PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9939(XX)0000-0

EXAMPLES OF SMOOTH MAPS WITH FINITELY MANY CRITICAL POINTS IN DIMENSIONS $(4,3)$, $(8,5)$ AND $(16,9)$

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Abstract. We consider manifolds M^{2n} which admit smooth maps into a connected sum of $S^1 \times S^n$ with only finitely many critical points, for $n \in \{2,4,8\}$, and compute the minimal number of critical points.

1. Introduction and statement of the main result

Let $\varphi(M^m, N^n)$ denote the minimal number of critical points of smooth maps between the manifolds M^m and N^n. When superscripts are specified they denote the dimension of the respective manifolds. We are interested below in the case when $m \geq n \geq 2$ and the manifolds are compact. The main problem concerning φ is to characterize those pairs of manifolds for which it is finite non-zero and then to compute its value (see [1]).

In [1] the authors found that, in small codimension $0 \leq m-n-1 \leq 3$, if $\varphi(M^m, N^{n+1})$ is finite then $\varphi(M^m, N^{n+1}) \in \{0,1\}$, except for the exceptional pairs of dimensions $(m, n+1) \in \{(2,2),(4,3),(4,2),(5,2),(6,3),(8,5)\}$. Notice that $(5,3)$ was inadvertently included in [1] among the exceptional pairs. Moreover, under the finiteness hypothesis, $\varphi(M,N) = 1$ if and only if M is the connected sum of a smooth fibration over N with an exotic sphere and not a fibration itself. There are two essential ingredients in this result. First, there are local obstructions to the existence of isolated singularities, namely the germs of smooth maps $\mathbb{R}^m \to \mathbb{R}^n$ having an isolated singularity at origin are actually locally topologically equivalent to a projection. Thus, these maps are topological fibrations. Second, singular points located in a disk cluster together.

The simplest exceptional case is that of (pairs of) surfaces, which is completely understood by elementary means (see [2] for explicit computations). Very little is known for the other exceptional and generic (i.e. $m-n-1 \geq 4$) cases and even the case of pairs of spheres is unsettled yet. In particular, it is not known whether φ is bounded in terms only of the dimensions, in general.

The aim of this note is to find non-trivial examples in dimensions $(4,3)$, $(8,5)$ and $(16,9)$ inspired by the early work of Antonelli ([3, 4]). The smooth maps considered in [4] are so-called Montgomery-Samelson fibrations with finitely many singularities where several fibers are pinched to points. According to [14] these maps should be locally topologically equivalent to a cone over the Hopf fibration, in a neighborhood of a critical point.

The main ingredient of our approach is the existence of global obstructions of topological nature to the clustering of genuine critical points in these dimensions. This situation seems rather exceptional and it permits us to obtain the precise value of φ using only basic algebraic topology.

Our computations show that φ can take arbitrarily large even values. Thus the behavior of φ is qualitatively different from what it was seen before in [1].

1991 *Mathematics Subject Classification.* 57 R 45, 55R 55, 58 K 05, 57 R 60, 57 R 70.
Key words and phrases. Critical point, isolated singularity, Hopf fibration, suspension, homotopy sphere.

Theorem 1.1. *Let $n \in \{2, 4, 8\}$, $e \geq c \geq 0$, with $c \neq 1$, and Σ^{2n} be a homotopy $2n$-sphere. If $n = 2$ assume further that $\Sigma^4 \setminus \mathrm{int}(D^4)$ embeds smoothly into S^4, where D^4 is a smooth 4-disk. Then*

$$\varphi(\Sigma^{2n} \#_e S^n \times S^n \#_c S^1 \times S^{2n-1}, \#_c S^1 \times S^n) = 2e - 2c + 2$$

Here $\#_c S^1 \times S^n = S^{n+1}$ if $c = 0$ and $\#_e S^n \times S^n \#_c S^1 \times S^{2n-1} = S^{2n}$ if $e = c = 0$.

The structure of the proof of the theorem is as follows. We prove Proposition 2.1 which yields a lower bound for the number of critical values derived from topological obstructions of algebraic nature. The existence of a non-trivial lower bound is not obvious since one might think that several singularities could combine into a single more complicated singularity. However, the proof uses only standard techniques of algebraic topology. The next step taken in section 3 is to construct explicit smooth maps with any even number of singularities. This follows by taking fiber sums of elementary blocks of maps coming naturally from Hopf fibrations. This construction is an immediate generalization of the one considered by Antonelli in the case of two elementary blocks in ([3], p.185-186). Then Proposition 3.1 concludes the proof. We warn the reader that our proof is narrative and key facts are singled out as lemmas in the process of unfolding the proof of these two propositions.

Remark 1.1. Observe that $S^1 \times S^{2n-1}$ fibers over $S^1 \times S^n$, when $n \in \{2, 4, 8\}$ so that the formula from Theorem 1.1 is still valid for $\Sigma^{2n} = S^{2n}$, $e = 0$ and $c = 1$. However, we do not know how to evaluate φ when $e \leq c - 1$. The present methods do not work for $e \geq c = 1$ either.

Acknowledgements. The authors are indebted to Dennis Sullivan, Yuli Rudyak and András Szűcs for useful discussions on this topic, during the M. M. Postnikov Memorial Conference at Bedlewo, June 2007. L. Funar was partially supported by the ANR Repsurf: ANR-06-BLAN-0311. C. Pintea was partially supported by the CNCSIS grant of type A, 8/1467/2007 and partially by the CNCSIS grant PN II, ID 523/2007. P. Zhang was partially supported by Eastern Mediterranean University Scientific Research Budget BAP-A-08-05. P. Zhang thanks Jianzhong Pan, Haibao Duan and Kaiming Zhao of Institute of Mathematics of CAS in Beijing for their hospitality when part of this note was written in the summer of 2007.

2. A lower bound for the number of critical values

Proposition 2.1. *For any dimension $n \geq 2$, homotopy $2n$-sphere Σ^{2n} and non-negative integers e and c, with $c \neq 1$ we have:*

$$\varphi(\Sigma^{2n} \#_e S^n \times S^n \#_c S^1 \times S^{2n-1}, \#_c S^1 \times S^n) \geq 2e - 2c + 2$$

Here $\#_c S^1 \times S^n = S^{n+1}$ if $c = 0$ and $\#_e S^n \times S^n \#_c S^1 \times S^{2n-1} = S^{2n}$ if $e = c = 0$.

We will prove, more generally, the following:

Proposition 2.2. *Let M^{2n} and N^{n+1} be closed connected orientable manifolds and $n \geq 2$. Assume that $\pi_1(M) \cong \pi_1(N)$ is a free group $\mathbb{F}(c)$ on c generators, $c \neq 1$ (with $\mathbb{F}(0) = 0$) , $\pi_j(M) = \pi_j(N) = 0$, for $2 \leq j \leq n-1$ and $H_{n-1}(M) = 0$. Then $\varphi(M, N) \geq \beta_n(M) - 2c + 2$, where β_k denotes the k-th Betti number.*

Proof. Let $B = B(f)$ denote the set of critical values of a smooth map $f : M \to N$. We will prove that the cardinality $|B|$ of $B(f)$ satisfies $|B| \geq \beta_n(M) - 2c + 2$, which will imply our claim. Set $V = f^{-1}(B(f)) \subset M$. We can assume that f has finitely many critical points, since otherwise the claim of Proposition 2.2 would be obviously verified.

The following two Lemmas do not depend on the homotopy assumptions of Proposition 2.2.

Lemma 2.1. *If A is a nonempty finite subset of a connected closed orientable manifold N^{n+1}, then $\beta_n(N \setminus A) = \beta_n(N) + |A| - 1$.*

Proof. Clear from the homology exact sequence of the pair $(N, N \setminus A)$. □

Lemma 2.2. *If M^{n+q+1} and N^{n+1} are smooth manifolds and $f : M \to N$ is a smooth map with finitely many critical points, then the inclusions $M \setminus V \hookrightarrow M$ and $N \setminus B \hookrightarrow N$ are n-connected.*

Proof. This is obvious for $N \setminus B \hookrightarrow N$. It remains to prove that $\pi_k(M, M \setminus V) \cong 0$ for $k \leq n$. Take $\alpha : (D^k, S^{k-1}) \to (M, M \setminus V)$ to be an arbitrary smooth map of pairs. Since the critical set $C(f)$ of f is finite and contained in V, there exists a small homotopy of α relative to the boundary such that the image $\alpha(D^k)$ avoids $C(f)$. By compactness there exists a neighborhood U of $C(f)$ consisting of disjoint balls centered at the critical points such that $\alpha(D^k) \subset M \setminus U$. We can arrange by a small isotopy that V becomes transversal to ∂U.

Observe further that $V \setminus U$ consists of regular points of f and thus it is a properly embedded submanifold of $M \setminus U$. General transversality arguments show that α can be made transverse to $V \setminus U$ by a small homotopy. By dimension counting this means that $\alpha(D^k) \subset M \setminus U$ is disjoint from V and thus the class of α in $\pi_k(M, M \setminus V)$ vanishes. □

The restriction of f to $M \setminus V$ is a proper submersion and thus the restriction $f|_{M\setminus V}$ is an open map. In particular, $f(M \setminus V) \subset N \setminus B$ is an open subset. On the other hand, the closed map lemma states that a proper map between locally compact Hausdorff spaces is also closed. Thus $f(M \setminus V)$ is also closed in $N \setminus B$ and hence $f(M \setminus V) = N \setminus B$. According to Ehresmann's theorem, the restriction $f|_{M\setminus V}$ is then a locally trivial smooth fibration over $N \setminus B$ with compact smooth fiber F^{n-1} (see [5]).

Lemma 2.3. *Assume that $c \neq 1$. Then the generic fiber F is homotopy equivalent to the $(n-1)$-sphere.*

Proof. When $c = 0$ the claim is a simple consequence of the homotopy sequence of the fibration $M \setminus V \to N \setminus B$.

Let us assume henceforth that $c \geq 2$. Consider the last terms of the homotopy exact sequence of this fibration:

$$\to \pi_1(M \setminus V) \xrightarrow{f_*} \pi_1(N \setminus B) \xrightarrow{p} \pi_0(F) \to \pi_0(M \setminus V) \to \pi_0(N \setminus B)$$

From Lemma 2.2 $M \setminus V$ and $N \setminus B$ are connected and $\pi_1(M \setminus V) \cong \pi_1(N \setminus B) \cong \mathbb{F}(c)$. If F has $d \geq 2$ connected components then the kernel $\ker p$ of p is a finite index proper subgroup of the free non-abelian group $\mathbb{F}(c)$. The Nielsen-Schreier theorem states that a subgroup of a free group is free. Moreover, the rank of an index d subgroup of $\mathbb{F}(c)$ is $d(c-1)+1$, by the Schreier index formula. In particular $\ker p$ is a free group of rank $d(c-1)+1$, where d is the number of components of F, and hence its rank is larger than c. On the other hand, by exactness of the sequence above, $\ker p$ is also the image of f_* and thus it is a group of rank at most c. This contradiction shows that F is connected.

If $n = 2$ then F is a circle, as claimed.

Let now $n > 2$. We obtained above that f_* is surjective. Since finitely generated free groups are Hopfian any surjective homomorphism $\mathbb{F}(c) \to \mathbb{F}(c)$ is also injective. Since $\pi_2(N \setminus B) \cong \pi_2(N) = 0$ and f_* is injective we derive that $\pi_1(F) = 0$. The remaining terms of the homotopy exact sequence of the fibration and Lemma 2.2 show then that $\pi_j(F) = 0$ for $2 \leq j \leq n-2$. Thus F is a homotopy sphere. □

Lemma 2.4. *Suppose that $B \neq \emptyset$.*

1. *We have $H_1(N \setminus B) \cong \mathbb{Z}^c$, $H_n(N \setminus B) = \mathbb{Z}^{|B|+c-1}$ and $H_{n+1}(N \setminus B) = 0$.*
2. *If $n > 2$ then $H_{n-1}(M \setminus V) = 0$.*
3. *The homomorphism $H_n(M \setminus V) \to H_n(M)$ induced by the inclusion map is surjective.*

Proof. The first two assertions are consequences of Lemma 2.1, Lemma 2.2 and standard algebraic topology. For instance, $H_1(N \setminus B) \cong H_1(N) = \mathbb{Z}^c$. The last claim follows from Lemma 2.2 and the long exact sequence in homology of the pair $(M, M \setminus V)$. □

Lemma 2.5. *If $B \neq \emptyset$ and $c \neq 1$ then the rank of $H_n(M \setminus V)$ is $2c + |B| - 2$.*

Proof. The Gysin sequence of the fibration $M \setminus V \to N \setminus B$ (whose fiber is a homotopy sphere) reads:

$$\to H_m(M \setminus V) \to H_m(N \setminus B) \to H_{m-n}(N \setminus B) \to H_{m-1}(M \setminus V) \to$$

Consider the exact subsequence

$$H_{n+1}(N \setminus B) \to H_1(N \setminus B) \to H_n(M \setminus V) \to H_n(N \setminus B) \to H_0(N \setminus B) \to H_{n-1}(M \setminus V)$$

If $n > 2$ then the first and the last terms vanish.

The Euler characteristic of this subsequence is zero by exactness and thus the rank of $H_n(M \setminus V)$ is $2c + |B| - 2$ by Lemma 2.4.

When $n = 2$, we can complete the exact sequence above by adding one more term to its right, namely $H_1(M \setminus V) \xrightarrow{f_*} H_1(N \setminus B)$. However, f_* is actually the map induced in homology by the isomorphism $f_* : \pi_1(M) \to \pi_1(N)$ and thus an isomorphism itself. The argument with the Euler characteristic can be applied again and yields the claimed result. □

From Lemma 2.5 and Lemma 2.4 (3) we derive that

$$2c + |B| - 2 \geq \beta_n(M)$$

and the proposition is proved. □

Corollary 2.1. If M^{2n} is a smooth $(n-1)$-connected closed manifold, then

$$\varphi(M, \Sigma^{n+1}) \geq \beta_n(M) + 2,$$

where Σ^{n+1} is a homotopy sphere.

Remark 2.1. The present approach does not work for $c = 1$. In fact, fibers might have several connected components, each one being a homotopy sphere. In the absence of an upper bound of the number of components the Leray-Serre spectral sequence leads only to a trivial lower bound for the number of critical values.

3. Fiber sums of suspensions of Hopf fibrations

Proposition 3.1. *Let $n \in \{2, 4, 8\}$, $e \geq c \geq 0$, with $c \neq 1$, and Σ^{2n} be a homotopy $2n$-sphere. If $n = 2$ assume further that $\Sigma^4 \setminus \mathrm{int}(D^4)$ embeds smoothly into S^4, where D^4 is a smooth 4-disk. Then*

$$\varphi(\Sigma^{2n} \#_e S^n \times S^n \#_c S^1 \times S^{2n-1}, \#_c S^1 \times S^n) \leq 2e - 2c + 2$$

Proof. Recall from [1] that $\varphi(S^{2n}, S^{n+1}) = 2$ if $n = 2, 4$ or 8. This is realized by taking suspensions of both spaces in the Hopf fibration $h : S^{2n-1} \to S^n$, where $n = 2, 4$ or 8, and then smoothing the new map at both ends. The extension $H : S^{2n} \to S^{n+1}$ has precisely two critical points. This is also the basic example of a Montgomery-Samelson fibration with finitely many singularities, as considered in [4]. Antonelli has considered in [3] manifolds which admit maps with two critical points into spheres, by gluing together two copies of H.

Our aim is to define fiber sums of Hopf fibrations leading to other examples of pairs of manifolds with finite φ using Antonelli's construction for more general gluing patterns. Identify S^{n+1} (and respectively S^{2n}) with the suspension of S^n (respectively S^{2n-1}) and thus equip it with the coordinates (x, t), where $|x|^2 + t^2 = 1$, and $t \in [-1, 1]$. We call the coordinate t the height of the respective point. The suspension H is then given by:

$$H(x, t) = \left(\psi(|x|) h \left(\frac{x}{|x|} \right), t \right)$$

where $\psi : [0, 1] \to [0, 1]$ is a smooth increasing function infinitely flat at 0 such that $\psi(0) = 0$ and $\psi(1) = 1$.

Pick up a number of points $x_1, x_2, \ldots, x_k \in S^{n+1}$ and their small enough disk neighborhoods $x_i \in D_i \subset S^{n+1}$, such that:

1. the projections of D_i on the height coordinate axis are disjoint;
2. the D_i's do not contain the two poles, i.e. their projections on the height axis are contained in the open interval $(-1, 1)$.

Let A_k be the manifold with boundary obtained by deleting from S^{n+1} of the interiors of the disks D_i, for $1 \leq i \leq k$. Let also $B_k \subset S^{2n}$ denote the preimage of A_k by the suspended Hopf map H. Since H restricts to a trivial fibration over the disks D_i it follows that B_k is a manifold, each one of its boundary components being diffeomorphic to $S^{n-1} \times S^n$. Moreover, the boundary components are endowed with a natural trivialization induced from D_i.

Let now Γ be a finite connected graph. To each vertex v of valence k we associate a block $(B_v, A_v, H|_{B_v})$, which will be denoted $(B_k, A_k, H|_{B_k})$, when we want to emphasize the dependence on the number of boundary components. Each boundary component of A_v or B_v corresponds to an edge incident to the vertex v. We define the fiber sum along Γ as the following triple $(B_\Gamma, A_\Gamma, H_\Gamma)$:

(1) A_Γ is the result of gluing the manifolds with boundary A_v, associated to the vertices v of Γ, by identifying, for each edge e joining the vertices v and w (which might coincide) the pair of boundary components in A_v and A_w corresponding to the edge e. The identification is made by using an orientation-reversing diffeomorphism of the boundary spheres.
(2) B_Γ is the result of gluing the manifolds with boundary B_v, associated to the vertices v of Γ, by identifying, for each edge e joining the vertices v and w (which might coincide) the boundary components in B_v and B_w corresponding to the pair of boundary components in A_Γ associated to e. Gluings in B_Γ are realized by some orientation-reversing diffeomorphisms which respect the product structure over boundaries of A_v and A_w. We choose the identification diffeomorphism $\nu : \partial B_v \to \partial B_w$ to be the one from the construction of the double of B_v.
(3) As the boundary components are identified the natural trivializations of the boundary components of B_v agree in pairs. Thus the maps H_v induce a well-defined map $H_\Gamma : B_\Gamma \to A_\Gamma$.

In the case where the graph Γ consists of two vertices joined by an edge this construction is essentially that given in ([3], p.185-186).

Proposition 3.2. *The map $H_\Gamma : B_\Gamma \to A_\Gamma$ has $2m$ critical points, where m is the number of vertices of Γ.*

Proof. Clear, by construction. □

We say that Γ has c independent cycles if the rank of $H_1(\Gamma)$ is c. This is equivalent to ask Γ to become a tree only after removal of at least c edges. Moreover, $c = e - m + 1$ where e denotes the number of edges.

Proposition 3.3. *If Γ has e edges and c cycles, i.e. $e - c + 1$ vertices, then B_Γ is diffeomorphic to $\Sigma^{2n} \#_e S^n \times S^n \#_c S^1 \times S^{2n-1}$ (where Σ^{2n} is a homotopy sphere, which is trivial when $n = 2$), while A_Γ is diffeomorphic to $\#_c S^1 \times S^n$. Here $\#_c S^1 \times S^n$ states for S^{n+1} when $c = 0$.*

Proof. The sub-blocks A_k are diffeomorphic to the connected sum of k copies of disks D^{n+1} out of their boundaries. When gluing together two such distinct sub-blocks (since there is an edge in Γ joining the corresponding vertices) the respective pair of disks leads to a factor $D^{n+1} \cup_\mu D^{n+1}$, where $\mu : S^n \to S^n$ is the identification map. If μ is a reflection then the factor $D^{n+1} \cup_\mu D^{n+1}$ is the double of D^{n+1} and hence diffeomorphic to S^{n+1}.

When gluing all sub-blocks in the pattern of the graph Γ the only non-trivial contribution comes from the cycles. Each cycle of Γ introduces a 1-handle. Thus the manifold A_Γ is diffeomorphic to $\#_c S^1 \times S^n$.

Further we have a similar result for the sub-blocks B_k:

Lemma 3.1. *The sub-blocks B_k are diffeomorphic to the connected sum of k copies of the product $S^n \times D^n$ out of their boundaries.*

Proof. One obtains B_k by deleting out k copies of $H^{-1}(D_i)$; each $H^{-1}(D_i)$ is a tubular neighborhood of the (generic) fiber of H and thus diffeomorphic to $S^{n-1} \times D^{n+1}$.

When $k = 1$ the generic fiber of H is an S^{n-1} embedded in S^{2n}, namely the image of the fiber of the Hopf fibration in the suspension sphere S^{2n}. The generic fiber is unknotted in S^{2n}, as an immediate consequence of Haefliger's classification of smooth embeddings. In fact, according to [7], any smooth embedding of S^k in S^m is unknotted, i.e. isotopic to the boundary of a standard ball, if the dimensions satisfy the meta-stable range condition $k < \frac{2}{3}m - 1$. This implies that the complement of a regular neighborhood of the fiber is diffeomorphic to the complement of a standard sphere and thus to $S^n \times D^n$.

When $k \geq 2$ we remark that the fibers over the points $x_i \in D_i$ lie at different heights and thus they are contained in disjoint slice spheres of the suspension S^{2n}. This implies that these fibers are unlinked, i.e. isotopic to the boundary of a set of disjoint standard balls. Thus the complement of a regular neighborhood of their union is diffeomorphic to the connected sum of their individual complements, and therefore to the connected sum of k copies of the product $S^n \times D^n$ out of their boundaries. □

Let us stick for the moment to the case when $k = 1$ and we have two diffeomorphic sub-blocks B_v and B_w, each one having one boundary component, to be glued together. Recall that the identification diffeomorphism $\nu : \partial B_v \to \partial B_w$ is the one from the construction of the double of B_v. Observe that the maps $B_v \to A_v$ and $B_w \to A_w$ glue together to form a well-defined smooth map $B_v \cup_\nu B_w \to A_v \cup_\mu A_w$, as already noticed in ([3], p.185).

Lemma 3.2. *The factor $B_v \cup_\nu B_w$ is diffeomorphic to $\Sigma^{2n} \# S^n \times S^n$, where $\Sigma^4 = S^4$.*

Proof. Consider first the case $n = 2$, which is the most interesting one since the result cannot follow from general classification results. The sub-block $D^2 \times S^2$ can be easily described by a Kirby diagram (see [6], chapter 4), which encodes its handlebody structure. As $D^2 \times S^2$ is obtained from D^4 by throwing away the regular neighborhood of an unknotted circle (i.e. a 1-handle) it can be described as the result of attaching the dual 2-handle on an unknotted circle with framing 0. There is also a dual handlebody decomposition of $D^2 \times S^2$ in which each j-handle generates a $4 - j$ handle. The double of $D^2 \times S^2$ is then described by putting together the two handlebody descriptions (the usual one and the dual one) and thus is made of D^4 with two 2-handles and finally a 4-handle capping off the boundary component.

Attaching maps of 4-handles are orientation preserving diffeomorphisms of S^3, and by a classical result of Cerf these are isotopic to identity. Thus there exists a unique way to attach a 4-handle to a 4-manifold with boundary S^3. By the way, recall that a theorem of Laudenbach and Poenaru ([11]) shows that there is only one way up to global diffeomorphism to attach 3-handles and 4-handles to a 4-manifold with boundary $\#_k S^1 \times S^2$ in order to obtain a closed manifold.

Now it is easy to see that the new 2-handle (in the handlebody structure of the double of $D^2 \times S^2$) is attached along a meridian circle of the former 2-handle with 0 framing. Thus a Kirby diagram of the double of $D^2 \times S^2$ consists of a Hopf link with both components having framing 0, and it is well-known that this diagram is also that of $S^2 \times S^2$. See also ([6], Example 4.6.3) for more details.

This argument applies as well for $n \geq 3$. We have a handle decomposition of $D^n \times S^n$ as D^{2n} with one n-handle attached. The set of framings on a sphere S^{n-1} in ∂D^{2n} is acted upon freely transitively by $\pi_{n-1}(O(n))$. Moreover $\pi_3(O(4)) \cong \pi_7(O(8)) \cong \mathbb{Z} \oplus \mathbb{Z}$ (see [12]). Then the n-handle is attached on an unknotted $(n-1)$-sphere with trivial framing, i.e. the $(0,0)$-framing. Observe that this is the canonical framing associated to the identity attaching map $\mathrm{id}_{S^{n-1} \times D^n}$ (see e.g. [6] Example 4.1.4.(d)). Further the double of $D^n \times S^n$ is obtained by putting together the usual handlebody and its dual. As above we can describe the double as the result of attaching two n-handles and one $2n$-handle. The dual n-handle is attached on a meridian $(n-1)$-sphere which links once the former attaching $(n-1)$-sphere and has trivial framing. The union of the two spheres is the analogue of the Hopf link in $S^{2n-1} = \partial D^{2n}$. As it is well-known $S^n \times S^n$ can also be obtained by adding two n-handles along this high-dimensional trivially-framed Hopf link and a $2n$-handle.

The only difference between the cases $n > 2$ and $n = 2$ is that the result of attaching a $2n$-handle for $n > 2$ is not unique, as there might exist diffeomorphisms of S^{2n-1} which are not isotopic to identity. However, detaching and then re-attaching a $2n$-handle with a reflection diffeomorphism as gluing map will create an exotic sphere (for $n \geq 4$) and thus the double is diffeomorphic to $\Sigma^{2n} \# S^n \times S^n$ for some homotopy sphere Σ^{2n}. □

When gluing all sub-blocks in the pattern of the graph Γ such that each identification map is ν then each pair of sub-blocks determines a factor $\Sigma^{2n} \# S^n \times S^n$. If there are no cycles in Γ then we obtain a connected sum of such factors, namely $\Sigma^{2n} \#_e S^n \times S^n$. Finally, the only additional non-trivial contribution comes from the cycles. Each cycle of Γ introduces an extra 1-handle. Thus the manifold B_Γ is diffeomorphic to $\Sigma^{2n} \#_e S^n \times S^n \#_c S^1 \times S^{2n-1}$. □

In order to prove Proposition 3.1 it suffices now to show that one can attach a homotopy sphere Σ to the manifolds B_Γ and still have the same number of critical points.

Every homotopy m-sphere Σ^m, for $m \neq 4$, can be obtained as the union of two disks glued together along their boundaries using some diffeomorphism f of the $(m-1)$-sphere. Therefore, by removing a small disk centered at a critical point and then gluing it back using the diffeomorphism f the manifold changes by means of a connected sum with the homotopy sphere Σ^m. When $m = 4$ it is unknown whether all homotopy 4-spheres can be obtained as the union of two disks. Actually, if this were true then any homotopy 4-sphere would be diffeomorphic to the standard 4-sphere. But we can obtain any homotopy 4-sphere as the gluing of one disk and a homotopy 4-disk. Consider a homotopy 4-sphere Σ^4 for which the associated homotopy 4-disk Δ^4 embeds smoothly into S^4. Then, as above, by removing a small disk centered at a critical point and then gluing the homotopy 4-disk Δ^4 along the boundary 3-sphere the 4-manifold changes by means of a connected sum with the homotopy sphere Σ^4.

Since the homotopy spheres form a finite abelian group under the connected sum one can obtain this way all manifolds of the form $\Sigma^{2n}\#_e S^n \times S^n \#_c S^1 \times S^{2n-1}$, when $n \neq 2$, and respectively those for which $\Sigma^4 \setminus D^4$ embeds smoothly into S^4, when $n = 2$.

One shows (see [1] where this argument is carried out in detail) that we can glue together the two restrictions of the smooth map to the disk and respectively to its complementary in order to obtain a smooth map on the connected sum $B_\Gamma \# \Sigma$ with the same (non-zero) number of critical points, namely $2e - 2c + 2$. When $n = 2$ we need a theorem of Huebsch and Morse for $n = 2$ (see [8]) concerning the existence of smooth maps $\Delta^4 \to D^4$ with one critical point, for a homotopy 4-disk Δ^4 which embeds smoothly into S^4. □

Remark 3.1. Recall that the group Θ^k of homotopy k-spheres is $\Theta^k = \mathbb{Z}/2\mathbb{Z}$, when $k \in \{8, 16\}$.

Remark 3.2. By twisting μ by a diffeomorphism of S^n which is not isotopic to identity (e.g. when $n = 8$) one could obtain exotic spheres factors in A_Γ. More interesting examples correspond to twisting ν by some orientation preserving diffeomorphism $\eta : S^{n-1} \times S^n \to S^{n-1} \times S^n$ which still respect the product structure. For instance we can consider some η induced from a map $S^{n-1} \to SO(n+1)$ whose homotopy class is an element of $\pi_{n-1}(SO(n+1))$. It seems that all examples obtained by twisting are still diffeomorphic to $\Sigma^{2n}\#_e S^n \times S^n \#_c S^1 \times S^{2n-1}$.

4. Examples with $\varphi = 1$

The result of [1] shows that if $\varphi(M^m, N^{n+1})$ is finite non-zero (small codimension non-exceptional dimensions) then $\varphi(M^m, N^{n+1}) = 1$ and M^m should be diffeomorphic to $\Sigma^m \# \widehat{N}$, where Σ^m is an exotic sphere and $\widehat{N}$ is the total space of a smooth fibration, such that M^m is not fibered over N. Actually this construction might produce non-trivial examples in any codimension.

Proposition 4.1. *If Σ^m is an exotic sphere (for $m = 4$ we assume that $\Sigma^4 \setminus \mathrm{int}(D^4)$ embeds smoothly in S^4) and $\widehat{N} \to N$ a smooth fibration then $\varphi(\Sigma^m \# \widehat{N}, N) \in \{0, 1\}$.*

Proof. We obtain $\Sigma^m \# \widehat{N}$ from $\widehat{N}$ by excising a ball D^{n+1} and gluing it (or a homotopy 4-disk when $m = 4$) back by means of a suitable diffeomorphism h of its boundary. By a classical result of Huebsch and Morse ([8]), there exists a smooth homeomorphism $\Sigma^m \# \widehat{N} \to \widehat{N}$ which has only one critical point located in the ball D^{n+1}. This provides a smooth map $\Sigma^m \# \widehat{N} \to N$ with one critical point. □

Remark 4.1. Notice however that $\Sigma^m \# \widehat{N}$ might still be fibered over N, although not diffeomorphic to $\widehat{N}$. This is so when $\widehat{N} \to N$ is the Hopf fibration $S^7 \to S^4$ and $\Sigma^7 \# \widehat{N}$ is a Milnor exotic sphere, namely a S^3-fibration over S^4 with Euler class ± 1.

Remark 4.2. The manifold $M^m = \Sigma^m \# S^{m-n-1} \times S^{n+1}$ is not diffeomorphic to $S^{m-n-1} \times S^{n+1}$ if Σ^m is an exotic sphere (see [13]). Thus, the proposition above yields effective examples where $\varphi = 1$.

If Σ^8 is the exotic 8-sphere which generates the group $\Theta^8 = \mathbb{Z}/2\mathbb{Z}$ then $\varphi(\Sigma^8 \# S^3 \times S^5, S^5) = 1$. In fact $M^8 = \Sigma^8 \# S^3 \times S^5$ is homeomorphic but not diffeomorphic to $S^3 \times S^5$. Assume the contrary, namely that

M^8 smoothly fibers over S^5. Then the fiber should be a homotopy 3-sphere and hence S^3, by the Poincaré Conjecture. The S^3-fibrations over S^5 are classified by the elements of $\pi_4(SO(4)) \cong \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. There exist precisely two homotopy types among the S^3-fibrations over S^5 which admit cross-sections (see [9], p.217). If M^8 is a S^3-fibration then it should have a cross-section because it is homotopy equivalent to $S^3 \times S^5$ and the existence of a cross-section is a homotopy invariant (see [9], p.196, [10], p.164). However the two homotopy types correspond to two distinct isomorphism types as spheres bundles. In fact they are classified by the image of $\pi_4(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$ into $\pi_4(SO(4))$. This means that a S^3-fibration having a cross-section is either homotopy equivalent to the trivial fibration and then it is isomorphic to the trivial fibration or else it has not the same homotopy type as $S^3 \times S^5$. Observe also that there is only one $O(4)$-equivalence class and thus precisely two isomorphism classes of such S^3-fibrations without cross-sections ([10], p.164) . In particular, non-trivial S^3-fibrations over S^5 cannot be homeomorphic to M^8 and this contradiction shows that M^8 cannot smoothly fiber over S^5.

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