LECTURES ON THE TOPOLOGY OF SYMPLECTIC FILLINGS OF CONTACT 3-MANIFOLDS

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ABSTRACT. These are some lecture notes for my mini-course in the Graduate Workshop on 4-Manifolds, August 18-22, 2014 at the Simons Center for Geometry and Physics. Most of this manuscript is an adaptation of my survey article *On the topology of fillings of contact 3-manifolds* (http://home.ku.edu.tr/ bozbagci/SurveyFillings.pdf) that will appear in the Proceedings of the Conference on "Interactions between low dimensional topology and mapping class groups" that was held in July 1-5, 2013 at the Max Planck Institute for Mathematics, Bonn.

1. LECTURE 1: SYMPLECTIC FILLINGS OF CONTACT MANIFOLDS

1.1. Topology of a Stein manifold.

Definition 1.1. *A Stein manifold is an affine complex manifold, i.e., a complex manifold that admits a proper holomorphic embedding into some $\mathbb{C}^N$.*

An excellent reference for Stein manifolds in the context of symplectic geometry is the recent book of Cieliebak and Eliashberg [14]. In the following we give an equivalent definition of a Stein manifold.

Definition 1.2. *An almost-complex structure on an even-dimensional manifold X is a complex structure on its tangent bundle TX, or equivalently a bundle map $J : TX \to TX$ with $J \circ J = -id_{TX}$. The pair (X, J) is called an almost complex manifold. It is called a complex manifold if the almost complex structure is integrable, meaning that J is induced via multiplication by i in any holomorphic coordinate chart.*

Example. The sphere S^n admits an almost complex structure if and only if $n \in \{2, 6\}$; S^2 is complex and it is not known whether or not S^6 admits a complex structure.

Let $\phi : X \to \mathbb{R}$ be a smooth function on an almost complex manifold (X, J). We set $d^{\mathbb{C}}\phi := d\phi \circ J$ (which is a 1-form) and hence $\omega_\phi := -dd^{\mathbb{C}}\phi$ is a 2-form which is skew-symmetric (by definition). In general, ω_ϕ may fail to be J-invariant, i.e, the condition $\omega_\phi(Ju, Jv) = \omega_\phi(u, v)$ may not hold for an arbitrary almost complex structure J. However,

Lemma 1.3. *If J is integrable, then ω_ϕ is J-invariant.*

Proof. [14, Section 2.2] The claim can be verified by a local computation: The Euclidean space $\mathbb{R}^{2n}$ with linear coordinates $(x_1, y_1, \ldots, x_n, y_n)$ has a standard complex structure J defined as

$$J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j} \text{ and } J(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}.$$

The space $(\mathbb{R}^{2n}, J)$ can be identified $(\mathbb{C}^n, i)$ via $z_j = x_j + iy_j$, where we use linear coordinates $(z_1, \ldots, z_n)$ for $\mathbb{C}^n$ and $i = \sqrt{-1}$ denotes the complex multiplication on $\mathbb{C}^n$. Let $\phi : \mathbb{R}^{2n} = \mathbb{C}^n \to \mathbb{R}$ be a smooth function. We calculate that

$$\begin{aligned} d\phi &= \sum_j \big(\frac{\partial \phi}{\partial x_j} dx_j + \frac{\partial \phi}{\partial y_j} dy_j\big) \\ &= \sum_j [\frac{1}{2}\big(\frac{\partial \phi}{\partial x_j} - i\frac{\partial \phi}{\partial y_j}\big)(dx_j + idy_j) + \frac{1}{2}\big(\frac{\partial \phi}{\partial x_j} + i\frac{\partial \phi}{\partial y_j}\big)(dx_j - idy_j)] \\ &= \sum_j \big(\frac{\partial \phi}{\partial z_j} dz_j + \frac{\partial \phi}{\partial \bar{z}_j} d\bar{z}_j\big) \\ &= \partial\phi + \overline{\partial}\phi. \end{aligned}$$

Since $\boxed{dz_j \circ i = idz_j}$ and $\boxed{d\bar{z}_j \circ i = -id\bar{z}_j}$ we have

$$d^{\mathbb{C}}\phi = \sum_j \big(\frac{\partial \phi}{\partial z_j} dz_j \circ i + \frac{\partial \phi}{\partial \bar{z}_j} d\bar{z}_j \circ i\big) = \sum_j \big(i\frac{\partial \phi}{\partial z_j} dz_j - i\frac{\partial \phi}{\partial \bar{z}_j} d\bar{z}_j\big) = i\partial\phi - i\bar{\partial}\phi.$$

Using $d = \partial + \bar{\partial}$ we get

$$dd^{\mathbb{C}}\phi = (\partial + \bar{\partial})(i\partial\phi - i\bar{\partial}\phi) = -2i\partial\overline{\partial}\phi$$

and hence $\boxed{\omega_\phi = 2i\partial\overline{\partial}\phi}$ where more explicitly we can write

$$\partial\overline{\partial}\phi = \sum_{j,k} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

The form $\partial\overline{\partial}\phi$ is i-invariant since for all j, k, we observe that

$$\begin{aligned} dz_j \wedge d\bar{z}_k \ (iu, iv) &= dz_j(iu)d\bar{z}_k(iv) - dz_j(iv)d\bar{z}_k(iu) \\ &= iu_j(-i)\bar{v}_k - iv_j(-i)\bar{u}_k \\ &= u_j\bar{v}_k - v_j\bar{u}_k \\ &= dz_j \wedge d\bar{z}_k \ (u, v). \end{aligned}$$

It follows that ω_ϕ is i-invariant. □

Definition 1.4. *Let (X, J) be an almost complex manifold. A smooth function $\phi : X \to \mathbb{R}$ is called J-convex if $\omega_\phi(u, Ju) > 0$ for all nonzero vectors $u \in TX$.*

The condition $\omega_\phi(u, Ju) > 0$ is often described as ω_ϕ being positive on the complex lines in TX, since for any $u \neq 0$, the linear space spanned by u and Ju can be identified with $\mathbb{C}$ with its usual orientation.

Now let g_ϕ be the 2-tensor defined by $g_\phi(u, v) := \omega_\phi(u, Jv)$. The J-convexity condition in Definition 1.4 is indeed equivalent to g_ϕ being positive definite, i.e., $g_\phi(u, u) > 0$ for any nonzero vector $u \in TX$.

Lemma 1.5. *If ω_ϕ is J-invariant, then g_ϕ is symmetric and $H_\phi := g_\phi - i\omega_\phi$ is a Hermitian form.*

Proof. The 2-tensor g_ϕ is symmetric since:

$$\begin{aligned} g_\phi(u, v) &= \omega_\phi(u, Jv) \quad \text{(by definition)} \\ &= \omega_\phi(-J^2u, Jv) \quad (J^2 = -Id) \\ &= \omega_\phi(-Ju, v) \quad (\omega_\phi \text{ is } J\text{-invariant}) \\ &= -\omega_\phi(v, -Ju) \quad (\omega_\phi \text{ is skew-symmetric}) \\ &= -g_\phi(v, -u) \quad \text{(by definition)} \\ &= g_\phi(v, u) \quad (g_\phi \text{ is bilinear}). \end{aligned}$$

It is clear that H_ϕ is $\mathbb{R}$-bilinear, since g_ϕ and ω_ϕ are both $\mathbb{R}$-bilinear. Now we verify that H_ϕ is complex linear in the first variable:

$$\begin{aligned} H_\phi(Ju, v) &= g_\phi(Ju, v) - i\omega_\phi(Ju, v) \\ &= g_\phi(v, Ju) + i\omega_\phi(v, Ju) \\ &= \omega_\phi(v, -u) + ig_\phi(v, u) \\ &= i\big(g_\phi(u, v) - i\omega_\phi(u, v)\big) \\ &= iH_\phi(u, v) \end{aligned}$$

and we check that

$$\begin{aligned} \overline{H_\phi(v, u)} &= g_\phi(v, u) + i\omega_\phi(v, u) \\ &= g_\phi(u, v) - i\omega_\phi(u, v) \\ &= H_\phi(u, v). \end{aligned}$$

$\square$

By Lemma 1.3 and Lemma 1.5, it follows that,

For any *complex* manifold (X, J), a smooth function $\phi : X \to \mathbb{R}$ is J-convex if and only if the Hermitian form H_ϕ is positive definite.

Lemma 1.6. *A smooth function $\phi : \mathbb{C}^n \to \mathbb{R}$ is i-convex if and only if the Hermitian matrix $\big(\dfrac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}\big)$ is positive definite.*

Proof. We set $h_{jk} := \dfrac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}$ and compute

$$\begin{aligned}
\omega_\phi(u, v) &= 2i \sum_{j,k} h_{j,k} dz_j \wedge d\bar{z}_k(u, v) \\
&= 2i \sum_{j,k} h_{j,k}\big(dz_j(u) \wedge d\bar{z}_k(v) - dz_j(v) \wedge d\bar{z}_k(u)\big) \\
&= 2i \sum_{j,k} h_{j,k}(u_j \bar{v}_k - v_j \bar{u}_k) \\
&= 2i \sum_{j,k} h_{j,k} u_j \bar{v}_k - 2i \sum_{j,k} h_{j,k} v_j \bar{u}_k \\
&= 2i \sum_{j,k} h_{j,k} u_j \bar{v}_k - 2i \sum_{j,k} \bar{h}_{k,j} v_j \bar{u}_k \quad (\text{used } \bar{h}_{k,j} = h_{j,k}) \\
&= 2i \sum_{j,k} h_{j,k} u_j \bar{v}_k - 2i \sum_{j,k} \bar{h}_{j,k} \bar{u}_j v_k \quad (\text{switched } j \leftrightarrow k \text{ in the second sum}) \\
&= -4 \ \mathrm{Im}\big(\sum_{j,k} h_{jk} u_j \bar{v}_k\big)
\end{aligned}$$

and hence it follows that

$$H_\phi(u, v) = g_\phi(u, v) - i\omega_\phi(u, v) = 4 \sum_{j,k} h_{jk} u_j \bar{v}_k.$$

Therefore we conclude that the Hermitian form H_ϕ is positive definite (i.e.,$H_\phi(u, u) > 0$ for all $u \neq 0$) if and only if the hermitian matrix $\big(\dfrac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}\big)$ is positive definite. □

Definition 1.7. *Any real valued smooth function on X is called exhausting if it is proper and bounded below.*

Lemma 1.8. *Every Stein manifold admits an exhausting J-convex function.*

Proof. We claim that the map $\phi : \mathbb{C}^N \to \mathbb{R}$ defined as $\phi(z) = |z|^2$ is an exhausting i-convex function on $\mathbb{C}^N$ with respect to the standard complex structure $i : \mathbb{C}^N \to \mathbb{C}^N$. To see that ϕ is i-convex we simply observe that

$$\phi(z) = \sum z_j \bar{z}_j \text{ and } \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} = \frac{\partial z_k}{\partial z_j} = \delta_{jk}.$$

Thus $\left(\frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}\right)$ is the identity matrix which is obviously positive definite. Since ϕ is proper and $|\phi(z)| \geq 0$, its restriction to any properly embedded holomorphic submanifold of $\mathbb{C}^N$ is an exhausting i-convex function. $\square$

The converse of Lemma 1.8 is due to Grauert [40]:

Theorem 1.9. *A complex manifold (X, J) is Stein if and only if it admits an exhausting J-convex function $\phi : X \to \mathbb{R}$.*

Remark. The classical definition of a Stein manifold originates from the concept of holomorphic convexity. We refer to [14, Section 5.3] for an extensive discussion on the equivalence of the affine definition, the definition using J-convex functions (Theorem 1.9) and the classical definition of a Stein manifold.

Every exhausting J-convex function on a Stein manifold (X, J) becomes an exhausting J-convex *Morse* function by a C^2-small perturbation. The following result of Milnor puts strong restrictions on the topology of the Stein manifolds.

Proposition 1.10 (Milnor)**.** *If (X, J) is a Stein manifold of real dimension $2n$, then the index of each critical point of a J-convex Morse function on X is at most equal to n.*

Therefore, if X is a smooth manifold of real dimension $2n$, a necessary condition for X to carry a Stein structure is that its handle decomposition does not include any handles of indices greater than n. Note that there is another obvious necessary condition—the existence of an almost complex structure on X. Eliashberg [20] proved that, for $n > 2$, these two necessary conditions are also sufficient for the existence of a Stein structure:

Theorem 1.11 (Eliashberg [20])**.** *Let X be a $2n$-dimensional smooth manifold, where $n > 2$. Suppose that X admits an almost complex structure J, and there exists an exhausting Morse function $\phi : X \to \mathbb{R}$ without critical points of index $> n$. Then J is homotopic through almost complex structures to a complex structure J' such that ϕ is J'-convex. In particular, the complex manifold (X, J') is Stein.*

For the case $n = 2$, the corresponding result is described in Theorem 3.5.

1.2. **Symplectic geometry of Stein manifolds.** In the following, we briefly explain how symplectic geometry is built into Stein manifolds.

Definition 1.12. *A symplectic form on a $2n$-dimensional manifold X is a differential 2-form ω that is closed $(d\omega = 0)$ and non-degenerate, meaning that for every nonzero vector $u \in TX$ there is a vector $v \in TX$ such that $\omega(u, v) \neq 0$. The pair (X, ω) is called a symplectic manifold. A submanifold $S \subset X$ is called symplectic if $\omega|_S$ is non-degenerate and it is called isotropic if for all $p \in S$, T_pS is contained in its ω-orthogonal complement in T_pX.*

Remark. The non-degeneracy condition in Definition 1.12 is equivalent to $\omega^n \neq 0$, where ω^n denotes the n-fold wedge product $\omega \wedge \ldots \wedge \omega$. A symplectic manifold (X^{2n}, ω) has a natural orientation defined by the non-vanishing top form ω^n. We will always assume that a symplectic manifold (X^{2n}, ω) is oriented such that $\omega^n > 0$. It follows that an orientable closed manifold X^{2n} can carry a symplectic form ω only if $H^2(X, \mathbb{R})$ is non-trivial, since any $[\omega] \neq 0 \in H^2(X, \mathbb{R})$. The sphere S^n, for example, is not symplectic for $n > 2$.

Definition 1.13. *We say that a symplectic form ω on an even dimensional manifold X is* compatible *with an almost complex structure J if ω is J-invariant and ω* tames *J, i.e., $\omega(u, Ju) > 0$ for all nonzero vectors $u \in TX$.*

It is well-known (see, for example, [52, 13]) that

Theorem 1.14. *For any symplectic manifold (X, ω), there exists an almost complex structure on X compatible with ω and the space of compatible almost complex structures is contractible.*

Remark. Since the condition $d\omega = 0$ is not used in the proof, this statement in fact holds for any symplectic vector bundle over a smooth manifold.

Suppose that (X, J) is a *complex* manifold. Since for any $\phi : X \to \mathbb{R}$ the 2-tensor g_ϕ is symmetric as we showed above, ϕ is J-convex if and only if g_ϕ defines a Riemannian metric on V. This is indeed equivalent to requiring that ω_ϕ is non-degenerate. But since ω_ϕ is closed (by definition), and taming condition implies non-degeneracy, we conclude that

> For any *complex* manifold (X, J), a smooth function $\phi : X \to \mathbb{R}$ is J-convex if and only if ω_ϕ tames J.

Definition 1.15. *A vector field V on a symplectic manifold (X, ω) is called a Liouville vector field if $\mathcal{L}_V \omega = \omega$, where $\mathcal{L}$ stands for the Lie derivative.*

Suppose that (X, J, ϕ) is a Stein manifold. Let $\nabla\phi$ denote the gradient vector field with respect to the metric g_ϕ, which is uniquely determined by the equation

$$d\phi(u) = g_\phi(\nabla\phi, u).$$

Define the 1-form $\alpha_\phi := \iota_{\nabla\phi}\omega_\phi$, that is, $\alpha_\phi(v) = \omega_\phi(\nabla\phi, v)$. (The 1-form α_ϕ is ω_ϕ-dual to the vector field $\nabla\phi$.) It follows that

Lemma 1.16. *The gradient vector field $\nabla\phi$ is a Liouville vector field for ω_ϕ.*

Proof. To see this we first observe that

$$\boxed{(\iota_{\nabla\phi}\omega_\phi)(v) = \omega_\phi(\nabla\phi, v) = -g_\phi(\nabla\phi, Jv) = -d\phi(Jv) = -(d^{\mathbb{C}}\phi)(v).}$$

Thus, by Cartan's formula, we have

$$\boxed{\mathcal{L}_{\nabla\phi}\omega_\phi = d(\iota_{\nabla\phi}\omega_\phi) + \iota_{\nabla\phi}d\omega_\phi = d(\iota_{\nabla\phi}\omega_\phi) = -d(d^{\mathbb{C}}\phi) = \omega_\phi.}$$

□

Note that a generic J-convex function is a Morse function. Moreover, for an exhausting J-convex Morse function $\phi : X \to \mathbb{R}$ on a Stein manifold (X, J), the gradient vector field $\nabla\phi$ may be assumed to be complete, after composing ϕ by a suitable function $\mathbb{R} \to \mathbb{R}$.

Definition 1.17. *A Weinstein structure on a $2n$-dimensional manifold X is a triple (ω, V, ϕ), where ω is a symplectic form, $\phi : X \to \mathbb{R}$ is an exhausting Morse function and V is a complete Liouville vector field which is gradient-like for ϕ. The quadruple (X, ω, V, ϕ) is called a Weinstein manifold.*

We conclude that

> Every Stein manifold (X, J, ϕ) is a Weinstein manifold $(X, \omega_\phi, \nabla\phi, \phi)$.

Moreover, the symplectic structure defined above on a Stein manifold (X, J) is independent of the choice of the J-convex function in the following sense:

Theorem 1.18. [14, Chapter 11] *Let ϕ_j be an exhausting J-convex Morse function on a Stein manifold (X, J) such that $\nabla\phi_j$ is complete for $j = 1, 2$. Then (X, ω_{ϕ_1}) is symplectomorphic to (X, ω_{ϕ_2}).*

Definition 1.19. *Two symplectic manifolds (X_1, ω_1) and (X_2, ω_2) are said to be symplectomorphic if there exists a diffeomorphism $\varphi : X_1 \to X_2$ such that $\varphi^*\omega_2 = \omega_1$.*

Remark. We would like to point out that *a Stein manifold is non-compact*. In fact, no *compact* complex manifold of complex dimension at least one can be a complex analytic submanifold of any Stein manifold. This is because if M is a compact analytic submanifold of a Stein manifold, then each coordinate function on $\mathbb{C}^N$ restricts to a nonconstant holomorphic function on M which is a contradiction unless M is zero-dimensional.

1.3. **Contact manifolds.** The reader is advised to turn to [35] for a thorough discussion about the topology of contact manifolds.

Definition 1.20. *A contact structure on a $(2n + 1)$-dimensional manifold Y is a tangent hyperplane field $\xi = \ker\alpha \subset TY$ for some 1-form α such that $\alpha \wedge (d\alpha)^n \neq 0$. The 1-form α is called a contact form and the pair (Y, ξ) is called a contact manifold.*

Note that the condition $\alpha \wedge (d\alpha)^n \neq 0$ is independent of the choice of α defining ξ, since any other 1-form defining ξ must be of the form $h\alpha$, for some non-vanishing real valued smooth function h on Y and we have:

$$\boxed{(h\alpha) \wedge (d(h\alpha))^n = (h\alpha) \wedge (hd\alpha + dh \wedge \alpha)^n = h^{n+1}(\alpha \wedge (d\alpha)^n) \neq 0.}$$

In these lectures, we assume that α is global 1-form, which is equivalent to the quotient line bundle TY/ξ being trivial. In this case, the contact structure $\xi = \ker\alpha$ on Y is said to be *co-orientable* and Y is necessarily orientable since $\alpha \wedge (d\alpha)^n$ is a non-vanishing top-dimensional form, i.e., a volume form on Y. Moreover ξ is called *co-oriented* if an orientation for TY/ξ is fixed. When Y is equipped with a specific orientation, one can speak of a *positive* or a *negative* co-oriented contact structure ξ on Y, depending on whether the orientation induced by ξ agrees or not with the given orientation of Y.

In terms of the defining 1-form α, the contact condition in Definition 1.20 is equivalent to $d\alpha|_\xi$ being non-degenerate. In particular, $(\xi, d\alpha|_\xi)$ is a symplectic vector bundle, where for any co-oriented contact structure ξ, the symplectic structure on ξ_p is defined uniquely up to a positive conformal factor.

All the contact structures in these notes are assumed to be positive and co-oriented.

Definition 1.21. *Two contact manifolds* (Y_1, ξ_1) *and* (Y_2, ξ_2) *are said to be contactomorphic if there exists a diffeomorphism* $\varphi : Y_1 \to Y_2$ *such that* $\varphi_*(\xi_1) = \xi_2$.

Example 1.22. *In the coordinates* $(x_1, y_1, \dots, x_n, y_n, z)$, *the standard contact structure* ξ_{st} *on* $\mathbb{R}^{2n+1}$ *can be given, up to contactomorphism, as the kernel of any of the* 1*-forms*

$$dz + \sum_{i=1}^{n} x_i\, dy_i$$

$$dz - \sum_{i=1}^{n} y_i\, dx_i$$

$$dz + \sum_{i=1}^{n} x_i\, dy_i - y_i\, dx_i = dz + \sum_{i=1}^{n} r_i^2 d\theta_i,$$

where, for the last equality, we used the polar coordinates (r_i, θ_i) *in the* (x_i, y_i)*-plane.*

An important class of submanifolds of contact manifolds is given by the following definition.

Definition 1.23. *A submanifold* L *of a contact manifold* (Y^{2n+1}, ξ) *is called an isotropic submanifold if* $T_pL \subset \xi_p$ *for all* $p \in L$. *An isotropic submanifold of maximal dimension* n *is called a Legendrian submanifold.*

Remark. Although every closed oriented 3-manifold admits a contact structure, there is an obstruction to the existence of contact structures on odd-dimensional manifolds of dimension ≥ 5. If (Y^{2n+1}, ξ) is a contact structure, then the tangent bundle of Y has a splitting as $TY = \xi \oplus \underline{\mathbb{R}}$. The contact structure $\xi = \ker\alpha$ is a symplectic vector bundle on X since $d\alpha$ is symplectic on ξ. Therefore ξ admits a compatible complex vector bundle structure. Such a splitting of TY is called an almost contact structure and it reduces the structure group of TY to $U(n) \times 1 \subset GL(2n+1, \mathbb{R})$.

For example, the simply-connected closed 5-manifold $SU(3)/SO(3)$ does not admit a contact structure since it does not admit an almost contact structure (see [35]).

1.4. **What is a Stein/symplectic filling?**

Definition 1.24. *A closed contact manifold* (Y, ξ) *is said to be strongly symplectically fillable if there is a compact symplectic manifold* (W, ω) *such that* $\partial W = Y$ *as oriented manifolds,* ω *is exact near the boundary and its primitive* α *can be chosen in such a way that* $\ker(\alpha|_Y) = \xi$. *In this case we say that* (W, ω) *is a strong symplectic filling of* (Y, ξ).

Definition 1.25. *We say that a compact symplectic manifold (W, ω) is a convex filling of closed contact manifold (Y, ξ) if $\partial W = Y$ as oriented manifolds and there exists a Liouville vector field V defined in a neighborhood of Y, pointing out of W along Y, satisfying $\xi = \ker(\iota_V \omega|_Y)$. In this case, (Y, ξ) is said to be the convex boundary of (W, ω). If V points into W along Y, on the other hand, then we say that (W, ω) is a concave filling of (Y, ξ) and (Y, ξ) is said to be the concave boundary of (W, ω).*

It is easy to see that the notion of a convex filling is the same as the notion of a strong symplectic filling: Given a convex filling, define the 1-form $\alpha := \iota_V \omega$ near Y and observe that $d\alpha = \omega$ by Cartan's formula. Conversely, given a strong symplectic filling, one solves the equation $\alpha := \iota_V \omega$ for V near the boundary Y, and observes that V is a Liouville vector field again by Cartan's formula.

Lemma 1.26. *If V is a Liouville vector field for a symplectic form ω on a manifold X, then the 1-form $\alpha := \iota_V \omega|_Y$ is a contact form when restricted to any hypersurface Y in X transverse to V.*

Proof. The form

$$\boxed{\alpha \wedge (d\alpha)^n = \iota_V \omega \wedge \omega^n = \frac{1}{n+1} \iota_V (\omega^{n+1})}$$

restricts to a volume form on any hypersurface Y in X transverse to V. □

Suppose that (X, J, ϕ) is a Stein manifold. Then, a regular level set $\phi^{-1}(t)$ is a compact hypersurface in X which is transverse to the Liouville vector field $\nabla \phi$ for the symplectic form ω_ϕ. Therefore α_ϕ restricts to a contact form on α_ϕ and the sublevel set $\phi^{-1}(-\infty, t]$ is a special kind of strong symplectic filling of the contact manifold $(\phi^{-1}(t), \ker(\alpha_\phi))$—which leads to the following definition.

Definition 1.27. *A compact complex manifold (W, J) with boundary $\partial W = Y$ is a Stein domain if it admits an exhausting J-convex function $\phi : W \to \mathbb{R}$ such that Y is a regular level set. Then we say that the contact manifold $(Y, \xi = \ker(\alpha_\phi|_Y))$ is Stein fillable and (W, J) is a called a Stein filling of it.*

Remark. A Stein filling is a strong symplectic filling, where the symplectic form is exact, because $\nabla \phi$ is a Liouville vector field for ω_ϕ as was shown in Lemma 1.16.

We can describe the contact structure $\ker(\alpha_\phi)$ on the hypersurface $\phi^{-1}(t)$ with another point of view as follows.

Let Y be a oriented smooth real hypersurface in a *complex* manifold (X, J). The complex tangencies $\xi := TY \cap J(TY)$ along Y form a unique complex hyperplane distribution in TY. The complex orientation of ξ, together with the orientation of Y gives a co-orientation to ξ, and hence $\xi = \ker \alpha$ for some 1-form α, where α defines the given co-orientation. The *Levi form* of Y is defined as $\omega_Y(u,v) := d\alpha|_\xi(u, Jv)$. Note that by taking the co-orientation of Y into account, ω_Y is defined up to multiplication by a positive function. The hypersurface Y is called *J-convex* if its Levi form is positive definite, i.e., $\omega_Y(u, Ju) > 0$ for every non-zero $u \in \xi$. This implies that ξ is a contact structure on Y since $d\alpha$ is non-degenerate on ξ.

We say that a compact complex manifold (W, J) with J-convex boundary Y is a *holomorphic filling* of its contact boundary (Y, ξ). It turns out that, if (X, J, ϕ) is any Stein manifold, and t is a regular value of $\phi : X \to \mathbb{R}$, then $\omega_Y(u,v) = \omega_\phi(u,v)$, where $Y = \phi^{-1}(t)$. This implies that a Stein filling is a holomorphic filling.

The upshot is that every regular level set of a J-convex function on a Stein manifold is a J-convex hypersurface equipped with a contact structure given by the complex tangencies to the hypersurface.

Not every holomorphic filling is a Stein filling in higher dimensions, but we have

Theorem 1.28 (Bogomolov and de Oliveira [12]). *If (W, J) is a minimal compact complex manifold of* complex dimension 2 *with J-convex boundary $(\partial W, \xi)$, then J can be deformed to J' such that (W, J') is a Stein filling of $(\partial W, \xi)$.*

In particular,

A closed contact 3-manifold is Stein fillable if and only if it is holomorphically fillable.

There is also the notion of a weak symplectic filling which we will discuss only for 3-dimensional contact manifolds. We refer the reader to [50], for the detailed study of weak versus strong symplectic fillings of higher dimensional contact manifolds.

Definition 1.29. *A contact 3-manifold (Y, ξ) is said to be weakly symplectically fillable if there is a compact symplectic 4-manifold (W, ω) such that $\partial W = Y$ as oriented manifolds and $\omega|_\xi > 0$. In this case we say that (W, ω) is a weak symplectic filling of (Y, ξ).*

2. Lecture 2: Basic results about fillings of contact 3-manifolds

In the following, we assume that the reader is familiar with smooth and contact surgery as well as Weinstein handle attachments (cf. [35, 39, 56]).

2.1. The standard contact S^3.

We start with describing the standard contact structure on S^3: Let $\omega_{st} := dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ denote the standard symplectic 2-form on $\mathbb{R}^4$ in the coordinates (x_1, y_1, x_2, y_2). Let

$$\lambda := \frac{1}{2} x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2$$

be the standard primitive of ω_{st}. The standard contact structure on $S^3 \subset \mathbb{R}^4$ is defined as $\xi_{st} = \ker \alpha$, where $\alpha = \lambda|_{S^3}$. The vector field

$$V = x_1 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial y_2}$$

is a Liouville vector field for ω_{st} which is transverse to S^3 (pointing outward), which shows that (D^4, ω_{st}) is a strong symplectic filling of the standard tight contact 3-sphere (S^3, ξ_{st}). In other words, (S^3, ξ_{st}) is the convex boundary of (D^4, ω_{st}).

Consider the standard complex structure J_{st} on $\mathbb{R}^4$ given by

$$J_{st}(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j} \text{ and } J_{st}(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j} \quad \text{for} \quad j = 1, 2.$$

Note that J_{st} is just the complex multiplication by i when $\mathbb{R}^4$ is identified with $\mathbb{C}^2$. Let $\phi : \mathbb{R}^4 \to \mathbb{R}$ be defined by

$$\phi(x_1, y_1, x_2, y_2) = x_1^2 + y_1^2 + x_2^2 + y_2^2.$$

Then ϕ is an exhausting J_{st}-convex function on $\mathbb{R}^4$, where S^3 is regular a level set. It is easy to check that $\alpha_{st} = -\frac{1}{2}(d\phi \circ J_{st})|_{S^3}$. This shows that D^4 equipped with the restriction of standard complex structure J_{st} on $\mathbb{R}^4$ is a Stein filling of (S^3, ξ_{st}). There is yet another description of ξ_{st} as the *complex tangencies*, i.e.,

$$\xi = TS^3 \cap J_{st}(TS^3).$$

We should also point out, in surgery theory, (S^3, ξ_{st}) is commonly defined as the extension of the standard contact structure on $\mathbb{R}^3$ which is given as the kernel of

$$dz + x\, dy - y\, dx = dz + r^2 d\theta$$

in the coordinates (x, y, z) or using polar coordinates (r, θ) for the xy-plane. In other words, for any $p \in S^3$, $(S^3 \setminus \{p\}, \xi_{st}|_{S^3 \setminus \{p\}})$ is contactomorphic to the standard contact $\mathbb{R}^3$. Note that the standard contact structure on $\mathbb{R}^3$ can also be defined as $\ker(dz + x\, dy)$ or

$\ker(dz - y\,dx)$, up to isomorphism.

Definition 2.1. *We say that two symplectic* 4*-manifolds* (W_1, ω_1) *and* (W_2, ω_2) *with convex boundary are symplectically deformation equivalent if there is a diffeomorphism* $\varphi : W_1 \to W_2$ *such that* $\varphi^*\omega_2$ *can be deformed to* ω_1 *through a smooth 1-parameter family of symplectic forms that are all convex at the boundary.*

The following result is due to Gromov [41] (see also [21, Theorem 5.1], [51, Theorem 1.7], [14, Theorem 16.6]).

Theorem 2.2. *Any weak symplectic filling of* (S^3, ξ_{st}) *is symplectically deformation equivalent to a blow-up of* (D^4, ω_{st})*.*

Remark. This result can be obtained [66, Corollary 5.7] as an easy corollary to Theorem 3.12 since (S^3, ξ_{st}) admits a planar open book whose page is an annulus and whose monodromy is a single positive Dehn twist along the core circle. This monodromy admits a unique positive factorization which proves Theorem 2.2.

2.2. **Legendrian knots and contact surgery.** Recall that a knot in a contact 3-manifold is called Legendrian if it is everywhere tangent to the contact planes. Now consider a Legendrian knot $L \subset (\mathbb{R}^3, \xi_{st} = \ker(dz + x\,dy))$ and take its *front* projection, i.e., its projection to the yz-plane. Notice that the projection has no vertical tangencies (since $-\frac{dz}{dy} = x \neq \infty$), and for the same reason at a crossing the strand with smaller slope is in front. It turns out that L can be C^2-approximated by a Legendrian knot for which the projection has only transverse double points and *cusp* singularities (see [35], for example). Conversely, a knot projection with these properties gives rise to a unique Legendrian knot in $(\mathbb{R}^3, \xi_{st})$ by defining x from the projection as $-\frac{dz}{dy}$. Since any projection can be isotoped to satisfy the above properties, every knot in S^3 can be isotoped (non uniquely) to a Legendrian knot.

For a Legendrian knot L in (S^3, ξ_{st}), the Thurston-Bennequin number $\mathrm{tb}(L)$ is the contact framing of L (measured with respect to the Seifert framing in S^3) which can be easily computed from a front projection of L. Define $w(L)$ (the *writhe* of L) as the sum of signs of the double points. For this to make sense we need to fix an orientation on the knot, but the result is independent of this choice. If $c(L)$ is the number of cusps, then $\mathrm{tb}(L) = w(L) - \frac{1}{2}c(L)$.

Theorem 2.3 (Ding and Geiges [16])**.** *Every closed contact* 3*-manifold can be obtained by a contact* (± 1)*-surgery on a Legendrian link in the standard contact* S^3*.*

Lemma 2.4 (The Cancellation Lemma, [16, 17])**.** *Suppose that (Y, ξ) is a given contact manifold, $L \subset (Y, \xi)$ is a Legendrian knot and L' is its contact push-off. Perform contact (-1)-surgery on L and $(+1)$-surgery on L', resulting in the contact manifold (Y', ξ'). Then the contact 3-manifolds (Y, ξ) and (Y', ξ') are contactomorphic. The contactomorphism can be chosen to be the identity outside of a small tubular neighborhood of the Legendrian knot L.*

Proposition 2.5 (Weinstein [66])**.** *Let (W, ω) be a compact symplectic 4-manifold with a convex boundary component (Y, ξ). A 2-handle can be attached symplectically to (W, ω) along a Legendrian knot $L \subset (Y, \xi)$ in such a way that the symplectic structure extends to the 2-handle and the new symplectic 4-manifold (W', ω') has a convex boundary component (Y', ξ'), where (Y', ξ') is given by contact (-1)-surgery (i.e., Legendrian surgery) along $L \subset (Y, \xi)$.*

Remark 2.6. *A Weinstein 2-handle can also be attached along a Legendrian knot in the boundary of a weak symplectic filling so that symplectic structure extends over the 2-handle and weakly fills the resulting contact 3-manifold. In short, Legendrian surgery preserves weak fillability* [30, Lemma 2.6].

2.3. **The fundamental dichotomy: Tight versus overtwisted.** An embedded disk D in a contact 3-manifold (Y, ξ) is called overtwisted if at each point $p \in \partial D$ we have $T_pD = \xi_p$. A contact 3-manifold which contains such an overtwisted disk is called *overtwisted*, otherwise it is called *tight*—which is the fundamental dichotomy in 3-dimensional contact topology. Note that ∂D of an overtwisted disk is a Legendrian unknot with $tb(\partial D) = 0$. If (Y, ξ) admits a topologically unknotted Legendrian knot K with $tb(K) = 0$, then (Y, ξ) is overtwisted. This can be taken as the definition of an overtwisted manifold.

For any null-homologous Legendrian knot K in an arbitrary contact 3-manifold, we can find a C^0-small isotopy that decreases $tb(K)$ by any integer, but it is not always possible to increase $tb(K)$. If (Y, ξ) is overtwisted, however, any null-homologous knot K can be made Legendrian with $tb(K)$ realizing any preassigned integer (see [38, p. 625]).

The following is due to Eliashberg and Gromov [25]:

Theorem 2.7. *A weakly symplectically fillable contact 3-manifold (Y, ξ) is tight.*

Proof. Here we give a sketch of a proof (cf. [56, Thm. 12.1.10]) of Theorem 2.7 which is very different from the original proof. Suppose that (W, ω) is a symplectic filling of an overtwisted contact 3-manifold (Y, ξ). Then, by the discussion above, there is an embedded disk $D \subset Y$ such that ∂D is Legendrian and the framing on ∂D induced by the contact planes differs by $+2$ from the surface framing induced by D, i.e., $tb(\partial D) = 2$. By attaching a Weinstein 2-handle along ∂D to (W, ω) we obtain a weak symplectic filling (W', ω') of the surgered contact 3-manifold (Y', ξ') (see Remark 2.6).

Now we claim that (W', ω') contains an essential sphere S with self-intersection $(+1)$. The sphere S is obtained by gluing D with the core disk of the 2-handle, and $[S]^2 = 1$ follows from the fact that the Weinstein 2-handle is attached with framing $tb(\partial D) - 1$.

By Theorem 2.11, (W', ω') can be (symplectically) embedded into a closed symplectic 4-manifold X with $b_2^+(X) > 1$—which contradicts to the combination of the following two results.

Lemma 2.8 (Fintushel-Stern [32]). *If X is a smooth closed 4-manifold with $b_2^+(X) > 1$ and S is an essential sphere in X of nonnegative self-intersection, then $SW_X \equiv 0$.*

Theorem 2.9 (Taubes [59]). *If (X, ω) is a closed symplectic 4-manifold with $b_2^+(X) > 1$, then $SW_X(c_1(X, \omega)) \neq 0$.*

□

2.4. **Brieskorn spheres.** We digress here to introduce a useful family of closed and oriented 3-manifolds $\Sigma(p, q, r)$ ($p, q, r \in \mathbb{Z}_{\geq 2}$) known as the *Brieskorn spheres*. The 3-manifold $\Sigma(p, q, r)$ is defined as

$$\Sigma(p, q, r) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^p + z_2^q + z_3^r = 0,\ |z_1|^p + |z_2|^q + |z_3|^r = 1\}.$$

In other words, $\Sigma(p, q, r)$ can be identified with the link of the isolated complex surface singularity $\{z_1^p + z_2^q + z_3^r = 0\}$ and it is the oriented boundary of the compactified Milnor fiber $V(p, q, r)$, where

$$V(p, q, r) = \{(x, y, z) \in \mathbb{C}^3 \mid z_1^p + z_2^q + z_3^r = \epsilon,\ |z_1|^p + |z_2|^q + |z_3|^r \leq 1\}$$

for sufficiently small positive ϵ. Here we list some facts:

- The diffeomorphism type of $V(p, q, r)$ does not depend on ϵ.
- The 4-manifold $V(p, q, r)$ has a natural orientation as a complex manifold, and its *oriented* boundary is the Brieskorn sphere $\Sigma(p, q, r)$
- $\Sigma(p, q, r)$ is homeomorphic to the r-fold cyclic branched covering of $\mathbb{S}^3$, branched along a torus link of type (p, q).
- $\Sigma(p, q, r)$ admits a Seifert fibration.
- $V(p, q, r)$ admits a plumbing description—which is the minimal resolution of the corresponding singularity.
- $\Sigma(p, q, r)$ is a spherical manifold provided that $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} > 1$. A spherical 3-manifold is a closed orientable manifold which is obtained as $\mathbb{S}^3/\Gamma$ where Γ is a

finite subgroup of $SO(4)$ which acts freely on $\mathbb{S}^3$ by rotations. In particular, $\mathbb{S}^3$ is the universal cover of $\Sigma(p,q,r)$ and $\pi_1(\Sigma(p,q,r)) = \Gamma$. As a consequence, in this case, $\Sigma(p,q,r)$ admits a metric of positive scalar curvature induced from the standard round metric on $\mathbb{S}^3$.

- The Brieskorn sphere $-\Sigma(2,3,5)$ with its usual orientation reversed is diffeomorphic to the boundary of the positive E_8 plumbing—which is the plumbing of disk bundles over the sphere of Euler number $+2$ according to the E_8-diagram.

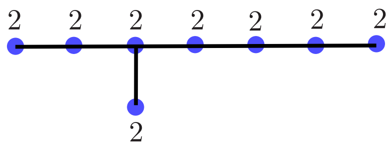

FIGURE 1. The positive E_8 diagram

Moreover $-\Sigma(2,3,5) \cong (+1)-$surgery on the right-handed trefoil $\cong$ Poincaré homology sphere with reversed orientation $\cong M(-\frac{1}{2}, \frac{1}{3}, \frac{1}{5})$ as a Seifert fibered manifold.

- $-\Sigma(2,3,4) \cong \partial(+E_7$ plumbing$) \cong (+2)-$surgery on the right-handed trefoil $\cong M(-\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$ as a Seifert fibered manifold.

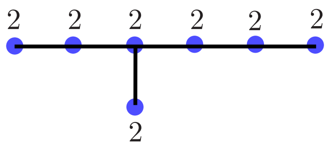

FIGURE 2. The positive E_7 diagram

- $-\Sigma(2,3,3) \cong \partial(+E_6$ plumbing$) \cong (+3)-$surgery on the right-handed trefoil $\cong M(-\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ as a Seifert fibered manifold.

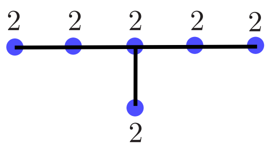

FIGURE 3. The positive E_6 diagram

Theorem 2.10 (Lisca [45, 46]). *The Brieskorn sphere $-\Sigma(2,3,n)$ does not admit any fillable contact structures for $n \in \{3,4,5\}$.*

Proof. We first give the proof for $-\Sigma(2,3,3)$. Let E denote the 4-manifold with boundary obtained by plumbing oriented disk bundles of Euler number 2 over the sphere according to the positive E_6 diagram. Suppose that W is a symplectic filling of $-\Sigma(2,3,3) \cong \partial E$. Then W can be symplectically embedded into a *closed symplectic* 4-manifold X such that $b_2^+(X \setminus \text{int}(W)) > 0$, by Theorem 2.11. Since $\Sigma(2,3,3)$ admits a positive scalar curvature metric, a standard result from gauge theory implies that $b_2^+(W) = 0$. Thus $Z = W \cup (-E)$ is a negative definite closed smooth 4-manifold, which by Donaldson's diagonalizability theorem [18] must have a standard intersection form $Q_Z \simeq \oplus\, m\, [-1]$, where $m = b_2(Z) = b_2^-(Z)$.

This gives a contradiction since the intersection lattice $(\mathbb{Z}^6, -E_6)$ of the negative E_6 plumbing does not admit an isometric embedding into any standard negative diagonal intersection form $(\mathbb{Z}^N, \oplus_i(-1))$ for any N. The result follows for $-\Sigma(2,3,4)$ and $-\Sigma(2,3,5)$ since $(\mathbb{Z}^6, -E_6)$ clearly embeds in $(\mathbb{Z}^7, -E_7)$ and $(\mathbb{Z}^8, -E_8)$. □

2.5. **The embedding theorem.** The following theorem was proved by Eliashberg [24] and also independently by Etnyre [28].

Theorem 2.11. *Any weak filling of a contact 3-manifold can be symplectically embedded into a closed symplectic 4-manifold.*

There are also proofs that appeared in [56, 35, 29]. The following result turned out to be an essential ingredient in some of the proofs.

Proposition 2.12. [22] [54] [24, Prop. 4.1] *Suppose that (W, ω) is a weak symplectic filling of (Y, ξ), where Y is a rational homology sphere. Then ω can be modified to a new symplectic form $\widetilde{\omega}$, where this modification is supported in a neighborhood of ∂W so that $(W, \widetilde{\omega})$ becomes a strong symplectic filling of (Y, ξ).*

Proof. We will give here the argument in [35, Lem. 6.5.5]—which is essentially the same as in [24, Prop. 4.1]. We know that $\partial W = Y$ has a collar neighborhood $N \cong [0,1] \times Y$ of Y in W with $Y \equiv \{1\} \times Y$. Since $H^2(N) = H^2(M) = 0$, we can write $\omega = d\eta$ for some 1-form η defined in N. Then we can find a 1-form α on Y such that $\xi = \ker \alpha$ and $\alpha \wedge \omega|_{TY} > 0$, by the assumption that (W, ω) is a weak symplectic filling of (Y, ξ).

We would like to construct a symplectic form $\widetilde{\omega}$ on $[0,1]\times Y$ which agrees with ω in a neighborhood of $\{0\}\times Y$ and strongly fills $\{1\}\times Y$. Let

$$\boxed{\widetilde{\omega} = d(f\eta)+d(g\alpha) = f'dt\wedge\eta + f\omega + g'dt\wedge\alpha + gd\alpha}$$

be a 2-form on $[0,1]\times Y$, where we will impose some conditions on the smooth functions $f:[0,1]\to[0,1]$ and $g:[0,1]\to\mathbb{R}_{\geq 0}$ to fit our purposes. First of all we require that $\widetilde{\omega}$ agrees with ω in a neighborhood of $\{0\}\times Y$. We simply fix a small $\varepsilon>0$ and set $f\equiv 1$ on $[0,\varepsilon]$ and $g\equiv 0$ on $[0,\frac{\varepsilon}{2}]$. More importantly $\widetilde{\omega}$ needs to be a symplectic form, and thus we compute

$$\begin{gathered}\widetilde{\omega}^2 = f^2\omega^2 + 2fg'\,dt\wedge\alpha\wedge\omega + 2gg'dt\wedge\alpha\wedge d\alpha \\ +2ff'dt\wedge\eta\wedge\omega + 2f'g\,dt\wedge\eta\wedge d\alpha + 2fg\,\omega\wedge d\alpha\end{gathered}$$

Since ω is symplectic, $\alpha\wedge\omega|_{TY}>0$ and α is contact, we know that the first three terms are positive volume forms on $[0,1]\times Y$ provided that they have positive coefficients. Hence we impose the condition $g'>0$ on $(\frac{\varepsilon}{2},1]$. Moreover, we can ensure that these positive terms dominate the other three terms, by choosing g to be very small on $[0,\varepsilon]$ (where $f'\equiv 0$) and g' very large compared to $|f'|$ and g on $[\varepsilon,1]$.

Finally, to verify that $\widetilde{\omega}$ strongly fills the contact boundary $(\{1\}\times Y,\xi)$, we impose that $f\equiv 0$ near $t=1$. By setting $s:=\log g(t)$, we see that $\widetilde{\omega}$ looks like $d(e^s\alpha)$ near the boundary. It follows that $\frac{\partial}{\partial s}$ is a Liouville vector field for $\widetilde{\omega}$ near the boundary:

$$\boxed{\mathcal{L}_{\frac{\partial}{\partial s}}d(e^s\alpha) = d\big(\iota_{\frac{\partial}{\partial s}}(e^sds\wedge\alpha + e^sd\alpha)\big) = d(e^s\alpha)}$$

and clearly we have $\xi=\ker\alpha=\ker(e^s\alpha)=\iota_{\frac{\partial}{\partial s}}d(e^s\alpha)=\iota_{\frac{\partial}{\partial s}}\widetilde{\omega}$ on $Y\equiv\{1\}\times Y$. □

Proof. **of Theorem 2.11**. We will describe here the proof appeared in [56]. This argument uses the same steps as in Etnyre's proof [28], but slightly different techniques in proving these steps. The crux of the argument is that the problem of embedding a weak filling reduces to the problem of embedding a strong filling as follows:

Let (W,ω) be a weak symplectic filling of (Y,ξ). By Theorem 2.3, (Y,ξ) can be given by a contact (± 1)-surgery on a Legendrian link $\mathbb{L}$ in the standard contact S^3. Consider the right-handed Legendrian trefoil knot K as depicted in Figure 4 in the standard contact S^3, having $\mathrm{tb}(K)=1$. Now for every knot L_i in $\mathbb{L}$ add a copy K_i of K into the diagram linking L_i once, not linking the other knots in $\mathbb{L}$. Adding symplectic 2-handles along K_i we get (W',ω'), which is a weak symplectic filling of the resulting 3-manifold (Y',ξ').

We claim that Y' is an integral homology sphere (cf. [58, Lemma 3.1]). To see this just convert the contact surgery diagram into a smooth handlebody diagram and calculate the first homology. Observe that the topological framing of K is 0. Denote by μ_i a small circle meridional to K_i and μ_i' a small circle meridional to L_i for $i=1,\ldots,n$. Recall that $H_1(Y',\mathbb{Z})$ is generated by $[\mu_i]$ and $[\mu_i']$ and the relations are $[\mu_i']=0$ and $[\mu_i]+\sum_{j\neq i}lk(L_i,L_j)[\mu_j']=0$. It follows that $H_1(Y',\mathbb{Z})=0$.

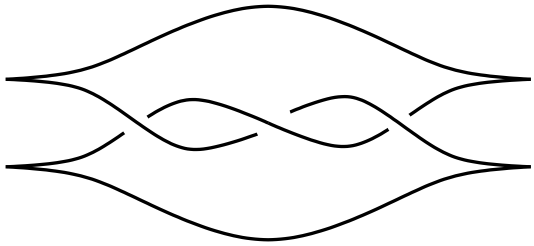

FIGURE 4. A right-handed Legendrian trefoil knot K

By Proposition 2.12 we can assume that (W', ω') is a strong filling of (Y', ξ'). To finish the proof of the embedding theorem, we need to embed any strong symplectic filling into a closed symplectic 4-manifold. In other words, we need to find a *concave* symplectic filling of the given contact 3-manifold, and glue it to the given strong symplectic filling to cap it off from the "other side". The fact that *every* contact 3-manifold admits (infinitely many distinct) concave symplectic fillings was proved by Etnyre and Honda [31] based on Theorem 2.13. (See also [34] for an alternative proof without using Theorem 2.13.)

Theorem 2.13 (Lisca-Matic [47]). *A Stein filling (W, ω_ϕ) admits a holomorphic embedding as a domain inside a minimal complex surface X of general type, with $b_2^+(X) > 1$, such that $\omega_X|_W = \omega_\phi$, where ω_X denotes the Kähler form on X.*

As a matter of fact, the problem of embedding a strong filling reduces to the problem of embedding a Stein filling as follows:

Suppose now that (W, ω) be a strong symplectic filling of (Y, ξ) and let $\mathbb{L} = \mathbb{L}^+ \cup \mathbb{L}^- \subset (S^3, \xi_{st})$ be a surgery presentation of (Y, ξ). Moreover let $\widehat{\mathbb{L}}^+$ denote the Legendrian link obtained by considering Legendrian push-offs of the knots of $\mathbb{L}^+$. Attaching Weinstein handles to (W, ω) along the components of $\widehat{\mathbb{L}}^+$ we get a strong filling (W', ω') of a contact 3-manifold (Y', ξ'). Observe that the (Y', ξ') is obtained by Legendrian surgery along $\mathbb{L}^-$ by Lemma 2.4 and hence Stein fillable by Theorem 3.5. This finishes the proof since (Y', ξ') admits a concave symplectic cap as we mentioned above. $\square$

In order to prove Theorem 2.11 Eliashberg attaches a symplectic 2-handle along the binding of an open book compatible with the given weakly fillable contact structure such that the other end of the cobordism given by this symplectic 2-handle attachment symplectically fibres over S^1. Then he fills in this symplectic fibration over S^1 by an appropriate symplectic Lefschetz fibration over D^2 to obtain a symplectic embedding of a weak filling into a closed symplectic 4-manifold. Note that the method of construction in [24] takes its roots from the one considered in [3].

2.6. **The adjunction inequality for Stein surfaces.** An immediate corollary of Theorem 2.13 and Seiberg-Witten theory is an adjunction inequality :

Theorem 2.14 ([1] [48]). *If W is a Stein domain, and $\Sigma \subset W$ is a closed, connected, oriented, embedded surface of genus g, then*

$$[\Sigma]^2 + |\langle c_1(W), [\Sigma]\rangle| \leq 2g - 2$$

unless Σ is a null-homologous sphere, where $c_1(W) := c_1(W, J) \in H^2(W, \mathbb{Z})$ denotes the first Chern class.

Corollary 2.15. *A Stein surface cannot contain a homologically essential smoothly embedded sphere S with $[S]^2 \geq -1$.*

2.7. **The standard contact structure on $S^1 \times S^2$.** The standard tight contact structure ξ_{st} on $S^1 \times S^2 \subset S^1 \times \mathbb{R}^3$ is given by $\ker \alpha$ where

$$\alpha := zd\theta + xdy - ydx.$$

The contact structure ξ_{st} is strongly symplectically fillable by the standard symplectic form on $S^1 \times D^3 \subset S^1 \times \mathbb{R}^3$ and in fact it is Stein fillable as it is given by the complex tangencies at the boundary of the Stein domain $S^1 \times D^3 \subset \mathbb{C}^2$.

In [38], Gompf gives yet another description of $(S^1 \times S^2, \xi_{st})$ as the contact 3-manifold obtained by removing two disjoint contact 3-balls from (S^3, ξ_{st}) and identifying the corresponding boundaries. This point of view turns out to be extremely useful if one considers $(S^1 \times S^2, \xi_{st})$ as the boundary of the Stein domain obtained by extending the standard Stein structure on D^4 over a 1-handle whose feet is two disjoint 3-balls in $S^3 = \partial D^4$.

The following theorem was implicit in [20] (see also [36]).

Theorem 2.16. *Any weak symplectic filling of $(S^1 \times S^2, \xi_{st})$ is diffeomorphic to $S^1 \times D^3$.*

The standard contact structure ξ_{st} on $S^1 \times S^2$ as the contact structure supported by the standard open book given as follows: The page is the annulus and the monodromy is the identity. Note that $(S^1 \times S^2, \xi_{st})$ is Stein fillable by Theorem 3.10 and it is well-known (cf. [35, Section 4.10]) that $S^1 \times S^2$ admits a unique tight contact structure, up to isotopy. Any Stein filling of $(S^1 \times S^2, \xi_{st})$ is deformation equivalent to the canonical Stein structure on

$S^1 \times D^3 \cong D^4 \cup$ 1-handle given by Theorem 3.5. Since $(S^1 \times S^2, \xi_{st})$ is planar, Wendl's Theorem 3.12 can be applied here to yield the next result as an immediate consequence:

Theorem 2.17. [66] *The strong symplectic filling of* $(S^1 \times S^2, \xi_{st})$ *is unique up to symplectic deformation equivalence and blow-up.*

In fact, using Theorem 3.13, "strong" can be replaced by "weak" in Theorem 2.17. Note that Theorem 2.12 does not apply here.

Remark. [14, Theorem 16.9] The standard contact $(\#_m S^1 \times S^2, \xi_{st})$ is defined as the contact connected sum of m copies of $(S^1 \times S^2, \xi_{st})$. Any Stein filling of $(\#_m S^1 \times S^2, \xi_{st})$ is deformation equivalent to the canonical Stein structure on $\natural_m S^1 \times D^3 \cong D^4 \cup k$ 1-handles.

2.8. **Some important results.** We just include here a few selected important results about 3-dimensional contact manifolds:

- Darboux: All contact structures look the same near a point, i.e., any point in a contact 3-manifold has a neighborhood isomorphic to a neighborhood of the origin in the standard contact $\mathbb{R}^3$.
- Martinet: Every closed oriented 3-manifold admits a contact structure.
- Lutz: In every homotopy class of oriented plane fields on a closed oriented 3-manifold there is an overtwisted contact structure.
- Eliashberg: Two overtwisted contact structures are isotopic if and only if they are homotopic as oriented plane fields.
- Martinet + Lutz + Eliashberg: There is a unique overtwisted contact structure in every homotopy class of oriented plane fields.
- Etnyre & Honda: The Poincaré homology sphere with its non-standard orientation does not admit a tight contact structure.
- Colin & Giroux & Honda: Only finitely many homotopy classes of oriented plane fields carry tight contact structures on a closed oriented 3-manifold.
- Colin & Giroux & Honda + Honda & Kazez & Matić: A closed, oriented, irreducible 3-manifold carries infinitely many tight contact structures (up to isotopy or up to isomorphism) if and only if it is toroidal.

3. Lecture 3: Contact 3-manifolds admitting infinitely many fillings

We will denote a positive Dehn twists along a curve γ by $D(\gamma)$, and we will use the usual composition of functions for expressing the products of Dehn twists. In addition, we will use $D^n(\gamma)$ to denote $(D(\gamma))^n$ for any integer n.

The *mapping class group* $\Gamma_{g,r}$ of an oriented compact surface F of genus $g \geq 0$ with $r \geq 0$ boundary components is defined to be the group of isotopy classes of orientation-preserving self diffeomorphisms of F fixing the points on the boundary. The isotopies are also assumed to fix the boundary pointwise. If $r = 0$, we sometimes drop r from the notation and use Γ_g to denote the mapping class group of a closed genus g surface.

3.1. **Lefschetz fibrations and open books.** Suppose that W and Σ are smooth oriented manifolds possibly with nonempty boundaries of dimensions four and two, respectively.

Definition 3.1. *A smooth map $f\colon W \to \Sigma$ is called a Lefschetz fibration if f has finitely many critical points in the interior of W, and there are orientation preserving complex charts U, V around each critical point p and $q = f(p)$, respectively, on which f is of the form $(z_1, z_2) \to z_1^2 + z_2^2$.*

For each critical value $q \in \Sigma$, the fiber $f^{-1}(q)$ is called a *singular* fiber, while the other fibers are called *regular*. Throughout this paper, we will assume that a regular fiber is connected and each singular fiber contains a unique critical point. It is a classical fact (see, for example, [19]) that for any loop a in Σ that does not pass through any critical values and that includes a unique critical value in its interior, $f^{-1}(a)$ is a surface bundle over a, which is diffeomorphic to

$$(F \times [0,1])/((1,x) \sim (0, D(\gamma)(x))$$

where γ denotes the *vanishing cycle* on a smooth fiber F over a point on the loop a. The singular fiber which is the inverse image of an interior point of a is obtained by collapsing the vanishing cycle to a point.

In this paper, we will mainly use Lefschetz fibrations with $\Sigma = \mathbb{D}^2$ or $\mathbb{S}^2$. Suppose first that $\Sigma = \mathbb{D}^2$ and choose an identification of the regular fiber, say over a fixed base point b near $\partial\mathbb{D}^2$, with an (abstract) oriented connected surface F of genus $g \geq 0$ with $r \geq 0$ boundary components. Now choose an arc that connects the point b to each critical value so that these arcs are pairwise disjoint in $\mathbb{D}^2$. Label these arcs by the set $\{c_1, c_2, \ldots, c_n\}$ in the increasing order as you go counterclockwise direction around a small loop around the base point b, and label the critical values as $\{q_1, q_2, \ldots, q_n\}$ corresponding to the labeling of the arcs as depicted in Figure 5. Consider a loop a_i around the critical value q_i, which does not pass through or include in its interior any other critical values and let γ_i denote the corresponding vanishing cycle.

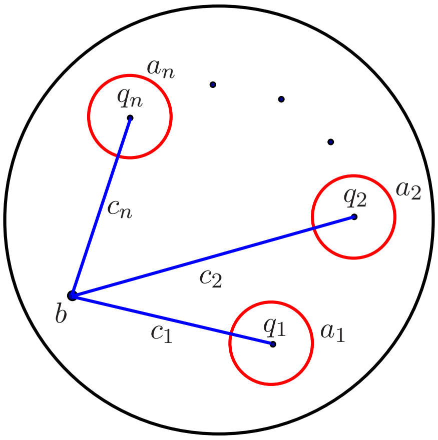

FIGURE 5.

Note that $f^{-1}(\partial\mathbb{D}^2)$ is an F-bundle over $\mathbb{S}^1 = \partial\mathbb{D}^2$ which is diffeomorphic to

$$(F \times [0,1])/((1,x) \sim (0,\psi(x))$$

for some self-diffeomorphism ψ of the fiber F preserving ∂F pointwise. It follows that

$$\psi = D(\gamma_n)D(\gamma_{n-1})\cdots D(\gamma_1) \in \Gamma_{g,r}.$$

The product of positive Dehn twists above is called a *monodromy factorization* or a *positive factorization* of the monodromy $\psi \in \Gamma_{g,r}$ of the Lefschetz fibration over $\mathbb{D}^2$. Conversely, a positive factorization of an element in $\Gamma_{g,r}$ determines a Lefschetz fibration over $\mathbb{D}^2$, uniquely up to some natural equivalence relations which we explain next.

If one chooses a different identification of the reference fiber over the base point with the abstract oriented surface F, then the monodromy of the Lefschetz fibration takes the form $\varphi\psi\varphi^{-1}$, where φ is the appropriate element of $\Gamma_{g,r}$. In this case, the monodromy factorization appears as

$$\begin{aligned}\varphi\psi\varphi^{-1} &= \varphi\big(D(\gamma_n)D(\gamma_{n-1})\cdots D(\gamma_1)\big)\varphi^{-1}\\ &= \varphi D(\gamma_n)\varphi^{-1}\varphi D(\gamma_{n-1})\varphi^{-1}\varphi\cdots\varphi^{-1}\varphi D(\gamma_1)\varphi^{-1}\\ &= D(\varphi(\gamma_n))D(\varphi(\gamma_{n-1}))\cdots D(\varphi(\gamma_1)),\end{aligned}$$

where the last equality follows by the fact that the conjugation $\varphi D(\gamma)\varphi^{-1}$ of a positive Dehn twist $D(\gamma)$ is isotopic to the positive Dehn twist $D(\varphi(\gamma))$.

Note that the trivial identity

$$D(\gamma_{i+1})D(\gamma_i) = \big(D(\gamma_{i+1})D(\gamma_i)D^{-1}(\gamma_{i+1})\big)D(\gamma_{i+1})$$

would also allow us to modify the monodromy factorization of a Lefschetz fibration by switching the order of two consecutive positive Dehn twists, where we conjugate one by the other. Such a modification is called a *Hurwitz move* and obtained by switching the order of two consecutive arcs connecting the base point to critical values that we chose to describe the monodromy factorization. The *isomorphism class* of a Lefschetz fibration (over $\mathbb{D}^2$) is determined up to global conjugation and Hurwitz moves. For further details we refer to [39, Chapter 8].

Now suppose that $\partial W = \emptyset$, and $f : W \to \mathbb{S}^2$ is Lefschetz fibration, where the genus g fiber F is necessarily closed. We may assume that all the critical values of f lie on a disk in the base $\mathbb{S}^2$, and the fibration is trivial on the complementary disk. It follows that in this case, the monodromy factorization satisfies

$$D(\gamma_n)D(\gamma_{n-1})\cdots D(\gamma_1) = 1 \in \Gamma_g.$$

We now turn our attention to the case $\partial W \neq \emptyset$ and $\Sigma = \mathbb{D}^2$. Under the assumption that $\partial F \neq \emptyset$, the boundary ∂W consists of two parts: The "vertical" boundary $f^{-1}(\partial\mathbb{D}^2)$ and the "horizontal" boundary $\partial F \times \mathbb{D}^2$ that meet each other at the corner $\partial F \times \partial\mathbb{D}^2$. After smooothing out the corners, we see that ∂W acquires an open book decomposition by which we mean the following:

Definition 3.2. *An open book decomposition of a closed and oriented* 3*-manifold* Y *is a pair* (B, f) *consisting of an oriented link* $B \subset Y$*, and a locally-trivial fibration* $f\colon Y - B \to \mathbb{S}^1$ *such that each component of* B *has a trivial tubular neighborhood* $B \times \mathbb{D}^2$ *in which* f *is given by the angular coordinate in the* $\mathbb{D}^2$*-factor.*

Here B is called the binding and the closure of each fiber, which is a Seifert surface for B, is called a page. We orient each page so that the induced orientation on its boundary agrees with that of fixed orientation of the binding B.

The (geometric) *monodromy* of an open book is defined as the self-diffeomorphism of an arbitrary page —identified with an abstract oriented genus $g \geq 0$ surface F with $r \geq 1$ boundary components—which is given by the first return map of a vector field that is transverse to the pages and meridional near B. Note that, up to conjugation, the monodromy of an open book is determined as element in $\Gamma_{g,r}$. It is clear that the monodromy of the open book on the boundary of a Lefschetz fibration can be identified with the monodromy of the Lefschetz fibration.

3.2. **Open books and contact structures.**

Definition 3.3. *A contact structure* ξ *on a closed oriented* 3*-manifold* Y *is said to be supported by the open book* (B, f) *if there is a contact form* α *for* ξ *such that* $\alpha|_{TB} > 0$ *and* $d\alpha|_{f^{-1}(\theta)} > 0$*, for each* $\theta \in \mathbb{S}^1$*.*

Remark. A contact 1-form α satisfying the conditions above is sometimes called a Giroux form.

In [61], Thurston and Winkelnkemper constructed a contact form on a 3-manifold Y using an open book decomposition of Y. Their construction was refined by Giroux showing that an open book supports a unique contact structure, up to isotopy.

Conversely, for any given contact structure ξ in a 3-manifold, Giroux [37] constructed an open book supporting ξ. As a matter of fact, Giroux established a bijection between the set of isotopy classes of contact structures on a closed 3-manifold Y and the set of open book decompositions of Y, up to positive stabilization/destabilization.

Giroux's correspondence is of central importance in the subject at hand, and we refer to Etnyre's elaborate lecture notes [29] for details.

Definition 3.4. *A contact 3-manifold (Y, ξ) is said to be planar if Y admits a planar open book supporting ξ.*

3.3. **Stein domains and Lefschetz fibrations.** Before we state a topological characterization of Stein domains due to Eliashberg and Gompf, we make some simple preliminary observations: By attaching m 1-handles to a 0-handle we obtain $\natural_m S^1 \times D^3$ whose boundary is $\#_m S^1 \times S^2$. Eliashberg [20] showed that $\natural_m S^1 \times D^3$ admits a Stein structure so that it is a Stein filling of $\#_m S^1 \times S^2$ equipped with its standard contact structure. The following theorem is a key result in the subject which made the study of Stein surfaces/domains accessible to low-dimensional topologists.

Theorem 3.5. [20, 38] *A smooth handlebody consisting of a 0-handle, some 1-handles and some 2-handles admits a Stein structure if the 2-handles are attached to the Stein domain $\natural_m S^1 \times D^3$ along Legendrian knots in the standard contact $\#_m S^1 \times S^2$ such that the attaching framing of each Legendrian knot is -1 relative to the framing induced by the contact planes. Conversely, any Stein domain admits such a handle decomposition.*

Similar to the handle decomposition of a Stein domain described in Theorem 3.5, there is a handle decomposition of a Lefschetz fibration over $\mathbb{D}^2$ consisting a 0-handle, some 1-handles and some 2-handles as follows: A neighborhood $F \times D^2$ of a regular fiber F is given by attaching appropriate number of 1-handles to a 0-handle. This is because the surface F can be described by attaching 2-dimensional 1-handles to a 2-dimensional disk, and $F \times D^2$ is a thickening of this handle decomposition in 4-dimensions.

Then, since each singularity of a Lefschetz fibration is modeled on complex Morse function $(z_1, z_2) \to z_1^2 + z_2^2$, for each singular fiber, a 2-handle is attached to $F \times D^2$ along the corresponding vanishing cycle. The crux of the matter is that the attaching framing of each such 2-handle is -1 relative to the framing induced by the fiber. Therefore if $W \to \mathbb{D}^2$ is a Lefschetz fibration, then W has a handle decomposition

$$W = (F \times D^2) \cup H_1 \cup \cdots \cup H_n$$

where, for each $1 \leq i \leq n$, the 2-handle H_i is attached along the vanishing cycle γ_i. One can easily compute some basic topological invariants of the 4-manifold W, using its corresponding cell-decomposition. Let χ denote the Euler characteristic.

Lemma 3.6. *The first integral homology group $H_1(W, \mathbb{Z})$ is isomorphic to the quotient of $H_1(F, \mathbb{Z})$ by the normal subgroup $\langle [\gamma_1], \dots, [\gamma_n] \rangle$ generated by the homology classes of the vanishing cycles. Moreover, $\chi(W) = \chi(F) + n$.*

Definition 3.7. *We say that a Lefschetz fibration over $\mathbb{D}^2$ is allowable if the regular fiber has nonempty boundary and each vanishing cycle is homologically nontrivial on the fiber.*

Next we show that if $W \to \mathbb{D}^2$ is an allowable Lefschetz fibration then W admits a Stein structure (cf. [2, 49]) so that the induced contact structure on ∂W is supported by the open book induced by the Lefschetz fibration. Suppose that W admits a handle decomposition as in the previous paragraph and let $W_i \to \mathbb{D}^2$ denote the Lefschetz fibration so that

$$W_i = (F \times D^2) \cup H_1 \cup \cdots \cup H_i.$$

We will show that W admits a Stein structure by induction. Suppose that W_{i-1} admits a Stein structure so that the induced contact structure on ∂W_{i-1} is supported by the open book induced by the Lefschetz fibration $W_{i-1} \to \mathbb{D}^2$. By the work of Torisu [62], we can assume the open book has a *convex* page that contains the attaching curve γ_i of the 2-handle H_i. Moreover, by the *Legendrian Realization Principle* [42], γ_i can be made Legendrian so that the framing induced by the contact planes agrees with that of induced from the page of the open book. This is precisely where we require the Lefschetz fibration to be allowable since Legendrian Realization Principle only works for homologically nontrivial simple closed curves. As a consequence, $W_i = W_{i-1} \cup H_i$ admits a Stein structure, by Theorem 3.5.

Furthermore, the induced contact structure on ∂W_i is supported by the induced open book by Proposition 3.8, since the effect of attaching a *Weinstein* 2-handle along γ_i corresponds to Legendrian surgery along the same curve on the contact boundary ∂W_i.

Lemma 3.8. [34] *Suppose (Y, ξ) is a contact 3-manifold supported by the open book with page F and monodromy ψ. Then the contact manifold obtained by performing a Legendrian surgery on a knot L contained in some page is is supported by the open book with the same page F and monodromy $\psi \circ D(L)$.*

For the initial step of the induction we just observe that $F \times D^2 \cong \natural_m S^1 \times D^3$ admits a Stein structure so that it is the Stein filling of the standard contact structure on its boundary $\#_m S^1 \times S^2$ (see Section 2.7).

Conversely, a Stein domain admits an allowable Lefschetz fibration over $\mathbb{D}^2$ which was proved in [2] and [49]. By a refinement of the algorithm in [2], Plamenevskaya showed, in addition, that the induced contact structure on the boundary is supported by the resulting open book [57, Appendix A]. This leads to the following topological characterization of Stein domains.

Theorem 3.9. *A Stein domain admits an allowable Lefschetz fibration over $\mathbb{D}^2$ and conversely an allowable Lefschetz fibration over $\mathbb{D}^2$ admits a Stein structure. Moreover the contact structure induced by the Stein structure on the boundary is supported by the open book induced by the Lefschetz fibration.*

In an other direction, the culmination of the work in [2, 37, 49] leads to one useful characterization of Stein fillable contact 3-manifolds:

Theorem 3.10. *A contact 3-manifold (Y, ξ) is Stein fillable if and only if ξ is supported by some open book in Y whose monodromy admits a factorization into a product of positive Dehn twists.*

Remark. The characterization above does not hold for *every* open book supporting the given contact structure (cf. [8, 63]).

Nevertheless, Stein/symplectic fillings of contact 3-manifolds supported by *planar* open books are understood much better due to the recent work of Wendl. To describe his work, we give a few basic necessary definitions here and refer to [66] for the details. In our discussion leading to Theorem 3.9 in Section 3.3, we gave a short proof of the fact that an allowable Lefschetz fibration over $\mathbb{D}^2$ admits a Stein structure, but we did not pay attention to how the Stein structure, or more precisely the exact symplectic form, restricts to the fibers of the Lefschetz fibration. However, there is a long history of the study of symplectic Lefschetz fibrations in the literature.

Suppose that Σ is a closed, connected and oriented surface, and $f : X \to \Sigma$ is a smooth fibre bundle whose fibers are also closed, connected and oriented surfaces. Thurston [60] showed that X admits a symplectic form ω such that all fibers are symplectic submanifolds of (X, ω), provided that the homology class of the fibre is non-zero in $H_2(X, \mathbb{R})$. Moreover, the space of symplectic forms on X having this property is connected. This result of Thurston was generalized to Lefschetz fibrations by Gompf.

Theorem 3.11 (Gompf [39])**.** *Suppose that $f : X^4 \to \Sigma^2$ is a Lefschetz fibration such that homology class of the fiber is non-zero in $H_2(X, \mathbb{R})$, where both X and Σ are* *closed, connected and oriented manifolds. Then the space of symplectic forms on X that are supported by f is nonempty and connected.*

We say that a symplectic form ω on X is *supported* by $f : X \to \Sigma$ if every fiber is a symplectic submanifold at its smooth points, and in a neighborhood of each critical point, ω tames some almost complex structure J that preserves the tangent spaces of the fibers.

In [66], Wendl defines a *bordered Lefschetz fibration* $f : E \to \mathbb{D}^2$ with a supported symplectic form ω_E such that, in addition to the conditions above, $\omega_E = d\lambda$ in a neighborhood of ∂E for some Giroux form λ. A symplectic filling (W, ω) of a contact 3-manifold (Y, ξ) is said to admit a symplectic Lefschetz fibration over $\mathbb{D}^2$ if there exists a bordered Lefschetz fibration $f : E \to \mathbb{D}^2$ with a supported symplectic form ω_E such that, after smoothing the corners on ∂E, (E, ω_E) is symplectomorphic to (W, ω).

Theorem 3.12 (Wendl [65, 66])**.** *Suppose that (W, ω) is a strong symplectic filling of a contact 3-manifold (Y, ξ) which is supported by a planar open book $f : Y \setminus B \to \mathbb{S}^1$. Then (W, ω) admits a symplectic Lefschetz fibration over $\mathbb{D}^2$, such that the induced open book at the boundary is isotopic to $f : Y \setminus B \to \mathbb{S}^1$. Moreover, the Lefschetz fibration is allowable if and only if (W, ω) is minimal.*

In this case, the Lefschetz fibration determines a supporting open book on (Y, ξ) uniquely up to isotopy. Moreover, the isotopy class of the Lefschetz fibration produced on (W, ω) depends only on the deformation class of the symplectic structure. The punch line is that the problem of classifying symplectic fillings up to symplectic deformation reduces to the problem of classifying Lefschetz fibrations that fill a given planar open book supporting the contact structure.

The following generalization of Theorem 3.12 was proved in [53]:

Theorem 3.13 (Niederkrüger-Wendl [53])**.** *If (Y, ξ) is a planar contact 3-manifold, then every weak symplectic filling (W, ω) of (Y, ξ) is symplectically deformation equivalent to a blow up of a Stein filling of (Y, ξ).*

Next we turn our attention to some examples of contact 3-manifolds each of which has been shown to admit infinitely many *distinct* Stein fillings. We will clarify what we mean by distinct for each of the examples we consider below.

Definition 3.14. *Let $Y_{g,m}$ denote the oriented 3-manifold obtained by plumbing of the disk bundle over a genus g surface with Euler number 0 and the disk bundle over a sphere with Euler number $2m$. The 3-manifold $Y_{g,m}$ admits an open book whose page is a genus g-surface with connected boundary and whose monodromy is $D^{2m}(\gamma)$, where γ is a boundary parallel curve. Let $\xi_{g,m}$ denote the contact structure supported by this open book.*

3.4. **Infinitely many pairwise non-homeomorphic Stein fillings.** The first example of a contact three manifold which admits infinitely many distinct Stein fillings was discovered by the author and Stipsicz:

Theorem 3.15. [55] *For each odd integer $g \geq 3$, the contact 3-manifold $(Y_{g,1}, \xi_{g,1})$ admits infinitely many pairwise non-homeomorphic Stein fillings.*

In the following, we outline the construction of these fillings, which is based on the following result (see, for example, [3]):

Lemma 3.16. *Let $f : X \to \mathbb{S}^2$ be an allowable Lefschetz fibration that admits a section. Let U denote the interior of a regular neighborhood of the union of this section and a regular fiber of f, and let $W = X \setminus U$. Then $f|_W : W \to \mathbb{D}^2$ is an allowable Lefschetz fibration and hence W carries a Stein structure such that the induced contact structure on ∂W is supported by the induced open book.*

For $g = 2h + 1 \geq 3$, consider the allowable Lefschetz fibration $f_g : X_g \to \mathbb{S}^2$ whose fiber is a closed oriented surface of genus g and whose monodromy factorization is given by the word [44]

$$\big(D(\beta_0)D(\beta_1)\cdots D(\beta_g)D^2(\alpha)D^2(\beta)\big)^2 = 1 \in \Gamma_g$$

where these curves are depicted in Figure 6.

Remark. The Lefschetz fibration $f_g : X_g \to \mathbb{S}^2$ admits a sphere section of self-intersection -1, which is equivalent to the fact that

$$\big(D(\beta_0)D(\beta_1)\cdots D(\beta_g)D^2(\alpha)D^2(\beta)\big)^2 = D(\delta) \in \Gamma_{g,1}$$

where δ is a boundary parallel curve on a genus g surface with one boundary component.

Note that the total space X_g is diffeomorphic to $\Sigma_h \times S^2 \# 8\overline{\mathbb{C}P^2}$, where Σ_h denotes a closed oriented surface of genus $h = \frac{1}{2}(g-1)$. In particular, the first homology group $H_1(X_g;\mathbb{Z})$ contains no torsion.

Let $f_g(n) : X_g(n) \to \mathbb{S}^2$ denote the *twisted* fiber sum of two copies of the Lefschetz fibration $f_g : X_g \to \mathbb{S}^2$, where the gluing diffeomorphism, i.e., a self-diffeomorphism of a

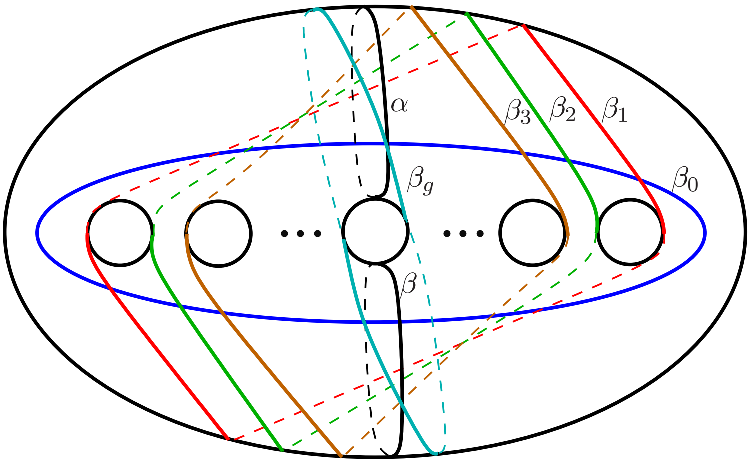

FIGURE 6. Vanishing cycles of the genus g Lefschetz fibration $f_g : X_g \longrightarrow \mathbb{S}^2$.

generic fiber, is an n-fold power of a right-handed Dehn twist along a certain homologically nontrivial curve on the fiber. We observe that

- $H_1(X_g(n);\mathbb{Z}) \cong \mathbb{Z}^{g-2} \oplus \mathbb{Z}_n$, and
- $f_g(n) : X_g(n) \longrightarrow \mathbb{S}^2$ admits a sphere section with self-intersection number -2.

The crux of the matter is that although $H_1(X_g;\mathbb{Z})$ has no torsion, $H_1(X_g(n);\mathbb{Z})$ has torsion $\mathbb{Z}_n$ depending on the power of the Dehn twist we use for the fiber sum. Let $U_g(n)$ denote the interior of a regular neighborhood of the union of the (-2)-sphere section above and a regular fiber of $f_g(n)$. It is easy to see that, for each positive integer n, the boundary $\partial U_g(n)$ is diffeomorphic to $Y_{g,1}$ with the *opposite* orientation. Let $W_g(n) := X_g(n) \backslash U_g(n)$. By Proposition 3.16, for fixed odd $g \geq 3$, the set

$$\{W_g(n) \mid n \in \mathbb{Z}^+\}$$

gives an infinite family of pairwise non-homeomorphic Stein fillings of the contact 3-manifold $(Y_{g,1}, \xi_{g,1})$, since one can see that

$$H_1(W_g(n);\mathbb{Z}) \cong H_1(X_g(n);\mathbb{Z}) \cong \mathbb{Z}^{g-2} \oplus \mathbb{Z}_n.$$

Remark. From the mapping class group point of view, the infinite set of pairwise non-homeomorphic fillings above owes its existence to the infinitely many distinct factorizations of $D^2(\delta) \in \Gamma_{g,1}$ as

$$\big(D(\beta_0)\cdots D(\beta_g)D^2(\alpha)D^2(\beta)\big)^2\big(D(\varphi^n(\beta_0))\cdots D(\varphi^n(\beta_g))D^2(\varphi^n(\alpha))D^2(\varphi^n(\beta))\big)^2$$

where δ denotes a boundary parallel curve and φ^n denotes $D^n(\gamma)$ for some homologically nontrivial curve γ on the genus g surface with one boundary component.

3.5. **Infinitely many exotic Stein fillings.** The first example of a contact 3-manifold which admits infinitely many *exotic* (i.e., homeomorphic but pairwise non-diffeomorphic) simply-connected Stein fillings was constructed in [5].

Theorem 3.17. [5] *For each integer $g > 4$ and $m \geq 1$, the contact 3-manifold $(Y_{g,m}, \xi_{g,m})$ admits infinitely many exotic Stein fillings.*

The essential ingredient in the proof of Theorem 3.17 is the Fintushel-Stern knot surgery [33] along a homologically essential torus using an infinite family of fibered knots in S^3 of fixed genus with distinct Alexander polynomials. The infinite family of Stein fillings are obtained—as in the previous section—by removing the interior of a regular neighborhood of the union of a section and a regular fiber of a certain allowable Lefschetz fibration over $\mathbb{S}^2$ *after* applying knot surgery along a torus T so that

- T is disjoint from the section, and
- T intersects each fiber of the Lefschetz fibration twice.

The Stein fillings are pairwise non-diffeomorphic since before the removal of the union of the section and the regular fiber, the closed 4-manifolds are already pairwise non-diffeomorphic. This is because they have different Seiberg-Witten invariants based on the choice of the infinite family of fibered knots with distinct Alexander polynomials. The fact that these fillings are all homeomorphic is essentially guaranteed by Freedman's Theorem.

Recently, Akhmedov and the author were able to generalize Theorem 3.17 as follows:

Theorem 3.18. [7] *For any finitely presentable group G, there exists a contact 3-manifold which admits infinitely many exotic Stein fillings such that the fundamental group of each filling is isomorphic to G.*

Remark. The contact 3-manifolds in Theorem 3.18 are the links of some isolated complex surface singularities, equipped with their canonical contact structures (see also [6]).

Moreover, Akbulut and Yasui [4] showed that there exists an infinite family of contact 3-manifolds each of which admits infinitely many simply connected exotic Stein fillings with $b_2 = 2$. Their approach to construct exotic Stein fillings is drastically different from what we outlined above for all the other previous constructions based on Proposition 3.16. The infinite family of exotic Stein fillings are obtained by a p-log transform ($p \geq 1$) along a single torus with trivial normal bundle in a certain 4-manifold with boundary. The Stein structures are described by Legendrian handle diagrams—as opposed to using Lefschetz fibrations—and the smooth structures on the fillings are distinguished by a clever use of the adjunction inequality (see Section 2.6).

3.6. **Stein fillings with arbitrarily large Euler characteristics.** Let (Y,ξ) be a closed contact 3-manifold and let

$$\chi_{(Y,\xi)} = \{\chi(W) \mid (W,J) \text{ is a Stein filling of } (Y,\xi)\}$$

where χ denotes the Euler characteristic. It was conjectured [55] that the set $\chi_{(Y,\xi)}$ is finite for every (Y,ξ). This conjecture holds true for planar contact 3-manifolds (see, Kaloti [43])—a theorem of Etnyre [27] implies that any Stein filling of a planar contact 3-manifold has $b_2^+ = 0$ and by [58, Corollary 1.5], $\chi_{(Y,\xi)}$ is finite for any contact 3-manifold such that every Stein filling of it has $b_2^+ = 0$.

Recently, the conjecture was disproved by Baykur and Van Horn Morris [9, 10] who showed that there are vast families of contact 3-manifolds each member of which admits infinitely many Stein fillings with arbitrarily large Euler characteristics.

In the following we describe an element in $\Gamma_{2,1}$ which has arbitrarily long positive factorizations (cf. [15]). The existence of such an element indeed provides a counterexample to the aforementioned conjecture. We refer to Figure 7 for the curves that appear in the following text. It is well-known that

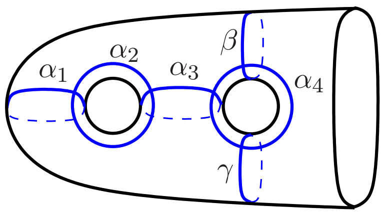

FIGURE 7. A genus two surface with connected boundary

$$D(\beta)D(\gamma) = \big(D(\alpha_1)D(\alpha_2)D(\alpha_3)\big)^4$$

and by applying braid relations we obtain

$$\begin{aligned}
D(\beta)D(\gamma) &= \big(D(\alpha_1)D(\alpha_2)D(\alpha_3)\big)^4\\
&= \big(D(\alpha_1)D(\alpha_2)D(\alpha_3)\big)^2 D(\alpha_1)D(\alpha_2)D(\alpha_3)D(\alpha_1)D(\alpha_2)D(\alpha_3)\\
&= \big(D(\alpha_1)D(\alpha_2)D(\alpha_3)\big)^2 D(\alpha_1)D(\alpha_2)D(\alpha_1)D(\alpha_3)D(\alpha_2)D(\alpha_3)\\
&= \big(D(\alpha_1)D(\alpha_2)D(\alpha_3)\big)^2 D(\alpha_2)D(\alpha_1)D(\alpha_2)D(\alpha_3)D(\alpha_2)D(\alpha_3)\\
&= \big(D(\alpha_1)D(\alpha_2)D(\alpha_3)\big)^2 D(\alpha_2)D(\alpha_1)D(\alpha_3)D(\alpha_2)D(\alpha_3)D(\alpha_3)
\end{aligned}$$

Now we define

$$T := D(\beta)D(\gamma)D^{-1}(\alpha_3)D^{-1}(\alpha_3) = \big(D(\alpha_1)D(\alpha_2)D(\alpha_3)\big)^2 D(\alpha_2)D(\alpha_1)D(\alpha_3)D(\alpha_2).$$

By taking the m-th power for any m, we have

$$T^m = D^m(\beta)D^{-m}(\alpha_3)D^m(\gamma)D^{-m}(\alpha_3).$$

We follow [15] to construct the desired element with arbitrarily long positive factorizations, although similar arguments appeared in [11] and also [10, Lemma 3.4]. Let

$$\varphi = D(\alpha_4)D(\alpha_3)D(\alpha_2)D(\alpha_1)D(\alpha_1)D(\alpha_2)D(\alpha_3)D(\alpha_4)D(\alpha_4)D(\beta)D(\alpha_3)D(\alpha_4).$$

It can be shown by a direct calculation that $\varphi(\alpha_3) = \gamma$ and $\varphi(\beta) = \alpha_3$. Therefore

$$\begin{aligned}
T^m &= D^m(\beta)D^{-m}(\alpha_3)D^m(\gamma)D^{-m}(\alpha_3)\\
&= D^m(\beta)D^{-m}(\alpha_3)D^m(\varphi(\alpha_3))D^{-m}(\varphi(\beta))\\
&= D^m(\beta)D^{-m}(\alpha_3)D^m\varphi D^m(\alpha_3)\varphi^{-1}\varphi D^{-m}(\beta)\varphi^{-1}\\
&= D^m(\beta)D^{-m}(\alpha_3)D^m\varphi D^m(\alpha_3)D^{-m}(\beta)\varphi^{-1}\\
&= [D^m(\beta)D^{-m}(\alpha_3), \varphi]
\end{aligned}$$

where brackets in the last line denote the commutator. Hence

$$\begin{aligned}
\varphi &= \varphi D^{-m}(\beta)D^m(\alpha_3)T^m D^{-m}(\gamma)D^m(\alpha_3)\\
&= \varphi D^{-m}(\beta)D^m(\alpha_3)\varphi^{-1}\varphi T^m D^{-m}(\gamma)D^m(\alpha_3)\\
&= D^{-m}(\alpha_3)D^m(\gamma)\varphi T^m D^{-m}(\gamma)D^m(\alpha_3).
\end{aligned}$$

Thus φ is a conjugation of φT^m by $D^{-m}(\alpha_3)D^m(\gamma)$. But since both φ and T admit positive factorizations, the product φT^m admits a positive factorization. Therefore we conclude that φ admits a factorization into $12 + 10m$ positive Dehn twists for arbitrary non-negative integer m.

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