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TP Systèmes dynamiques et chaos
2017-2018

Contents

The laboratory course consists in one experimental session (4 hours), and two numerical sessions (3 hours each). In addition to the theoretical lectures, a (non exhaustive) bibliography includes the two basic textbooks:

- *Non linear Dynamics and Chaos*, Strogatz (1994)
- *L'ordre dans le chaos*, Bergé, Pommeau, Vidal

The numerical experiments will be based on Python routines. However, you may want to use other softwares with built-in mathematical routines needed to solve ordinary differential equations problems (e.g. Matlab, etc.)

1 Evaluation

- study of a non-linear dynamic system, not analyzed during the lab session
- individual work
- in english
- 4 page article (including bibliographic references)
- appendix may be added, although must not be needed to understand the 4 pages
- typical layout : introduction on the system (physical meaning, or analogy to some real-life situation), dimensionless equations, parameters and characteristic scales, analysis of fixed points, numerical exploration. You may want to also provide ideas of some experimental device.
- PDF format only, to be sent by e-mail to your tutor, by the end of Christmas break

Experimental session

The experimental session splits into two experiments: one is dedicated to the Rayleigh-Bénard instability, the other to a chaotic pendulum.

2 Rayleigh-Bénard instability

2.1 Qualitative analysis

Consider a fluid in hydrostatic equilibrium in the local terrestrial gravitational field.

In a first instance, consider that the fluid is confined between two *vertical* plates, with one plate being heated. The fluid located close to the cold plate tends to sink while the fluid close to the warmer plate moves upwards, such that a torque is exerted on the fluid which becomes convective. In the process, heat is conducted from one plate to the other, essentially by convection (the latter being, by orders of magnitudes, more effective than diffusion). Because this torque is created for arbitrarily small temperature difference between the two plates, such a situation is globally unstable.

Now let us confine the fluid between two *horizontal* plates. Consider a fluid particle at altitude z where the temperature varies by $\delta T > 0$. In general, the density of a fluid decreases with temperature

$$d\rho/dT < 0$$

and the fluid particle will moves upwards (buoyancy)[1] due to Archimede force. What happens next depends on the competition between thermal conduction and viscosity. If the motion is so slow that thermal conduction has time to equilibrate its temperature with the surrounding, its temperature will decrease, and the buyoancy will stop. The fluid is stable. The upwards velocity is driven by the competition between viscosity— which slows down the motion— and the density contrast between the heated fluid particle and its surrounding, hence by the temperature gradient. Intuitively, one anticipates that sufficiently large δT would result in an ascending motion on a characteristic timescale shorter than those of the two stabilizing effects, namely thermal conduction and viscosity. The fluid is here *not* globally unstable. Rather, one talks about a threshold instability.

Let us draw our attention on the characteristic timescales involved, by considering a setup in which a negative vertical temperature gradient ($dT/dy < 0$ with y the altitude) exists between the two horizontal plates. A fluid particle of radius a is moved upward from an initial position y_1 to a new position $y_2 = y_1 + \delta y > y_1$. Its initial temperature is $T(y_1) = T_1$ and the fluid particle is now surrounded by fluid at a temperature $T_2 = T_1 + \delta T < T_1$. On the other hand, at height y_2, the fluid density is $\rho_2 = \rho_1 + \delta\rho$, with

$$\frac{\delta\rho}{\rho} \approx -\alpha\delta T > 0$$

where α is the isobaric dilatation coefficient [2]. The net force, which is the sum of the weight and the Archimede's force, is directed upward:

$$F_A = \frac{4}{3}\pi a^3 g\delta\rho$$

[1] Buoyancy=flottabilité

[2] Isothermal compressibility, $\kappa_T\delta P$, has been neglected.

For the convective instability to set in, this force must overcome the Stokes drag, $F_{\mathrm{visc}} = 6\pi\nu\rho_1 a v$:

$F_A > F_{\mathrm{visc}}$

The characteristic timescales associated to this net force is

$$\tau_A^2 \sim \frac{a\rho}{g\delta\rho} = \frac{a}{\alpha g\delta T}$$

On the other hand, the characteristic timescales associated to the stabilizing effects are:

- molecular diffusion[3] (viscosity) timescale: $\tau_\mu \sim a^2/\mu$
- thermal diffusion timescale: $\tau_\kappa \sim a^2/\kappa$

A dimensionless number, the Rayleigh number Ra, can be obtained that combines the three aforementioned timescales, which compares the relative strength of stabilizing and destabilizing processes:

$$Ra = \frac{\tau_\mu \, \tau_\kappa}{\tau_A^2} = \frac{g \, a^3 \, \alpha \, \delta T}{\kappa\mu} \qquad (1)$$

When Ra is larger than a critical value Ra_c, buoyancy takes over the stabilizing effects (diffusion) and convective motion sets in. A thorough analysis leads to

$Ra_c = 1708$

In other words, when $Ra > Ra_c$, hydrostatic equilibrium becomes unstable and the system becomes convective (Guyon, E. and Hulin, J.-P. and Petit, L. 2001, p. 461). This transition is in fact a supercritical bifurcation (voir Fig. 1). Nevertheless, for values of Ra not too large, the convective motion is stationary because of the third order term which is negative. When Ra increases further, new bifurcations appear and the motion becomes highly complex. The second bifurcation, at $Ra \sim 25000$, is a Hopf bifurcation, and the convective rolls appear to oscillate. Transition to chaos takes place for $Ra \sim 10^5$ which leads to weak turbulent regime at even higher Ra numbers (Guyon, E. and Hulin, J.-P. and Petit, L. 2001, annexe A-1 et A-2).

2.2 Transition to convection through bifurcation

2.3 Experiments

2.3.1 Material

- plaque chauffante électrique ;
- support télescopique ;
- transformateur variable ;
- disque d'aluminium ;
- boîte transparente ; plusieurs épaisseurs sont disponibles.
- encens et une boîte d'allumettes (ou un briquet).

[3] The dynamic viscosity μ is related to the kinematic viscosity ν by $\nu = \mu/\rho$.

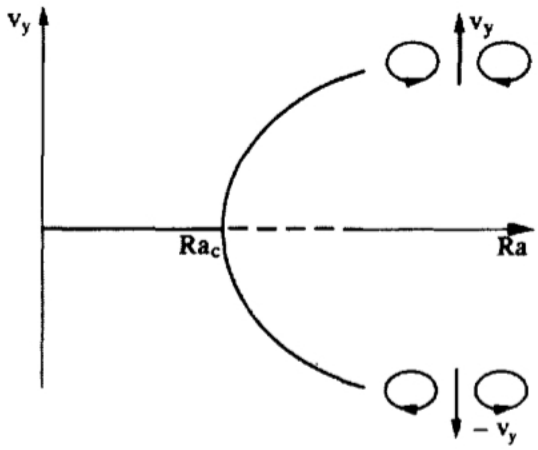

Figure 1: The vertical velocity as a function of the control parameter, here the Rayleigh number *Ra* (from Guyon, E. and Hulin, J.-P. and Petit, L. 2001).

- thermocouple type K pour la mesure de la différence de température entre la plaque l'aluminium et le dessus de la boîte. La sensibilité est 40μV/K.
- voltmètre
- scotch
- laser He-Ne et une lentilles cylindriques.

2.3.2 Experimental setup

(Voir Fig. 2)

- Mettre la plaque d'aluminium sur la plaque chauffante électrique, et branchez le transformateur en amont de la plaque électrique ;
- Installer la boîte sur la plaque d'aluminium ;
- Installez les thermocouples (une extrémité sur la plaque, l'autre sur la face supérieure de la boîte, vers le centre) ;
- Placer le laser et la lentille cylindrique de manière à avoir une nappe (réfléchissez à l'orientation de cette nappe, horizontale ou verticale) ;
- Introduire de la fumée d'encens dans la boîte ;
- Chauffer très progressivement (utiliser pour cela le transformateur).
- Recommencer l'expérience en disposant une nappe de lumière vous permettant de mettre en évidence la forme en rouleaux des cellules convectives.

2.3.3 Numerical application

Air properties at 20°C

- isobaric dilatation coefficient: $\alpha = 3.34 \times 10^{-3}$ K^{-1}

- thermal diffusion: $\kappa = 0.202 \times 10^{-4}\ \mathrm{m^2\,s^{-1}}$
- dynamic viscosity: $\nu = 1.81 \times 10^{-5}$ Pa s
- kinematic viscosity: $\mu = \nu/\rho$
- masse volumique: $\rho = 1.205\ \mathrm{kg\,m^{-3}}$

2.3.4 Observations

Voici une liste (non exhaustive) d'observations et mesures que vous pourriez effectuer :

- observer la première bifurcation : pour cela, partir d'une situation statique, puis chauffer très lentement ;
- établir la valeur du nombre de Rayleigh critique ; comparer à la prédiction théorique ;
- tailles caractéristiques, nombre, orientation, sens de rotation des rouleaux ;
- quelles sont les incertitudes de votre montage ?
- en utilisant une boîte plus épaisse, mettre en évidence les bifurcations suivantes, par exemple mettre en évidence l'oscillation des rouleaux. Cela nécessite de chauffer très lentement. Il faut refroidir la plaque avant de commencer.

2.3.5 How to improve the experiment ?

We note that our qualitative derivation of the R-B instability implicitely assumes that the Prandtl number[4], $Pr = \nu/\kappa$, is much greater than unity, which means that the difference of temperature between the fluid particle and its surrounding is driven by thermal conduction. Is this verified in our experiment ? What would you propose to satisfy this requirement ?

[4]This dimensionless number compares the momentum and thermal diffusivity.

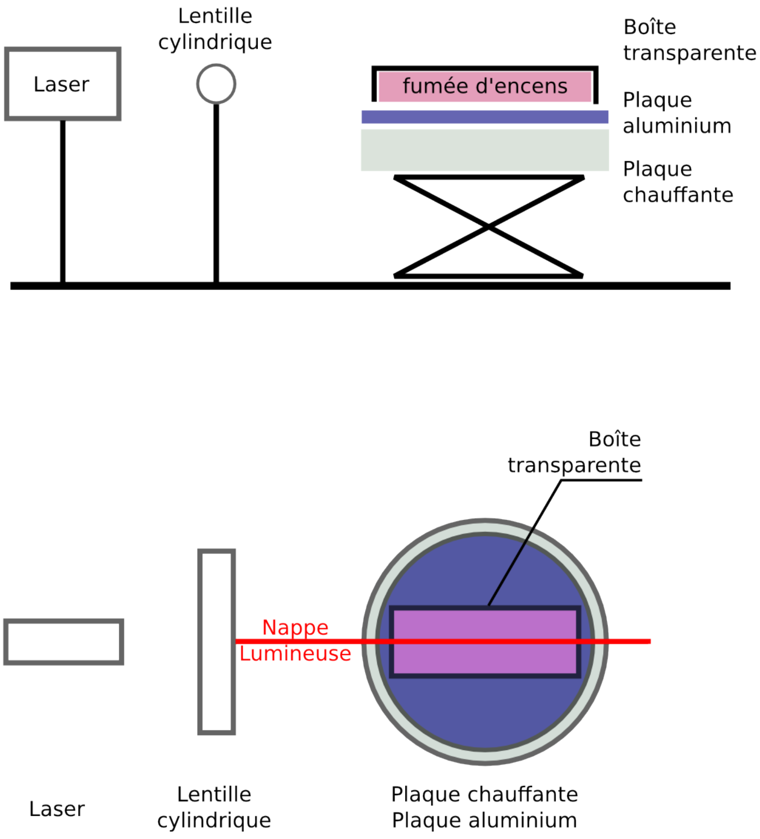

Figure 2: Suggested experimental setup (side and top views).

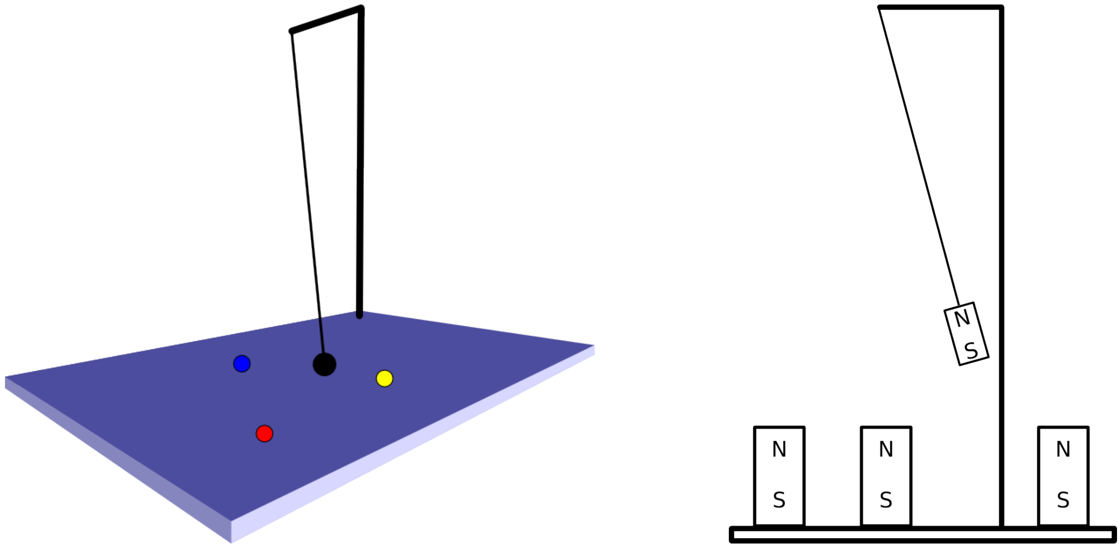

Figure 3: Experimental setup of the chaotic pendulum: a magnet is freely swinging in the static magnetic fields created by three magnets at rest.

3 Chaotic pendulum

3.1 Introduction

Chaos is difficult to identify quantitatively. One way could be to analyse the spectral content of any variable, e.g. by computing a Fourier Transform which, for a chaotic behaviour, would look like a noisy spectrum with no peak clearly outstanding, although possibly evidencing sharp lines at isolated frequencies (see the Lorentz convection in Numerical Experiment). Another way of characterizing chaos is based on the fact that chaotic systems show extreme sensitivity to initial conditions: the method consists in analysing the time evolution of trajectories from very close initial conditions. This is what we propose to do here.

3.2 Experimental setup

Consider a pendulum made of a small magnet at its end, which moves freely above the static magnetic field created by three magnets at rest (see Fig. 3). This system presents several characteristics: 1/ it has three fixed points, close to each static magnet, and 2/ it is chaotic.

The first goal of the experiment is to evidence the sensitivity to initial conditions.

1. place the pendulum at a height where it becomes sensitive to the three magnets;

2. perform several experiments starting with, close initial conditions; observe the final state.

The second goal is to try and quantify this sensitivity, by recording a set of trajectories, and compute the increase, with time, of the distance between phase-space $(\theta, \dot{\theta})$ trajectories. One expects that the distance δ grows exponentially with time after an *horizon time* τ_h

$$\delta \sim \delta_0 \exp(\lambda t) \qquad t \gtrsim \tau_h$$

Here, δ_0 is the initial distance between two trajectories and λ the Lyapunov exponent. A chaotic system will possess at least one positive Lyapounov exponent, and we wish to compute the largest of these, which defines the principal direction of maximum expansion in phase space.

3.3 Experiments

3.3.1 Précautions

En pratique, il faut partir initialement de zones qui présentent des structures fractales. Pour contrôler les conditions initiales prendre une pince non métallique (type pince à linge) ou en aluminium et faire un lâché sans vitesse initiale.

La caméra peut être posée sous les aimants, ou au-dessus du dispositif.

Essayer de trouver une méthode pour repérer les aimants.

Les films sont enregistrés avec VLC, et l'acquisition des trajectoirs sera effecutée avec IMAGEJ. Leur analyse pourra se faire dans SCIDAVIS.

Trajectoires à un aimant

Il peut être intéressant d'étudier l'évolution du système lorsqu'on en change le nombre de degrés de liberté. Pour ce faire, on peut étudier le pendule en présence d'un seul aimant fixe, placé à l'aplomb de la position d'équilibre.

Observer la régularité des trajectoires dans l'espace des phases. Dans ce cas la force est centrale et réduit le nombre de degrés de liberté par conservation du moment cinétique. Comparer aux trajectoires chaotiques du cas précédent. De même, caractériser les écarts entre deux trajectoires ayant des conditions initiales identiques à notre échelle.

4 Numerical experiments: foreword

In the following, we will use numerical tools to investigate the behaviour of non-linear dynamic systems. Our approach is general:

1. We are physicists ! Begin with a physical understanding of the system
2. Make equations dimensionless; in the process, characteristic timescales, length-scales, etc., show up.
3. Analytical approach:
   (a) look for the fixed points if any
   (b) investigate the system by hand, possibly in the linear regime (typically close to the fixed points)
4. write a program to solve the Ordinary Differential Equations describing the evolution of the system
5. check the program ! compare your expectations with model predictions
6. once validated, your program can be trusted:
   (a) explore the non-linear regime
   (b) explore the behaviour of the fixed points
   (c) is there any chaotic behaviour ?
   (d) etc...

5 One-dimensional systems

5.1 Budworms population

5.1.1 Description

The population of *budworms*[5] will be described using a modification of the logistic equation (Strogatz 1994):

$$\dot{x} = f(x) = rx(1 - x/k) - H(x) \quad (2)$$

in which $x(t)$ is the time-dependent population at time t. The parameter r describes the rate of growth of the budworms population, and k is the carrying capacity [6]. Here, k is the amount of foliage. Both parameters are assumed to be constant over time. The function $h(x)$ describes the mortality of the budworms, e.g. due to predators. We will examine the particular case

$H(x) = x^2/(1 + x^2)$.

[5] *Tordeuses de bourgeons*
[6] In french: capacité d'accueil

5.1.2 Goal of the work

1. analytic study:
   - discuss the number of fixed points for different parameters (r, k)
   - discuss the stability of the fixed points
   - how many initial conditions are needed ?
   - examine the influence of the initial condition
2. numerical study:
   - fixed points ?
   - can you catch an unstable fixed points ?
   - find (r, k) parameters in the bistable regime
3. Perform a numerical experiment to evidence the hysteresis

5.2 Van der Pol oscillator: Hopf bifurcation

5.2.1 Description

Amplification or damping of oscillators is usually described by a forcing term proportional to $\dot{x}$:

$$\ddot{x} + \mu\dot{x} + x = 0$$

with $\mu < 0$ and $\mu > 0$ for amplified and damped[7] oscillations, respectively. However, in the case of an amplified oscillator, this implies that the energy of the system goes to infinity, which is unphysical. To circumvent this problem, Van der Pol introduced a forcing $\mu(x)$ which depends on the amplitude of the oscillations themselves. This forcing is such that μ is negative for small amplitudes which are then amplified, and positive for large amplitudes. In a dimensionless form, the Van der Pol oscillator differential equation becomes:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \qquad (3)$$

5.2.2 Behaviour at the origin

The nature of the equilibrium solutions, or stability of the fixed points, is given by the eigenvalues of the Jacobian matrix of the system. The stability is given by the sign of the real part of the eigenvalues. Adopting $u = x$ and $v = \dot{x}$, the system can be written as

$$\begin{aligned} \dot{u} &= f_1(u, v) = v \\ \dot{v} &= f_2(u, v) = \mu(1 - u^2)v - u \end{aligned}$$

such that the Jacobian matrix is

$$J = \begin{pmatrix} 0 & 1 \\ -1 - 2\mu uv & \mu(1 - u^2) \end{pmatrix}$$

[7] damped=*amorti*

The two eigenvalues are such that their sum is the trace τ of J, and their product its determinant D. If we now restrict ourselves to the case when u and v remain small, we find that $\tau = \mu$ and $D = 1$, and the eigenvalues are of the same sign, and are solutions of

$$\lambda^2 - \mu\lambda + 1 = 0$$

If $\Delta = \mu^2 - 4 \geq 0$, the eigenvalues are real and, being of the same sign, lead to a node:

$$\lambda_{\pm} = 1/2[\mu \pm \sqrt{\mu^2 - 4}]$$

For $\mu \geq 2$, both solutions $\lambda_{\pm}$ are real positive, giving rise to an unstable node fixed points. For $\mu \leq 2$, $\lambda_{\pm}$ are real negative, giving rise to a stable node fixed point.

For $|\mu| < 2$, the solutions are complexe conjugate, their real part begin $\mu/2$. Therefore, the case $0 < \mu < 2$ leads to *unstable focus*, that is, unstable oscillatory fixed points or diverging spirals. On the other hand, $-2 < \mu < 0$ leads to *stable focus*, that is stable oscillatory fixed points or converging spirals. For $\mu = 0$, one recovers the harmonic oscillator. In summary,

- $\mu \geq 2$: unstable node
- $\mu \leq -2$: stable node (stable equilibria)
- $0 < \mu < 2$: unstable focus
- $-2 < \mu < 0$: stable focus
- $\mu = 0$: family of periodic solutions

5.2.3 Goal of the work

You will have to solve the second order ordinary differential equations (ODE), using a system of two, coupled, first-order ODE. The usual way is to define a rank-2 vector, $[u, v] = [x, \dot{x}]$.

1. Write the routine. What value of μ can be used to check that your program is not wrong ? How many initial conditions does the system require ?
2. Explore the behaviour of the system: influence of μ, influence of the initial conditions. Plot $\theta(t)$, $\dot{\theta}$, and phase portraits of the system.
3. Verify that the shape of the limit cycle depends only on μ.
4. Study of the limit cycle: what is the amplitude on the limit cycle ? What is the period and how does it vary with μ ?
5. Compute and display a Poincaré section.

5.3 Parametric oscillator

Let us consider a damped oscillator but with an additional, time dependent, forcing, described by the dimensionless equation:

$$\ddot{\theta} + \gamma\dot{\theta} + \theta\omega_0^2(1 + h\cos\omega t) = 0 \qquad (4)$$

This system thus possesses four parameters, γ, h, ω_0, and ω. When h and γ are both zero, we recover the harmonic oscillator. The forcing is *not* proportional to $\dot{\theta}$, like in the Van der Pol oscillator. This is rather like a swinging oscillator where the anchoring point moving back and forth vertically, with a pulsation ω and amplitude h. The damping term is $\gamma\dot{\theta}$.

The damping term $\gamma > 0$ decreases the amplitude of the oscillations, and acts as a stabilizing term, whereas the forcing term $h \cos \omega t$ tends to unstabilize it.

- Rewrite the equation by defining two parameters $\Omega = \omega/\omega_0$, and $\Gamma = \gamma/\omega_0$.
- Chose one set of initial conditions with no initial velocity, and explore the behaviour of the oscillator for different value of γ, h, and ω.
- Fix Γ and Ω, and vary h. What happens for sufficiently large values of h ?

$$\ddot{\theta} + \Gamma\dot{\theta} + \theta(1 + h\cos\Omega\tau) = 0 \tag{5}$$

where we introduced the dimensionless time variable $\tau = \omega_0 t$, and where $\dot{X} = \mathrm{d}X\,\mathrm{d}\tau$. In fact, the oscillator will become unstable if h becomes larger than h_{min}:

$$h_{\text{lim}}^2 = 4(1 - \Omega^2)^2 + 4\Gamma^2\Omega^2$$

which may be rewritten

$$h_{\text{lim}}^2 = 4\left[1 + (\Gamma^2 - 2)\Omega^2 + \Omega^4\right] \tag{6}$$

The stability domain thus depends on Ω and Γ.

- discuss qualitatively the behaviour of the oscillator
- try to determine the stability domain of the oscillator for different values of ω_0 and γ;

6 Chaotic systems

In this example, we will try to find tools to characterize chaotic systems.

6.1 Description of the system

We propose to study the Lorenz model of atmospheric convection.

$$\dot{x} = \sigma(y - x) \tag{7}$$
$$\dot{y} = \rho x - y - xz \tag{8}$$
$$\dot{z} = xy - \beta z \tag{9}$$

where σ, r, b are three positive parameters, r being the Rayleigh number and σ the Prandtl number.

6.2 Goal of the work

1. Try different sets of parameters. A particular set of values leading to chaotic trajectories is the historical values found by Lorenz, $(\sigma, \beta, \rho) = (10, 8/3, 28)$.
2. Find a set of parameters leading to a stable fixed point.
3. For a given set of parameters, compute the trajectories corresponding to arbitrarily close initial conditions (IC). Conclusions ?
4. Consider a non-chaotic set of parameters: compute the distance between two trajectories from close IC. Compare with the case of a chaotic set of parameters. Conclusions ?
5. Consider a non-chaotic set of parameters: compute the power spectrum of the x, y, and z variables. Compare with the case of a chaotic set of parameters. Conclusions ?
6. Repeat the study by computing the Fourier transform of one coordinate (x, y, or z); what are the qualitative differences between a non-chaotic and a chaotic system ?

References

Guyon, E. and Hulin, J.-P. and Petit, L. 2001, Hydrodynamique Physique, ed. CNRS Editions (EDP Sciences)

Strogatz, S. H. 1994, Nonlinear dynamics and chaos : with applications to physics, biology, chemistry, and engineering, Studies in nonlinearity (Cambridge (Mass.): Westview Press), autre(s) tirage(s) : 2000

Appendix

A Rayleigh-Bénard instability

- Reading the voltage: upper scale 200mV etc.

B Chaotic pendulum

B.1 Recording the video in VLC

1. Media → capture device
2. Open properties before starting the capture
3. Chose MJPG compression for the colors
4. Adjust the resolution/frame rate such that the files are not too big; you may want to disable the audio channel;
5. Where are the files ? By default, in the "Mes vidéos" folder; but you may change this in "Outils ¿ Preferences ¿ lecture/codecs"

B.2 Data extraction in ImageJ

- `File → Import → AVI...`
- Translate to grayscale (for memory purpose)
- Chose the *crosshair* tool from the icon bar
- Double click to configure: chose `auto-measure` and `auto-next slice`
- start the measurements
- In the measurement window: `File → Save as`

B.3 Data analysis in SciDavis

- `File → Import ASCII`
- Chose the "," separator for the `.csv` file created in ImageJ.

Or any other language/software that allows to read columns, make precises calculations, and graphs.