GENERALIZED HADWIGER NUMBERS
FOR SYMMETRIC OVALS

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ABSTRACT. Some estimations for the "juxtaposition function" $h_F$ and an asymptotic formula for the function $h_F/h_G$, where $F, G$ are central symmetric convex bodies, are given. Hadwiger and Grünbaum gave for $h_F(1)$ the bounds $n^2 + n < h_F(1) < 3n - 1$. Grünbaum conjectured (and proved for $n = 2$ in Pacific J. Math. 11 (1961), 215-219) that for every even $r$ between these bounds there exists in $E^n$ an oval $F$ such that $h_F(1) = r$. Lower bounds for $h_F$ could be derived in the same way as in Theorems 1 and 2 from a good estimate of packing numbers on a Minkowski sphere, that is, from solutions to a Tammes-type problem in a Banach space.

For a topological disk $F \subseteq \mathbb{E}^n$ we shall denote by $h_F : (0, 1] \to \mathbb{N}$ the "juxtaposition function" introduced by the first author [2, 3] as follows. Let $A_{F, \lambda}$ denote the family of all sets, homothetic to $F$ in the ratio $\lambda$, which have only boundary points in common with $F$. Then $h_F(\lambda)$ is the greatest integer $k$ such that $A_{F, \lambda}$ contains $k$ sets with pairwise disjoint interiors. In particular, $h_F(1)$ is just the Hadwiger number of $F$.

In case of convex $F$, Hadwiger [11] and Grünbaum [8] gave for $h_F(1)$ the bounds $n^2 + n < h_F(1) < 3n - 1$. Grünbaum [8] conjectured (and proved for $n = 2$; see also Boltyanski and Gohberg [4]) that for every even $r$ between these bounds there exists in $E^n$ an oval $F$ such that $h_F(1) = r$.

Unless explicitly stated otherwise, throughout this paper $F, G$ will denote symmetric plane ovals. Any such $F$ determines a norm $\| \|$ by $\|x - y\|_F = \|x - y\|/\|o - z\|$, where $\| \|$ is the Euclidean norm, $o$ is the center of $F$, and $z$ is a point on the boundary $\partial F$ of $F$ such that $oz$ and $xy$ are parallel. With this norm $\mathbb{E}^2$ becomes a Banach space, with unit disk isometric to $F$. Set $p(F)$ for the perimeter of $\partial F$ in its inner norm.

**Theorem 1.** For a symmetric oval $F$ in the plane

$$p(F) = 2 \lim_{\lambda \to 0} \lambda h_F(\lambda).$$

**Proof.** Let $x, y$ be points of $\partial F$, and let points $x', y'$ be given by $ox' = (1 + \lambda)ox$ and $oy' = (1 + \lambda)oy$. Denote by $F_x, F_y$ those sets in $A_{F, \lambda}$ which have centers at $x'$ and $y'$, respectively. If $F_x \cap F_y \neq \emptyset$, it follows from the
symmetry and convexity of $F$ that $x'y' \subset F_x \cap F_y$. We put $x'y' \cap \partial F_x = \{a, b\}$, $x'y' \cap \partial F_y = \{c, d\}$, and $z \in \partial F$ such that $oz$ is parallel to $x'y'$. Then
\[
\|x' - y'\| \leq \|x' - b\| + \|c - y'\| = 2\|x' - b\| = 2 \lambda \|o - z\|
\]
hence
\[
\|x - y\|_F = \|x' - y'\|/(1 + \lambda) \|o - z\| \leq 2 \lambda/(1 + \lambda).
\]
Reversing the reasoning we obtain
(2) \hspace{1cm} \text{int} F_x \cap \text{int} F_y = \emptyset \text{ if and only if } \|x - y\|_F \leq 2 \lambda/(1 + \lambda).

Now consider a maximal collection $\{F_i: i = 1, \ldots, k\} \subset A_{f, \lambda}$ of sets with disjoint interiors and the points $x_i \in \partial F$, $i = 1, \ldots, k$, for which $F_i = Fx_i$. From (2) it follows that $\|x_i - x_{i+1}\|_F \leq 2 \lambda/(1 + \lambda)$ and thus
\[
\sum_{1 \leq i \leq k} \|x_i - x_{i+1}\|_F \leq 2k \lambda/(1 + \lambda);
\]
however,
\[
p(k, F) = \sup \left\{ \sum_{1 \leq i \leq k} \|x_i - x_{i+1}\|_F, \ x_i \in \partial F \right\} \leq p(F).
\]
These inequalities yield
(3) \hspace{1cm} h_F(\lambda) = k \leq (1 + \lambda)p(k, F)/2 \lambda < (1 + \lambda)p(F)/2 \lambda.

Conversely, let $P_\lambda$ be an inscribed polygon with $2k$ vertices $u_1, \ldots, u_{2k}$ such that $P_\lambda$ is symmetric about $o$ and
\[
\|u_1 - u_2\|_F = \|u_2 - u_3\|_F = \cdots = \|u_{k-2} - u_{k-1}\|_F = 2 \lambda/(1 + \lambda),
\]
\[
2 \lambda/(1 + \lambda) \leq \|u_{k-1} - u_k\|_F < 4 \lambda/(1 + \lambda).
\]
Then the sets $F_{u_i}$ have disjoint interiors and
\[
4(k + 1) \lambda/(1 + \lambda) > \sum_{1 \leq i \leq 2k} \|u_i - u_{i+1}\|_F \geq 4k \lambda/(1 + \lambda).
\]
Since $h_F(\lambda) \geq 2k$, it follows that
(4) \hspace{1cm} 2 + h_F(\lambda) \geq (1 + \lambda) \left( \sum_{1 \leq i \leq 2k} \|u_i - u_{i+1}\|_F \right)/2 \lambda.

If $p(\lambda)_F$ denotes the perimeter of $P_\lambda$ in the $\|\cdot\|_F$ norm, then (see [1, 11])
(5) \hspace{1cm} \lim_{\lambda \to 0} p(\lambda)_F = p(F).

For symmetric ovals $F, G$ relations (3)-(5) imply
\[
\lim_{\lambda \to 0} h_F(\lambda)/h_G(\lambda) \geq \lim_{\lambda \to 0} (-2 + (1 + \lambda)p(\lambda)_F)/2 \lambda)/(1 + \lambda)p(G)/2 \lambda
\]
\[
= p(F)/p(G),
\]
and similarly the reverse inequality. Therefore, taking for $G$ a square we obtain the claim which was to be proved.

Denote by $\lfloor t \rfloor$ the integer part of $t \in \mathbb{R}$. 

Theorem 2. For every symmetric oval $F$ in the plane
\begin{equation}
  3 + \frac{3}{\lambda} \leq h_F(\lambda) \leq 4(1 + \lambda)/\lambda,
\end{equation}
with equality on the left if and only if $1/\lambda \in \mathbb{N}$, and $F$ is an affine-regular hexagon and equality on the right if and only if $1/\lambda \in \mathbb{N}$ and $F$ is a parallelogram.

Proof. A result of Golab [6] and Reshetnyak [14], generalized by Schäffer [15], asserts that $6 \leq p(F) \leq 8$. Hence we have
\[ h_F(\lambda) \leq 4(1 + \lambda)/\lambda, \]
and, using the existence of an affine-regular hexagon inscribed in $F$ [13], we obtain $h_F(2/(1 + k)) \geq 6k$. Since $h_F(\lambda)$ is a decreasing function of $\lambda$, we are done.

If the dimension of $F$ is greater than two, the situation is essentially different. We shall prove (see also [7])

Theorem 3. Any symmetric convex body $F \subset \mathbb{E}^n$ satisfies the inequality
\begin{equation}
  h_F(\lambda) \leq ((1 + \lambda)^n - 1)/\lambda^n,
\end{equation}
with equality if and only if $1/\lambda \in \mathbb{N}$ and $F$ is a parallelohedral body.

Proof. Let $B_{\lambda} = \bigcup_{H \in A_F, \lambda} H$. We shall prove that
\begin{equation}
  B_{\lambda} \subset (1 + 2\lambda)F.
\end{equation}
Indeed, let $x$ be a point on the boundary of $F_v$, $|ox| \cap \partial F = \{a\}$, $|ov| \cap \partial F_v = \{q\}$, and let $ux''$ be parallel to $qx$ with $x'' \in \partial F$. Then $\zeta v'x = \zeta v'xo + \zeta v'ox \geq \zeta v'ox$, which yields $\zeta vox = \zeta vox'' = \zeta qv'x$. Since $F$ is convex, we can take a point $b$ in the nonempty intersection $|oa| \cap |ux''|$. Then $|ux''| \subset F$, $b \in F$, $b \in |ox|$. Since
\[ \|o + a\|/\|o - x\| \geq \|o - b\|/\|o - x\| \geq \|o - v\|/\|o - q\| = 1/(1 + 2\lambda), \]
the point $x$ belongs to $(1 + 2\lambda)F$, and (9) is proved.

If $\{F_i, i = 1, \ldots, k\} \subset A_{F, \lambda}$ have disjoint interiors, then
\[ \bigcup_{1 \leq i \leq k} F_i \subset B_{\lambda} \subset (1 + 2\lambda)F; \]
therefore,
\[ \text{vol}(F) + \text{vol}(F_1) + \cdots + \text{vol}(F_k) \leq (1 + 2\lambda)^n \text{vol}(F) \]
where $\text{vol}(F)$ denotes the volume of $F$. This gives the desired estimation on $h_F(\lambda)$. The equality case is treated in [7].

Lower bounds for $h_F(\lambda)$ could be derived in the same way as in Theorems 1 and 2 from a good estimate of packing numbers on a Minkowski sphere, that is, from solutions to a Tammes-type problem in a Banach space.

Grünbaum asked what happens to relation (1) in case $F$ is not centrally symmetric. We recall that for an arbitrary oval $F$ and $z \in \text{int} F$ a norm (nonsymmetric, in general) is defined by the Minkowski functional
\[ \|x\|_{F, z} = \inf\{\lambda > 0 : x - z \in \lambda(F - z)\}. \]
Using the (possibly nonsymmetric) distance derived from this norm it is possible to define arc-length for oriented arcs. For an oriented closed curve \( C \) let the length of \( C \) in the metric derived from \( \| \|_{F,z} \) be denoted by \( p_{F,z}(C) \). The intrinsic perimeter (self-circumference [6, 10]) of \( F \) is \( P(F) = \inf\{p_{F,z}(\partial F) : z \in \text{int} F\} \). Then it follows that

\[
g(F) = \lim_{\lambda \to 0} \lambda h_F(\lambda)/P(F)
\]

is a measure of symmetry (see [8]). By the same method as used above, it is possible to show that \( g(F) \leq \frac{1}{2} \), with equality if \( F \) is centrally symmetric. If \( F \) is a triangle then \( g(F) = \frac{1}{3} \), and we conjecture that \( g(F) \geq \frac{1}{3} \) for any oval \( F \).

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**REFERENCES**


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