Diffeomorphisms groups of Cantor sets and Thompson-type groups

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Abstract

The group of $\mathcal{C}^1$-diffeomorphisms groups of any sparse Cantor subset of a manifold is countable and discrete (possibly trivial). Thompson's groups come out of this construction when we consider central ternary Cantor subsets of an interval. Brin's higher dimensional generalizations nV of Thompson's group V arise when we consider products of central ternary Cantor sets. We derive that the $\mathcal{C}^2$-smooth mapping class group of a sparse Cantor sphere pair is a discrete countable group and produce this way versions of the braided Thompson groups.

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1 Introduction

Differentiable structures on Cantor sets have first been considered by Sullivan in [31]. Our aim is to consider groups of diffeomorphisms of Cantor sets, mapping class groups of Cantor punctured spheres and their relations with Thompson-like groups. In particular, the usual Thompson groups (see [9]) can be retrieved as diffeomorphisms groups of Cantor subsets of suitable spaces (a line, a circle or a 2-sphere).

Let M be a compact manifold and $C \subset M$ be a Cantor set, namely a *compact totally disconnected subset without isolated points*. Any two Cantor sets are homeomorphic as topological spaces. But if M has dimension $m \geq 3$ there exists Cantor sets C_1, C_2 embedded into M so that there is no ambient homeomorphism of M carrying C_1 into C_2. One says that C_1 and C_2 are not *topologically equivalent* Cantor set embeddings.

A Cantor set in $\mathbb{R}^m$ is *tame* if it is topologically equivalent to the *standard ternary Cantor set*, namely when there is a homeomorphism which sends it within a standard interval. All Cantor sets in $\mathbb{R}^m$, for $m \leq 2$ are tame, but there exists uncountably many *wild* (i.e. not tame) Cantor sets in $\mathbb{R}^m$, for every $m \geq 3$ (see [2]).

One defines similarly *smooth equivalence* and *smoothly tame* Cantor sets. The analogous story for diffeomorphisms is already interesting for $m = 1$, as Cantor subsets of $\mathbb{R}$ might be differentiably non-equivalent. Our main concern is the image of the group of diffeomorphisms of M which preserve a Cantor set C into the automorphism group of C. Under fairly general conditions we are able to prove that this is a countable group, thereby providing an interesting class of discrete groups. For Cantor sets obtained from a topological iterated function system the associated groups are non-trivial, while for many self-similar Cantor sets these are versions of Thompson's groups.

A. General countability statements

1.1 Pure mapping class groups

Definition 1. *We denote by* $\mathrm{Diff}^k(M,C)$ *the group of diffeomorphisms of class* $\mathcal{C}^k$ *of* M *sending* C *to itself. The* $\mathcal{C}^k$*-mapping class group of the pair* (M,C) *is the group* $\pi_0(\mathrm{Diff}^k(M,C))$ *of* $\mathcal{C}^k$*-isotopy classes of diffeomorphisms of class* $\mathcal{C}^k$ *of* (M,C)*, which we denote by* $\mathcal{M}^k(M,C)$*.*

We denote by $P\mathrm{Diff}^k(M,C)$ *the group of diffeomorphisms of class* $\mathcal{C}^k$ *of* M *which are pure, namely they preserve pointwise* C*. The pure* $\mathcal{C}^k$*-mapping class group of the pair* (M,C) *is the group* $\pi_0(P\mathrm{Diff}^k(M,C))$ *of* $\mathcal{C}^k$*-isotopy classes of diffeomorphisms of class* $\mathcal{C}^k$ *of* (M,C)*, which we denote by* $P\mathcal{M}^k(M,C)$*.*

In a similar vein but a different context, the group of homeomorphisms $\mathrm{Diff}^0(M,A)$ associated to a manifold M and a countable dense set $A \subset M$ was studied recently in [13]. The authors proved there that $\mathrm{Diff}^0(M,A)$ is either isomorphic to a countably infinite product of copies of $\mathbb{Q}$, when M is 1-dimensional, or the Erdös subgroup of l^2 elements, otherwise. In the present setting, when A is closed and the smoothness is at least $\mathcal{C}^1$ the situation is fundamentally different.

For the sake of simplicity we focus on the case when M is a compact surface, so that any Cantor subset C is tame, and we suppose that C is contained within an embedded interval E. We will write $C = \cap_{j=1}^{\infty} I_j$ as an infinite nested sequence of subsets $I_j \subset E$, each one being a finite union of closed intervals. Consider next small enough pairwise disjoint neighborhoods C_j of I_j in M, such that $C = \cap_{j=1}^{\infty} C_j$ and now each C_j is a finite union of disks. The family of nested sets $\{C_j\}$ will be called a *defining sequence* of the Cantor set C.

Definition 2. *A diffeomorphism* $\varphi \in P\mathrm{Diff}^k(M,C)$ *has compact support if there exists some presentation* $\{C_j\}$ *of* C *and some* n *for which the restriction of* φ *to* C_n *is identity.*

The class of φ *in* $P\mathcal{M}^k(M,C)$ *is compactly supported if there exists some presentation* $\{C_j\}$ *of* C *and some* n *for which the restriction of* φ *to* C_n *is isotopic to identity rel* C*, i.e. by an isotopy which is identity on* C*.*

Notice that the property of being compactly supported is actually independent on the choice of the defining sequence of C.

Our first result is the following:

Theorem 1. *When* $k \geq 2$ *and* M *is a compact surface all classes in the group* $P\mathcal{M}^k(M,C)$ *are compactly supported. In particular, the group* $P\mathcal{M}^k(M,C)$ *is countable.*

In contrast, the topological mapping class group $P\mathcal{M}^0(S^2,C)$ obtained for $k=0$ is an uncountable non-discrete topological group.

The following is an easy consequence:

Corollary 1. *If* $k \geq 2$ *and* C *is a Cantor subset of the compact surface* M *then* $P\mathcal{M}^k(M,C)$ *coincides with the inductive limit* $\lim_{j\to\infty} P\mathcal{M}(M - \mathrm{int}(C_j))$ *of pure mapping class groups of an ascending exhaustion by compact subsurfaces of* $M - C$*.*

1.2 $\mathcal{C}^1$-diffeomorphism groups of Cantor sets

We turn now to the full mapping class groups. Several groups which arised recently in the literature could be thought to play the role of the mapping class groups for some infinite type surfaces, for instance the group $\mathcal{B}$ from [15] and its version BV, which was defined by Brin [5] and Dehornoy [11], independently. These two groups are braidings of the Thompson group V (see [9]). Geometric constructions of the same sort permitted the authors of [16] to derive two braidings T^* and $T^\sharp$ of the Thompson group T.

Our next goal is to show that these groups are indeed smooth mapping class groups in the usual sense and that most (if not all) smooth mapping class groups are related to some Thompson-like groups.

Assume now that $C \subset M$, where M can be either one or two dimensional. Set then $\mathfrak{Diff}_M^k(C)$ for the group of induced transformations of C arising as restrictions of elements of $\mathrm{Diff}^k(M,C)$. The $\mathcal{C}^k$ topology on $\mathrm{Diff}^k(M,C)$ induces a topology on $\mathfrak{Diff}_M^k(C)$.

Notice now that we have the exact sequence:

$$1 \to P\mathcal{M}^k(M,C) \to \mathcal{M}^k(M,C) \to \mathfrak{Diff}_M^k(C) \to 1 \tag{1}$$

By Theorem 2 the group $\mathcal{M}^k(S^2, C)$ is discrete countable if and only if $\mathfrak{Diff}_M^k(C)$ does, when $k \geq 2$.

Classical Thompson groups can be realized as groups of dyadic piecewise linear homeomorphisms (or bijections) of an interval, circle or a Cantor set (see [9, 17]) or as groups of automorphisms at infinity of graphs (respecting or not the planarity), as in [26]. Notice that the more involved construction from [17] provides embeddings of Thompson groups into the group of diffeomorphisms of the circle, admitting invariant minimal Cantor sets. In particular, Ghys and Sergiescu obtained embeddings as discrete subgroups of the group of diffeomorphisms (see [17], Thm. 2.3).

In our setting we see that whenever it is discrete and countable the group $\mathcal{M}^k(S^2, C)$ is the braiding of $\mathfrak{Diff}_M^k(C)$ according to Corollary 1, as in the cases studied in [5, 11, 15, 16]. This pops out the question whether $\mathfrak{Diff}_M^k(C)$ is a Thompson-like group, in general. We were not able to solve this question in full generality and actually when C is a generic Cantor set of the interval we expect the group $\mathfrak{Diff}_M^k(C)$ be small, if not trivial. To this purpose we introduce the following property of Cantor sets.

Definition 3. *The Cantor set $C \subset \mathbb{R}$ is σ-sparse if for any $a, b \in C$ there is a complementary interval $(\alpha, \beta) \subset (a, b) \cap \mathbb{R} \setminus C$ such that*

$$(\beta - \alpha) \geqslant \sigma(b - a) \tag{2}$$

Moreover C is sparse if it is σ-sparse for some $\sigma > 0$.

Set $\mathfrak{Diff}^k(C) = \mathfrak{Diff}_{\mathbb{R}}^k(C)$, for the sake of notational simplicity.

Theorem 2. *If C is sparse then the group $\mathfrak{Diff}^1(C)$ is countable.*

Theorem 2 cannot be extended to all Cantor sets C, without additional assumptions, as we can see from the examples given in section 5.

Corollary 2. *Let C be a sparse Cantor set on the circle S^1. Then the group $\mathfrak{Diff}_{S^1}^k(C)$ is countable and discrete, for $k \geq 1$. Moreover, if M is a compact planar surface and C is a finite union of sparse Cantor sets in ∂M, then the group $\mathfrak{Diff}_M^k(C)$ is countable and discrete, for $k \geq 1$. In particular, under the same conditions $\mathcal{M}^k(M, C)$ is countable and discrete, for $k \geq 2$.*

We have the following more general version of the previous result:

Theorem 3. *If C is a sparse Cantor set of an interval smoothly embedded into a compact orientable manifold M of dimension at least 2 then the group $\mathfrak{Diff}_M^1(C)$ is countable and discrete. In particular, $\mathcal{M}^k(S^2, C)$ is countable and discrete when $k \geq 2$.*

The relationship between $\mathfrak{Diff}_{S^2}^1(C)$ and $\mathfrak{Diff}_{S^1}^1(C)$ is similar to that between the Thompson groups V and T. Things might be more complicated when considering $\mathfrak{Diff}_{S^3}^k(C)$, as now the topological type of the Cantor set embedding might be wild. There is nevertheless a large supply of nice Cantor subsets in any dimensions for which we can prove the countability:

Theorem 4. *Let C_i be sparse Cantor sets in $\mathbb{R}$ and $C = C_1 \times C_2 \times \cdots C_n \subset \mathbb{R}^n$. Then the group $\mathfrak{Diff}_{\mathbb{R}^n}^1(C)$ is countable and discrete.*

The key point is to show that the stabilizer of a point in this group is a finitely generated abelian group (see Lemma 6, Proposition 2). The discreteness of the stabilizers seems to be the counterpart to the following unpublished theorem of G. Hector (see [25]): If the subgroup G of the group $\mathrm{Diff}^\omega(S^1)$ of analytic diffeomorphisms of the circle has an exceptional minimal set then the stabilizer G_a of any point a of the circle in G is either trivial or $\mathbb{Z}$. As a corollary every subgroup of $\mathrm{Diff}^\omega(S^1)$ having a minimal Cantor set is countable. This of course is not true for subgroups of $\mathrm{Diff}^\infty(S^1)$. The proof of our result is also inspired and closely related to Thurston's generalization of Reeb's stability theorem from [32].

B. Specific families of Cantor sets

1.3 Iterated functions systems

Definition 4. *A* contractive iterated function system *(abbreviated contractive* IFS*) is a finite family* $\Phi = \{\phi_0, \phi_1, \ldots, \phi_N\}$ *of contractive maps* $\phi_j : \mathbb{R}^d \to \mathbb{R}^d$*. Recall that a map* ϕ *is* contractive *if its Lipschitz constant is smaller to unit, namely:*

$$\sup_{x,y \in \mathbb{R}^d} \frac{d(\phi(x), \phi(y))}{d(x,y)} < 1.$$

According to Hutchinson (see [21]) there exists an unique non-empty compact $C = C_\Phi \subset \mathbb{R}^d$, called the *attractor* of the IFS Φ, such that $C = \cup_{j=0}^{N} \phi_j(C)$.

Example 1. *The central Cantor set* C_λ*, with* $\lambda > 2$*, is the attractor of the IFS* $\{\phi_0, \phi_1\}$ *on* $\mathbb{R}$ *given by*

$$\phi_0(x) = \frac{1}{\lambda} x, \ \phi_1(x) = \frac{1}{\lambda} x + \frac{\lambda - 1}{\lambda}.$$

Although the IFS makes sense also when $1 < \lambda \leq 2$*, in this case the attractor is not a Cantor set but the whole interval* $[0, 1]$*.*

Consider now the following type of IFS of topological nature.

Definition 5. *Let* U *be a manifold (possibly non-compact) and* $\varphi_j : U \to U$ *be finitely many homeomorphisms on their image. We say that* $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ *has a* strict attractive basin M *if* M *is a compact submanifold* $M \subset U$ *with the following properties:*

1. $\varphi_j(M) \subset \mathrm{int}(M)$*, for all* $j \in \{1, \ldots, n\}$*;*
2. $\varphi_i(M) \cap \varphi_j(M) = \emptyset$*, for any* $j \neq i \in \{1, \ldots, n\}$*.*

We say that the pair (Φ, M) *is an* invertible *IFS if* M *is a strict attractive basin for* Φ*. If moreover,* φ_j *are* $\mathcal{C}^k$*-diffeomorphisms then we say that the IFS is of class* $\mathcal{C}^k$*.*

Although the metric is not present in this definition, it seems rather clear that the existence of an attractive basin is a topological version of uniform contractivity of φ_j. There exists then a unique invariant non-empty compact $C_\Phi \subset M$ with the property that $C_\Phi = \cup_{i=1}^{n} \varphi_i(C_\Phi)$. In general, C_Φ might not be a Cantor set.

We have the following general statement:

Theorem 5. *Consider a* $\mathcal{C}^1$ *contractive invertible IFS* (Φ, M) *whose strict attractive basin* M *is diffeomorphic to a d-dimensional ball. Then, the group* $\mathfrak{Diff}^1(C_\Phi)$ *contains the Thompson group* F_n*, when* M *is of dimension* $d = 1$ *and the Thompson group* V_n*, when* $d \geq 2$*, respectively.*

In particular, the groups $\mathfrak{Diff}^1_M(C_\Phi)$ are (highly) nontrivial.

For a clear introduction to the classical Thompson groups F, T, V we refer to [9]. The generalized versions F_n, T_n, V_n were considered by Higman ([20] and further extended and studied by Brown and Stein (see [30]) and Laget [23]. We will recall their definitions in section 2.

The result of the theorem does not hold when the attractive basin M is not a ball. For instance, when M is a 3-dimensional solid torus, by taking nontrivial (linked) embeddings $\cup_{j=1}^{n} \phi_j(M) \subset M$ we can provide examples of wild Cantor sets, some of them being topologically rigid, in which case the group $\mathfrak{Diff}^1(C)$ is trivial.

On the other hand the theorem also holds without the contractivity assumption, with the same proof. This additional condition will only be used for proving that C_Φ is a Cantor set.

1.4 Self-similar Cantor subsets of the line

The second part of this paper is devoted to concrete examples of groups arising by these constructions, for particular choices of the Cantor sets.

We will be concerned in this section with self-similar Cantor sets, namely attractors of IFS which consist only of similitudes. The typical example is the central ternary Cantor set $C_\lambda \subset [0,1]$ (respectively $C_\lambda \subset S^1$) of parameter $\lambda > 2$ (respectively, its image in $S^1 = [0,1]/0 \sim 1$). A direct construction of C_λ is to start from the interval $[0,1]$ and then, to iteratively remove at the n-th step an open central interval of length $\left(1 - \frac{2}{\lambda}\right)^n$ from each of the intervals obtained at $(n-1)$-th step.

We consider the group F_{C_λ} (respectively T_{C_λ}) defined as follows. Set first $PL(\mathbb{R}, C_\lambda)$ (and $PL(S^1, C_\lambda)$) be the group of piecewise linear homeomorphisms φ of $\mathbb{R}$ (and S^1) preserving C_λ, namely given by a finite collection of intervals with disjoint interiors $\{I_j\}$, integers $k_j \in \mathbb{Z}$ and $a_j, b_j \in C_\lambda$, such that the restriction of the map φ to each segment has the form

$$\varphi(x) = b_j \pm \lambda^{k_j}(x - a_j), \qquad \text{for any} \quad x \in I_j. \tag{3}$$

The intervals I_j will moreover be supposed to be either gaps or else *standard* intervals, namely I_j is obtained after finitely many steps of the previous construction. Then F_{C_λ} (T_{C_λ} respectively) is the image of $PL(\mathbb{R}, C_\lambda)$ ($PL(S^1, C_\lambda)$ respectively) in the group of homeomorphisms of C_λ. Let further consider the group of piecewise exchanges $PE(C_\lambda)$ which are piecewise linear left continuous bijections of φ of S^1 preserving C_λ, given as above by a finite collection of intervals. We denote by V_{C_λ} its image into the group of homeomorphisms of C_λ.

We first formulate the main result of this section for the central ternary Cantor set, for the sake of simplicity:

Theorem 6. *Let $C_\lambda \subset [0,1]$ be the central ternary Cantor set of parameter $\lambda > 2$. Let $\varphi \in \mathfrak{Diff}^1(C_\lambda)$. Then there is a covering of C_λ by a finite collection of disjoint standard segments $\{I_j\}$, whose images are also standard intervals, integers $k_j \in \mathbb{Z}$ and $a_j, b_j \in C_\lambda$, such that the restriction of the map φ to each segment has the form*

$$\varphi(x) = b_j \pm \lambda^{k_j}(x - a_j), \qquad \text{for any} \quad x \in I_j \cap C_\lambda. \tag{4}$$

In particular, $\mathfrak{Diff}^1(C_\lambda)$ is isomorphic to F_{C_λ} which is an extension of the Thompson group F by $\mathbb{Z}/2\mathbb{Z}$. Similarly, $\mathfrak{Diff}^1_{S^1}(C_\lambda)$ is isomorphic to T_{C_λ} which is an extension by $\mathbb{Z}/2\mathbb{Z}$ of the Thompson group T. Eventually, $\mathfrak{Diff}^1_{S^2}(C_\lambda)$ is isomorphic to V_{C_λ} which is an extension by $(\mathbb{Z}/2\mathbb{Z})^\infty$ of the Thompson group V.

We derive easily now the following interpretation for the Thompson groups and their braided versions:

Corollary 3.

1. *Let C be the image of the standard ternary Cantor subset into the equatorial circle of the sphere S^2 and $k \geq 2$.*
   (a) *The smooth mapping class group $\mathcal{M}^k(D^2_+, C)$ is the Thompson group T, where D_+ is the upper hemisphere;*
   (b) *The smooth mapping class group $\mathcal{M}^k(S^2, C)$ is the braided Thompson group $\mathcal{B}$ from [15] (see section 2 for definitions).*
2. *Let C be the standard ternary Cantor subset of an interval contained in the interior of a 2-disk D^2 and $k \geq 2$. Then $\mathcal{M}^k(D^2, C)$ coincides with the braided Thompson group BV of Brin and Dehornoy.*

Remark 1. *The central ternary Cantor sets C_λ are pairwise non-diffeomorphic, i.e. there is no $\mathcal{C}^1$ diffeomorphism of $\mathbb{R}$ sending C_λ into $C_{\lambda'}$ for $\lambda \neq \lambda'$. Indeed, if there were such a diffeomorphism then the Hausdorff dimensions of the two Cantor sets would agree, while the Hausdorff dimension of C_λ is $\frac{\log 2}{\log \lambda}$. Nevertheless, the groups $\mathfrak{Diff}^1(C_\lambda)$ are all isomorphic, for $\lambda > 2$, according to Theorem 6.*

The general case of self-similar Cantor subsets of the line is slightly more delicate to state, because of some technical conditions which we are not able to get rid of.

Let $\Phi = \{\phi_1, \ldots, \phi_n\}$ be a set of affine transformations of $[0,1]$, given by:

$$\phi_j(x) = \lambda_j x + a_j,$$

where

$$0 = a_1 < \lambda_1 < a_2 < \lambda_2 + a_2 < a_3 \cdots < \lambda_{n-1} + a_{n-1} < a_n < \lambda_n + a_n = 1.$$

The last condition means that segments $\phi_j([0,1])$ are mutually disjoint, so that the attractor $C = C_\Phi$ is a sparse Cantor subset of $[0,1]$. The positive reals $g_j = a_{j+1} - a_j - a_j$ are the initial *gaps* as they represent the distance between consecutive intervals $\phi_j([0,1])$ and $\phi_{j+1}([0,1])$. The image of $[0,1]$ by the elements of the monoid generated by Φ are called standard intervals.

We consider the group $F_{C,N}$ (respectively $T_{C,N}$) defined as follows. Set first $PL(\mathbb{R}, C, N)$ (and $PL(S^1, C, N)$) be the group of piecewise linear homeomorphisms φ of $\mathbb{R}$ (and S^1) given by finite collection of intervals with disjoint interiors $\{I_j\}$, $\mathbf{k}_j \in \mathbb{Z}^n$ and $a_j, b_j \in C$, such that the restriction of the map φ to each segment has the form

$$\varphi(x) = b_j \pm \Lambda_{\mathbf{k}_j,N}(x - a_j), \qquad \text{for} \quad \text{any} \quad x \in I_j, \tag{5}$$

where for each multi-index $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ we put:

$$\Lambda_{\mathbf{k},N} = \lambda_1^{k_1/N} \prod_{j=2}^{n} \lambda_j^{k_j}. \tag{6}$$

We drop the subscript N in $\Lambda_{\mathbf{k},N}$ when $N = 1$.

The set of those N such that $PL(\mathbb{R}, C, N)$ preserves C invariant is non-empty, as for instance $N = 1$ has this property. We are not presently able to prove that larger N cannot arise for particular values of the parameters.

Then $F_{C,N}$ ($T_{C,N}$ respectively) is defined as the image of $PL(\mathbb{R}, C, N)$ ($PL(S^1, C, N)$ respectively) in the group of homeomorphisms of C. Let further consider the group of piecewise exchanges $PE(C, N)$ which are piecewise linear left continuous bijections of φ of S^1 preserving C, given as above by a finite collection of intervals. We denote by $V_{C,N}$ its image into the group of homeomorphisms of C.

Definition 6. *The self-similar Cantor set $C \subset [0,1]$ satisfies the genericity condition (C) if*

1. *either all homothety ratios λ_i are equal and the initial generation gaps g_i are equal;*
2. *or the factors λ_i and the gaps g_j are incommensurable, in the following sense:*
   (a) *$\Lambda_{\mathbf{k}} g_i = g_j$ implies that $\mathbf{k} = 0$ and $i = j$;*
   (b) *there exists no permutation σ different from identity and $\mathbf{k}, \mathbf{k}_i \in \mathbb{Z}_+^{n+1}$ such that for all i we have:*

$$\frac{g_{\sigma(i)}}{g_i} = \Lambda_{-\mathbf{k} + \frac{1}{n}\sum_{i=1}^{n} \mathbf{k}_i}.$$

Theorem 7. *Let $C \subset [0,1]$ be a self-similar Cantor set satisfying the genericity condition (C). Then there exists some $N \in \mathbb{Z}_+$ such that for every $\varphi \in \mathfrak{Diff}^1(C)$ we can find a covering of C by a finite collection of disjoint standard intervals $\{I_j\}$, whose images are also standard intervals, integers $\mathbf{k}_j \in \mathbb{Z}^n$ and $a_j, b_j \in C$, such that the restriction of the map φ to each segment has the form*

$$\varphi(x) = b_j \pm \Lambda_{\mathbf{k}_j,N}(x - a_j), \qquad \text{for} \quad \text{any} \quad x \in I_j \cap C. \tag{7}$$

In particular, $\mathfrak{Diff}^1(C)$ is isomorphic to $F_{C,N}$, $\mathfrak{Diff}^1_{S^1}(C)$ is isomorphic to $T_{C,N}$ and $\mathfrak{Diff}^1_{S^2}(C)$ is isomorphic to $V_{C,N}$.

The main points in the statement of the theorem is the finiteness of the covering and the fact that the intervals are standard. If we drop the requirement that the intervals be standard then a similar result holds with the same proof without the genericity condition (C) in the hypothesis. However, we don't know whether the group obtained this way is isomorphic to some generalized Thompson group, even when $N = 1$.

Notice that the denominator $N = N(\Phi)$ arising in Theorem 7 is strongly constrained by the condition that $PL(\mathbb{R}, C, N)$ preserves C invariant. In all cases in which we were able to compute it we found $N(\Phi) = 1$, for instance as in the following:

Proposition 1. *Let $C = C_\Phi \subset \mathbb{R}$ be the attractor of an affine IFS satisfying the genericity condition (C) and the following additional hypothesis:*

$$\max_j \frac{a_{j+1}}{\lambda_j + a_j} > \sqrt{\lambda_1}. \tag{8}$$

Then $N(\Phi) = 1$, and thus the group $\mathfrak{Diff}^{1,+}(C_\Phi)$ is isomorphic to the generalized Thompson group F_n.

Although it might be possible to have $N(\Phi) > 1$, the group $\mathfrak{Diff}^{1,+}(C_\Phi)$ is sandwiched between two isomorphic copies of generalized Thompson group F_n.

We notice that a weaker version of our Theorem 6 concerning the form of C^1-diffeomorphisms of the central Cantor sets C_λ, was already obtained in ([1], Proposition 1).

A case which attracted considerable interest is that of bi-Lipschitz homeomorphisms of Cantor sets (see [10, 14] and the recent [27, 33]). In particular, the results of Falconer and Marsh [14] imply that every bi-Lipschitz homeomorphism of a Cantor set is given by a pair of possibly infinite coverings of the Cantor set by disjoint intervals and affine homeomorphisms between the corresponding intervals. It seems that the finiteness of the covering needs additional hypotheses beyond the bi-Lipschitz condition. In this sense our results are specific to the C^k-smoothness, with $k \geq 1$ and cannot be extended too much if we are seeking for countable groups. Notice that any countable subgroup of $\mathrm{Diff}^0(S^1)$ (or $\mathrm{Diff}^0([0,1])$ can be conjugated (by a homeomorphism) into the group of bi-Lipschitz homeomorphism (see [12], Thm. D).

1.5 Self-similar Cantor dusts

The next step is to go to higher dimensions. Examples of Blankenship (see [2]) show that there exist wild Cantor sets in $\mathbb{R}^n$, for every $n \geq 3$. A Cantor set C is tame if and only if for every $\varepsilon > 0$ there exist finitely many disjoint piecewise linear cells of diameter smaller than ε whose interiors cover C. In particular, products of tame Cantor sets are tame. More generally, the product of a Cantor subset of $\mathbb{R}^n$ with any compact 0 dimensional subset $Z \subset \mathbb{R}^m$ is a tame Cantor subset of $\mathbb{R}^{m+n}$ (see [24], Cor.2).

In order to emphasize the role of the embedding we will consider now the simplest Cantor subsets, which although tame they are not smoothly tame. Let $C^n_\lambda \subset \mathbb{R}^n$ be the Cartesian product of n copies of C_λ, where $n \geq 2$, which is itself a Cantor set.

Theorem 8. *Let $C^n_\lambda \subset \mathbb{R}^n$ be the product of $n \geq 2$ copies of the central ternary Cantor set of parameter $\lambda > 2$. Let $\varphi \in \mathfrak{Diff}^1_{\mathbb{R}^n}(C^n_\lambda)$. Then there is a covering of C^n_λ by a finite collection of disjoint standard parallelepipeds $\{I_j\}$, integers $k_{j,i} \in \mathbb{Z}$ and $a_{j,i}, b_{j,i} \in C_\lambda$, such that the restriction of the map φ to each of them has the form:*

$$\varphi(x) = (b_{j,i} \pm \lambda^{k_j}(x_i - a_{j,i}))_{i=1,n} R^{m_j}_{\mathbf{b}}, \qquad \text{for any} \quad x \in I_j \cap C^n_\lambda. \tag{9}$$

where R_b is a linear symmetry of a product cube fixing b. In particular, $\mathfrak{Diff}^1_{\mathbb{R}^n}(C^n_\lambda)$ is isomorphic to $V_{C^n_\lambda}$ which is an extension by D_n^∞ of Brin's higher dimensional Thompson group nV, where D_n is the symmetry group of the n-dimensional cube.

Notice that in a series of papers (see [4, 6, 3, 19]) by Brin, Bleak and Lanoue, Hennig and Matucci the authors proved that nV are pairwise non-isomorphic finitely presented simple groups (see also [28, 29]).

Remark 2. *The group $\mathfrak{Diff}^1_{[0,1]^n}(C^n_\lambda)$ is also discrete countable but different from nV, which is simple. There is an obvious homomorphism*

$$\mathfrak{Diff}^1_{[0,1]^n}(C^n_\lambda) \to \mathfrak{Diff}^1_{\partial[0,1]^n}(C^n_\lambda \cap \partial[0,1]^n)$$

The group on the right hand side is isomorphic to $\mathfrak{Diff}^1_{S^{n-1}}(SC_\lambda)$, where we identified $C^n_\lambda \cap \partial[0,1]^n$ with a Cantor subset SC_λ in S^{n-1}. This homomorphism is not surjective, its image is the subgroup which preserves each boundary cell of the cubical complex associated to $\partial[0,1]^n$.

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2 Definition of Thompson-like groups

The standard reference for the classical Thompson groups is [9]. For the sake of completeness we provide here the basic definitions from several different perspectives, which lead naturally the path to the generalizations considered by Brown and Stein and further to the high dimensional Brin groups.

2.1 Groups of piecewise affine homeomorphisms/bijections

Thompson's group F is the group of continuous and nondecreasing bijections of the interval $[0,1]$ which are piecewise dyadic affine. Namely, for each $f \in F$, there exist two dyadic subdivisions of $[0,1]$, $a_0 = 0 < a_1 < \ldots < a_n = 1$ and $b_0 = 0 < b_1 \ldots < b_n$, with $n \in \mathbb{N}^*$, i.e. such that $a_{i+1} - a_i$ and $b_{i+1} - b_i$ belong to $\{\frac{n}{2^k},\ n,k \in \mathbb{N}\}$, so that the restriction of f to $[a_i, a_{i+1}]$ is the unique nondecreasing affine map onto $[b_i, b_{i+1}]$.

Therefore, an element of F is completely determined by the data of two dyadic subdivisions of $[0,1]$ having the same cardinality.

Let us identify the circle to the quotient space $[0,1]/0 \sim 1$. *Thompson's group* T is the group of continuous and nondecreasing bijections of the circle which are piecewise dyadic affine. In other words, for each $g \in T$, there exist two dyadic subdivisions of $[0,1]$, $a_0 = 0 < a_1 < \ldots < a_n = 1$ and $b_0 = 0 < b_1 \ldots < b_n$, with $n \in \mathbb{N}^*$, and $i_0 \in \{1, \ldots, n\}$, such that, for each $i \in \{0, \ldots, n-1\}$, the restriction of g to $[a_i, a_{i+1}]$ is the unique nondecreasing map onto $[b_{i+i_0}, b_{i+i_0+1}]$. The indices must be understood modulo n.

Therefore, an element of T is completely determined by the data of two dyadic subdivisions of $[0,1]$ having the same cardinality, say $n \in \mathbb{N}^*$, plus an integer i_0 mod n.

Finally, *Thompson's group* V is the group of bijections of $[0,1[$, which are right-continuous at each point, piecewise nondecreasing and dyadic affine. In other words, for each $h \in V$, there exist two dyadic subdivisions of $[0,1]$, $a_0 = 0 < a_1 < \ldots < a_n = 1$ and $b_0 = 0 < b_1 \ldots < b_n$, with $n \in \mathbb{N}^*$, and a permutation $\sigma \in \mathfrak{S}_n$, such that, for each $i \in \{1, \ldots, n\}$, the restriction of h to $[a_{i-1}, a_i[$ is the unique nondecreasing affine map onto $[b_{\sigma(i)-1}, b_{\sigma(i)}[$. It follows that an element h of V is completely determined by the data of two dyadic subdivisions of $[0,1]$ having the same cardinality, say $n \in \mathbb{N}^*$, plus a permutation $\sigma \in \mathfrak{S}_n$. Denoting $I_i = [a_{i-1}, a_i]$ and $J_i = [b_{i-1}, b_i]$, these data can be summarized into a triple $((J_i)_{1 \le i \le n}, (I_i)_{1 \le i \le n}, \sigma \in \mathfrak{S}_n)$.

We have obvious inclusions $F \subset T \subset V$. R.J. Thompson proved in 1965 that F, T and V are finitely presented groups and that T and V are simple (cf. [9]). The group F is not perfect, as $F/[F,F]$ is isomorphic to $\mathbb{Z}^2$, but $F' = [F,F]$ is simple. However, F' is not finitely generated (this is related to the fact that an element f of F lies in F' if and only if its support is included in $]0,1[$).

2.2 Groups of diagrams of finite binary trees

A *finite binary rooted planar tree* is a finite planar tree having a unique 2-valent vertex, called the *root*, a set of monovalent vertices called the *leaves*, and whose other vertices are 3-valent. The planarity of the tree provides a canonical labelling of its leaves, in the following way. Assuming that the plane is oriented, the leaves are labelled from 1 to n, from left to right, the root being at the top and the leaves at the bottom.

There exists a bijection between the set of dyadic subdivisions of $[0,1]$ and the set of finite binary rooted planar trees. Indeed, given such a tree, one may label its vertices by dyadic intervals in the following way. First, the root is labelled by $[0,1]$. Suppose that a vertex is labelled by $I = [\frac{k}{2^n}, \frac{k+1}{2^n}]$, then its two descendant vertices are labelled by the two halves I: $[\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}]$ for the left one and $[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}]$ for the right one. Finally, the dyadic subdivision associated to the tree is the sequence of intervals which label its leaves.

Thus, an element h of V is represented by a triple (τ_1, τ_0, σ), where τ_0 and τ_1 have the same number of leaves $n \in \mathbb{N}^*$, and $\sigma \in \mathfrak{S}_n$. Such a triple will be called a *symbol* for h. It is convenient to interpret the permutation σ as the bijection φ_σ which maps the i-th leaf of the source tree τ_0 to the $\sigma(i)$-th leaf of the target tree τ_1. When h belongs to F, the permutation σ is identity and the symbol reduces to a pair of trees (τ_1, τ_0).

Now, two symbols are equivalent if they represent the same element of V and one denotes by $[\tau_1, \tau_0, \sigma]$ the equivalence class. The composition law of piecewise dyadic affine bijections is pushed out on the set of equivalence classes of symbols in the following way. In order to define $[\tau_1', \tau_0', \sigma'] \cdot [\tau_1, \tau_0, \sigma]$, one may suppose, at the price of refining both symbols, that the tree τ_1 coincides with the tree τ_0'. Then the product of the two symbols is

$$[\tau_1', \tau_1, \sigma'] \cdot [\tau_1, \tau_0, \sigma] = [\tau_1', \tau_0, \sigma' \circ \sigma].$$

It follows that V is isomorphic to the group of equivalence classes of symbols endowed with this internal law.

2.3 Partial automorphisms of trees

The beginning of the article [18] formalizes a change of point of view, consisting in considering, not the finite binary trees, but their complements in the infinite binary tree.

Let $\mathcal{T}_2$ be the infinite binary rooted planar tree (all its vertices other than the root are 3-valent). Each finite binary rooted planar tree τ can be embedded in a unique way into $\mathcal{T}_2$, assuming that the embedding maps the root of τ onto the root of $\mathcal{T}_2$, and respects the orientation. Therefore, τ may be identified with a subtree of $\mathcal{T}_2$, whose root coincides with that of $\mathcal{T}_2$.

Definition 7 (cf. [22]). *A* partial isomorphism *of $\mathcal{T}_2$ consists of the data of two finite binary rooted subtrees τ_0 and τ_1 of $\mathcal{T}_2$ having the same number of leaves $n \in \mathbb{N}^*$, and an isomorphism $q : \mathcal{T}_2 \setminus \tau_0 \to \mathcal{T}_2 \setminus \tau_1$. The complements of τ_0 and τ_1 have n components, each one isomorphic to $\mathcal{T}_2$, which are enumerated from 1 to n according to the labeling of the leaves of the trees τ_0 and τ_1. Thus, $\mathcal{T}_2 \setminus \tau_0 = T_0^1 \cup \ldots \cup T_0^n$ and $\mathcal{T}_2 \setminus \tau_1 = T_1^1 \cup \ldots \cup T_1^n$ where the T_j^i's are the connected components. Equivalently, the partial isomorphism of $\mathcal{T}_2$ is given by a permutation $\sigma \in \mathfrak{S}_n$ and, for $i = 1, \ldots, n$, an isomorphism $q_i : T_0^i \to T_1^{\sigma(i)}$.*

Two partial automorphisms q and r can be composed if and only if the target of r coincides with the source of r. One gets the partial automorphism $q \circ r$. The composition provides a structure of inverse monoid on the set of partial automorphisms.

Let $\partial\mathcal{T}_2$ be the boundary of $\mathcal{T}_2$ (also called the set of “ends” of $\mathcal{T}_2$) endowed with its usual topology, for which it is a Cantor set. Although a partial automorphism does not act (globally) on the tree, it does act on its boundary. One has therefore a morphism from the monoid of partial isomorphism into the homeomorphisms of $\partial\mathcal{T}_2$, whose image N is the *spheromorphism group of Neretin* (see [26]).

Thompson's group V can be viewed as the subgroup of N which is the image of those partial automorphisms which respect the local orientation of the edges.

2.4 Generalizations following Brown and Stein

Brown considered in [8] similar groups $F_{n,r} \subset T_{n,r} \subset V_{n,r}$, extending previous work of Higman, which were defined as in the last two constructions above but using instead of binary trees forests of r copies of n-ary trees so that F, T, V correspond to $n = 2$ and $r = 1$. It appears that the isomorphism type of $V_{n,r}$ and $T_{n,r}$ only depends on $r \pmod n$ while $F_{n,r}$ depends only on n. These groups are finitely presented and of type FP_∞ according to [7] for the case of F and T and then ([8], thm. 4.17) for its extension to all other groups from this family. Moreover, Higman have proved (see [20]) $V_{n,r}$ has a simple subgroup of index g.c.d$(2, n-1)$, and this was extended by Brown who showed that F_n have simple commutator and $T_{n,r}$ have simple double commutator groups (see [8] for more details and refinements).

One can obtain these groups also by considering n-adic piecewise affine homeomorphisms (or bijections) of [0,r] (with identified endpoints for $T_{n,r}$) i.e. having singularities in $\mathbb{Z}\left[\frac{1}{n}\right]$ and derivatives in $\{n^a, a \in \mathbb{Z}\}$. This point of view was taken further by Stein in [30]. Specifically, given a multiplicative subgroup $P \subset \mathbb{R}$, a $\mathbb{Z}[P]$-submodule $A \subset \mathbb{R}$ satisfying $P \cdot A = A$, and a positive $r \in A$, one can consider the group $F_{A,P,r}$ of those PL homeomorphisms of $[0, r]$ with finite singular set in A and all slopes in P. There are similar families $T_{A,P,r}$ and $V_{A,P,r}$. Brown and Stein proved that $F_{\mathbb{Z}\left[\frac{1}{n_1 n_2 \cdots n_k}\right], \langle n_1, n_2, \ldots, n_k\rangle, r}$ is finitely presented of FP_∞ type. Furthermore $F_{A,P,r}$ and $V_{A,P,r}$ have simple commutator subgroups, while $T_{A,P,r}$ have simple second commutator subgroup.

2.5 Mapping class groups of infinite surfaces and braided Thompson groups

Let $\mathscr{S}_{0,\infty}$ be the oriented surface of genus zero, which is the following inductive limit of compact oriented genus zero surfaces with boundary $\mathscr{S}_n$. Starting with a cylinder $\mathscr{S}_1$, one gets $\mathscr{S}_{n+1}$ from $\mathscr{S}_n$ by gluing a pair of pants (i.e. a three-holed sphere) along each boundary circle of $\mathscr{S}_n$. This construction yields, for each $n \geq 1$, an embedding $\mathscr{S}_n \hookrightarrow \mathscr{S}_{n+1}$, with an orientation on $\mathscr{S}_{n+1}$ compatible with that of $\mathscr{S}_n$. The resulting inductive limit (in the topological category) of the $\mathscr{S}_n$'s is the surface $\mathscr{S}_{0,\infty}$:

$$\mathscr{S}_{0,\infty} = \lim_{\vec{n}} \mathscr{S}_n$$

By the above construction, the surface $\mathscr{S}_{0,\infty}$ is the union of a cylinder and of countably many pairs of pants. This topological decomposition of $\mathscr{S}_{0,\infty}$ will be called the *canonical pair of pants decomposition.*

The set of isotopy classes of orientation-preserving homeomorphisms of $\mathscr{S}_{0,\infty}$ is an *uncountable* group. By restricting to a certain type of homeomorphisms (called asymptotically rigid), we shall obtain countable subgroups.

Any connected and compact subsurface of $\mathscr{S}_{0,\infty}$ which is the union of the cylinder and finitely many pairs of pants of the canonical decomposition will be called an *admissible subsurface* of $\mathscr{S}_{0,\infty}$. The *type* of such a subsurface S is the number of connected components in its boundary.

Definition 8 (following [22, 15]). *A homeomorphism φ of $\mathscr{S}_{0,\infty}$ is* asymptotically rigid *if there exist two admissible subsurfaces S_0 and S_1 having the same type, such that $\varphi(S_0) = S_1$ and whose restriction $\mathscr{S}_{0,\infty} \setminus S_0 \to \mathscr{S}_{0,\infty} \setminus S_1$ is rigid, meaning that it maps each pants (of the canonical pants decomposition) onto a pants.*

The asymptotically rigid mapping class group *of $\mathscr{S}_{0,\infty}$ is the group of isotopy classes of asymptotically rigid homeomorphisms.*

The *asymptotically rigid mapping class group* of $\mathscr{S}_{0,\infty}$ is a finitely presented group $\mathcal{B}$ (see [15]) which fits into the exact sequence:

$$1 \to P\mathcal{M}(\mathscr{S}_{0,\infty}) \to \mathcal{B} \to V \to 1.$$

Some very similar versions of the same group (using a Cantor disk instead of a Cantor sphere or a more combinatorial framework) were obtained independently by Brin ([5]) and Dehornoy ([11]). We will call any version of them as *braided Thompson groups.*

Notice that this extension of V splits over the subgroup $T \subset V$. In [22] one gave an explicit section over T, as follows. Let us choose an involutive homeomorphism j of $\mathscr{S}_{0,\infty}$ which reverses the orientation, stabilizes each pair of pants of its canonical decomposition, and has fixed points along lines which decompose the pairs of pants into hexagons. The surface $\mathscr{S}_{0,\infty}$ can be disconnected along those lines into two planar surfaces with boundary, one of which is called the *visible side* of $\mathscr{S}_{0,\infty}$, while the other is the *hidden side* of $\mathscr{S}_{0,\infty}$. The involution j maps the visible side of $\mathscr{S}_{0,\infty}$ onto the hidden side, and vice versa. The data consisting of the canonical pants decomposition of $\mathscr{S}_{0,\infty}$ together with the above decomposition into a visible and a hidden side is called the *canonical rigid structure* of $\mathscr{S}_{0,\infty}$.

The tree $\mathcal{T}_2$ may be embedded into the visible side of $\mathscr{S}_{0,\infty}$, as the dual tree to the pants decomposition. The *tree of S* is the trace of $\mathcal{T}_2$ on S. An isotopy class belongs to the image of the embedding if it may be represented by an asymptotically rigid homeomorphism of $\mathscr{S}_{0,\infty}$ which globally preserves the decomposition into visible/hidden sides. The visible side of $\mathscr{S}_{0,\infty}$ is a planar surface which inherits from the canonical decomposition of $\mathscr{S}_{0,\infty}$ a decomposition into hexagons (and one rectangle, corresponding to the visible side of the cylinder). We could restate the above definitions by replacing pairs of pants by hexagons and the surface $\mathscr{S}_{0,\infty}$ by its visible side. Then T is the *asymptotically rigid mapping class group* of the visible side of $\mathscr{S}_{0,\infty}$, namely the group of mapping classes of those homeomorphisms which map all but finitely many hexagons onto hexagons.

2.6 Brin's groups nV

A rather different direction was taken in the seminal paper [4] of Brin, where the author constructed a family of countable groups nV acting as homeomorphisms of the product of n-copies of the standard triadic Cantor, generalizing the group V which occurs for $n = 1$.

Let $I^n \subset \mathbb{R}^n$ denote the unit cube. A *numbered pattern* is a finite dyadic partition of I^n into parallelepipeds along with a numbering. A dyadic partition is obtained from the cube by dividing at each step of the process one parallelepiped into two equal halves by a cutting hyperplane parallel to one of the coordinates hyperplane.

One definition of nV is as the group of piecewise affine (not continuous!) transformations associated to pairs of numbered patterns. Given the numbered patterns $P = (L_1, L_2, \ldots, L_n)$ and $Q = (R_1, R_2, \ldots, R_n)$, we set $\varphi_{P,Q}$ for the unique piecewise affine transformation of the cube sending affinely each L_i into R_i and preserving the coordinates hyperplanes. Thus nV is the group of piecewise affine transformations of the form $\varphi_{P,Q}$, with P, Q running over the set of all possible dyadic partitions.

Another description is as a group of homeomorphisms of the product C^n of the standard triadic Cantor set C. Parallelepipeds in a dyadic partition correspond to a closed and open (clopen) subset of C^n. Every dyadic cutting hyperplane H subdividing some parallelepiped R into two halves determines a parallel shadow (open) parallelepiped in R whose width is one third of the width of R in the direction orthogonal to H. Notice then that the complement of the union of all shadow parallelepipeds is C^n. Every pattern $P = (R_1, R_2, \ldots, R_n)$ determines a numbered collection of parallelepipeds $X_P = (X(R_1), X(R_2), \ldots, X(R_n))$ whose complementary is the set of shadows parallelepipeds of those cutting hyperplanes used to built P. Then $A(R_i) = X(R_i) \cap C^n$ form a clopen partition of C^n. We further define for a pair of patterns P, Q as above the homeomorphism $h_{P,Q}$ of C^n is the unique one which sends affinely $A(L_i)$ into $A(R_i)$ so as to preserve the orientation in each coordinate. This amounts to say that $h_{P,Q}$ is the restriction to C^n of the piecewise affine transformation sending affinely $X(L_i)$ into $X(R_i)$ and and preserving the coordinates hyperplanes.

The groups nV are simple (see e.g. [4, 6]) and finitely presented (see [19]). The stabilizer at some $a \in C^n$ of the (germs of) homeomorphisms in nV is isomorphic to $\mathbb{Z}^{r(a)}$, where $r(a)$ is the number of rational coordinates of a. This implies that the groups nV are pairwise non-isomorphic (see [3] for details).

We could of course extend this construction to arbitrary products of central Cantor sets C_λ in the spirit of the Brown and Stein, as above.

3 Proof of general countability statements

3.1 Proof of Theorem 1

For the sake of simplicity we will only consider here $M = S^2$, but the proof below goes on for general surfaces without essential modifications.

We parameterize E by the curve $\gamma : [0,1] \to S^2$ and denote by $A \subset [0,1]$ the preimage of C, which is still a Cantor set. We may assume that $0 \in A$. Let $\varphi \in \mathrm{Diff}^k(S^2, C)$ and denote by $\xi(t) = \varphi \circ \gamma(t)$. We can compose φ by a $\mathcal{C}^k$-isotopy such that φ is identity in a neighborhood of the north pole of the sphere. Thus it will be enough to consider the similar problem for the disk B_2^2 of radius 2 in the plane. The image of the equator can be taken to be the unit circle.

Assume for the moment that A is just an infinite set without isolated points. The set of those t for which $\gamma(t) = \xi(t)$ is a closed subset of $[0,1]$ containing A and hence its closure $\overline{A}$. Let now $t_0 \in \overline{A}$. Then, since γ and ξ are differentiable at t_0 we have:

$$\dot{\gamma}(t_0) = \lim_{t \in A, t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \lim_{t \in A, t \to t_0} \frac{\xi(t) - \xi(t_0)}{t - t_0} = \dot{\xi}(t_0) \tag{10}$$

If φ is twice differentiable then the same argument shows that:

$$\ddot{\gamma}(t_0) = \ddot{\xi}(t_0) \tag{11}$$

Suppose that γ is parameterized by arc length, namely that $|\dot{\gamma}| = 1$. We can also take a chart in which γ is linear so that $\ddot{\gamma} = 0$.

Since φ is of class $\mathcal{C}^2$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $s_1, s_2 \in A$, with $|s_1 - s_2| < \delta$ we have:

$$1 - \varepsilon < \langle \dot{\gamma}(t), \dot{\xi}(t) \rangle \leq 1, \text{for all } t \in [s_1, s_2] \tag{12}$$

$$|\ddot{\xi}(t)| < \varepsilon, \text{for all } t \in [s_1, s_2] \tag{13}$$

Set further $\gamma_s(t) = (1-s)\gamma(t) + s\,\xi(t)$, for $t \in [s_1, s_2]$ and $s \in [0,1]$.

Lemma 1. *$\gamma_s|_{[s_1,s_2]}$ provides a $\mathcal{C}^k$-isotopy between the restrictions $\gamma|_{[s_1,s_2]}$ and $\xi|_{[s_1,s_2]}$ to the interval $[s_1, s_2]$.*

Proof. We have to prove that for any $s \in [0,1]$ the curve $\gamma_s|_{[s_1,s_2]}$ is simple. This follow immediately from the fact that whenever $\varepsilon < \frac{1}{3}$ we have:

$$\langle \dot{\gamma}_s(t), \dot{\gamma}(t) \rangle \geq 1 - s + s\langle \dot{\xi}(t), \dot{\gamma}(t) \rangle \geq 1 - \varepsilon s > 0 \tag{14}$$

for any $t \in [0,1]$, $s \in [0,1]$. □

Let now $I_\delta = \cup_{s_1,s_2 \in A; |s_1 - s_2| \leq \frac{\delta}{3}} [s_1, s_2] \subset [0,1]$. Let then

$$\eta(t) = \begin{cases} \xi(t), & \text{if } t \notin I_\delta; \\ \gamma(t), & \text{if } t \in I_\delta; \end{cases} \tag{15}$$

Lemma 2. *The curves ξ and η are isotopic.*

Proof. We prove that $\gamma_s|_{I_\delta}$ is such an isotopy between the two curves. From Lemma 1 it suffices to show that there are not intersections between the segments of curves $\gamma_s|_{[s_1,s_2]}$ and $\gamma_s|_{[s_3,s_4]}$, when $s_i \in A$ and $[s_1, s_2], [s_3, s_4] \subset I_\delta$.

Let $p = \gamma_s|_{[s_1,s_2]}(t_0)$ be a point on the first curve segment. We want to estimate the angle β of the Euclidean triangle with vertices $p, \gamma(s_1), \gamma(s_2)$ at $\gamma_s(s_1)$. We can write then:

$$\langle \gamma_s(t_0) - \gamma_s(s_1), \dot{\gamma}(0) \rangle = \int_0^{t_0 - s_1} \langle \dot{\gamma}_s(s_1 + x), \dot{\gamma}(0) \rangle \, dx = \int_0^{t_0 - s_1} 1 - s + s\langle \dot{\xi}(s_1 + x), \dot{\gamma}(0) \rangle \, dx \tag{16}$$

Then (3.1) implies:

$$|\gamma_s(t_0) - \gamma_s(s_1)| \cos(\beta) = \langle \gamma_s(t_0) - \gamma_s(s_1), \dot{\gamma}(0) \rangle \geq (t_0 - s_1)(1 - s\varepsilon) \tag{17}$$

On the other hand from (3.1) we derive

$$|\dot{\xi}(x) - \dot{\xi}(s_1)| \leq \varepsilon(x - s_1) \tag{18}$$

and then:

$$|\gamma_s(t_0) - \gamma_s(s_1)| \leq \int_0^{t - s_1} |\dot{\gamma}_s(x)| dx \leq \int_0^{t - s_1} (s|\dot{\xi}(x)| + (1 - s)) dx \leq t - s_1 + \frac{\varepsilon}{2}(t - s_1)^2 \tag{19}$$

From (17) we obtain

$$\cos(\beta) \geq \frac{1 - s\varepsilon}{1 + \frac{\varepsilon}{2}(t_0 + s_1)} \geq \frac{1 - \varepsilon}{1 + \varepsilon} \tag{20}$$

If we choose $\varepsilon = \frac{1}{3}$ then $\beta \in [-\frac{\pi}{3}, \frac{\pi}{3}]$.

Assume now the contrary of our claim, namely that there exists some intersection point p between $\gamma_s|_{[s_1,s_2]}$ and $\gamma_s|_{[s_3,s_4]}$. Up to a symmetry of indices we can assume that the Euclidean triangle with vertices at p and at $\gamma_s(s_1), \gamma_s(s_2)$ has the angle β at $\gamma_s(s_1)$ within the interval $[\frac{\pi}{2}, \pi)$. This contradicts our estimates for β. □

We assume now that $A = \cap_{j=1}^{\infty} A_j$ is the infinite nested intersection of the closed finite unions of intervals $A_j \supset A_{j+1} \supset \cdots$.

It follows from the description of I_δ that the curve η is compactly supported in the sense that it coincides with γ outside.

Lemma 3. *Assume that there exists an isotopy of class $\mathcal{C}^k$ between γ and ξ rel some closed set C. Then φ is $\mathcal{C}^k$-isotopic to the identity in $\mathrm{Diff}^k(S^2, C)$.*

Proof. Assume γ be the boundary of the northern hemisphere. We delete a small ball of radius from the south hemisphere and denote by B the closure of its complement in the sphere.

Observe that it is enough to prove the claim when γ and ξ are isotopic within B. In fact, if γ_s, $s \in [0,1]$ is an isotopy between them then we can cover the interval $[0,1]$ by finitely many small enough intervals J_j such that curves γ_s , with $s \in J$ all belong to some disk $B_j \subset S^2$.

Thus the claim reduces to the case when γ is the boundary of the unit disk in the plane. Define then:

$$\varphi_s(az) = \lambda(a)\gamma_s(t(z)), \text{ if } a \in [0,1], \ |z| = 1, \text{ so that } z = \gamma(t(z)), \text{for some } t(z) \in [0,1], \quad (21)$$

where $\lambda : [0,1] \to \mathbb{R}_+$ is a smooth function infinitely flat at 0 which is identically 1 outside a small open neighborhood of 0. Then φ_s is the desired isotopy. □

Remark 3. *Let $\mathcal{C}^k(S^1, S^2; C)$ be the set of non-parameterized $\mathcal{C}^k$* simple *curves on S^2 passing through C. The claim follows from the more general fact that the map $\mathrm{Diff}^k(S^2, C) \to \mathcal{C}^k(E, S^2; C)$ sending φ to ξ is a homotopy equivalence.*

3.2 Sparse sets and proofs of Theorems 2 and 3

3.2.1 Preliminaries

Let $\mathcal{N}_\varepsilon(a)$ denote the ε-neighborhood $|x - a| < \varepsilon$ of a in $\mathbb{R}$, $\dot{\mathcal{N}}_\varepsilon^\pm(a)$ the punctured right and left semi-neighborhoods of a, i.e., $a < x < a + \varepsilon$ and $a - \varepsilon < x < a$, respectively.

We say that a is a *left point* of C if there is a left semi-neighborhood $\dot{\mathcal{N}}^-(a)$ such that $\dot{\mathcal{N}}^-(a) \cap C = \varnothing$. In the same way we define *right points*.

For $a \in C$ denote by Diff_a^k the stabilizer of a in $\mathrm{Diff}^k(\mathbb{R}, C)$, and by $\mathfrak{diff}_a^k$ the group of germs of elements of the stabilizer of a in $\mathfrak{diff}^k(C)$. The superscript $+$ in $\mathrm{Diff}_a^{k,+}$ and $\mathfrak{diff}_a^{k,+}$ means that we only consider those diffeomorphisms that preserve the orientation of the interval, i.e. increasing.

Let φ be a diffeomorphism with $\varphi(a) = a$. We say that φ is *N-flat* at a if:

$$\varphi(x) - x = o\left((x - a)^N\right), \qquad \text{as} \quad x \to a. \quad (22)$$

Moreover φ is *flat* if it is N-flat for every $N \geq 0$.

Lemma 4. *Let $\varphi \in \mathrm{Diff}_a^1$ be 1-flat at a. Then $\varphi\Big|_C$ is the identity in a small neighborhood of a.*

Proof. We can assume without loss of generality that a is not a right point of C. Suppose that φ is nontrivial on $\mathcal{N}_\delta^+(a) \cap C$ for any $\delta > 0$.

We first claim that fixed points of φ accumulate from the right to a. Otherwise, there exists some δ such that $\varphi(x) - x$ keeps constant sign for all $x \in \mathcal{N}_\delta^+(a)$. Assume that this sign is positive and choose $b \in \mathcal{N}_\delta^+(a) \cap C$. Let $(\alpha, \beta) \subset (a, b)$ be a maximal complementary interval of length at least $\sigma(b - a)$. By maximality $\alpha \in C$. Since $\varphi(\alpha) \in C$ and $\varphi(\alpha) > \alpha$ we have $\varphi(\alpha) \geq \beta$, so that:

$$\frac{\varphi(\alpha) - \alpha}{\alpha - a} \geq \frac{\beta - \alpha}{\alpha - a} \geq \frac{\sigma(b - a)}{\alpha - a} \geq \sigma \quad (23)$$

But this inequality contradicts the 1-flatness condition $\frac{\varphi(\alpha) - \alpha}{\alpha - a} = o(1)$ for small δ. When the sign of $\varphi(x) - x$ is negative we reach the same conclusion by considering $\varphi(\beta) - \beta$. This proves the claim.

Therefore there is a decreasing sequence u_k accumulating on a, such that $\varphi(u_k) = u_k$. As $\varphi\Big|_{C \cap \mathcal{N}_\delta^+(a)}$ is not identity for any $\delta > 0$ there exists a decreasing sequence $v_k \in C$ accumulating on a, such that all $\varphi(v_k) - v_k$ are of the same sign, say positive. Therefore, up to passing to a subsequence, we obtain a sequence of disjoint intervals (α_j, β_j) such that $\beta_{j+1} \leqslant \alpha_j$, $\varphi(\alpha_j) = \alpha_j$, $\varphi(\beta_j) = \beta_j$, and $v_j \in (\alpha_j, \beta_j)$.

Since φ is monotone, it has to be monotone increasing, by above. Thus $\varphi^k(v_j) \in [\alpha_j, \beta_j]$, for any $k \in \mathbb{Z}$, where φ^k denotes the k-th iterate of φ. The bi-infinite sequence $\varphi^k(v_j)$ is increasing and so:

$$\alpha_j \leq \lim_{k \to -\infty} \varphi^k(v_j) < \lim_{k \to \infty} \varphi^k(v_j) \leq \beta_j \quad (24)$$

Now $\lim_{k \to -\infty} \varphi^k(v_j)$ and $\lim_{k \to \infty} \varphi^k(v_j)$ are fixed points of φ and we can assume, without loss of generality that our choice of intervals is such that $\alpha_j = \lim_{k \to -\infty} \varphi^k(v_j)$, $\lim_{k \to \infty} \varphi^k(v_j) = \beta_j$. In particular $\alpha_j, \beta_j \in C$.

As C is σ-sparse there is a complementary interval $(\gamma_j, \delta_j) \subset (\alpha_j, \beta_j)$ of length at least $\sigma(\beta_j - \alpha_j)$. The interval (γ_j, δ_j) cannot contain any point $\varphi^k(v_j)$ and thus there exists some $k_j \in \mathbb{Z}$ such that

$$\varphi^{k_j}(v_j) \leq \gamma_j < \delta_j \leq \varphi^{k_j+1}(v_j). \tag{25}$$

Denote $\varphi^{k_j}(v_j) = \eta_j$. We have then

$$\frac{\varphi(\eta_j) - \varphi(\alpha_j)}{\eta_j - \alpha_j} - 1 = \frac{\varphi(\eta_j) - \eta_j}{\eta_j - \alpha_j} \geq \frac{\sigma(\beta_j - \alpha_j)}{\eta_j - \alpha_j} \geq \sigma, \tag{26}$$

By the mean value theorem there exists $\xi_j \in (\alpha_j, \eta_j)$ such that

$$\frac{\varphi(\eta_j) - \varphi(\alpha_j)}{\eta_j - \alpha_j} = \varphi'(\xi_j). \tag{27}$$

and thus such that $\varphi'(\xi_j) \geq 1 + \sigma$. As φ' is continuous at a, by letting j go to infinity we derive $\varphi'(a) \geq 1 + \sigma$ which contradicts the 1-flatness. □

Lemma 5. *If C is σ-sparse and $\varphi \in \mathrm{Diff}^1_a$ is not 1-flat then*

$$|\varphi'(a) - 1| \geq \sigma. \tag{28}$$

Proof. By Lemma 4 we can assume that there exists some $\varphi \in \mathrm{Diff}^1_a$ which is not 1-flat, so that $\varphi'(a) \neq 1$. Let us further suppose that $\varphi'(a) > 1$, the other situation being similar. For arbitrary δ we can choose $b \in \mathcal{N}^+_\delta(a) \cap C$. There is then a maximal complementary interval $(\alpha, \beta) \subset (a, b)$ of length at least $\sigma(b - a)$. By maximality $\alpha \in C$.

We claim that for small enough δ we have $\varphi(\alpha) > \alpha$. Assume the contrary. By the mean value theorem there exists $\xi \in (a, \alpha) \subset (a, b)$ such that

$$\varphi'(\xi) = 1 + \frac{\varphi(\alpha) - \alpha}{\alpha - a} \leq 1 \tag{29}$$

and letting δ go to 0 we would obtain $\varphi'(a) \leq 1$, contradicting our assumptions. Thus $\varphi(\alpha) > \alpha$, and hence $\varphi(\alpha) \geq \beta$. As above, the mean value theorem provides us $\xi \in (a, \alpha)$ so that

$$\varphi'(\xi) = 1 + \frac{\varphi(\alpha) - \alpha}{\alpha - a} \geq 1 + \delta \tag{30}$$

Letting δ go to zero we obtain $\varphi'(a) \geq 1 + \sigma$. When $\varphi'(a) < 1$ we can use similar methods or pass to φ^{-1} in order to obtain $\varphi'(a) \leq 1 - \sigma$. □

Lemma 6. *Let $a \in C$. Then one of the following alternative holds:*

1. *either for any $\varphi \in \mathrm{Diff}^1_a$, the restriction $\varphi\Big|_C$ is identity in a small neighborhood of a, so that $\mathfrak{diff}^1_a = 1$;*
2. *or else, there is $\psi_a \in \mathrm{Diff}^1_a$ such that for any $\varphi \in \mathrm{Diff}^1_a$ the restriction of φ to a small neighborhood $\mathcal{N}_\delta(a) \cap C$ coincides with the iterate $\psi^k\Big|_C$ for some $k \in \mathbb{Z} \setminus \{0\}$. Moreover, any such ψ_a is of the form:*

$$\psi_a(x) = a + p(x - a) + o(x - a), \qquad \text{as} \quad x \to a, \tag{31}$$

where $|p - 1| \geq \sigma$. Thus $\mathfrak{diff}^1_a = \mathbb{Z}$.

Proof. By Lemma 4 we can assume that there exists some $\varphi \in \mathrm{Diff}^1_a$ which is not 1-flat.

The map $\chi : \mathrm{Diff}^1_a \to \mathbb{R}^*$ given by $\chi(\varphi) = \varphi'(a)$ is easily seen to be a group homomorphism. By Lemma 5 the subgroup $\chi(\mathrm{Diff}^1_a)$ of $\mathbb{R}^*$ is discrete and non-trivial and thus it is isomorphic to $\mathbb{Z}$.

The kernel of χ consists of those $\varphi \in \mathrm{Diff}^1_a$ which are 1-flat. By Lemma 4 the germ at a of the restriction of φ to C is identity. □

Remark 4. *If $C = C_\lambda$ is the ternary central Cantor set in $\mathbb{R}$, then from the proof of theorem 6 we derive that $\mathfrak{Diff}^1_a(C_\lambda)$ is not always $\mathbb{Z}$. An element a of C_λ is said to be λ-rational if it has an eventually periodic developpement*

$$a = \sum_{i=1}^{\infty} a_i \lambda^i,$$

where $a_i \in \{0, \lambda - 1\}$. Therefore $\mathfrak{Diff}^1_a(C_\lambda)$ is $\mathbb{Z}$ if and only if a is λ-rational and trivial, otherwise.

3.2.2 End of proof of theorem 2

We need to show that the identity is an isolated point of the group $\mathfrak{Diff}^1(C)$, if C is σ-sparse. To this purpose consider an element $\mathfrak{Diff}^1(C)$ having a representative in $\psi \in \mathrm{Diff}^1(\mathbb{R}, C)$ satisfying

$$1 - \sigma < \psi'(x) < 1 + \sigma, \quad \text{for} \quad \text{any} \quad x \in C. \tag{32}$$

We consider the subgroup $\mathrm{Diff}^{1,+}(\mathbb{R}, C)$ of index 2 in $\mathrm{Diff}^1(\mathbb{R}, C)$ consisting of those diffeomorphisms preserving the orientation of $\mathbb{R}$. There is no loss of generality in assuming that $\psi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C)$. The elements of $\mathrm{Diff}^{1,+}(\mathbb{R}, C)$ are monotone increasing. The minimal element $\min C$ of C should therefore be fixed by any element of $\mathrm{Diff}^{1,+}(\mathbb{R}, C)$, in particular by ψ. By Lemma 5, $\psi \in \mathrm{Diff}^1_{\min C}(\mathbb{R}, C)$ must be 1-flat at $\min C$.

Consider the set

$$U = \{x \in C; \psi(z) = z \quad \text{for} \quad \text{any} \quad z \in C \cap (-\infty, x]\}. \tag{33}$$

The set U is nonempty, as $\min C \in U$. Let $\xi = \sup U$.

Assume first that ξ is not a right point of C. Since ψ is continuous $\xi \in U$, so that $\psi \in \mathrm{Diff}^1_\xi$. From Lemma 5 $\psi'(\xi) = 1$ and ψ is 1-flat at ξ. According to Lemma 4 there is some $\delta > 0$ such that the restriction $\psi\Big|_{C \cap \mathcal{N}^+_\delta(\xi)}$ is the identity, which contradicts the maximality of ξ.

If ξ is a right point of C, then there is some maximal complementary interval $(\xi, \eta) \subset \mathbb{R} \setminus C$. Since $\psi\Big|_{C \cap [\min C, \xi]}$ is identity it follows that $\psi(C \cap [\xi, \infty)) \subset C \cap [\xi, \infty)$. As η is the minimal element of $C \cap [\xi, \infty)$ it should be a fixed point of $\psi\Big|_{[\xi, \infty)}$ and so $\eta \in U$. This contradicts the maximality of ξ. Hence ψ is identity on C.

3.2.3 Proof of Corollary 2

Let V_δ be the set of those elements in $\mathfrak{Diff}_{S^1}(C)$ having a representative $\psi \in \mathrm{Diff}^1(S^1, C)$ such that

$$1 - \delta < \psi'(x) < 1 + \delta, \quad \text{for} \quad \text{any} \quad x \in C. \tag{34}$$

Here elements of $\mathrm{Diff}^1(S^1)$ are identified with real periodic functions on $\mathbb{R}$. We choose $\delta < \min(\sigma, 0.3)$. It is enough to prove that V_δ is finite.

Consider a complementary interval $J \subset S^1 - C$ of maximal possible length, say $|J|$. Consider its right end η, with respect to the cyclic orientation. If $\psi \in V_\delta$ is such that $\psi(\eta) = \eta$, then the arguments from the proof of Theorem 2 show that $\psi(x) = x$ when $x \in C$.

We claim that the set of intervals of the form $\psi(J)$, for $\psi \in V_\delta$ is finite. Each $\psi(J)$ is a maximal complementary interval, because if it were contained in a larger interval J', then $\psi^{-1}(J)$ would be a complementary interval strictly larger than J. This shows that any two such intervals $\psi(J)$ and $\varphi(J)$ are either disjoint or they coincide, for otherwise their union would contradict their maximality. Further, each $\psi(J)$ has length at least $(1 - \delta)|J|$. This shows that the set of intervals is a finite set $\{J_1, J_2, \ldots, J_k\}$.

Assume that $\psi(J) = \varphi(J)$ and both ψ and φ preserve the orientation of the circle. If the right end of J is η, with respect to the cyclic orientation, then $\varphi \circ \psi^{-1}$ sends J to J and hence fixes η. Then the arguments from the proof of Theorem 2 show that $\varphi \circ \psi^{-1}(x) = x$ when $x \in C$. It follows that there are at most $2k$ elements in V_δ, finishing the proof of the first part.

For the second part it suffices to remark that $\mathfrak{Diff}_M(C)$ can be embedded in a product of $\mathfrak{Diff}_{S^1}(C)$, up to finite index.

3.2.4 Proof of Theorem 3

Let C be a Cantor set contained within a simple closed curve L on the orientable manifold M. For the sake of simplicity we will suppose from now on that M is a surface, but the proof goes on without essential modifications in higher dimensions. Let φ be a diffeomorphism of M sending C into C. Fix a parameterization of a collar N such that (N, L) is identified with $(L \times [-1,1], L \times \{0\})$. Denote by $\pi : N \to L$ the projection on the first factor and by $h : N \to [-1,1]$ the projection on the second factor.

There exists an open neighborhood U of C in L so that $\varphi(U) \subset N$. In particular, the closure $\overline{U}$ is a finite union of closed intervals. The map $\varphi : \overline{U} \to N = L \times [-1,1]$ has the property $h \circ \varphi(a) = 0$, for each $a \in C$. Therefore the differential $D_a\,(h \circ \varphi) = 0$, for each $a \in C$. Since φ is a diffeomorphism $D_a(\pi \circ \varphi) \neq 0$, for every $a \in C$.

For each $a \in C$ consider an open interval neighborhood U_a within L, so that $D_x(\pi\circ\varphi) \neq 0$ and $|D_x\,(h\circ\varphi)| < 1$, for every $x \in \overline{U_a}$. We obtain an open covering $\{U_a; a \in C\}$ of C. As C is compact there exists a finite subcovering by intervals $\{U_1, U_2, \ldots, U_n\}$. Without loss of generality one can suppose that $\underline{U_j} \subset U$, for all j. We consider such a covering having the minimal number of elements. This implies that $\overline{U_j}$ are disjoint intervals.

For every j the map $\pi\Big|_{\varphi(\overline{U_j})} : \varphi(\overline{U_j}) \to \pi(\varphi(\overline{U_j})) \subset L$ is a diffeomorphism on its image, since $\varphi(\overline{U_j})$ is connected and $D_x(\pi \circ \varphi) \neq 0$, for any $x \in \overline{U_j}$.

Consider a slightly smaller closed interval $I_j \subset U_j$ such that $I_j \cap C = U_j \cap C$.

Let μ be a positive smooth function on $\sqcup_{j=1}^n \overline{U_j}$ such $\mu(t)$ equals 1 near the boundary points and vanishes on $\sqcup_{j=1}^n I_j$. Define $\phi_s : \sqcup_{j=1}^n \overline{U_j} \to N$ by:

$$\phi_s(x) = (\pi \circ \varphi(x), (s\mu(x) + 1 - s) \cdot h \circ \varphi(x)) \tag{35}$$

Then $\phi_0(x) = \varphi(x)$ and for each $s \in [0,1]$ we have $\phi_s(x) = \varphi(x)$, for x near the boundary points of $\sqcup_{j=1}^n \overline{U_j}$. Furthermore $\phi_1(x) = \pi \circ \varphi(x) \in L$, when $x \in \sqcup_{j=1}^n I_j$. One should also notice that $\psi_s(x) = \varphi(x)$, for each $x \in C$ and $s \in [0,1]$.

Let now denote $J_j = \pi \circ \varphi(I_j)$. It is clear that $C = \varphi(C) \subset \cup_{j=1}^n J_j$. We claim that we can assume that J_j are disjoint. Indeed, since φ is bijective we have $\varphi(I_j \cap C) \cap \varphi(I_k \cap C) = \emptyset$, for any $j \neq k$. Since $\varphi(I_j \cap C)$ sets are closed subsets of L there exists $\varepsilon > 0$ so that $d(\varphi(I_j \cap C), \varphi(I_k \cap C)) \geq \varepsilon$, for $j \neq k$, where d is a metric on L. Since $\phi_1(I_j \cap C) = \varphi(I_j \cap C)$, we have $d(\phi_1(I_j \cap C), \phi_1(I_k \cap C)) \geq \varepsilon$, for $j \neq k$. Thus there exists some open neighborhoods J'_j of $\phi_1(I_j \cap C)$ within L so that $d(J'_j, J'_k) \geq \frac{1}{2}\varepsilon$, for $j \neq k$. As ϕ_1 is a diffeomorphism there exists an open neighborhood I'_j of $I_j \cap C$ with the property that $\phi_1(I'_j) \subset J'_j$, for all j. Now I'_j and J'_j are finite unions of open intervals. We can replace them by closed intervals with the same intersection with C. This produces two new families of disjoint closed intervals related by ϕ_1, as the initial situation. This proves the claim.

We obtained that there exist two coverings $\{I_1, I_2, \ldots, I_n\}$ and $\{J_1, J_2, \ldots, J_n\}$ of C by disjoint closed intervals and a diffeomorphism $\phi_1 : \sqcup_{j=1}^n I_j \to \sqcup_{j=1}^n J_j$ such that $\phi_1(x) = \varphi(x)$, for any $x \in C$.

Notice that the sign of $D_a(\pi \circ \varphi)$ might not be the same for all intervals.

Every partition of C induced by a covering $\{I_1, I_2, \ldots, I_n\}$ as above is determined by the choice of complementary intervals, namely the $n-1$ connected components of $L \setminus \cup_{j=1}^n I_j$. It follows that there are only countably many finite partitions of C of the type considered here. Next, the set of those elements of $\mathfrak{Diff}_M(C)$ which arise from partitions induced by the coverings $\{I_1, I_2, \ldots, I_n\}$ and $\{J_1, J_2, \ldots, J_n\}$ of C is acted upon transitively by the stabilizer of one partition. The stabilizer of one partition embeds into the product of $\mathfrak{Diff}^1_{I_j}(C \cap I_j)$. Theorem 2 then implies that $\mathfrak{Diff}^1_M(C)$ is countable.

3.2.5 Proof of Theorem 4

Before to proceed we need some preparatory material. Let $A \subset \mathbb{R}^n$ be a set without isolated points. Let $T_p\mathbb{R}^n$ denote the tangent space at p on $\mathbb{R}^n$ and $UT_p\mathbb{R}^n \subset T_p\mathbb{R}^n$ the sphere of unit vectors. For any $p \in A$ one defines the *unit tangent spread* $UT_pA \subset UT_p\mathbb{R}^n$ at p as the set of vectors $v \in UT_p\mathbb{R}^n$ for which there exists a sequence of points $a_i \in A$ with $\lim_{i\to\infty} a_i = p$ and

$$\lim_{i\to\infty} \frac{a_i - p}{|a_i - p|} = v.$$

Vectors in UT_pA will also be called *(unit) tangent vectors* at p to A. We also set $T_pA = \mathbb{R}_+ \cdot UT_pA \subset T_pA$ for the tangent spread at p.

A differentiable map $\varphi : (R^n, A) \to (R^n, B)$ induces a tangent map $T_p\varphi : T_pA \to T_{\varphi(p)}B$. Specifically, let $D_p\varphi : T_p\mathbb{R}^n \to T_{\varphi(p)}\mathbb{R}^n$ be the differential of φ; then we have

$$T_p\varphi = U(D_p\varphi)$$

where for a linear map $L : V \to W$ between vector spaces we denoted by $U(L) : U(V) \to U(W)$ the map induced on the unit spheres, namely

$$U(L)v = \frac{L(v)}{\parallel L(v) \parallel}.$$

As the unit tangent spread UT_pA is a subset of the unit sphere, it inherits the spherical geometry and metric. In particular, it makes sense to consider the convex hull $Hull(UT_pA) \subset UT_p\mathbb{R}^n$ in the sphere.

Although tangent spreads to product Cantor sets might depend on the particular factors, their convex hulls have a simple description. Let $C = C_1 \times C_2 \times \cdots \times C_n \subset \mathbb{R}^n$ be a product of Cantor sets $C_i \subset \mathbb{R}$. The usual cubical complex underlying the n-dimensional cube $[0,1]^n$ will be denoted by $\square^n$. Let then denote by $Lk(p)$ the spherical link of $p \in \square^n$. If p belongs to a k-dimensional cube but not to a $k+1$-dimensional cube of $\square^n$ then $Lk(p)$ is isometric to the link $\mathcal{L}_{k,n}$ of the origin in $\mathbb{R}^k \times \mathbb{R}_+^{n-k}$. Thus there are precisely $n+1$ different isometry types of links of points.

Now a direct inspection shows that for each $p \in C$ there exists some k so that the convex hull $Hull(UT_pA)$ is isometric to $\mathcal{L}_{k,n}$.

When the diffeomorphism $\varphi : (\mathbb{R}^n, C) \to (\mathbb{R}^n, C)$ is also *conformal*, then the tangent maps are isometries between the unit tangent spreads, because the spherical distance is given by angles between the corresponding vectors. However this is not true for general diffeomorphisms.

Nevertheless the spherical links $\mathcal{L}_{k,n}$ are quite particular. There exist $n+k$ vectors along the coordinates axes which are extremal points of UT_pC, such that their convex hull is $Hull(UT_pC)$, so isometric to $\mathcal{L}_{k,n}$. These are vectors of the form $e_i, -e_i, e_j$, where e_i correspond to the coordinates axes in $\mathbb{R}^k$ and e_j to those in $\mathbb{R}^{n-k}$. Now, any diffeomorphism $\varphi : (\mathbb{R}^n, C) \to (\mathbb{R}^n, C)$ should send an unit tangent spread of type $\mathcal{L}_{k,n}$ into one of the same type, since $\mathcal{L}_{k,n}$ is not affinely equivalent to $\mathcal{L}_{k',n}$, for $k \neq k'$. Moreover, the extremal vectors are sent into extremal vectors of the same type.

Let further φ be such that $\|D_a\varphi - \mathbf{1}\| \leq \varepsilon$ for all $a \in C$. Assume now that the unit tangent spread UT_aC is isometric to $\mathcal{L}_{0,n}$, namely it is of *corner* type. In this case $U(D_a\varphi)$ should permute the n coordinate vectors, which are the extremal vectors of $\mathcal{L}_{0,n}$. Therefore either $U(D_a\varphi) = \mathbf{1}$, or else

$$\|U(D_a\varphi) - \mathbf{1}\| \geq \sqrt{2},$$

which yields

$$\|D_a\varphi - \mathbf{1}\| \geq \sqrt{2}.$$

In other words, taking $\varepsilon < \sqrt{2}$ any diffeomorphism φ as above should satisfy $U(D_a\varphi) = \mathbf{1}$. Now, if φ is of class $\mathcal{C}^1$ then $U(D_a\varphi)$ is continuous. Since the set of corner points is dense in C we derive $U(D_a\varphi) = \mathbf{1}$, for any $a \in C$. This is the same as saying that for any $a \in C$ the linear map $D_a\varphi$ is represented by a diagonal matrix, with respect to the standard coordinate system of $\mathbb{R}^n$.

Proposition 2. *Let $a \in C$ be a corner point. The map $\chi : \mathfrak{diff}^1_{\mathbb{R}^n,a}(C) \to (\mathbb{R}^*)^n$, which associates to the germ φ the eigenvalues of $D_a\varphi$ is an isomorphism onto a discrete subgroup of $(\mathbb{R}^*)^n$.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be the standard coordinates functions on $\mathbb{R}^n$ and $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denote the projection onto the hyperplane $H_j = \{x_j = 0\}$. For the sake of simplicity we assume that $a = (0, 0, \ldots, 0)$, and that the convex hull of the unit tangent spread is the union of of the sets $H_j^+ = H_j \cap \{x_i \geq 0, i = 1, \ldots, n\}$. We will use induction on n. The claim was proved in Lemma 6 for $n = 1$. Assume it holds for all dimensions at most $n-1$.

Let $\varphi \in \mathrm{Diff}^1(\mathbb{R}^n, C)$ such that $\varphi(a) = a$. Assume that $\|D_x\varphi - \mathbf{1}\| < \frac{1}{2}\sigma < \frac{1}{2}$ for all x in a neighborhood V of a in $\mathbb{R}^n$. We will prove that $\varphi\Big|_C$ is a trivial germ at a. This shows that the image of χ is a discrete subgroup of $(\mathbb{R}^*)^n$ and the kernel of χ is trivial.

Consider the maps $\varphi_j : H_j \to H_j$ given by $\varphi_j(x) = \pi_j \circ \varphi(x)$. The determinant of $D_a\varphi_j$ is the product of all eigenvalues of $D_a\varphi$ but the j-th eigenvalue, and hence it is non-zero. Moreover, we have $\|D_a\varphi_j - \mathbf{1}\| < \frac{1}{2}\sigma$. We claim that

Lemma 7. *The map* $\varphi_j : H_j \cap V \to H_j$ *is injective.*

Proof. Assume the contrary, namely that there exist two points $p, q \in H_j \cap V$ such that $\pi_j(\varphi(p)) = \pi_j(\varphi(q))$. Consider the first non-trivial case $n = 2$, when H_j^+ are half-lines issued from a. The mean value theorem and the previous equality proves that there exists some $\xi \in H_j^+ \cap V$ between p and q so that $(\pi_j \circ \varphi)'(\xi) = 0$. This amounts to the fact that the image of $D_\xi\varphi$ is contained in the kernel of $D_{\varphi(\xi)}\pi_j$, namely that

$$\langle D_\xi\varphi(v_j), v_j\rangle = 0,$$

where v_j is an unit tangent vector at H_j^+ at ξ. We derive $\|D_\xi\varphi_j - \mathbf{1}\| \geq 1$, contradicting our assumptions.

In the general case $n > 2$ we will use a trick to reduce ourselves to $n = 2$, because we lack a multidimensional mean value theorem. Let P a generic affine 2-dimensional half-plane whose boundary line passes through p and q . We can find arbitrarily small $\mathcal{C}^1$-isotopy deformations ψ of φ_j so that $\psi(H_j)$ is transversal to P and $\|D_a\psi - \mathbf{1}\| < \sigma$. It follows that $\psi(H_j) \cap P$ is a 1-dimensional manifold Z with boundary containing both p and q. Now either there exist two distinct points of the boundary ∂Z joined by an arc within Z, or else there is an arc of Z issued from p which returns to p, contradicting the transversality of the intersection $\psi(H_j) \cap P$. In any case there exists the mean value argument above shows that it should exist a point $\psi(\xi)$ of Z for which the tangent vector v is orthogonal to H_j. We can write $v = D_\xi\psi(w)$, for some tangent vector $w \in H_j$ at ξ. It follows that

$$\langle D_\xi\psi(w), w\rangle = 0,$$

which implies $\|D_\xi\psi - \mathbf{1}\| \geq 1$, contradicting our assumptions.

□

It follows that $\varphi_j : H_j \cap V \to H_j$ is an injective map of maximal rank in a neighborhood V of a, and hence a diffeomorphism on its image. The projection π_j sends C into $C \cap H_j$, so that

$$\varphi_j(C \cap H_j \cap V) \subset C \cap \varphi(H_j \cap V) \subset C \cap H_j.$$

Our aim is to use the induction hypothesis for φ_j. In order to do that we need to show that the class of φ_j defines indeed an element of $\mathfrak{Diff}^1_{\mathbb{R}^{n-1},a}(C)$, where we identified H_j with $\mathbb{R}^{n-1}$.

We assume from now on that the neighborhood V is a parallelepiped, all whose vertices being corner points. Its boundary ∂V will consist then in the union of the faces $V_j = \partial V_j \cap H_j^+$s with their respective parallel faces V_j'. The parallelepiped V is surrounded by gaps, whose smaller width is some $\delta > 0$. Let V^δ be the δ-neighborhood of V. If φ is Lipschitz with Lipschitz constant $1 + \varepsilon$ and

$$(1 + \varepsilon)l_i < \delta + l_i$$

where l_i are the edges lengths of V then the image $\varphi(V)$ is contained in V^δ, so that $\varphi_j(V_j) \subset V^\delta \cap H_j$.

Further $\varphi_j(\partial V_j)$ bounds $\varphi_j(V_j)$ and thus there are no points of $C \cap H_j$ accumulating on $\varphi_j(V_j)$, as their unit tangent spread cannot be of the type $\mathcal{L}_{n-1,n-1}$. Thus $C - \varphi_j(V_j)$ is a closed subset of C and hence its distance to $\varphi_j(V_j)$ is strictly positive. There exists then an open set $U \subset V^\delta$ which contains $\varphi_j(V_j)$ and $U \cap (C - \varphi_j(V_j)) = \emptyset$. It follows that there exists an extension of φ_j to a diffeomorphism Φ_j of (H_j, C) which is identity outside U , and hence on $V_\delta \cap H_j \cup (C - \varphi(V_j))$.

It only remains to check that $\Phi_j^{-1}(C)$ is also contained in C, as needed for $\Phi_j \in \mathrm{Diff}^1(\mathbb{R}^{n-1}, C)$. This follows from the following:

Lemma 8. *The map* φ_j *has the property*

$$\varphi_j(C \cap V_j) = C \cap \varphi(V_j).$$

Proof. Assume that there exists some point p in $\varphi(V_j)\cap C$ which does not belong to V_j. Then the line issued from p which is orthogonal to V_j intersects $\varphi(V_j)$ only once, from Lemma 7. On the other hand there are points of C on this line, as C is a product and $p \notin V_j$. By Jordan's theorem there exist points of C which belong to different components of $R^n - \varphi(\partial V)$ which contradicts the fact that φ is surjective on C.

Thus $\varphi(C\cap V_j) \subset C \cap V_j$. The same argument for φ^{-1} yields the opposite inclusion and hence $\varphi(C\cap V_j) = C \cap V_j$. Our claim follows. □

Lemma 8 tells us that φ_j defines a germ in $\mathfrak{Diff}^1_{H_j,a}(C\cap H_j)$, namely both φ_j and φ_j^{-1} sends $C\cap H_j$ into itself. By the induction hypothesis $\varphi_j\Big|_{C\cap H_j}$ must be identity in a neighborhood of a within H_j.

Notice that this implies already that $D_a\varphi = \mathbf{1}$, and hence establishing the first claim of the proposition.

For the second claim we consider the distance $d(C-V,V)=\mu>0$, as V is surrounded by gaps. We suppose further that

$$\|D_x\varphi - \mathbf{1}\| < \min(\sigma/2, \frac{\mu}{1+\sigma}).$$

We know that $\varphi(y,0)=(y,u(y))$, for $y\in C\cap V\cap H_n$ and some function $u\geq 0$. The next step is to show that $u\Big|_{C\cap V\cap H_n} = 0$.

Assume that there exists some $x\in C\cap V\cap H_n$ so that $u(x)>0$. Observe that $u(x)\in C_n$, since $\varphi(C)\subset C$. Since points of C_n which are not endpoints are dense in C_n there should exist $x\in C$ for which $u(x)$ is not an endpoint of C_n. Set $z=(x,u(x))\in C$.

Then for each $\nu>0$ there exist points $z_+, z_- \in C$ with $\pi_n(z_+) = \pi_n(z_-) = x$, so that the distances $d(z_+,z), d(z_-,z) < \nu$.

Observe that the segment z_+z_- intersects just once $\varphi(H_n^+)$, namely at z. One might expect to use Jordan's theorem in order to derive that $z_+\in C$ and $z_-\in C$ could not belong to the same connected component of $\varphi(\partial V)$. This is not exactly true, as the segment z_+z_- could possibly intersect other sheets $\varphi(H_j^+)$ or $\varphi(H_j^{++})$ which are part of $\varphi(\partial V)$.

Set r for the distance between $x\in H_n^+$ and the union of the other $2n-1$ faces $\cup_{j=1}^{n-1} H_j^+ \cup_{i=1}^{n} H_i^{++}$ of ∂V. By the induction hypothesis we can assume that $r>0$. Choose now ν so that $\nu < \min((1-\sigma)r/2, \mu(1-\sigma)/2)$.

Suppose that there exists $x_+, x_- \in C\cap V$ such that $\varphi(x_+) = z_+$ and $\varphi(x_-) = z_-$. By Jordan's theorem the segment z_+z_- intersects at least once $\varphi(\partial V - H_n^+)$, say in a point $\tilde{z} = \varphi(\tilde{x})$.

We have then $d(x,\tilde{x})\geq r$ while

$$d(\varphi(x),\varphi(\tilde{x})) \leq d(z_+,z_-) \leq 2\nu$$

On the other hand since $\mathcal{C}^1$ the diffeomorphism φ^{-1} is Lipschitz with Lipschitz constant bounded by $\sup_{x\in V}\|D_x\varphi^{-1}\|$. Now, by standard functional calculus we have:

$$\|D_x\varphi^{-1}\| \leq \sum_{k=0}^{\infty} \|\mathbf{1} - D_x\varphi\|^k < \frac{1}{1-\sigma}.$$

Therefore the Lipschitz constant of φ^{-1} is bounded by above by $\frac{1}{1-\sigma}$ so that

$$d(x,\tilde{x}) \leq \frac{1}{1-\sigma} d(\varphi(x),\varphi(\tilde{x})) \leq \frac{2\nu}{1-\sigma}.$$

This contradicts our choice of ν.

Furthermore if one of x_+, x_-, say x_+ belongs to $C-V$ then we have $d(x,x_+)\geq\mu$ while

$$d(\varphi(x),\varphi(x_+)) \leq \nu$$

and the argument above still lead to a contradiction.

This shows that φ cannot be surjective on C. On the other hand a diffeomorphism of $\mathbb{R}^n$ which preserves C restricts to a bijection on C. If it were not surjective then its inverse would send points of C outside.

In particular $u(x)\Big|_{C\cap H_n^+}=0$ and so $\varphi\Big|_{C\cap H_n^+}$ is identity. The same proof shows that $\varphi\Big|_{C\cap H_j^+}$ is identity, for all j.

By using the same argument when a runs over the points of $V\cap C\cap \cup_{j=1}^n H_j^+$ we derive that $\varphi\Big|C\cap V$ is identity, as claimed. □

End of the proof of theorem 4. The proof will be by induction on n. For $n=1$ this was already proved above. Let V denote now the smallest parallelepiped containing C, in order to match previous notations and constructions. Suppose that $\varphi\in \mathrm{Diff}^1(\mathbb{R}^n, C)$ is such that $\|D_x\varphi-\mathbf{1}\|<\varepsilon$, for all $x\in V^{\delta}$. Then $\varphi(\partial V)$ surrounds C and the proof of Lemma 8 gives us $\varphi(C\cap\partial V)=C\cap\partial V$. Moreover, each φ_j preserves the associated face V_j. By the induction hypothesis φ_j is identity. It follows that $\varphi\Big|_{C\cap\partial V}$ is identity. We can therefore use Proposition 2 to derive that around every corner point of $C\cap\partial V$ the map $\varphi\Big|_C$ is identity. The same argument works for all corner points of V.

Remark 5. *If $C=C_\lambda^n$, then $\mathfrak{Diff}_a^1(C_\lambda)$ is isomorphic to $\mathbb{Z}^{r(a)}$, where $r(a)$ is the number of coordinates of a which are λ-rational (compare with [3]).*

4 Diffeomorphisms groups of specific Cantor sets

4.1 Proof of Theorem 5

Observe first that C_Φ is a Cantor set. Indeed the contractivity assumption implies that an infinite intersection $\lim_{p\to\infty}\phi_{i_1}\phi_{i_2}\cdots\phi_{i_p}(M)$ cannot contain but a single point. Two such points which are distinct are separated by some smoothly embedded sphere, which is the image of ∂M by an element of the semigroup generated by Φ, so that the set C_Φ is totally disconnected. The perfectness follows the same way.

We will draw a rooted $(n+1)$-valent tree $\mathcal{T}$, with edges directed downwards. When $M=[0,1]$ there is an extra structure, as all edges issued from a vertex are enumerated from left to the right.

There is a one-to-one correspondence between the points of the boundary at infinity of the tree and the points of the Cantor set $C=C_\Phi$ associated to the strict regular IFS (Φ, M). To each point $\xi\in C$ we can assign an infinite sequence $I=i_1i_2\ldots i_p\ldots$, so that $\xi=\xi(I)$ where we denoted:

$$\xi(I)=\bigcap_{p=1}^{\infty}\phi_{i_1}\phi_{i_2}\cdots\phi_{i_p}(M)$$

The vertices of the tree are endowed with a compatible labeling by means of finite multi-indices I, where the root is associated the empty index and the vertex v_I is the one reached after traveling along the edges labeled $i_1, i_2,\ldots,i_p$. We also put

$$\phi_I(x)=\phi_{i_1}\phi_{i_2}\cdots\phi_{i_p}(x)$$

for finite I. This extends obviously to the case of infinite multi-indices I.

We further need to introduce a special class of germs, as follows:

Definition 9. *The standard germ associated to the finite multi-indices I and J is the diffeomorphism $\phi_{I/J}:\phi_I(M)\to\phi_J(M)$ given by*

$$\phi_{I/J}(\phi_I(x))=\phi_J(x). \tag{36}$$

Standard germs preserve the Cantor set C as germs, namely $\phi_{I/J}(C\cap\phi_I(M)\subset C\cap\phi_J(M)$. In fact if S is an infinite multi-index then

$$\phi_{I/J}(\xi(IS))=\xi(JS).$$

Graphically we can realize this map as a partial isomorphism of the tree $\mathcal{T}$ which maps the subtree hanging at v_I onto the subtree hanging at v_J.

Consider a pair (t_1,t_2) of finite labeled subtrees of the same degree of $\mathcal{T}$ both containing the root, and whose leaves are enumerated $v_{I_1}, v_{I_2},\ldots,v_{I_p}$ and $v_{J_1}, v_{J_2},\ldots,v_{J_p}$.

Lemma 9. *The map*

$$\phi(x) = \phi_{I_k/J_k}(x), \quad \text{if} \quad x \in \phi_{I_k}(C) \tag{37}$$

defines an element $\phi_{(t_1,t_2)} \in \mathfrak{diff}^+(C)$.

Proof. We know that $C = \cup_{j=1}^{p} \phi_i(C)$, since C is the attractor of Φ. By recurrence on the number of leaves we show that

$$C = \cup_{j=1}^{p} \phi_{I_i}(C)$$

for any finite subtree t of $\mathcal{T}$ containing the root and having leaves v_{I_i}, $i = 1, p$. Now ϕ is a smooth map defined on $\cup_{j=1}^{p} \phi_{I_i}(M)$, and so its domain of definition contains C.

When the dimension $d = 1$, the complementary $M \setminus \cup_{j=1}^{p} \phi_{I_i}(M)$ is the union of finitely many intervals, which we call gaps and there exists an extension of ϕ to a diffeomorphism of $M = [0, 1]$ sending gaps into gaps.

When the dimension $d > 1$, the complementary gap $M \setminus \cup_{j=1}^{p} \phi_{I_i}(M)$ is now connected and diffeomorphic to the standard sphere with p holes. Taking a suitable smoothing at the singular vertex of the conical extension of $\phi\Big|_{\cup_{j=1}^{p} \phi_{I_i}(\partial M)}$ we obtain an extension of ϕ by diffeomorphisms to the ball M, possibly non-trivial on ∂M.

This extension preserves C invariant as gaps are disjoint from C and therefore defines an element $\phi_{(t_1,t_2)} \in \mathfrak{diff}^+(C)$. □

Assume now that we stabilize the pair of trees (t_1, t_2) to a pair (t'_1, t'_2), where t'_j is obtained from t_j by adding the first descendants at vertex v_{I_s}, for $j = 1$ and v_{J_s}, when $j = 2$. The new vertices come with a compatible labeling. Moreover, an orientation preserving diffeomorphism of C induces a monotone map of the boundary of the tree, when $d = 1$.

By direct inspection using the explicit form of ϕ we find that:

$$\phi_{(t_1,t_2)} = \phi_{(t'_1,t'_2)}.$$

Thus the map which associates to the pair (t_1, t_2) of labeled trees the element $\phi_{(t_1,t_2)}$ factors through a map $F_n \to \mathfrak{diff}^+(C)$, for $d = 1$, and $V_n \to \mathfrak{diff}^+(C)$, for $d = 2$, respectively. This is easily seen to be a homomorphism. When $I \neq J$ the map $\varphi_{I/J}\Big|_C$ is not identity since $\varphi_I(M) \cap \varphi_J(M) = \emptyset$. This proves that the homomorphism defined above is injective thereby ending the proof of Theorem 5.

Remark 6. *There is a more general setting in which we allow basins to have boundary fixed points. We say that the compact submanifold M is an* attractive basin *if:*

1. $\varphi_j(\text{int}(M)) \subset \text{int}(M)$, *for all* j;
2. $\text{int}(\varphi_j^{-1}(\varphi_j(\partial M) \cap \partial M)) \supset \text{int}(\varphi_j(\partial M) \cap \partial M)$.
3. $\varphi_i(M) \cap \varphi_j(M) = \emptyset$, *for any* $j \neq i$;
4. $\text{int}(\varphi_j(\partial M) \cap \partial M) \subset \text{int}(\varphi_j^{-1}(\varphi_j(\partial M) \cap \partial M)))$, *for all* j.

Using similar arguments one can show that $\mathfrak{diff}(C_\Phi)$ *contains* F_n *whenever* Φ *has an attractive basin.*

4.2 Proof of Theorem 6

Our strategy is to give first a detailed proof of Theorem 6, and then to explain the necessary changes needed to achieve the more general Theorem 7 in the next section.

Let a be a left point of C_λ. We first claim that:

Lemma 10. $\chi(\text{Diff}_a^{1,+})$ *is the subgroup* $\langle \lambda \rangle \subset \mathbb{R}^*$.

Proof. By the homogeneity of C it is enough to prove it for $a = 0$.

Let $L(C_\lambda)$ denote the set of left points of C_λ. Now left points of C_λ have to be sent into left points by any $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C_\lambda)$ and moreover 0, which is the minimal left point should be fixed. Elements of $L(C_\lambda)$ can be described explicitly, as:

$$L(C_\lambda) = \bigcup_{n=1}^{\infty} \{x \in [0,1]; x = \sum_{j=1}^{n} a_j \lambda^{-j}, \quad \text{where} \quad a_j \in \{0, \lambda - 1\}\}. \tag{38}$$

Therefore there exists δ such that the multiplication by $\lambda \in \mathbb{R}^*$ sends $C_\lambda \cap \mathcal{N}_\delta(0)$ into C_λ. This easily implies that $\chi(\mathrm{Diff}_a^{1,+})$ contains the subgroup $\langle \lambda \rangle$.

For the reverse inclusion we need a sharpening of Lemma 5. As the Cantor set C_λ is $\frac{1}{\lambda}$-sparse, the estimate from Lemma 5 is not strong enough. Notice first that the set of lengths of gaps in C_λ is $\{(\lambda - 1)\lambda^{-n}, n \in \mathbb{Z}_+ \setminus \{0\}\}$. In particular, the quotients of the lengths of any two gaps belongs to $\langle \lambda \rangle$.

But we can use the same argument as in its proof. Let $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C)$ which stabilizes 0. If $\varphi'(0) > 1$, then $\varphi\left(\frac{1}{\lambda^n}\right) \geq \frac{\lambda - 1}{\lambda^n}$, for large enough n. In particular this leads to $\varphi'(0) \geq \lambda - 1$.

Let now $\alpha > 1$ minimal which occurs in $\chi(\mathrm{Diff}_0^1(C_\lambda)) \subset \mathbb{R}^*$. By Lemma 6 there exists $k \in \mathbb{Z}_+$ such that $\lambda^{-1} = \alpha^k$. Assume that $k > 1$ and that $\varphi'(0) = \lambda^{-1/k}$. Consider a set of maximal gaps (x_n, y_n) accumulating to 0. Then $(\varphi(x_n), \varphi(y_n))$ is also a maximal gap, so that

$$\frac{\varphi(x_n) - \varphi(y_n)}{x_n - y_n} \in \langle \lambda \rangle.$$

On the other hand

$$\lim_{n \to \infty} \frac{\varphi(x_n) - \varphi(y_n)}{x_n - y_n} = \varphi'(0) = \lambda^{-1/k}$$

This contradicts the previous relation, so that $k = 1$ and hence $\alpha = \lambda$. □

We next observe that for each left point a of C there exists a small neighborhood U_a of a such that the affine map $\psi_a = a + \lambda(x - a)$ sends $U_a \cap C$ into C, defining therefore a germ in $\mathfrak{diff}_a^{1,+}$. Then Lemmas 10 and 6 imply together that $\mathfrak{diff}_a^{1,+}$ is generated by $\psi_a = a + \lambda(x - a)$.

Let now a and b be two left points of C. Denote by $D(a, b)$ the set of germs at a of classes in $\mathfrak{diff}^1(C_\lambda)$ having representatives $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C_\lambda)$ such that $\varphi(a) = b$. This set is acted upon transitively by $\mathfrak{diff}_a^{1,+}$, so that using a similar argument to the one from above concerning stabilizers $D(a, b)$ consists of the germs of maps of the form $\psi_{a,b,k} = b + \lambda^k(x - a)$, with $k \in \mathbb{Z}$.

Now left points of C_λ have to be sent into left points by any $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C_\lambda)$. Therefore, for any left point $a \in C_\lambda$ we have $\varphi'(a) \in \langle \lambda \rangle$. But left points of C_λ are dense in C_λ. Since φ' is continuous and $\langle \lambda \rangle$ has no other accumulation points in $\mathbb{R}^*$ we obtain $\varphi'(a) \in \langle \lambda \rangle$, for any $a \in C_\lambda$ and any $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C_\lambda)$.

For a given $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C_\lambda)$ its derivative φ' is continuous on the whole interval $[0,1]$ and hence is bounded. Moreover, the same argument for φ^{-1} shows that φ' is also bounded from below away from 0, so that $\varphi'\Big|_{C_\lambda}$ can only take finitely many values of the form λ^n, $n \in \mathbb{Z}$. Moreover, the uniform continuity of φ' on $[0,1]$ implies that $\varphi'\Big|_{C_\lambda}$ is locally constant and moreover for any $\varepsilon > 0$ there exists some $\nu > 0$ with the property that $x, y \in C_\lambda$ with $|x - y| < \nu$ must have $\varphi'(x) = \varphi'(y)$.

The following is a key ingredient in the description of the group $\mathfrak{diff}^{1,+}(C_\lambda)$:

Lemma 11. *There is a covering of C_λ by a finite collection of disjoint standard intervals I_k, whose images are also standard intervals such that $\varphi\Big|_{C_\lambda \cap I_k}$ is the restriction of an affine function to $I_k \cap C_\lambda$. Specifically,*

$$\varphi(x) = \varphi(c_k) + \lambda^{j_k}(x - c_k), \qquad \text{for} \quad x \in I_k \cap C_\lambda, \tag{39}$$

where c_k is a left point of $C_\lambda \cap I_k$.

We postpone the proof of this lemma to the end of this section.

By passing to a subdivision we can assume that the intervals I_k are standard. Now the image of a standard interval by an affine map is a standard interval since the central gap has to be sent into a gap.

Consider the rooted binary tree $\mathcal{T}$ embedded in the plane so that its ends abut on the interval $[0,1]$. We label each edge e by $l(e) \in \{0, \lambda - 1\}$, such that the leftmost edge is always labeled 0. Let v be a vertex of $\mathcal{T}$ and $e_1, e_2, \ldots, e_n$ the sequence of edges representing the geodesic which joins the root to v. To the vertex v one associates then the number

$$r(v) = \sum_{j=1}^{n} l(e_j)\lambda^{-j} \tag{40}$$

Denote by $D(v)$ the set of all descendants of the vertex v. If I is a closed interval in $[0,1]$ we claim that $L(C_\lambda) \cap I$ coincide with the set $r(D(v))$, for some unique vertex $v \in \mathcal{T}$. We denote by V_I this vertex v. Furthermore, if $I_1, I_2, \ldots, I_k$ is a set of disjoint standard intervals covering C_λ then $v_{I_1}, v_{I_2}, \ldots, v_{I_k}$ are the leaves of a finite binary subtree $T(I_1, I_2, \ldots, I_k)$ of $\mathcal{T}$ containing the root. In particular, if $J_1, J_2, \ldots, J_k$ is another covering of C_λ by standard intervals then we have two finite trees $T(I_1, I_2, \ldots, I_k)$ and $T(J_1, J_2, \ldots, J_k)$. Furthermore we also have affine bijections $\varphi_j : I_j \to J_j$ which are of the form $\varphi_j(x) = b_j + \lambda^{k_j}(x - a_j)$, where $a_j, b_j \in L(C_\lambda)$. It is clear that $\varphi_j(I_j \cap L(C_\lambda)) = J_j \cap L(C_\lambda)$. The explicit form of $\varphi_j\Big|_{I_j \cap L(C_\lambda)}$ actually can be interpreted in terms of $r(v_{I_j})$, as follows. Let $\mathcal{D}(v)$ be the planar rooted subtree of $\mathcal{T}$ of vertices $D(v)$ and root v. There is a natural identification $\iota_{v,w}$ of the planar binary rooted trees $D(v)$ and $D(w)$, for any $v, w \in \mathcal{T}$. Then under the identification of $L(C_\lambda) \cap I_j$ coincide with the set $r(D(v_{I_j}))$ the induced action of φ_j on $w \in D(v_{I_j})$ coincide with $\iota_{v_{I_j}, v_{J_j}}$.

Consider now the operation of replacing an interval I_j by two disjoint intervals I_j' and I_j" whose union is disjoint from the other intervals I_k. Correspondingly we replace J_j by $J_j' = \varphi_j(I_j')$ and $J_j" = \varphi_j(I_j")$ and φ_j by its restrictions. This operation does not change the element in $\mathfrak{Diff}^1(C_\lambda)$. The immediate consequence of the description of φ_j is that the pairs of trees $T(I_1, \ldots, I_j', I_j", \ldots, I_k)$ and $T(J_1, \ldots, J_j', J_j", \ldots, J_k)$ are both obtained from $T(I_1, I_2, \ldots, I_k)$ and $T(J_1, J_2, \ldots, J_k)$ by adding one caret at the j-th leaf. This proves that this pair of trees is a well-defined element of the standard Thompson group F. It is rather clear that the map defined this way $\mathfrak{Diff}^{1,+}(C_\lambda) \to F$ is an isomorphism.

In a similar way we define an isomorphism $\mathfrak{Diff}^{1,+}_{S^1}(C_\lambda) \to T$, when we work with the infinite unrooted binary tree $\mathcal{T}$ embedded in the plane so that its ends abut to S^1.

In the case of $\mathfrak{Diff}_{S^2}(C_\lambda)$ we use Theorem 3 and the infinite unrooted binary tree $\mathcal{T}$ without any planar structure. The only difference is that the restrictions $\varphi\Big|_{I_j}$ are not having anymore a coherent orientation. Some of them might be orientation preserving and the others not. The orientation data is encoded into an element of $(\mathbb{Z}/2\mathbb{Z})^\infty$, which is the infinite direct sum of $\mathbb{Z}/2\mathbb{Z}$, namely the ascending union $\cup_{n=1}^{\infty}(\mathbb{Z}/2\mathbb{Z})^n$. This explains the isomorphism between $\mathfrak{Diff}_{S^2}(C_\lambda)$ and an extension by $(\mathbb{Z}/2\mathbb{Z})^\infty$ of the Thompson group V.

This ends the proof of Theorem 6, except for the:

Proof of Lemma 11. For $c \in C_\lambda$ there is some $m \in \mathbb{Z}$ such that $\varphi'(c) = \lambda^m$. We want to prove that there exists a neighborhood U of c such that:

$$\varphi(x) = \varphi(c) + \lambda^{j_k}(x - c), \qquad \text{for} \quad x \in U \cap C_\lambda. \tag{41}$$

Then such neighborhoods will cover C_λ and we can extract a finite subcovering by clopen (closed and open) subsets with the same property.

This claim is true for any left (and by similar argument for right) end points c of C_λ. It is then sufficient to prove that whenever we have a sequence of left points $a_n \to a_\infty$ contained in a closed interval $U \subset [0,1]$ and a $\mathcal{C}^1$-diffeomorphism $\varphi : U \to \varphi(U) \subset [0,1]$ with $\varphi(C \cap U) \subset C$, there exists a neighborhood U_{a_∞} of a_∞ and an affine function ψ such that for large enough n the following holds:

$$\varphi(x) = \psi(x), \text{ for } x \in C_\lambda \cap U_{a_\infty}.$$

Around each left point a_n there are affine maps $\psi_{a_n,k_n} : U_{a_n,k_n} \to [0,1]$ defining germs in $D(a_n, c_n)$, where $c_n = \varphi(a_n)$, such that c_n converge to $c_\infty = \varphi(a_\infty)$ and

$$\varphi(x) = \psi_{a_n,k_n}(x), \text{ for } x \in C_\lambda \cap U_{a_n,k_n}.$$

We can further assume that $U_{a_n,k_n} \cap C_\lambda$ are clopen sets (i.e. closed and open), and we can take $U_{a_n,k_n} = [a_n, b_n]$ where b_n are right points of C_λ, and the sequence a_n is monotone, say increasing.

There is no loss of generality to assume that $\psi'_{a_n,k_n}\Big|_{C\cap U_{a_n,k_n}}$ is independent on n, say it equals λ^m, namely $k_n = m$. Replacing φ by its inverse φ^{-1} we can also assume that $m \leq 0$. Since C_λ is invariant by the homothety of factor λ and center 0, we can further reduce the problem to the case where $m = 0$. We have then $\varphi'(a_\infty) = 1$, by continuity.

Choose n large enough so that $|\varphi'(x) - 1| < \varepsilon$, for any $x \in [a_n, a_\infty]$, where the exact value of ε will be chose later. Let now consider the maximal interval of the form $[a_n, b]$ to which we can extend $\psi_{a_n,0}$ to an affine function which coincides with φ on $C \cap [a_n, b]$.

If $b = a_\infty$, then the Lemma follows. Otherwise, it is no loss of generality in assuming that $b = b_n$ and thus b is a right point of C_λ. Then b_n is adjacent to some gap (b_n, d). Since d is a left point of C_λ and $\varphi'(d) = 1$, we can suppose that $d = a_{n+1}$.

Since φ preserves $C \cap U$, it should send the gap (b_n, a_{n+1}) into some gap contained into $[\varphi(a_n), \varphi(b_{n+1})]$. Now, gaps of C_λ have lengths of the form $(1 - 2\lambda)\lambda^m$, for $m \in \mathbb{Z}$. Thus the ratios of lengths of gaps is the discrete subset $\langle\lambda\rangle \subset \mathbb{R}^*$.

When $|\varphi'(x) - 1| < \varepsilon$, we derive that the ratio of the lengths of the gaps $\varphi(b_n, a_{n+1})$ and (b_n, a_{n+1}) is bounded by $1 + \varepsilon$. By taking $\varepsilon < 1 - \lambda$ we see that the only possibility is that the lengths of these two gaps coincide, namely that

$$\varphi(a_{n+1}) = \varphi(b_n) + a_{n+1} - b_n.$$

This implies that there is a smooth extension of $\psi_{a_n,0}$ to an affine function on $[a_n, b_{n+1}]$ which coincides with φ on points of C_λ, contradicting the maximality of $b = b_n$. This proves that $b = a_\infty$, proving the claim.

When a_∞ is not a right point we also have an affine extension of φ to a right neighborhood of a_∞, by the same argument. □

4.3 Proof of Theorem 7

Lemma 12. *There exists some $N = N(\Phi) \in \mathbb{Z}_+$ such that $\chi(\mathrm{Diff}_a^{1,+})$ is the subgroup $\langle\lambda_1^{1/N}\rangle \subset \mathbb{R}^*$, for any left point a of C.*

Proof. We obviously have $\langle\lambda_1\rangle \subset \chi(\mathrm{Diff}_0^{1,+})$, and by Lemma 5 there exists some $N \in \mathbb{Z}_+$ so that $\chi(\mathrm{Diff}_0^{1,+}) = \langle\lambda_1^{1/N}\rangle$. It remains to observe that for any left point a of C there exists a germ $\psi \in \mathrm{Diff}^1(\mathbb{R}, C)$ sending 0 to a. We can even take this germ to be affine. □

The only missing ingredient is the result generalizing Lemma 11 to the more general self-similar sets considered here, as follows:

Lemma 13. *Let $\varphi \in \mathrm{Diff}^1(\mathbb{R}, C)$. Then there is a covering of C by a finite collection of disjoint standard intervals I_k, whose images are also standard intervals such that $\varphi\Big|_{C\cap I_k}$ is is the restriction of an affine function to $I_k \cap C$. Specifically,*

$$\varphi(x) = \varphi(c_k) + \Lambda_{\mathbf{j}_k,N}(x - c_k), \qquad \text{for} \quad x \in I_k \cap C, \tag{42}$$

where c_k is a left point of $C \cap I_k$.

The proof of this lemma will occupy the next sections 4.3.1 and 4.3.2 and it will be divided into two parts, according to whether the parameters are commensurable or totally incommensurable.

Now, any φ in the group $\mathfrak{diff}^{1,+}(C_\Phi)$ corresponds to a pair of coverings of C by intervals $(I_1, I_2, \ldots, I_k)$ and $(J_1, J_2, \ldots, J_k)$ so that φ sends affinely I_j into J_j, for all j. These intervals could be chosen to be of the form $[a, b]$, where a is a left point of C and b is a right point of C. We call them *clopen* intervals. Particular examples of clopen intervals are the images of $[0, 1]$ by the semigroup generated by Φ, which will be called *standard (clopen)* intervals. Each standard clopen interval corresponds to a finite geodesic path descending from the root in the (regular rooted) tree of valence $N + 1$ associated to Φ. It remains to prove then:

Lemma 14. *We assume that Φ verifies the genericity condition (C) from Definition 6. Then any $\varphi \in \mathfrak{diff}^{1,+}(C_\Phi)$ corresponds to a pair of coverings of C by standard intervals $(I_1, I_2, \ldots, I_k)$ and $(J_1, J_2, \ldots, J_k)$ so that φ sends affinely I_j into J_j, for all j.*

Pairs of coverings by standard clopen intervals of C correspond to pairs of finite rooted subtrees. Subdividing the covering by standard subintervals is then equivalent to stabilizing the trees. This provides an isomorphism with the Thompson group F_{n+1}, ending the proof of Theorem 7.

Proof of Lemma 14. Every clopen interval is the disjoint union of finitely many standard intervals. We can therefore suppose that I_j are standard intervals.

We claim that the image J of a standard interval I by an affine map φ preserving C must be a standard interval.

We will need in the sequel more terminology. Standard intervals are associated to vertices of the $(n+1)$-ary tree, and one says that they belong to the k-th *generation of standard intervals* if the associated vertex is at distance k from the root. The complementary intervals to the union of all k-th generation of standard intervals will be the k-th *generation of gaps*. Moreover, given a standard interval I of the k-th generation, the gaps of the $k+1$-th generation lying in I will also be called the first generation of gaps in I.

Let us write $J = J^1 \cup J^2 \cup \cdots \cup J^m$, as the union of finitely many standard intervals. Suppose that J^u is the largest among the intervals J^i. By further composing φ with the affine map in $\mathfrak{diff}^1(C)$ sending J^u onto I we can assume that $I = J^u$. In particular, the homothety factor μ of the affine map $\varphi : J^u \to J$ is at least 1.

Assume that C is central, namely the IFS is homogeneous (i.e. all $\lambda_i = \lambda$) and the initial (i.e. first generation) gaps have the same length. Then the set of largest gaps in J^u consists of n equidistant gaps of the same size g. Their image by an affine map should be the set of largest gaps in J, so that there are n equidistant equal gaps in J. The only possibility for the size of these gaps is $\lambda^n g$, for some $n \in \mathbb{Z}_-$. Every such image gap determines uniquely a standard interval J_n of size λ^n to which it belongs. If two of these gaps determined distinct standard intervals, then they would be separated by another gap of size $\lambda^{n-1}g$, contradicting their maximality. Then all but possibly the leftmost and rightmost intervals of the complement of these n gaps in J are standard.

Now, the interval between two consecutive gaps in J^u is a standard interval of length λ, whose image by an affine map should have length λ^{1+n}. This shows that the leftmost and the rightmost intervals also should be standard intervals, having the same size as the other $n-2$ standard intervals between consecutive image gaps. Altogether this shows that $J = J_n$ is a standard interval.

The set of gaps of the same generation is totally ordered from the leftmost gap towards the right. The sequence of lengths of $(k+1)$-th generation gaps within a standard interval of the k-th generation is of the form $(\Lambda_{\mathbf{k}} g_1, \Lambda_{\mathbf{k}} g_2, \ldots, \Lambda_{\mathbf{k}} g_n)$, for some $\mathbf{k}$. Consider now a gap of the first generation, say $\Lambda_{\mathbf{k}} g_i$ of J^u. Its image by an affine map should be a gap of J. It follows that there exists some $\sigma(i) \in \{1, 2, \ldots, n\}$ and $\mathbf{k}_i \in \mathbb{Z}_+^{n+1}$, so that:

$$\mu \Lambda_{\mathbf{k}} g_i = \Lambda_{\mathbf{k}_i} g_{\sigma(i)},$$

where μ is the homothety factor of the map φ. Conversely, any gap of $J^u \subset J$ is the image by φ of some gap of J^u, and hence there exists some $\tau(i) \in \{1, 2, \ldots, n\}$ and $\mathbf{l}_i \in \mathbb{Z}_+^{n+1}$, so that:

$$\frac{1}{\mu} \Lambda_{\mathbf{k}} g_i = \Lambda_{\mathbf{l}_i} g_{\tau(i)}.$$

Getting rid of μ in the two equalities above we obtain the following identities, for all i, j:

$$\Lambda_{\mathbf{k}_i + \mathbf{l}_j - 2\mathbf{k}}\, g_{\sigma(i)} g_{\tau(j)} = g_i g_j.$$

By taking $j = \sigma(i)$ we derive:

$$\Lambda_{\mathbf{k}_i + \mathbf{l}_{\sigma(i)} - 2\mathbf{k}}\, g_{\tau(\sigma(i))} = g_i.$$

If g_i and λ_j satisfy the genericity condition (C) the last equality implies $\tau(\sigma(i)) = i$ and $\mathbf{k}_i + \mathbf{l}_{\sigma(i)} = 2\mathbf{k}$, for every i. A symmetric argument yields $\sigma(\tau(i)) = i$, so that σ and τ are bijections inverse to each other. Furthermore we derive:

$$\mu^n = \prod_{i=1}^n \Lambda_{\mathbf{k}_i - \mathbf{k}} \frac{g_{\sigma(i)}}{g_i} = \Lambda_{\sum_{i=1}^n (\mathbf{k}_i - \mathbf{k})}$$

so that

$$\mu = \Lambda_{-\mathbf{k}+\frac{1}{n}\sum_{i=1}^{n}\mathbf{k}_i}$$

Therefore, for each i we have:

$$\frac{g_{\sigma(i)}}{g_i} = \Lambda_{-\mathbf{k}+\frac{1}{n}\sum_{i=1}^{n}\mathbf{k}_i}$$

Then our assumptions of genericity imply that σ must be identity. It turns that $\mathbf{k}_i = \mathbf{k}$ and hence $\mu = 1$. Therefore $J = J^u$ and thus J is standard. □

4.3.1 Asymmetric Cantor sets with commensurable parameters

Let a and b be two left points of C and $D(a,b)$ be the set of germs at a of classes in $\mathfrak{Diff}^1(C)$ having representatives $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C)$ such that $\varphi(a) = b$. This set is acted upon transitively by $\mathfrak{diff}_a^{1,+}$, so that $D(a,b)$ consists of the germs of maps of the form $\psi_{a,b,k} = b + \Lambda_{k,N}(x-a)$, with $\mathbf{k} \in \mathbb{Z}$. The commensurability assumption implies that derivatives belong to some discrete subgroup of $\mathbb{R}^*$, namely there exists some $\lambda \in \mathbb{R}^*$ such that $\Lambda_{k,N} \in \langle\lambda\rangle$. Therefore, for any left point $a \in C$ we have $\varphi'(a) \in \langle\lambda\rangle$, and as before $\varphi'\Big|_C$ can only take finitely many values of the form λ^n, $n \in \mathbb{Z}$, so that $\varphi'\Big|_C$ is locally constant. The proof of lemma 11 extends now word by word to the present situation. We skip the details and prefer to work out below an explanatory example.

The simplest asymmetric Cantor set AC with commensurable parameters is obtained from the non-homogeneous IFS Φ given by:

$$\phi_0(x) = \frac{1}{4}x,\ \phi_1(x) = \frac{1}{2}x + \frac{1}{2}.$$

A more explicit description of AC is to start from the interval $[0,1]$ and remove first the open interval $(\frac{1}{4}, \frac{1}{2})$. At each further step we remove a gap interval $(a + \frac{1}{4}(b-a), a + \frac{1}{2}(b-a))$ from the interval $[a,b]$ which is a connected component of the previous stage. In the end we retrieve the asymmetric Cantor set AC.

To describe explicitly the set of left points $L(AC)$ is slightly more subtle than in the homogeneous case. One easy description is by recurrence. For each finite multi-index $I = i_1 i_2 \dots i_k$, with $i_j \in \{0,1\}$ we set $l_\emptyset = 0$ and define by induction:

$$l_{0I} = \frac{1}{4}l_I,\ l_{1I} = \frac{1}{2}l_I + \frac{1}{2}.$$

Thus $l_1 = \frac{1}{2}$, $l_{01} = \frac{1}{8}$, $l_{11} = \frac{3}{4}$. By induction one proves that:

$$l_{i_1 i_2 \dots i_k} = \sum_{1\le s\le k} i_s \cdot 4^{-\sum_{j=1}^{s}(1-i_j)} \cdot 2^{-\sum_{j=s}^{k} i_j} = \sum_{1\le s\le k} i_s \cdot 4^{-\sum_{\alpha=1}^{s}\delta_{0i_\alpha}} \cdot 2^{-\sum_{\beta=s}^{k} i_\beta}$$

Furthermore $L(AC)$ consists of the set $\{l_I; I \text{ finite and admissible}\}$, where $I = i_1 i_2 \dots i_k$ is admissible if it is either empty or else $i_k = 1$.

We put then for any infinite $I = i_1 i_2 \dots i_k \dots$

$$l_I = \lim_{k\to\infty} l_{i_1 i_2 \dots i_k}$$

It is not difficult to show that AC consists of the union of $L(AC)$ and the set of l_I, with infinite I. Moreover we can identify $L(AC)$ to the set of those l_I for which I is infinite and eventually 0.

Further $\frac{1}{4} \in \chi(\mathrm{Diff}_0^1(AC))$. We verify immediately that $\frac{1}{2} \notin \chi(\mathrm{Diff}_0^1(AC))$, as there are infinitely many $x \in L(AC)$ with $\frac{1}{2}x \notin AC$. This implies that $\chi(\mathrm{Diff}_0^1(AC)) = \langle 4\rangle$, so that $N(\Phi) = 1$.

The main difference with the previous cases is the description of $D(a,b)$, where $a, b \in L(AC)$. We have for instance:

$$D\left(\frac{1}{2}, \frac{3}{4}\right) = \left\{\frac{3}{4} + \frac{1}{2}\cdot 4^k\left(x - \frac{1}{2}\right), k \in \mathbb{Z}\right\}$$

while

$$D\left(\frac{1}{8}, \frac{1}{2}\right) = \left\{\frac{1}{2} + 4^k\left(x - \frac{1}{8}\right), k \in \mathbb{Z}\right\}$$

so that the coefficient of the linear part is not necessarily in $\chi(\mathrm{Diff}^1_0(AC)) = \langle 4 \rangle$.

In order to understand this discrepancy we consider the binary tree $\mathcal{T}(AC)$ associated to AC. As in the case of the central Cantor sets the vertices correspond with the elements of $L(AC)$, namely with the multi-indices I. There is one special vertex for $I = \emptyset$ and the root of the tree is now considered to be $I = \frac{1}{2}$. Edges of the tree descend from the root and they are labeled either 0 or 1. We have a vertex labeled l_I in the tree which is joined by a geodesic labeled I to the root, for any admissible I. The geodesic label corresponds to reading the labels of the edges encountered. The special vertex $\emptyset$ is a leaf of the tree, which will be ignored in the sequel, as it is not playing any role and the tree will be called reduced after that.

If a, b are labels of two vertices in the reduced tree then

$$D(a,b) = \left\{ \psi_{a,b,k} = b + \frac{1}{2^{n(a,b)}} 4^{-k}(x-a), k \in \mathbb{Z} \right\}, \tag{43}$$

where $n(a,b) \in \{0,1\}$ is the parity of the length of a the geodesic joining a to b in the reduced tree. This has an immediate analog for $a = 0$.

Eventually any element φ of $\mathfrak{Diff}^{1,+}(AC)$ determines a finite covering of AC by intervals I_j on which $\varphi|_{I_j}$ is of the form ψ_{a_j,b_j,k_j}, for some $a_j \in L(AC)$. It follows that φ is entirely determined by two sequences of pairs (a_j, b_j). The intervals associated to either the set of the a_j's or the set of the b_j's form two coverings of AC. This is equivalent to the fact that the the set of vertices a_j (respectively b_j) represent the leaves of some finite labeled rooted subtree of the reduced tree.

The map which associates to φ the class of the two finite labeled trees provides an isomorphism with the usual Thompson group F.

4.3.2 Proof of Lemma 13 for incommensurable parameters

We will use a much weaker restriction than the total incommensurability, see below.

Recall the notation from section 4.1 concerning the rooted $(n+1)$-valent labeled tree with edges directed downwards, and all edges issued from a vertex being enumerated from left to the right.

With this notation left points of C correspond to sequences which eventually end in 1, namely of the form

$$L(i_1 \ldots i_p) = i_1 i_2 \ldots i_p 111111 \ldots,$$

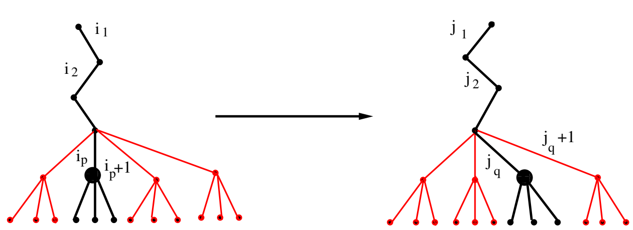
while right point correspond to sequences which eventually end in n:

$$R(i_1 \ldots i_p) = i_1 \ldots i_p nnnnnn \ldots.$$

The standard germs are in this case affine functions, which can be therefore extended to the whole line. Consider two finite multi-indices $I = i_1 \ldots i_p$ and $J = j_1 \ldots j_q$ and set $a = L(i_1 \ldots i_p)$, $b = R(i_1 \ldots i_p)$, $\alpha = L(j_1 \ldots j_q)$, $\beta = R(j_1 \ldots j_q)$. The *standard germ* associated to these indices is the affine map $\psi_{I,J} : [a,b] \to [\alpha, \beta]$ given by the formula:

$$\psi_{I,J}(x) = a + \left(\frac{\prod_{m=1}^{q} \lambda_{j_m}}{\prod_{k=1}^{p} \lambda_{i_k}^{-1}} \right)(x-a)$$

Each multi-index I determine a vertex v_I of the tree, which is the endpoint of the geodesic issued from the root which travels along the edges labeled $i_1, i_2, \ldots, i_p$. Then, at the level of trees a standard germ corresponds to a combinatorial map sending the subtree hanging at the vertex v_I onto the subtree issued from the vertex v_J, as in the figure below:



An *extension* of the standard germ $\psi : [a, b] \to [\alpha, \beta]$ is a standard germ defined on $[c, d] \supset [a, b]$ whose restriction to $[a, b]$ coincides with ψ. In this case $[c, d]$ must corresponds to a vertex $v_{I'}$ of the tree whose multi-index I' is a prefix of I, namely $I' = i_1 i_2 \dots i_r$ with $r \le p$. This shows that a non-trivial extension of ψ exists only if $i_p = j_q$.

A *multi-germ* is a finite collection of standard germs $\psi_j : [a_j, b_j] \to [\alpha_j, \beta_j]$ such that:

$$a_1 < b_1 < a_2 < b_2 < c \cdots < a_k < b_k, \quad \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_k < \beta_k$$

and $[b_j, a_{j+1}]$ and $[\beta_j, \alpha_{j+1}]$ are gaps of C, for all j.

Eventually an *extension of a multi-germ* $\{\psi_j\}_{j=1,k}$ is a multi-germ $\{\theta_j\}_{j=1,m}$ such that every standard germ ψ_j is extended by some θ_i. Notice that several elements of the multi-germ $\{\psi_j\}_{j=1,k}$ might have the same extension θ_i.

Lemma 15. *Let $\{\psi_j\}_{j=1,m}$ be a chain with the property that there exist constants $\mu, \nu > 0$ satisfying*

$$\frac{\mu}{\nu} > \frac{1}{\max(\lambda_1, \lambda_2, \dots, \lambda_n)},$$

such that:

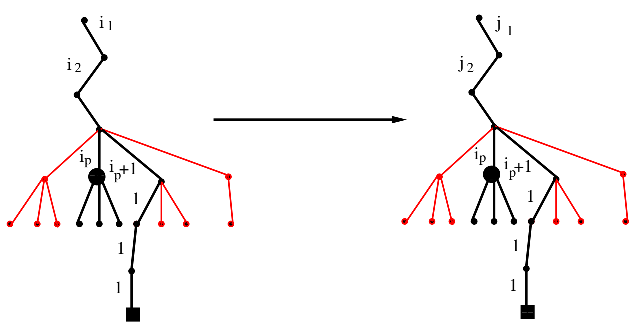
$$\mu \le \psi_j'(x) \le \nu, \quad \text{for every } x. \tag{44}$$

If the standard germ ψ_j admits an extension χ, then there exists an extension of the multi-germ $\{\psi_j\}_{j=1,m}$ containing the standard germ χ.

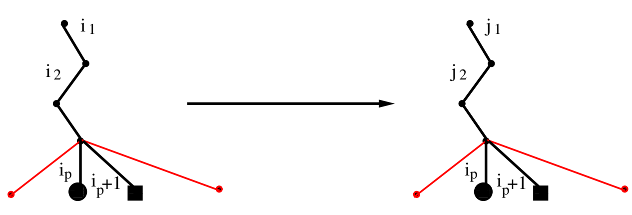
Moreover, if a diffeomorphism $\varphi \in \mathrm{Diff}^1(\mathbb{R}, C)$ whose derivative φ' verifies the condition for derivative (44) coincides with the multi-germ $\{\psi_j\}_{j=1,m}$ on $[a_1, b_m]$, then it coincides with χ on its domain of definiteness.

Proof. The standard germ ψ_j is of the form $\psi_j = \psi_{I,J}$, with $i_p = j_q = k$. We want to construct an increasing function extending the standard germ $\psi_{I,J}$ which satisfies the condition (44) for the derivative. Such a function will be called a *continuation* of ψ_j. Moving one step upward on the tree (i.e. the ancestor vertices) we arrive at the vertices $v_{I'}$ and $v_{J'}$, where $I = I'k$, $J = J'k$.

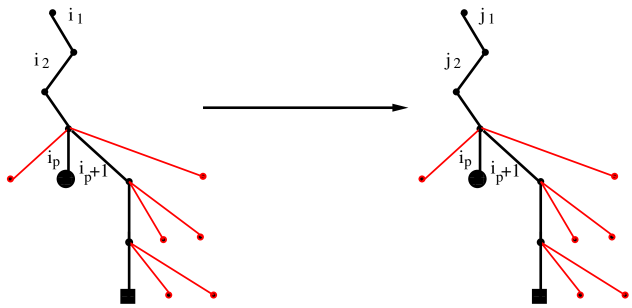
Let, for the sake of definiteness take first $k < n$ and seek for a continuation on the right side of the interval on which $\psi_{I,J}$ is defined. Therefore the continuation must have form drawn below, where points marked by squares correspond to each other:



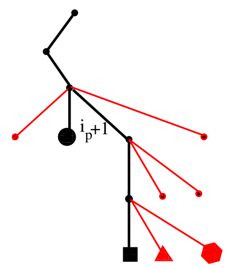
Since the ratio of the derivatives is uniformly bounded, the vertices corresponding to squares should be on the same level, namely at equal distance from the vertices $v_{I'}$ and $v_{J'}$, respectively. Consider the highest possible level of such squares for which the extended map is compatible with the standard germ ψ_{j+1}. We claim that this continuation has the following form, namely that squares sit on the vertices $v_{I'k+1}$ and $v_{J'k+1}$:



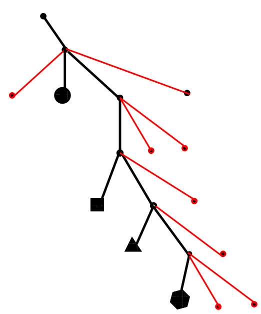
Assume the contrary holds, namely that the squares sit on lower levels, as in the figure below:



Consider further continuation to the right of this increasing function. We label points on the next branch issued from the ancestor of squares vertices by triangles and further by hexagons etc. Consider further the highest levels for which continuation is compatible. Then the picture



is impossible, since then the ancestor of the square vertex should also have been labeled by a square. Therefore we must continue along an infinite path down to a boundary point of the tree, as in the figure:



The boundary point corresponds to an infinite multi-index I. Then $\xi = \xi(I) \in [0, 1]$ cannot be a right point of the Cantor set, since this would give a continuation to a whole subtree issued from $v_{I'}$, contradicting the form of our path.

Now our continuation coincides with the multi-germ $\{\psi_j\}_{j=1,m}$ for values $x \in [a_j, \xi]$. Since ξ is not a right point, they coincide in a right semi-neighborhood of ξ and this contradicts the choice of our infinite path.

We summarize the discussion above as follows. Let $k_r < n$. Then the only possible right continuation (which satisfies the condition (44)) of $\psi_{Ik_1\ldots k_r,Jk_1\ldots k_r}$ is by the germ $\psi_{Ik_1\ldots k_{r-1}k_r+1,Jk_1\ldots k_{r-1}k_r+1}$. A similar argument shows that whenever $k_r > 1$ the only possible left continuation (which satisfies the condition (44)) of $\psi_{Ik_1\ldots k_r,Jk_1\ldots k_r}$ is by the germ $\psi_{Ik_1\ldots k_{r-1}k_r-1,Jk_1\ldots k_{r-1}k_r-1}$.

Repeating the same argument, we get the desired statement. □

Lemma 16. *There exists $\varepsilon > 0$ with the following property. Consider a standard germ $\psi_{I,J}$ with $i_p \neq j_q$ and $j_q \neq n \neq i_p$.*

Then any continuation of $\psi_{I,J}$ to a standard germ θ sending $L(i_1i_2\ldots i_{p-1}i_p+1)$ to $L(j_1j_2\ldots j_{q-1}j_q+1)$ which is defined in a right semi-neighborhood of $L(i_1i_2\ldots i_{p-1}i_p+1)$ is either an extension of the standard germ $\psi_{I,J}$, or else it verifies:

$$\left|\frac{\psi'_{I,J}}{\theta'} - 1\right| > \varepsilon.$$

Notice that θ is locally affine and hence we don't need to specify the point (of the corresponding domain of definiteness) in which we consider the derivative.

Proof. The ratio of the derivatives of the standard germs $\psi_{I,J}$ and $\theta = \psi_{i_1 i_2 \dots i_{p-1} i_p+1,\, j_1 j_2 \dots j_{q-1} j_q+1}$ is given by:

$$\frac{\psi'}{\theta'} = \frac{\lambda_{i_p}^{-1}\lambda_{i_q}}{\lambda_{i_p+1}^{-1}\lambda_{i_q+1}}\lambda_1^m,$$

where $m \in \mathbb{Z}$. This is a discrete subset of $\mathbb{R}^*$ and hence the claim. □

We can apply the same arguments when $i_p \neq 1 \neq j_q$. Specifically, we have:

Lemma 17. *Let $n \geqslant 3$. Then there exists $\varepsilon > 0$ such that any multi-germ $\{\psi_j\}_{j=1,m}$ with the property:*

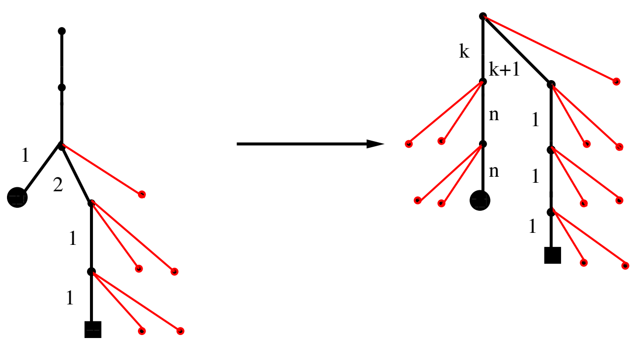
$$\left|\frac{\psi_i'}{\psi_j'} - 1\right| < \varepsilon$$

admits an extension containing with at most two elements.

Proof. It remains to examine the standard germs $\psi_{I,J}$ in following two cases:

$$(I,J) \in \{(i_1 \dots i_{p-1}1,\ J = j_1 \dots j_{q-1}n),\ (i_1 \dots i_{p-1}n,\ j_1 \dots j_{q-1}1)\}.$$

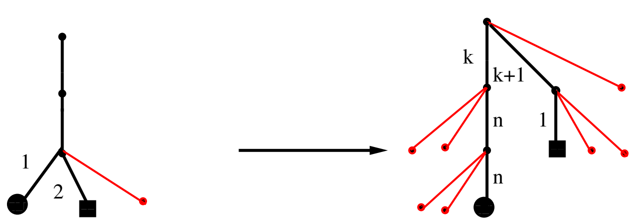
The corresponding picture depends on the number s of occurrences of n in the tail of $j_1 \dots j_{q-1}n$ and the positions of the the square vertices (having r and m respectively ancestors labeled 1) as below:



The ratio of derivatives is

$$\frac{\lambda_n^s\lambda_k\lambda_{k+1}^{-1}\lambda_1^{-r}}{\lambda_1\lambda_2^{-1}\lambda_1^{-m}} = \frac{\lambda_k\lambda_2}{\lambda_{k+1}}\cdot\frac{\lambda_n^s}{\lambda_1^{r-m+1}}$$

Letting s and $\mu = r - m + 1$ be large enough we can insure that $\lambda_n^s/\lambda_1^\mu$ is arbitrarily close to $\lambda_{k+1}/\lambda_k\lambda_2$. In this case $\mu > 0$, so that we can automatically extend the new standard germ obtained this way and get the figure below, where the position of the squared vertex is the highest possible:



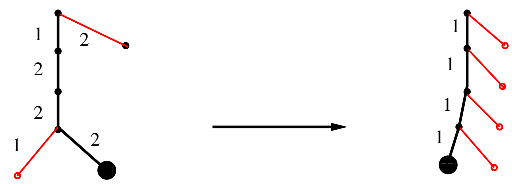
Now, as $n \geq 3$ we cannot find a non-trivial extension of the two standard germs corresponding to the labeled vertices. This means that there is an extension with at most two elements, thereby proving our statement. □

Lemma 18. *Let $n = 2$. Then there exists $\varepsilon > 0$ such that any chain $\{\psi_j\}_{j=1,m}$ verifying the condition:*

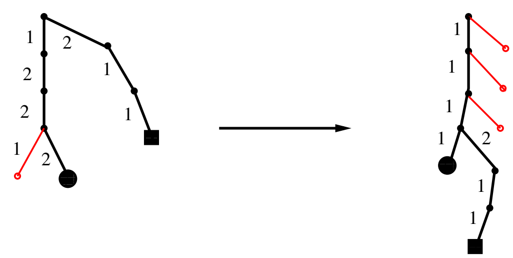
$$\left| \frac{\psi_i'}{\psi_j'} - 1 \right| < 1 + \varepsilon$$

admits an extension containing at most 4 elements.

Proof. The only possible situation is that pictured below:



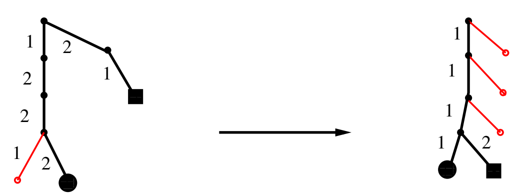
Consider a right continuation as follows:



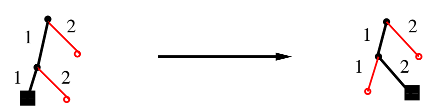
In the left hand side picture we have $r+1$ ancestors of the fat dotted vertex which are labeled 2 and s ancestors of the square vertex labeled 1, while in the right picture that there are v ancestors of the square vertex labeled 1. Then the ratio of derivatives of the two standard germs is:

$$\frac{\lambda_2^{-r-1}\lambda_2\lambda_1^s}{\lambda_2\lambda_1^v} = \frac{\lambda_1^{s-v}}{\lambda_2^{r+1}}$$

We can approximate arbitrarily close 1 by $\lambda_1^{s-v}/\lambda_2^{r+1}$, but then $s-v$ must be large, and in particular positive. This implies that we can automatically extend this to a standard germ as follows:



or, after removing nonessential information:



And we see now that a right continuation is impossible. Thus we get our claim. ∎

4.4 Proof of Proposition 1

Choose j which realizes the inequality in the hypothesis. We have sequences of smaller and smaller gaps (R_n, L_n) accumulating on 0, where $R_1 = \lambda_j + a_j$, $L_1 = a_{j+1}$ and R_n, L_n are there images by ϕ_1^n;

Since (R_n, L_n) is a gap of K we have $\varphi(R_n) \geq L_n$ for every increasing $\varphi \in \mathrm{Diff}^1(\mathbb{R}, K)$ which is nontrivial around $0 \in K$. It follows that

$$\varphi'(0) \geq \lim_n \frac{L_n}{R_n} = \frac{a_{j+1}}{\lambda_j + a_j} > \sqrt{\lambda_1}$$

If there is some k such that the quantity above is greater than $\sqrt{\lambda}$, it follows that $\chi(\mathrm{Diff}_0^1(K)) = \langle \lambda \rangle$.

We follow now the remaining steps in the proof of Theorem 6, without any essential modifications. The binary tree is now replaced by a rooted tree whose vertices have valence n. This yields an isomorphism with the generalized Thompson group F_n.

4.5 Proof of Theorem 8

Let $\mathrm{Diff}_a^1(\mathbb{R}^n, C)$ denotes the stabilizer of $a \in C$ in the group $\mathrm{Diff}^1(\mathbb{R}^n, C)$. We verify immediately that the map $\chi : \mathrm{Diff}_a^1(\mathbb{R}^n, C) \to GL(n, \mathbb{R})$, given by $\chi(\varphi) = D_a\varphi$ is a homomorphism. In the case when C is a product we can describe explicitly $\chi(\mathrm{Diff}_a^1(\mathbb{R}^n, C))$. For the sake of simplicity we restrict ourselves to the case $n = 2$. Consider $C = C_{\lambda_1} \times C_{\lambda_2}$.

Lemma 19. *Let $\lambda_i > 2$.*

1. *If $\lambda_1 \neq \lambda_2$ then*
$$\langle \lambda_1 \rangle \oplus \langle \lambda_2 \rangle \subset \chi(\mathrm{Diff}_a^1(\mathbb{R}^2, C)) \subset \langle \pm\lambda_1 \rangle \oplus \langle \pm\lambda_2 \rangle \tag{45}$$

2. *If $\lambda_1 = \lambda_2$ then*
$$\left\langle \langle \lambda_1 \rangle \oplus \langle \lambda_2 \rangle, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \subset \chi(\mathrm{Diff}_a^1(\mathbb{R}^2, C)) = \left\langle \langle \pm\lambda_1 \rangle \oplus \langle \pm\lambda_2 \rangle, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \tag{46}$$

Proof. From the first part of the proof of Theorem 4 we infer that whenever C is a product and $a \in C$ is fixed by φ its differential $D_a\varphi$ must send both horizontal and vertical vectors into horizontal or vertical vectors.

Moreover, when λ_i are distinct the horizontality/verticality of the segment should be preserved. Otherwise φ would induce a germ of $\mathcal{C}^1$-diffeomorphism $\phi : (\mathbb{R}, C_{\lambda_1}) \to (\mathbb{R}, C_{\lambda_2})$. The Hausdorff dimension of $C_\lambda \cap [0, \varepsilon)$ is $\frac{\log 2}{\log \lambda}$ for any $\varepsilon > 0$. A $\mathcal{C}^1$-diffeomorphism of intervals carrying a Cantor set into another should preserve the Hausdorff dimension and hence $\lambda_1 = \lambda_2$.

Therefore φ restricts to a germ of diffeomorphism $\phi_i : (\mathbb{R}, C_{\lambda_i}) \to (\mathbb{R}, C_{\lambda_i})$. By Lemma 10 $\chi(\phi_i) \subset \langle \pm\lambda_i \rangle$. This proves the right inclusion of the first item. Further let $a = (a_1, a_2)$. If a_i is an endpoint of C_i then $\chi(\phi_i) = \langle \lambda_i \rangle$, as the symmetry around a_i does not preserve C_i.

On the other hand when $\lambda_1 = \lambda_2$ the permutation R_a which exchanges two orthogonal axes meeting at $a \in C$ does belong to $\mathrm{Diff}_a^1(\mathbb{R}^2, C)$. Composing φ with the permutation R_a we arrive at the situation when $D_a\varphi$ is diagonal. Then we conclude as above. □

Lemma 20. *Let a be an endpoint of C. The group $\mathfrak{diff}^1_{a,\mathbb{R}^2}(C)$ is either isomorphic to $\mathbb{Z}^2$, when λ_i are distinct, or an extension of $\mathbb{Z}^2$ by $\mathbb{Z}/2\mathbb{Z}$, otherwise. For general $a \in C$ we have an isomorphism between $\mathfrak{diff}^1_{a,\mathbb{R}^2}(C)$ and $\chi(\mathrm{Diff}_a^1(\mathbb{R}^2, C))$.*

Proof. From Proposition 2 the kernel of χ consists of $\varphi \in \mathrm{Diff}_a^1(\mathbb{R}^2, C)$ for which the germ $\varphi\Big|_C$ is identity near a. □

Consider now that $\lambda_1 = \lambda_2$. Let now a and b be two left points of C. Denote by $D(a, b)$ the set of germs at a of classes in $\mathfrak{diff}^1_{\mathbb{R}^2}(C)$ having representatives $\varphi \in \mathrm{Diff}^1(\mathbb{R}^2, C)$ such that $\varphi(a) = b$. This set is acted upon transitively by $\mathfrak{diff}^1_{a,\mathbb{R}^2}(C)$, so that $D(a, b)$ consists of maps of the form:

$$\psi_{a,b}(x) = (b_{j,i} \pm \lambda^{k_j}(x_i - a_{j,i}))_{i=1,n} R_b^m, \qquad \text{for} \quad \text{any} \quad x \in I_j \cap C. \tag{47}$$

Now endpoints of C have to be sent into endpoints by any $\varphi \in \mathrm{Diff}^1(\mathbb{R}^2, C)$. Therefore, for any endpoint $a \in C$ we have $D_a\varphi \in \langle\, (\langle \pm\lambda \rangle \oplus \langle \pm\lambda \rangle)\, R_a \,\rangle$. But endpoints of C are dense in C. Since $D\varphi$ is continuous we have $D_a\varphi \in \langle\, (\langle \pm\lambda \rangle \oplus \langle \pm\lambda \rangle)\, R_a \,\rangle$, for any $a \in C$ and any $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C)$.

For a given φ both the norm $||D\varphi||$ and the determinant $\det(D\varphi)$ of its differential are continuous on interval $[0,1] \times [0,1]$ and hence these quantities are bounded. Moreover, the same argument for φ^{-1} shows that these quantities are also bounded from below away from 0, so that $D\varphi\Big|_C$ can only take finitely many values. The next point is the analogue of Lemma 11 to this situation:

Lemma 21. *There is a covering of C by a finite collection of disjoint standard rectangles I_k whose images are standard rectangles such that $\varphi\Big|_{C\cap I_k}$ is the restriction of an affine function and more precisely we have:*

$$\varphi(x) = (\varepsilon_k \lambda^{j_{k,1}} \oplus \varepsilon'_k \lambda^{j_{k,2}}) R_{b_k}^{m_k}(x - (\alpha_1, \alpha_2)) + \varphi(\alpha_1, \alpha_2), \text{ for } x \in I_k \cap C, \tag{48}$$

where $\varepsilon_k, \varepsilon'_k \in \{-1, 1\}$ and α_i are left points of C_i.

Proof. We can choose both I_k and their images to be standard rectangles, as in the case of central Cantor sets C_λ.

Let $c \in C$. Then $D_c\varphi = (\varepsilon_k \lambda^{j_{k,1}} \oplus \varepsilon'_k \lambda^{j_{k,2}}) R_{b_k}^{m_k}$, which we denote by A for simplicity in the proof. We have to prove that there exists a neighborhood U of c such that:

$$\varphi(x) = A(x - (\alpha_1, \alpha_2)) + \varphi(\alpha_1, \alpha_2), \qquad \text{for} \quad x \in U \cap C. \tag{49}$$

Such neighborhoods will cover C and we can extract a finite subcovering by clopen sets to get the statement.

This claim is true for end points $a = (\alpha_1, \alpha_2)$ of C. It is then sufficient to prove that whenever we have a sequence of end points $a_n \to a_\infty$ contained in a closed rectangle $U \subset [0,1]$ and a $\mathcal{C}^1$-diffeomorphism $\varphi : U \to \varphi(U) \subset [0,1]$ with $\varphi(C \cap U) \subset C$, there exists a neighborhood U_{a_∞} of a_∞ and an affine function ψ such that for large enough n the following holds:

$$\varphi(x) = \psi(x), \text{ for } x \in C_\lambda \cap U_{a_\infty}.$$

Around each left point a_n there are affine maps $\psi_{a_n} : U_{a_n,k_n} \to [0,1]$ defining germs in $D(a_n, c_n)$, where $c_n = \varphi(a_n)$, such that c_n converge to $c_\infty = \varphi(a_\infty)$ and

$$\varphi(x) = \psi_{a_n}(x), \text{ for } x \in C_\lambda \cap U_{a_n,k_n}.$$

We can further assume that $U_{a_n} \cap C_\lambda$ are clopen sets (i.e. closed and open), and we can take U_{a_n} to be standard rectangles $[\alpha_{n,1}, \beta_{n,1}] \times [\alpha_{n,2}, \beta_{n,2}]$ where $\beta_{n,i}$ are right points of C_i.

There is no loss of generality to assume that $D\psi_{a_n}\Big|_{C\cap U_{a_n}}$ is independent on n, say it equals $(\lambda^{m_1} \oplus \lambda^{m_2})R^j$. Replacing φ by its inverse φ^{-1} we can also assume that $m_1 \leq 0$. Since C_1 is invariant by the homothety of factor λ and center 0, we can further reduce the problem to the case where $m_1 = 0$. We can further assume that $m_2 \leq 0$ by the same trick and finally get rid of the second diagonal component of the differential. Then, by continuity, we have $D_{a_\infty}\varphi = R^j$.

Choose n large enough so that $|D_x\varphi(x) - \mathbf{1}| < \varepsilon$, for any x in a square centered at a_∞ and containing all U_{a_n}, with n large enough. where the exact value of ε will be chose later.

Let now consider the maximal standard rectangle of the form $U' = [\alpha_{n,1}, \beta_1] \times [\alpha_{n,2}, \beta_2]$ to which we can extend ψ_{a_n} to an affine function which coincides with φ on $C \cap U'$.

The endpoint (β_1, β_2) belongs to the closure of three maximal rectangular gaps: the rectangle Q which is opposite to U' is a product of two gaps, while the other two Q_v and Q_h are products of gaps with one (vertical or horizontal) side of U'. Since $D_x\varphi$ is close to identity the image of the rectangular gaps are closed to rectangular gaps of approximatively the same sizes. Now, the images by φ of the vertices of Q are points of C forming a rectangle, which is itself the product of two gaps. Thus the sizes of this rectangle belongs to the set $\{(1-2\lambda)\lambda^n; n \in \mathbb{Z}\} \times \{(1-2\lambda)\lambda^n; n \in \mathbb{Z}\}$. Since the ratios of two different lengths form a discrete set and $D_x\varphi$ is close to identity, the four points in the image form a rectangle congruent to Q. A similar argument holds now for the rectangles Q_v and Q_h. This implies that ψ_{a_n} can be extended to an affine function on a strictly larger rectangle, contradicting our assumptions. This proves the claim. □

This description shows that $\mathfrak{diff}_{\mathbb{R}^2}(C)$ is isomorphic to an extension by $\mathbb{Z}/2\mathbb{Z}$ of the 2-dimensional Thompson group $2V$ defined by Brin in [4].

5 Examples and counterexamples

5.1 Nonsparse Cantor sets with uncountable diffeomorphism group

Let $h(x) : \mathbb{R}_+ \to \mathbb{R}$ be C^∞-function satisfying the following conditions:

$$\begin{aligned} h(x) &= 0, && \text{for } 0 \leqslant x \leqslant 1,\ x > 2, \\ h(x) &> 0, && \text{for } 1 < x < 2, \\ h'(x) &> -1. && \end{aligned}$$

Since the maps $g_j : [0,1] \to [0,1]$ given by:

$$g_j = x + 2^{-2^j} h(2^j x) \tag{50}$$

are strictly increasing they are smooth diffeomorphisms of the interval. The support of g_j is $[2^{-j}, 2^{-j+1}]$ and hence the diffeomorphisms g_j pairwise commute. Their derivatives are of the form:

$$g_j'(x) = 1 + 2^{j-2^j} h'(2^j x),$$

and respectively

$$g_j^{(k)} = 2^{kj-2^j} h^{(k)}(2^j x), \qquad \text{for } k \geq 2.$$

Consider the group R consisting of bounded infinite sequences $\mathbf{m} = m_1, m_2, \ldots$ of integers, endowed with the term-wise addition.

There is a map $\Theta : R \to \mathrm{Diff}^0([0,1])$ given by:

$$\Theta(\mathbf{m}) = \lim_{n\to\infty} g_1^{m_1} \circ g_2^{m_2} \circ \cdots \circ g^{m_n}, \tag{51}$$

where g_j^m is the m-fold composition of g_j. The order in the previous definition does not matter, as the maps commute. The limit map $\Theta(m)$ is immediately seen to be a homeomorphism of $[0,1]$ which is a diffeomorphism outside 0.

Let first consider only those $\mathbf{m}$ where $m_j \in \{0,1\}$. Then we can compute first:

$$\lim_{x\to 0} \Theta(\mathbf{m})'(x) = 1,$$

and then

$$\lim_{x\to 0} \Theta(\mathbf{m})^{(k)}(x) = 1, \qquad \text{for } k \geq 2.$$

Therefore $\Theta(\mathbf{m})$ is a $\mathcal{C}^\infty$ diffeomorphism of $[0,1]$.

Moreover any element of R can be represented as a product of $\Theta(\mathbf{m})$, with $\mathbf{m}$ of having only 0 or 1 entries. This implies that $\Theta(R) \subset \mathrm{Diff}^\infty([0,1])$. Furthermore it is clear that Θ is injective, by looking at factor corresponding to the first place where two sequences disagree. This implies that Θ provides a faithful $\mathcal{C}^\infty$ action of R by $\mathcal{C}^\infty$ diffeomorphisms on $[0,1]$.

The dynamics of each g_j on its support $[2^{-j}, 2^{-j+1}]$ is of type north-south with repelling and attracting fixed points on the boundary. Pick up some $a_j \in (2^{-j}, 2^{-j+1})$, so that $b_j = g_j(a_j) > a_j$. Then the intervals $g_j^n((a_j, b_j))$ are all pairwise disjoint. If $C_j^0 \subset [a_j, b_j]$ is some Cantor set, then its orbit $C_j = \cup_{j=-\infty}^{\infty} g^j(C_j^0)$

is a g_j-invariant Cantor subset of $[2^{-j}, 2^{-j+1}]$. Moreover, for any $n \neq 0$ the restriction $g_j^n\Big|_{C_j}$ cannot be identity, since g_j^m is strictly increasing.

Then their union $C = \cup_{j=1}^{\infty} C_j$ is a Cantor subset of $[0,1]$ and for $\mathbf{m}$ not identically 0 we also have $\Theta(\mathbf{m})\Big|_C$ is not identity. This shows that the diffeomorphism group $\mathfrak{diff}^{\infty}(C)$ contains R. In particular, $\mathfrak{diff}^{\infty}(C)$ is uncountable.

5.2 Nonsparse Cantor set with trivial diffeomorphism group

Let X be obtained by removing a sequence of intervals, as follows. At the first step we remove from $[0,1]$ the central interval of length $\frac{1}{4}$. At the step m we have 2^m intervals which we label, starting from the leftmost to the rightmost as $I_1^{(m)}, I_2^{(m)}, \ldots, I_{2^m}^{(m)}$. We remove then from $I_j^{(m)}$ the central interval of length $2^{-2^{2^{m-1}-1+j}}$. The result of this procedure is a Cantor set X which is not sparse.

We claim that $\mathfrak{diff}^1(X) = 1$. Let first consider a point of X which is not a right point, for instance 0 and $\varphi \in \mathrm{Diff}_0^1(\mathbb{R}, X)$. If $\varphi'(0) \neq 1$ we can assume without loss of generality that $\varphi'(0) < 1$. Consider a sequence of gaps (x_n, y_n) approaching 0. We have either $\varphi(x_n, y_n) = (x_n, y_n)$ for all large enough n, or else $\varphi(x_n, y_n)$ is a different gap than (x_n, y_n).

If there exist infinitely many n such that the gap $\varphi(x_n, y_n)$ either coincides or is on the right side of (x_n, y_n) we have $\varphi(x_n) \geq x_n$ and taking the limit or $n \to \infty$ we would obtain $\varphi'(0) \geq 1$, contradicting our assumptions. Thus we can suppose that for all n the gap $\varphi(x_n, y_n)$ is different from (x_n, y_n) and lies to its left side. Moreover, for large enough n we have

$$\left| \frac{\varphi(x_n) - \varphi(y_n)}{x_n - y_n} \right| < 1$$

since otherwise we would obtain as above $\varphi'(0) \geq 1$. Now lengths of gaps belong to the discrete set $\{2^{-2^n}, n \in Z_+\}$ and there are not two gaps of the same length. Thus if $|x_n - y_n| = 2^{-2^{a_n}}$, for some increasing sequence a_n of integers then $|\varphi(x_n) - \varphi(y_n)| \leq 2^{-2^{1+a_n}}$. Therefore

$$\left| \frac{\varphi(x_n) - \varphi(y_n)}{x_n - y_n} \right| \leq 2^{-2^{1+a_n} - 2^{a_n}}$$

Taking $n \to \infty$ we derive that $\varphi'(0) = 0$, which contradicts the fact that φ was a diffeomorphism. This proves that $\mathfrak{diff}_0^1(X) = 1$.

Let now $a \in X$, with $a \neq 0$ and some germ $\varphi \in \mathrm{Diff}^1(\mathbb{R}, X)$ with $\varphi(0) = a$. As above we can suppose that $\varphi'(0) \leq 1$. Take a sequence of gaps (x_n, y_n) of length $|x_n - y_n| = 2^{-2^{a_n}}$, approaching 0 with increasing a_n. For infinitely many n the length of the image gap $|\varphi(x_n) - \varphi(y_n)|$ is smaller than $2|x_n - y_n|$, as otherwise $\varphi'(0) \geq 2$. It follows that for $a_n \geq 2$ we should have

$$\left| \frac{\varphi(x_n) - \varphi(y_n)}{x_n - y_n} \right| \leq 2^{-2^{1+a_n} - 2^{a_n}}$$

which leads again to a contradiction. This argument was not specific to $0 \in X$, and is valid for any point of X.

This shows that the only possibility left is that φ is identity.

5.3 Sparse Cantor set with trivial diffeomorphism group

We consider now the very sparse central Cantor C_0 obtained as follows. Start as above from the interval $I^{(0)} = [0,1]$ by removing a central gap $J_1^{(1)}$ of size $(1-\varepsilon)$. By recurrence at the n-th step we have 2^n intervals $I_j^{(n)}$, $j = 1, \ldots, 2^n$, numbered from the left to the right. To go further we remove a central gap $J_j^{(n+1)}$ from $I_j^{(n)}$ of length $|J_j^{(n+1)}| = (1 - \varepsilon^n)|I_j^{(n)}|$. The set so obtained is obviously a sparse Cantor set C_0.

Let $a \in C_0$. Let b_n be the right endpoint of the interval $I_{j_n}^{(n)}$ to which a belongs. Then set (x_n, y_n) for the gap $J_{j_n}^{(n)} \subset I_{j_n}^{(n)}$. There is no loss of generality of assuming that $a < x_n < y_n < b_n$. Given $\varphi \in \mathrm{Diff}_a^1(\mathbb{R}, C_0)$, with $\varphi'(a) \neq 1$, there are infinitely many n for which the gap $J_{j_n}^{(n)}$ is not fixed by φ. It follows that either $\varphi(y_n) < x_n$, or $\varphi(x_n) > y_n$, for infinitely many n. By symmetry we can assume that the second alternative holds. Then

$$\frac{\varphi(x_n) - x_n}{x_n - a} \geq \frac{|y_n - x_n|}{|x_n - a|} \geq \frac{(1-\varepsilon^n)|b_n - a|}{|x_n - a|} \geq \frac{(1-\varepsilon^n)|b_n - a|}{\varepsilon^n |b_n - a|} = \frac{1-\varepsilon^n}{\varepsilon^n} \tag{52}$$

Letting $n \to \infty$ we obtain that $\varphi'(a) = \infty$, contradiction. This proves that $\mathfrak{diff}^{1,+}(C_0) = 1$.

Consider now an arbitrary $\varphi \in \mathrm{Diff}^{1,+}(\mathbb{R}, C_0)$ and set $A = \{x \in C; \varphi(x) = x\}$. First A is nonempty because $0 \in A$ and A is closed as φ is continuous. By above, for any $a \in A$ we must have $\varphi'(a) = 1$. Since C_0 is sparse there exists an open neighborhood U_a of a such that $\varphi\Big|_{C_0 \cap U_a}$ is identity. Thus A is open and hence a clopen subset of C_0. Suppose that $C_0 - A \neq \emptyset$. Then it makes sense to consider $b = \inf\{x; x \in C_0 - A\}$. Since $C_0 - A$ is closed in C_0 the infimum is attained, namely $b \in C_0 - A \subset C_0$. Since φ is monotonic increasing and surjective and $\varphi(A) = A$ we must have $\varphi(b) = b$, so that $b \in A$, which is a contradiction. Hence $A = C_0$, so that $\varphi\Big|_{C_0}$ is identity.

Another proof can be given along the idea used for the nonsparse example X. If $\varphi'(0) < 1$, then φ must eventually contract towards 0 gaps. Let γ_n be a sequence of leftmost maximal gaps converging to 0: this means that for a positive sequence $b_n \to 0$ we consider among the gaps of maximal length within $[0, b_n]$ the one which is closest to 0. The same arguments as before show that infinitely many γ_n cannot be fixed by φ. Since $\varphi'(0) < 1$ infinitely many images $\varphi(\gamma_n)$ should get closer to 0 than γ_n. But then the quotients of the lengths of the two gaps is smaller than $\frac{\varepsilon^n(1-\varepsilon^{n+1}}{1-\varepsilon^n}$ which tends to 0 as n goes to infinity. This would imply that $\varphi'(0) = 0$ contradiction.

Another potential example. In order to convert the nonsparse example above X into a sparse Cantor set with the same properties, we have to mix ordinary gaps and very small gaps. Start as above from the interval $I^{(0)} = [0,1]$ by removing a central gap $LG^{(1)}$ of size $\frac{1}{3}$ and two very small gaps each one centered within an interval component of $I^{(0)} - LG^{(1)}$, namely $SG_1^{(1)}$ and $SG_2^{(1)}$ of lengths $2^{-2^{\alpha}}$ and $2^{-2^{\alpha+1}}$, respectively. Here α is chosen so that

$$\frac{1}{3} - 2^{-2^{\alpha}} > \frac{1}{6}$$

We obtain at the next stage four intervals $I_1^{(1)}, I_2^{(1)}, I_3^{(1)}, I_4^{(1)}$, labeled from the left to the right.

By recurrence at the n-th step we have 4^n intervals $I_j^{(n)}$, $j = 1, \ldots, 4^n$. To go further we remove first a central gap $LG_j^{(n+1)}$ from $I_j^{(n)}$ of length $|LG_j^{(n+1)}| = \frac{1}{3}|I_j^{(n)}|$. Further we remove two very small gaps each one centered within an interval component of $I_j^{(n)} - LG_j^{(n)}$, namely $SG_{2j+1}^{(n)}$ and $SG_{2j+2}^{(n)}$ of lengths $2^{-2^{\alpha+j+4^n}}$ and $2^{-2^{\alpha+j+1+4^n}}$,

Letting n go to ∞ we obtain a Cantor set MC. At each step we have that

$$|I_j^{(n+1)}| \geq \left(\frac{1}{3} - 2^{-2^{\alpha+4^n}}\right) \geq \frac{1}{6}|I_j^{(n)}|$$

so that

$$|I_j^{(n)}| \geq \left(\frac{1}{6}\right)^n$$

Now MC is $\frac{1}{3}$-sparse. In fact, let $a, b \in MC$. Let m be maximal such that there exists some j for which $a, b \in I_j^{(m)}$. Such an m clearly exists since the size of $I_j^{(m)}$ goes to 0 with m. On the other hand $LG_j^{(m+1)} \subset (a,b) \subset I_j^{(m)}$, since otherwise m would not be maximal with this property. This means that there is a gap of length at least $\frac{1}{3}$ in (a,b).

Further each point $a \in MC$ can be approached by a sequence of large gaps $LG_{j_n}^{(a_n)}$ as well as by a sequence of gaps $SG_{m_n}^{(c_n)}$.

We believe that $\mathfrak{diff}^1(MC) = 1$, but the arguments of the previous section alone do not suffice for that.

5.4 Split Cantor sets

Two Cantor sets $C_i \subset \mathbb{R}^n$ are *locally smoothly nonequivalent* if for any $p_i \subset C_i$ there is no $\mathcal{C}^1$-diffeomorphism germ $(\mathbb{R}^n, C_1, p_1) \to (\mathbb{R}^n, C_2, p_2)$.

A Cantor set in $C \subset \mathbb{R}^n$ is said to be *smoothly split* if we can write $C = C_1 \cup C_2$ as a union of two Cantor sets with C_1 and C_2 locally smoothly nonequivalent.

We have the following easy:

Proposition 3. *Let $n \geq 1$ and $C \subset \mathbb{R}^n$ be a Cantor set which is smoothly split as $C_1 \cup C_2$. Then $\mathfrak{Diff}^1(C) = \mathfrak{Diff}^1(C_1) \times \mathfrak{Diff}^1(C_2)$.*

Proof. In this situation C_i are contained into disjoint open sets U_i. Then diffeomorphisms preserving C should also send each C_i into itself. Furthermore all elements from $\mathfrak{Diff}^1(C_1) \times \mathfrak{Diff}^1(C_2)$ can be realized as classes of pairs of commuting diffeomorphisms supported in U_i. □

According to Remark 1 the central Cantor sets C_λ are pairwise locally smoothly nonequivalent. In particular the union $C_\lambda \cup 2 + C_\mu$ of two distinct Cantor sets, one of which is translated by 2 is a split Cantor set. It follows that

$$\mathfrak{Diff}^1(C_\lambda \cup 2 + C_\mu) = \mathfrak{Diff}^1(C_\lambda) \times \mathfrak{Diff}^1(C_\mu) \cong F \times F,$$

for distinct λ and μ.

It is not yet clear what would be $\mathfrak{Diff}^1(C_\lambda \cup C_\mu)$, for the moment.

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