ITERATIVE PROCESSES FOR \( Z_2 \)

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In the last fifty years the study of iterative processes suffered an important development. Beginning with the paper of M. H. Lyapunov [1] upon now, a lot of articles concerning this subject has been published. A retrospective look is given in the expository paper of J. J. St. Fleis [2] about iteration of number theoretic functions, where it can be found many bibliographical references. This great interest in the field of iterative processes motivated our paper.

Let \( G \) be a graph with labelled vertices from 1 to \( n \), and the set of edges \( E \). It induces a transformation \( f_0 : Z^n \rightarrow Z^n \) in the following manner: For \( X = (x_1, \ldots, x_n) \) let \( x_i \) denote the \( i \)-th component in the standard basis \( e_1, \ldots, e_n \). Then \( f_0 \) is defined by:

\[
(f_0 X)_i = \sum_{j=0}^{n-1} X_{i+j}\mod n
\]

We consider in our paper that \( p = 2 \).

Definition. 1 The graph \( G \) is \( p \)-nilpotent if exists \( h \) such that for every \( X \in Z_2^n \) we have:

\[
(f_0 \circ f_0 \circ \ldots \circ f_0)(x) = 0
\]

\( (0) \) denotes the null element of \( Z_2^n \).

We shall give a characterization of \( 2 \)-nilpotent \( \Pi_{2,1}, T_{2,1}, C_{2,1} \) graphs.

If we paint a table \( (n \times k) \) in the plane, on the torus or on the cylinder the corresponding graphs induced by the relations between neighbors are denoted \( \Pi_{2,1}, T_{2,1}, C_{2,1} \) respectively.

![Graphs](image)

\( \Pi_{2,1}, C_{2,1}, T_{2,1} \)

Figure

We want study 2-nilpotence for these graphs.

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Theorem 1. (i) $T_{n,k}$ is 2-nilpotent iff $T_{n,1}$ and $T_{k,1}$ are 2-nilpotent.
(ii) $C_{n,k}$ is 2-nilpotent iff $C_{n,1}$ and $T_{k,1}$ are 2-nilpotent.
(iii) $\Pi_{n,k}$ is 2-nilpotent iff $\Pi_{n,1}$ and $\Pi_{k,1}$ are 2-nilpotent.

Proof. We identify $Z_2^n$ with the set of matrices $M_{n,n}(Z_2)$, and denote with same letters $t_{n,k}$, $t_{n,1}$, $t_{k,1}$ the induced transformations. Let:

$$
E_2 = \begin{bmatrix}
010 & \cdots & 0 \\
01 & \cdots & 0 \\
010 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0000 & \cdots & 01 \\
00 & \cdots & 010
\end{bmatrix}, \quad E_2 = M_{n,n}(Z_2)
$$

$$
D_2 = \begin{bmatrix}
0100 & \cdots & 001 \\
1010 & \cdots & 000 \\
0101 & \cdots & 000 \\
\vdots & \ddots & \vdots \\
0000 & \cdots & 101 \\
1000 & \cdots & 010
\end{bmatrix}, \quad D_2 = M_{n,n}(Z_2)
$$

By direct computation it follows:

$$
t_{n,1}X = R_nX + XE_2 \quad (M_{n,1}(Z_2))
$$

(6)

$$
t_{n,1}X = E_2X + XD_2 \quad (M_{n,1}(Z_2))
$$

$$
t_{k,1}X = D_nX + XD_2 \quad (M_{n,1}(Z_2))
$$

Proposition 1. (i) $\Pi_{n,2}$ is 2-nilpotent iff there exists $q$ such that:

$$
E_2^pX = XE_2^p \text{ for every } X = M_{n,1}(Z_2)
$$

(7)

(ii) $C_{n,2}$ is 2-nilpotent iff there exists $q$ such that:

$$
E_2^pX = XD_2^p \text{ for every } X = M_{n,1}(Z_2)
$$

(iii) $T_{n,2}$ is 2-nilpotent iff there exists $q$ such that:

$$
D_2^pX = XD_2^p \text{ for every } X = M_{n,1}(Z_2)
$$

Proof (i). We have:

$$
t_{n,2}X = E_2(E_2X + XE_2) = (E_2X + XE_2)E_2 = E_2^2X + XE_2^2(M_{n,1}(Z_2)) \text{ and by induction;}
$$

$$
t_{n,2}^pX = E_2^pX + XE_2^p
$$
If there exists $h$ such that $t_{h,3}^X = 0(M_{a,h}(Z_2))$ then, for every $q, 2^r > h$

$$t_{h,3}^X = 0(M_{a,h}(Z_2))$$

so it follows (7)

Conversely if we have (7) then $t_{h,3}^X = 0(M_{a,h}(Z_2))$ for every $X \in M_{a,h}(Z_2)$ and $\Pi_{1,3}$ is 2-nilpotent.

In the same manner (i) and (iii) are proved.

Lemma 2. If $H = M_{n,n}(Z_2), S = M_{a,a}(Z_2)$ such that for every $X \in M_{a,a}(Z_2)$ we have $RX = XS$

then there exist $\beta$ such that $R = \beta I_n, S = \beta I_a$ (A Schur type lemma [3]).

Proof. Let $H = (h_{ij})_{1 \leq i, j \leq n}, S = (s_{ij})_{1 \leq i, j \leq a}$

| 00 ... 0 |
| 00 ... 0 | \( i \)
| 11 ... 1 |

Put

$$X = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \ddots & \cdots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}$$

Then:

\[ r_{ij} = s_{ij} + \cdots + s_{1j} - s_{11} - s_{22} + \cdots + s_{ii} - \cdots = s_{ij} + \]

and $r_{ij} = 0$ if $j \neq i$.

It follows, that $R = \beta I_n$ if $i \in \{1, \ldots, n\}$. In the same manner $S = \gamma I_a$ and because $RX = XS$ it follows $\beta = \gamma$ which proves lemma.

From Proposition 1 and Lemma 2 it follows:

Proposition 3. (i) $\Pi_{1,3}$ is 2-nilpotent iff there exist $\gamma$ such that:

\[ \Pi_{1,3}^q = \beta I_n, E_{3,3}^q = \beta I_a(\Pi_{1,3}(Z_2)) \]

(ii) $C_{1,3}$ is 2-nilpotent iff there exist $\gamma$ such that:

\[ \Pi_{1,3}^q = \beta I_n, C_{3,3}^q = \beta I_a(\Pi_{1,3}(Z_2)) \]

(iii) $T_{3,3}$ is 2-nilpotent iff there exist $\gamma$ such that:

\[ D_{3,3}^q = \beta I_n, D_{3,3}^q = \beta I_a(\Pi_{1,3}(Z_2)) \]

From Proposition 3 it results that Theorem 1 holds.

Proposition 4. $T_{3,3}$ is 2-nilpotent iff $n - 2^r, a = Z_2$.

Proof. We consider $a = Z_2$ and denote by

\[ a^{(r)} = t_{r,0,0} \ldots a_{r,1} \]

(8)
We prove by induction that \( a_{n+1} \equiv a_n \pmod{2} \) for \( k = 0 \) this is obvious. Also,
\[
\frac{a_{n+1}^{(i+1)}}{a_{n+1}^{(i)}} \equiv \frac{(a_{n+1})_i^{(i)}}{a_{n+1}^{(i)}} \equiv a_{n+1}^{(i)} + a_{n+1}^{(i-1)}
\]
\( a_{n+1} \equiv a_n + a_{n+2} \equiv a_{n+2} \pmod{2} \) (mod 2)
If \( n = 2^k \) then for \( k = d \)
\( a_{n+1}^{(2^k)} \equiv a_{n+1}^{(2^{k-1})} + a_{n+1}^{(2^{-1})} \equiv 0 \pmod{2} \)
so :
\[
\prod_{n=2^{k-1}}^{n=2^k} \equiv 0
\]
for every \( n \in M_n \). Also this follows from [4]. For the converse, let \( n \) odd, and \( k \) be the smallest integer such that \( a_{n-k} \equiv 0 \pmod{2} \), \( k \in \{1, \ldots, n\} \) so the \( a_{n-k-1} \equiv 1 \pmod{2} \) if \( h > 3 \) one deduces that \( a_{n-k}^{2^h} + a_{n-k-1}^{2^h} \equiv 1 \pmod{2} \), so :
\[
n = \sum_{i=1}^{n} (a_{n-i+1}^{(2^h)} + a_{n-k}^{(2^h)}) \equiv \sum_{i=1}^{n} a_{n-i+1} \equiv 0 \pmod{2}
\]
which is false because \( n \) is odd. Let now \( n = 2^h \), \( h \) odd \( h > 1 \), and \( a_n = \ldots = (a_{n+2}, a_{n+1}, \ldots, a_{n+1}, a_n) \). Then
\[
\prod_{n=2}^{n=n+1} \equiv \prod_{n=2}^{n=n+1} \equiv 0
\]
since :
\[
a_{n+1}^{2^{n+1}} \equiv a_{n+1} \pmod{2}
\]
Because \( h \) is odd, if \( u \neq \{(0, 0, \ldots, 0), (1, \ldots, 1)\} \) as below, it can be proved, \( t_{n+1}^{(u)} \equiv 0 \pmod{2} \) for any \( m \), which implies :
\[
\prod_{n=2}^{n=n+1} \equiv 0 \pmod{2}
\]
Proposition 5. \( \Pi_n \) is 2-nilpotent iff \( n = 2^d - 1 \), \( d \in \mathbb{Z}_+, \)

Proof: Let \( a \in \mathbb{Z}_2 \), \( fa \in \mathbb{Z}_2 \) such that
\[
(fa)_i = \begin{cases} a_i, & \text{if } i < n \\ 0, & \text{if } i = n + 1 \end{cases}
\]
if \( n+1 \neq 2^d \), let \( n = 1 \) odd, and let \( n = \mathbb{Z}_2 \) such that
\[
a_{n+1} - a_n, n \geq 3
\]
\( a_n \equiv 1 \pmod{2} \) and \( a \neq (1, 1, \ldots, 1) \).
From this relations it follows that
\[ f_{n+1, 1} f(a) = f(Z_2) \] for every \( m \) and also, \( f(\Pi_{n+1, 1}) = f_{n+1, 1}(f(a)). \)

But from Proposition 4 because \( f(a) \notin \{0, 0, \ldots, 0\}, (1, \ldots, 1) \) it follows: \( f_{n+1, 1}(f(a)) \neq 0 \) for any \( m \). So \( n \neq k \) must be a power of 2, or \( n = 2 \) in which case it is easy to verify that \( \Pi_{n, 1} \) is not 2-nilpotent.

Let \( \Omega_{k, b} \) be a square matrix of order \( n \) with elements \( a_{ij} \) defined as follows:

If \( k \leq \frac{n-3}{2} \):

\[
\begin{align*}
a_{1, 1} = 1, & \quad a_{1, 2} = a_{2, 1} = 1, \\
a_{2, 2} = a_{2, 1} = 1, & \quad a_{3, 2} = a_{2, 3} = 1, \\
& \quad \vdots \\
& \quad a_{k, k+1} = a_{k+1, k} = 1,
\end{align*}
\]

If \( k > \frac{n-3}{2} \), \( \Omega_{k, b} = \Omega_{k-1, b} \) and if \( k = \frac{n-3}{2} \):

\[
\begin{align*}
a_{1, 1} = 1, & \quad a_{1, 2} = a_{2, 1} = 1, \\
a_{2, 2} = a_{2, 1} = 1, & \quad a_{3, 2} = a_{2, 3} = 1, \\
& \quad \vdots \\
& \quad a_{k-1, k} = a_{k, k-1} = 1,
\end{align*}
\]

Then a direct computation gives us:

\[ \Omega_{k, b} = \Omega_{k-1, b} \cdot a_{1, b} \cdot (M_{k, b}(Z_2)) \]

for \( n \neq 2k + 3 \) and \( \Omega_{k, b} \equiv 0(M_{k, b}(Z_2)). \)

But we have:

\[ E_n^{a} = Z_0, \quad \text{so} \quad \Pi_n^{a} = \Omega_{b, b}(M_{a, b}(Z_2)), \]

where \( a_1 = 0, \quad a_{3, 1} = \min(2a_1 + 2, 2a_1 + 2a_2 - 4). \)

But for \( n = 2^d \) we have \( a_d(n) = 2^{d-1} - 2, \quad n = 2a_2 - 3 \) which imply

\[ E_n^{a_d-1} \equiv 0(M_{a, b}(Z_2)) \]

so after Proposition 3 \( \Pi_{n, 1} \) is 2-nilpotent.

Theorem 2 and Propositions 4, 5 give us:

(i) \( \Pi_{n, 1} \) is 2-nilpotent iff \( n = 2^d - 1, \quad k = 2^d - 1, \quad d, f = 2^d - 1 \),

(ii) \( C_{n, 1} \) is 2-nilpotent iff \( n = 2^d - 1, \quad k = 2^d, \quad d, f = 2^d \),

(iii) \( T_{n, 1} \) is 2-nilpotent iff \( n = 2^d, \quad k = 2^d, \quad d, f = 2^d \).

Corollary. Let the sequence \( a_d(n) \), defined by \( a_1(n) = 0, \)

\[ a_{d, 1}(n) = \min(2a_d(n) + 2, 2n - 2a_2(n) - 4) \]

Then there exists \( m \) such that \( a_d(n) = 0 - 1 \) iff \( n = 2^d - 1, \quad d \in Z \).
3. Unsolved problems

There are a lot of questions which naturally arise when we study the
\(p\)-nilpotent graphs.

We enumerate some of them without comments.

Problem 1. For what \(g, p\) there exist \(p\)-nilpotent graphs of genus \(g\)?

Problem 2. If \(p \geq 3\), for what \(n\), there exist \(p\)-nilpotent graphs with \(n\) vertices?

Problem 3. For what \(n, m\) the graphs \(\Pi_n\), \(C_m\), \(T_m\) are each
\(p\)-nilpotent? (\(p \geq 3\)).

BIBLIOGRAPHY


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