A Note on the Bonnet-Myers Theorem

V. Boju and L. Funar

Abstract. The aim of this note is to derive a compactness result for complete manifolds whose Ricci curvature is bounded from below. The classical result, usually stated as Bonnet-Myers theorem, provides an estimation of the diameter of a manifold whose Ricci curvature is greater than a strictly positive constant. Weaker assumptions that the Ricci curvature function tends slowly to zero (when the distance from a fixed point goes to infinity) were already considered in [2, 3]. We shall improve here their results.

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We will be concerned with the following analytic

Problem. Given a function \( a : [r_0, +\infty) \rightarrow (0, +\infty) \) we consider positive solutions \( y = y(r) \) of the differential equation

\[
y'' + ay = 0
\]

satisfying \( y(r_0) = 0 \). Obviously, \( y \) has to be concave. We have to determine the functions \( a = a(r) \) for which \( y \) has a further zero \( r_1 > r_0 \) which may be bounded from above.

It is clear that there is such a bound in case \( a \) is a positive constant, but this bound tends to infinity as \( a(r) \rightarrow 0 \). The above problem seems to be interesting for functions \( a \) satisfying \( \lim_{r \rightarrow +\infty} a(r) = 0 \). It turns out that the right asymptotic is \( a(r) \sim cr^{-2} \), with critical value \( c = \frac{1}{4} \). In fact, for \( c = \frac{1}{4} + v^2 \) one gets the solution \( y(r) = r^{\frac{1}{2}} \sin v \left( \log \frac{r}{r_0} \right) \), and hence there is a second zero. In this paper we show that in fact the constant \( v^2 \) may be replaced by a function which tends as weakly as an iterated logarithm to zero, which enters in our definition of some function \( A_{k,v} = A_{k,v}(r) \).

Let us first make some notations. For each natural number \( k \) we set

\[
\begin{align*}
\Log_0 (r) &= r \\
L_k (r) &= \log \ldots \log r \\
&\quad \text{under the k-fold logr}
\end{align*}
\]

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whenever is defined, and
\[
A_{k,v}(r) = \frac{1}{4r^2} \left( 1 + \frac{1}{L_1(r)^2} + \ldots + \frac{1}{L_{k-1}(r)^2} \right) \\
+ \frac{1 + 4v^2}{L_1(r)^2L_2(r)^2 \ldots L_k(r)^2}.
\]

For a Riemannian manifold \( M \), we denote by \( \text{Ric}_x(Y) \) the Ricci curvature in the direction \( Y \in T_x(M) \), for a point \( x \in M \) and \( T_x(M) \) being the tangent space to \( M \) in \( x \). The space \( M \) is said to have an almost positive asymptotic Ricci curvature (abbreviated to be an AP-Riemannian space) if there exist \( k, v, r_0 > 0 \) and \( p \in M \) such that
\[
\text{Ric}_x(Y) \geq (n - 1) A_{k,v}(r) |Y|^2
\]
holds for all \( x \in M \) whose distance from a fixed point \( p \) is \( r = \text{dist}(p, x) \geq r_0 \) and for all vectors \( Y \in T_x(M) \). Also \( |\cdot| \) stands for the norm in the tangent space induced by the metric, and \( n \) is the dimension of \( M \).

Our result can be stated now as follows.

**Theorem 1.** A complete AP-Riemannian manifold is compact, and its diameter \( d(M) \) is bounded by
\[
d(M) \leq e_{k-1} \left( L_{k-1} \left( \exp \frac{\pi}{v} \max\{r_0, e_k(0)\} \right) \right)
\]
where \( e_0(x) = x \) and \( e_{m+1}(x) = \exp e_m(x) \) for \( m > 0 \).

Notice that the case \( k = 0 \) is discussed in [2] and the case \( k = 1 \) is covered by [3]. Also, Dekster and Kupka [3] proved that the constant \( \frac{1}{4} \) is sharp, i.e. for any function \( A = A(r) \) using in the place of \( A_{k,v} \) so that Theorem 1 holds we must have
\[
\lim_{r \to +\infty} A(r)r^2 \geq \frac{1}{4} \quad \text{and} \quad \lim_{r \to +\infty} \left( A(r)r^2 - \frac{1}{4} \right) (\log r)^2 \geq 1.
\]
So our result identifies the higher order terms which might be added in spite to preserve the boundedness of the manifold. We think that the function \( A_{k,v} \) is sharp.

**Proof of Theorem 1.** We write the Jacobi equation associated to the sectional curvature function \( A_{k,v} \), namely
\[
y'' + A_{k,v}(r)y = 0.
\]
We claim that this equation admits the basic solutions
\[
\Psi_0 = \Phi_k(r) \cos vL_k(r) \quad \text{and} \quad \Psi_1 = \Phi_k(r) \sin vL_k(r)
\]
where
\[
\Phi_k(r) = r^{\frac{1}{2}} L_1(r)^{\frac{1}{2}} \ldots L_{k-1}(r)^{\frac{1}{2}}.
\]
For \( k = 1 \) it is easy to see that \( r^\frac{1}{2} \cos(u \log r) \) and \( r^\frac{1}{2} \sin(u \log r) \) are solutions for equation (2). By recurrence we prove first that the following relations are fulfilled (for \( k = 1 \) they are simply to check):

\[
\Phi_k'' + A_{k,0} \Phi_k = 0 \quad \text{and} \quad 2 \Phi_k' L_k' + \Phi_k L_k'' = 0.
\]

In fact we have

\[
\Phi_{k+1} = \Phi_k L_k^\frac{1}{2} \quad \text{and} \quad L_{k+1} = \log L_k,
\]

hence

\[
2 \Phi_{k+1}' L_{k+1}' + \Phi_{k+1} k L_{k+1}'' = (2 \Phi_k' L_k' + \Phi_k L_k') L_k^{-\frac{1}{2}} = 0.
\]

On the other hand

\[
\frac{\Phi''_{k+1}}{\Phi_{k+1}} = -A_{k,0} + (L_k')^2 L_k^{-2} = -A_{k+1,0}
\]

holds and the two relations stated above are proved.

Furthermore we verify that

\[
\Psi_0'' = \Phi_0'' \cos(v L_k) - v (2 \Phi_0' L_k' + \Phi_k L_k') \sin(v L_k) - v^2 \Phi_k L_k'^2 \cos(v L_k).
\]

The two relations stated above and the obvious identity

\[
L_k = L_0^{-1} L_1^{-1} \cdots L_{k-1}^{-1}
\]

complete the proof of our claim for \( \Psi_0 \) (the case of \( \Psi_1 \) is similar).

Both \( \Psi_0 \) and \( \Psi_1 \) are defined on the interval \([e_k(0), +\infty)\). Set \( r_1 = \max\{r_0, e_k(0)\} \).

Therefore, for each \( \lambda \geq e_k(0) \) the linear combination

\[
Y_{k,v,\lambda}(r) = -\sin(v L_k(\lambda)) \Psi_0(\lambda) + \cos(v L_k(\lambda)) \Psi_1(\lambda)
\]

is a solution for equation (2), which satisfies also \( Y_{k,v,\lambda}(\lambda) = 0 \). Also, we may write

\[
Y_{k,v,\lambda}(r) = \sin(v(L_k(r) - L_k(\lambda))) \Phi_k(r) L_k(r)
\]

so that \( Y_{k,v,\lambda} \) is positive on the interval \( (\lambda, \beta(\lambda)) \) where \( \beta(\lambda) = e_{k-1}(L_{k-1}(\lambda) \exp(\frac{\pi}{v})) \) and vanishes again in \( \beta(\lambda) \). This is a consequence of the straightforward formula

\[
L_k(\beta(\lambda)) - L_k(\lambda) = \frac{\pi}{v}.
\]

A standard argument (see, for instance, [1]) proves that the diameter of the manifold \( M \) is less than \( \beta(r_1) \). Since \( M \) is complete from the Hopf-Rinow theorem it follows that \( M \) is in fact compact and this ends the proof of the theorem.

**Remark 2.** The form of the function \( A_{k,v} \) is in some sense sharp. In fact, for \( v = 0 \) the analog result is false: We may choose on \( M = \mathbb{R}^n - K \), with \( K \) being a sufficiently large compact, the metric with radial symmetry \( dr + P_k(r)d\theta \) (in polar coordinates) where \( d\theta \) is the metric form on the standard sphere \( S^{n-1} \) and

\[
P_k(r) = r \left( \sum_{i=1}^{k} L_i(r)^{-2} \right)^{-\frac{1}{2}}.
\]

Then a straightforward computation shows that \( \text{Ric}_r(Y) = A_{k,0}(r)|Y|^2 \) for all points \( z \) outside the compact \( K \) and all tangent vectors \( Y \).

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References


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