On the cohomology of weighted complete intersections

By

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The weighted projective space \( P(a_0, a_1, \ldots, a_n) \) is defined as the quotient of \( CP^n \) by the following action of \( G = \mathbb{Z}/a_0 \mathbb{Z} \oplus \mathbb{Z}/a_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_n \mathbb{Z} \):

\[
(k_0, k_1, \ldots, k_n) (z_0, z_1, \ldots, z_n) = (z_0^{k_0}z_0^{-k_0}, z_1^{k_1}z_1^{-k_1}, \ldots, z_n^{k_n}z_n^{-k_n}),
\]

where \( \xi_i = \exp(2\pi i/l) \).

It is known that the integral homology groups of \( P(a_0, a_1, \ldots, a_n) \) are torsion free (see [6, 3]) so they are isomorphic to the homology groups of \( CP^n \). An entirely elementary computation was carried out for \( n = 2 \) in [4].

Let now \((V, 0)\) be an isolated singularity of complete intersection in \( \mathbb{C}^{n+k+1} \) defined by the weighted homogeneous polynomials \( f = (f_1, f_2, \ldots, f_k) \). We suppose that \( f_i \) has degree \( d_i \) with respect to the weights \( \omega(z_j) = a_j, j = 0, 1, \ldots, n + k \). There are two spaces naturally associated to the singularity \((V, 0)\), namely the link \( K = V \cap S^{2(n+k)+1} \) and the quasi-smooth weighted complete intersection \( Y_m \) defined by the polynomials \( f_i \) in \( P(a_0, a_1, \ldots, a_{n+k}) \). Notice that \( K \) is a smooth compact oriented \((2n+1)\)-dimensional manifold which is \((n-1)\)-connected (see [5]). The middle Betti numbers of \( K \) have been computed in terms of the \( a_j \)’s and the \( d_j \)’s by Dimca ([2]). The aim of this note is to give a brief insight into the cohomology of \( Y_m \). All the cohomology groups considered below have integer coefficients. We say that \((a_0, a_1, \ldots, a_{n+k+1})\) is \( m \)-prime if the greatest common divisor of any \( m \) of the \( a_j \)’s equals one.

**Proposition 1.** Suppose that \((a_0, a_1, \ldots, a_{n+k+1})\) is \( m \)-prime. Then the relative cohomology groups vanish:

\[
H^i(P(a_0, a_1, \ldots, a_{n+k}), Y_m) = 0 \quad \text{for } i \leq n - m + 1.
\]

**Proof.** Consider \( F_i(z) = f_i(z_0^{a_0}, z_1^{a_1}, \ldots, z_{n+k}^{a_{n+k}}) \) and set \( Z_\infty \) for the complete intersection defined by the polynomials \( F_i \) in \( CP^{n+k} \). Remark that the \( G \)-action on \( CP^{n+k} \) leaves \( Z_\infty \) invariant and we have \( Z_\infty / G = Y_m \). Let now \( P = \mathbb{Z}/p^m \mathbb{Z} \oplus \mathbb{Z}/p^m \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p^m \mathbb{Z} \subset G \) be a \( p \)-subgroup of \( G \). Therefore \( p^m \) divides \( a_i \) for all \( i = 0, 1, \ldots, n + k \). Then the \( P \)-invariant subsets are

\[
(CP^{n+k})^P = \{ z_i z_i = 0 \},
\]

and

\[
(Z_\infty)^P = Z_\infty \cap \{ z_i z_i = 0 \}.
\]
Since \( Z_{\infty} \) is a complete intersection \((Z_{\infty})^p\) is also a complete intersection, eventually using only part of the original equations \( F_i \). Next the number of non-zero \( a_i \)'s cannot exceed \((m-1)\) because \((a_0, a_1, \ldots, a_{n+k})\) is \( m \)-prime. Then Lefschetz’s theorem for complete intersections implies
\[
\pi_j((C\cdot P^{m+k})^p, (Z_{\infty})^p) = 0 \quad \text{for } j \leq n - m + 1.
\]
But this holds for all primes \( p \) and all maximal \( p \)-subgroups \( P \) so from [1] we derive our claim.

**Corollary 2.** For a prime number \( p \) write \( a_i = p^{r_i} \cdot c_i \) with \( r_i \) maximal. Choose a permutation \( \sigma \) of \([0, 1, 2, \ldots, n + k]\) such that
\[
r_{\sigma(1)} \geq r_{\sigma(2)} \geq \cdots \geq r_{\sigma(n+k)} \geq r_{\sigma(n+k+1)} = 0,
\]
and set:
\[
b_i = \prod_{l \leq j \leq n} p^{r_{\sigma(l)}} \quad \text{and} \quad b_i = \prod_{p} b_i(p).
\]
If \((a_0, a_1, \ldots, a_{n+k})\) is \( m \)-prime then the set of numbers
\[
R_{ij} = b_i b_j b_{i+j} \quad \text{with} \quad 0 \leq i, j, i + j \leq (n - m + 1)/2
\]
is a topological invariant of the isolated singularity \((V, 0)\).

**Proof.** The \( \mathbb{Z} \)-cohomology algebra of \( P(a_0, a_1, \ldots, a_{n+k}) \) is determined in [6]; if \( g_i \) is the generator of \( H^1 P(a_0, a_1, \ldots, a_{n+k}) \) then \( g_i \cup g_j = R_{ij} g_{i+j} \). But in low rank the cohomology algebra of \( Y_{\infty} \) is induced from that of \( P(a_0, a_1, \ldots, a_{n+k}) \) (according to Proposition 1) and we are done.

Set now
\[
F = (F_1, F_2, \ldots, F_k),
\]

\[
\bar{F} = (F_1 - z_{n+k+1}, F_2 - z_{n+k+1}, \ldots, F_k - z_{n+k+1}),
\]
and
\[
\bar{f} = (f_1 - z_{n+k+1}, f_2 - z_{n+k+1}, \ldots, f_k - z_{n+k+1}).
\]

The link of the singularity defined by \( \bar{f} \) will be denoted by \( K \). Let \( Z \) be the fibre of \( F \) over \( 1 \) (the global Milnor fibre) and \( \bar{Z} \) its projective closure. Observe that \( Z_{\infty} \) is in fact \( \bar{Z} - Z \). In fact \( P(a_0, a_1, \ldots, a_{n+k}) \) is the compactification of \( \mathbb{C}^{n+k+1} \) whose locus at infinity is precisely \( P(a_0, a_1, \ldots, a_{n+k}) \). If \( Y \) is the global Milnor fibre of \( f \) and \( \bar{Y} \) is the quasi-smooth weighted intersection in \( P(a_0, a_1, \ldots, a_{n+k}) \) associated to \( \bar{f} \) then \( Y \) may be identified with \( \bar{Y} - Y_{\infty} \). Otherwise we can look at the \( S^1 \)-action on \((S^{2n+k+3}, S^{2n+k+3})\) given by
\[
\theta \cdot z = (\theta^{a_0} z_0, \theta^{a_1} z_1, \ldots, \theta^{a_{n+k}} z_{n+k}, \theta z_{n+k+1}).
\]
Then \( (K, K)/S^1 = (\bar{Y}, Y_{\infty}) \). Then \( Y_{\infty} \) is called strongly smooth ([2]) if the \( S^1 \)-action on \( K \) is semi-free.
**Proposition 3.** Assume that \( Y_\omega \) is strongly smooth. Then \( H_+(K) \) is torsion free and the Milnor lattice of \( f \) is equivalent to the cup product
\[
H^{n+1}(\overline{K}, K) \times H^{n+1}(\overline{K}, K) \to H^{2n+2}(\overline{K}, K) \cong \mathbb{Z}.
\]
Moreover if \( k = 1 \) then this may be expressed also as the cup product
\[
H^{n+k}(S^{2n+k+1}, K) \times H^{n+k}(S^{2n+k+1}, K) \to H^{2n+k}(S^{2n+k+1}, K).
\]

**Proof.** From the Smith-Gysin sequence associated to the \( S^1 \)-action on \( K \) we derive that \( H_+(K) \) is torsion free and:
\[
H_j(Y_\omega) = H_j(CP^n) \quad \text{for } j \neq n, \quad H_n(Y_\omega) = H_n(K) \oplus H_n(CP^n).
\]
Now \( Y_\omega \) is strongly smooth if and only if \( Y \) is strongly smooth. The long exact sequence of the pair \( (\overline{K}, K) \) gives us:
\[
\begin{align*}
H^1(\overline{K}, K) &= 0 \quad \text{for } k \neq n + 1, n + 2, 2n + 2, 2n + 3, \\
H^{2n+2}(\overline{K}, K) &= H^{2n+3}(\overline{K}, K) = \mathbb{Z}.
\end{align*}
\]
But \( Y \) has the homotopy type of a bouquet of \( (n + 1) \)-spheres (see [7]) so using the Lefschetz's duality we find:
\[
H^j(\overline{Y}, Y_\omega) = 0 \quad \text{for } j \neq n + 1.
\]
Now from the Smith-Gysin sequence associated to the \( S^1 \)-action on \( (\overline{K}, K) \) we obtain:
\[
\begin{align*}
0 &= H^{2n}(\overline{Y}, Y_\omega) = \ker (H^{2n+2}(\overline{Y}, Y_\omega) \to H^{2n+2}(\overline{K}, K)), \\
0 &= H^{2n+1}(\overline{Y}, Y_\omega) = \ker (H^{2n+2}(\overline{Y}, Y_\omega) \to H^{2n+2}(\overline{K}, K)), \\
H^{n+2}(\overline{K}, K) &\cong H^{n+1}(\overline{Y}, Y_\omega), \\
H^{n+3}(\overline{Y}, Y) &\cong H^{n+1}(\overline{K}, K).
\end{align*}
\]
Using the functoriality of Lefschetz duality the first part of our claim follows. If \( k \) equals one then \( \overline{K} - K \) is a non-ramified \( \mathbb{Z}/d \mathbb{Z} \)-covering of \( S^{2n+k+1} \) and the Alexander duality gives the second claim.

References


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