Elasticité d'un brin de laine

(1) $$L = N_\alpha a + (N - N_\alpha) b = N_\alpha (a-b) + N b$$

(2) $$E = N_\alpha E_\alpha + (N - N_\alpha) E_\beta - F L$$

$$= N_\alpha (E_\alpha - E_\beta) + N E_\beta$$

$$- F N_\alpha (a-b) - F N b$$

$$= N_\alpha \left(E_\alpha - E_\beta - F(a-b) \right) + N (E_\beta - F b)$$

$$\left(\frac{\partial E}{\partial N_\alpha} \right)_N = E_\alpha - E_\beta - F(a-b)$$

(3) $$\mathcal{N} = \binom{N}{N_\alpha} = \frac{N!}{N_\alpha ! (N - N_\alpha)!}$$

(4) $$S = k \ln (\mathcal{N}) = k \left(\ln N! - \ln N_\alpha ! - \ln (N - N_\alpha)! \right)$$

$$= k \left(N \ln N - N_\alpha \ln N_\alpha - (N - N_\alpha) \ln (N - N_\alpha) \right)$$

$$\left(\frac{\partial S}{\partial N_\alpha} \right)_N = k \left(- \ln N_\alpha - \frac{N_\alpha}{N_\alpha} + \ln (N - N_\alpha) + \frac{(N - N_\alpha)}{N - N_\alpha} \right)$$

$$\underset{N_\alpha \gg 1}{=} k \left(- \ln N_\alpha + \ln (N - N_\alpha) \right)$$

$$= k \ln \left(\frac{N}{N_\alpha} - 1 \right)$$

⑤ $\dfrac{1}{T} = \dfrac{\partial S}{\partial E}$: définition de T

$$= \left(\frac{\partial S}{\partial N_\alpha}\right)\left(\frac{\partial N_\alpha}{\partial E}\right)$$

$$= k \ln\left(\frac{N}{N_\alpha} - 1\right) \frac{1}{\left(E_\alpha - E_\beta - F(a-b)\right)}$$

⑥ D'après ① $N_\alpha = \dfrac{L - Nb}{a-b}$,

et d'après ⑤,

$$\frac{1}{T} = \frac{k \ln\left(\frac{N(a-b)}{L-Nb} - 1\right)}{E_\alpha - E_\beta - F(a-b)}$$

on pose $\begin{cases} \Delta = \dfrac{E_\alpha - E_\beta}{kT} \\ y = \dfrac{F(a-b)}{kT} \end{cases}$: paramètre sans dimension qui compare kT à la différence d'énergie des monomères alors

$$\frac{1}{T} = \frac{k \ln\left(\frac{N(a-b)}{L-Nb} - 1\right)}{k T (\Delta - Y)}$$

$$\Leftrightarrow \quad (\Delta - Y) = \ln\left(\frac{(a-b)}{\frac{L}{N} - b} - 1\right)$$

$$\Leftrightarrow \quad (a-b) = \left(e^{\Delta - Y} + 1\right)\left(\frac{L}{N} - b\right)$$

$$\Leftrightarrow \quad L = \left(\frac{a-b}{e^{\Delta - Y} + 1} + b\right) N$$

$$\Leftrightarrow \quad \boxed{L = N\left(\frac{a + b e^{\Delta - Y}}{1 + e^{\Delta - Y}}\right)}$$

7 d'après 2,

si on pose $\mathcal{E}_\alpha = E_\alpha - Fa$, $\mathcal{E}_\beta = E_\beta - Fb$,

alors

$$E = N_\alpha \left(E_\alpha - E_\beta - F(a-b)\right) + N(E_\beta - Fb)$$

$$= N_\alpha (\mathcal{E}_\alpha - \mathcal{E}_\beta) + N \mathcal{E}_\beta$$

$$= N_\alpha \mathcal{E}_\alpha + (N - N_\alpha) \mathcal{E}_\beta$$

D'après la loi de Boltzmann,

$$P_\alpha = \frac{1}{Z} e^{-\mathcal{E}_\alpha / kT} \quad , \quad P_\beta = \frac{1}{Z} e^{-\mathcal{E}_\beta / kT}$$

alors en moyenne $\begin{cases} N_\alpha = P_\alpha N \\ N_\beta = P_\beta N \end{cases}$

et d'après (1),

$$L = N_\alpha a + N_\beta b = \frac{N}{Z} \left(e^{-\frac{\mathcal{E}_\alpha}{kT}} a + e^{-\frac{\mathcal{E}_\beta}{kT}} b \right)$$

on a $1 = P_\alpha + P_\beta$,

donc $Z = e^{-\frac{\mathcal{E}_\alpha}{kT}} + e^{-\frac{\mathcal{E}_\beta}{kT}}$

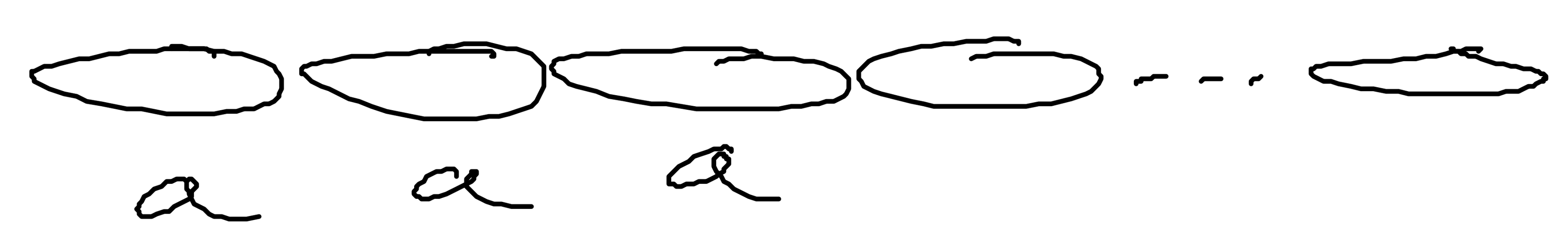
$$L = N \left(\frac{a e^{-\frac{\mathcal{E}_\alpha}{kT}} + b e^{-\frac{\mathcal{E}_\beta}{kT}}}{e^{-\frac{\mathcal{E}_\alpha}{kT}} + e^{-\frac{\mathcal{E}_\beta}{kT}}} \right)$$

$$= N \left(\frac{a + b e^{\frac{\mathcal{E}_\alpha - \mathcal{E}_\beta}{kT}}}{1 + e^{\frac{\mathcal{E}_\alpha - \mathcal{E}_\beta}{kT}}} \right)$$

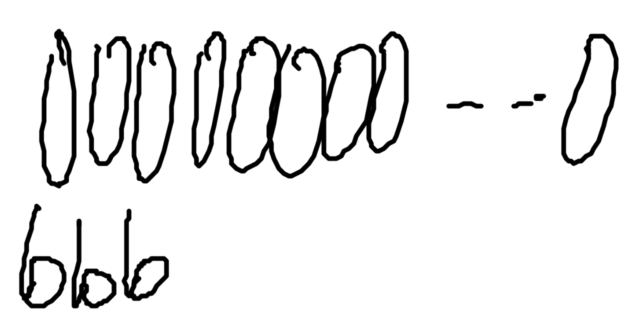
$$\boxed{L = N \left(\frac{a + b e^{\Delta - Y}}{1 + e^{\Delta - Y}} \right)}$$: idem 6

(8) on a $Y \geq 0$;

- si $\Delta - Y \longrightarrow -\infty$, alors $L \longrightarrow Na$

le plus long : 

- si $\Delta - Y \longrightarrow +\infty$, alors $L \longrightarrow Nb$

le plus court : 

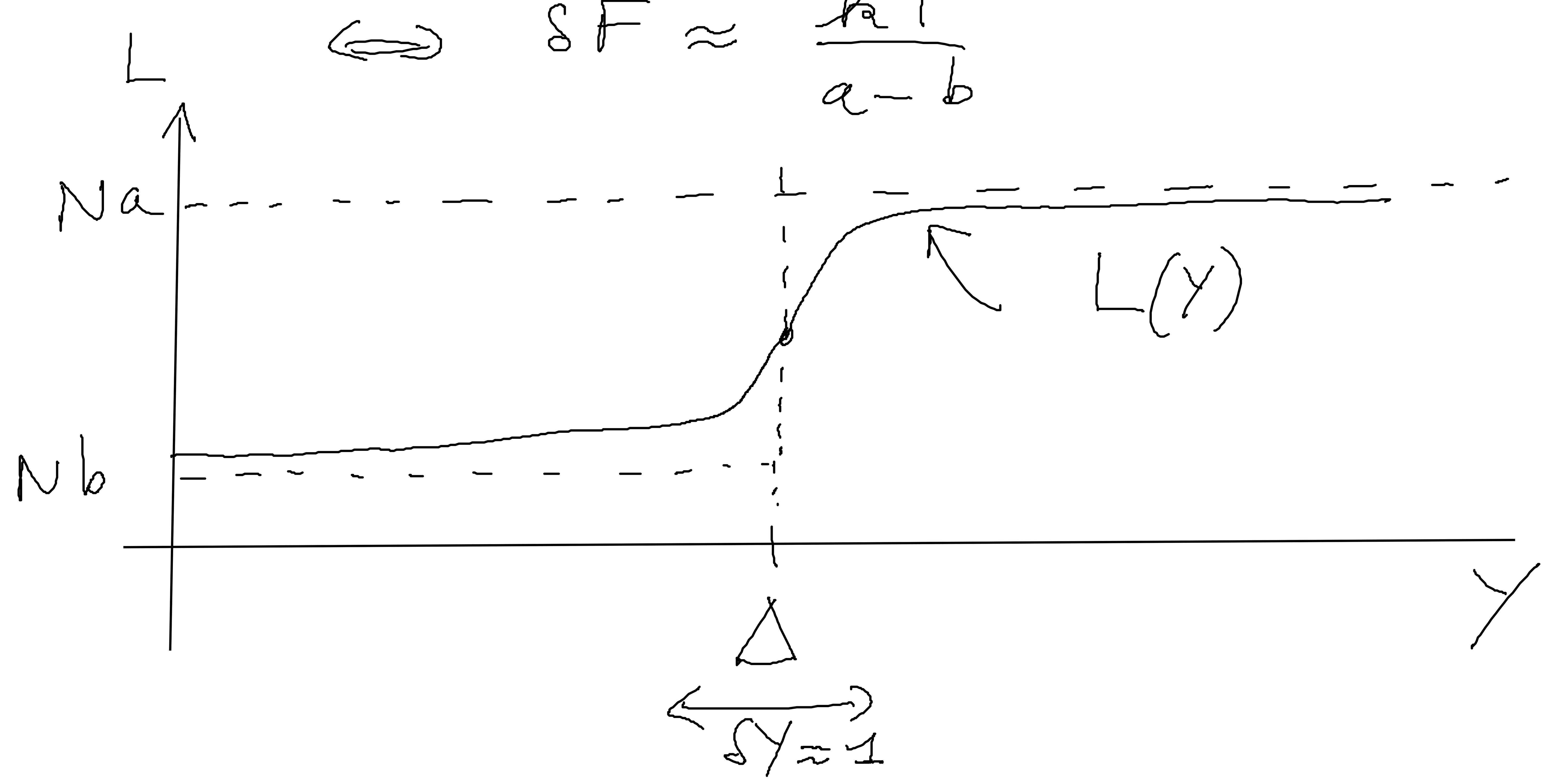
- Le changement est pour $\Delta - Y \simeq 0$

$$\iff \Delta \simeq Y$$

$$\iff F_0 \simeq \frac{E_\alpha - E_\beta}{a-b}$$

et pour des variations $\delta Y \simeq 1$

$$\iff \delta F \simeq \frac{kT}{a-b}$$



9 Connaissant kT, et mesurant F_0 et δF,
on déduit $(a-b)$, $(E_\alpha - E_\beta)$

et $L_{max} - L_{min} = N\,(a-b)$,

on déduit N.