Stokes equations with initial data in uniformly local L^p spaces, Diff. Int. Equations **19** (2006), 369–400.

[34] A. Mielke and G. Schneider, Attractors for modulation equations on unbounded domains — existence and comparison, Nonlinearity **8** (1995), 743–768.

[35] Y. Meyer, *Ondelettes et opérateurs. II. Opérateurs de Calderón-Zygmund*, Actualités Mathématiques, Hermann, Paris, 1990.

[36] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences **44**, Springer, New York, 1983.

[37] M. Protter and H. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

[38] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York, 1972.

[39] M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975.

[40] O. Sawada and Y. Taniuchi, A remark on L^∞ solutions to the 2-D Navier-Stokes equations, J. Math. Fluid Mech. **9** (2007), 533–542.

[41] M. Schonbek, Large time behaviour of solutions to the Navier-Stokes equations, Comm. Partial Differential Equations **11** (1986), 733–763.

[42] Ph. Serfati, Solutions C^∞ en temps, n-log Lipschitz bornées en espace et équation d'Euler, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), 555–558.

[43] E. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton, 1993.

[44] R. Temam, *Navier-Stokes equations. Theory and numerical analysis. Third edition*, Studies in Mathematics and its Applications **2**, North-Holland, Amsterdam, 1984.

[45] M. Wiegner, Decay results for weak solutions of the Navier-Stokes equations on $\mathbb{R}^n$, J. London Math. Soc. (2) **35** (1987), 303–313.

[46] S. Zelik, Infinite energy solutions for damped Navier-Stokes equations in $\mathbb{R}^2$, J. Math. Fluid Mech. **15** (2013), 717–745.