

Modèle d'Ising à une dimension

①

①. chaque site X a deux valeurs possible $\sigma_X = \pm 1$,
et il y a N sites.

Donc il y a $\underbrace{2 \times 2 \times \dots \times 2}_N = 2^N$ configurations
possibles

②. Si $B = 0$, $E(\sigma) = -\sum_X \sigma_X \cdot \sigma_{X+1}$

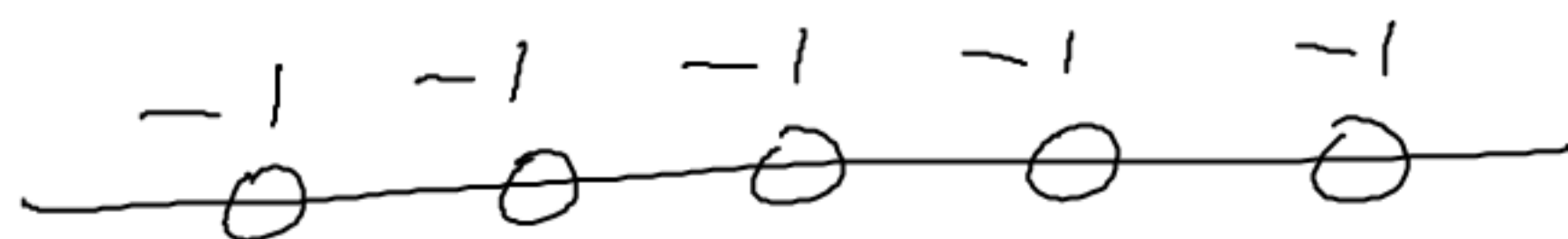
est minimale si $\sigma_X = \sigma_{X+1}, \forall X$,

donc si $\sigma_X = +1 \forall X$:



L'énergie $E(\sigma) = -N$, aimantation $M(\sigma) = +1$

ou $\sigma_X = -1, \forall X$:



L'énergie $E(\sigma) = -N$, aimantation $M(\sigma) = -1$

(c) l'énergie est maximale si $-f_x f_{x+1} = +1, \forall x,$
 soit $f_{x+1} = -f_x$: spins alternés.

• si N est pair, ce sont les 2 configurations :

$\uparrow \downarrow \uparrow \downarrow$ et $\downarrow \uparrow \downarrow \uparrow$ ($\because N=4$)

• si N est impair, il y a $2N$ configurations

l'énergie maximale :

ex $N=5$: $\uparrow \downarrow \uparrow \downarrow \uparrow, \uparrow \uparrow \downarrow \uparrow \downarrow, \downarrow \uparrow \uparrow \downarrow \uparrow, \uparrow \downarrow \uparrow \uparrow \downarrow, \downarrow \uparrow \downarrow \uparrow \uparrow$
 $\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

et les opposés.

(d) si $B \gg 1$, alors $E(f) \approx -B \sum_{x=0}^{N-1} f_x$

est minimale pour $f = \{+1, +1, \dots, +1\}$

et maximale pour $f = \{-1, -1, \dots, -1\}$

$$\textcircled{2} \langle E \rangle (T) = \sum_{\ell \in \{-1, +1\}^{\Lambda}} p(\ell) E(\ell)$$

$$\langle M \rangle (T) = \sum_{\ell \in \{-1, +1\}^{\Lambda}} p(\ell) M(\ell)$$

$$\textcircled{3} \text{On a } \sum_{\ell} p(\ell) = 1$$

$$\Leftrightarrow Z(\beta, B) = \sum_{\ell \in \{-1, +1\}^{\Lambda}} e^{-\beta E(\ell)}$$

$$\text{donc } \frac{\partial Z}{\partial \beta} = - \sum_{\ell} E(\ell) e^{-\beta E(\ell)}$$

$$\begin{aligned} \frac{\partial}{\partial \beta} \ln Z &= \frac{(\partial Z / \partial \beta)}{Z} = - \sum_{\ell} E(\ell) \underbrace{\left(\frac{e^{-\beta E(\ell)}}{Z} \right)}_{p(\ell)} \\ &= - \langle E \rangle \end{aligned}$$

$$\textcircled{a} \text{ donc } \boxed{\langle E \rangle = - \frac{\partial}{\partial \beta} \ln Z}$$

b) Capacité calorifique :

$$C = \frac{\partial \langle E \rangle}{\partial T} = \frac{\partial \langle E \rangle}{\partial \beta} \left(\frac{\partial \beta}{\partial T} \right) = - \left(\partial_{\beta}^2 \ln Z \right) \left(- \frac{1}{k T^2} \right)$$

donc :

avec $\beta = \frac{1}{k T}$

$$C = k \beta^2 \left(\partial_{\beta}^2 \ln Z \right)$$

c) fluctuation d'énergie :

$$\text{Var}(E) = \langle (E - \langle E \rangle)^2 \rangle$$

$$= \langle E^2 - 2E\langle E \rangle + \langle E \rangle^2 \rangle$$

$$= \langle E^2 \rangle - 2\langle E \rangle^2 + \langle E \rangle^2$$

$$= \langle E^2 \rangle - \langle E \rangle^2$$

d'après ci-dessus,

$$\frac{\partial^2 Z}{\partial \beta^2} = \sum_{\text{b}} E^2 e^{-\beta E} = Z \langle E^2 \rangle$$

donc $\langle E^2 \rangle = \frac{1}{Z} \frac{\partial^2 Z}{\partial \beta^2}$

or $\partial_{\beta}^2 \ln Z = \partial_{\beta} \left(\frac{\partial_{\beta} Z}{Z} \right) = \frac{(\partial_{\beta}^2 Z) Z - (\partial_{\beta} Z)^2}{Z^2}$
 $= \frac{1}{Z} (\partial_{\beta}^2 Z) - (\partial_{\beta} \ln Z)^2$

donc $\langle E^2 \rangle = \partial_{\beta}^2 \ln Z + (\partial_{\beta} \ln Z)^2$

$$\text{Var}(E) = \partial_{\beta}^2 (\ln Z) + (\partial_{\beta} \ln Z)^2 - (\partial_{\beta} \ln Z)^2$$

$$\text{Var}(E) = \partial_{\beta}^2 (\ln Z) = \frac{1}{k \beta^2} C$$

① Aimentation :

$$\langle M \rangle = \sum_{\ell} p(\ell) M(\ell)$$

$$\text{avec } M(\ell) = \frac{1}{N} \sum_x \ell_x$$

On observe que

$$Z = \sum_{\ell} e^{-\beta E(\ell)} = \sum_{\ell} e^{\beta \left(\sum_x \ell_x \ell_{x+1} - B \ell_x \right)}$$

$$= \sum_{\ell} e^{\beta \left(\sum_x \ell_x \ell_{x+1} \right) - \beta B N M(\ell)}$$

$$\text{donc } \frac{\partial Z}{\partial B} = \sum_{\ell} \beta N M(\ell) e^{-\beta E(\ell)}$$

$$= \beta N Z \langle M \rangle$$

$$\Leftrightarrow \langle M \rangle = \frac{1}{\beta N} \frac{1}{Z} (\partial_B Z)$$

$$\Leftrightarrow \boxed{\langle M \rangle = \frac{1}{\beta N} \partial_B (\ln Z)}$$

e) Susceptibilité magnétique :

$$\chi = \frac{\partial \langle M \rangle}{\partial B} = \frac{1}{\beta N} \left(\frac{\partial^2}{\partial B^2} \ln Z \right)$$

Fluctuation de l'aimantation :

$$\frac{\partial^2 Z}{\partial B^2} = \sum_b (\beta N M_b)^2 e^{-\beta E(b)} = Z (\beta N)^2 \langle M^2 \rangle$$

et $\text{Var}(M) = \langle M^2 \rangle - \langle M \rangle^2$

$$= \frac{1}{(\beta N)^2} \frac{1}{Z} \frac{\partial^2 Z}{\partial B^2} - \left(\frac{1}{\beta N} \frac{\partial \ln Z}{\partial B} \right)^2$$

$$\text{Var}(M) = \frac{1}{(\beta N)^2} \frac{\partial^2}{\partial B^2} (\ln Z)$$

On déduit que

$$\text{Var}(M) = \frac{1}{(\beta N)} \chi$$

⑥ Entropie :

$$S = -k \sum_b p(b) \ln p(b)$$

$$= -k \frac{1}{Z} \sum_b e^{-\beta E(b)} (-\ln Z - \beta E)$$

$$= k \frac{\ln Z}{Z} \underbrace{\left(\sum_b e^{-\beta E} \right)}_Z + k \beta \sum_b p(b) E(b)$$

$$= k \ln Z + k \beta \langle E \rangle$$

$$S = k \left(\ln Z - \beta \left(\frac{\partial}{\partial \beta} \ln Z \right) \right)$$

④ Si $A = (A_{b_0, b_1})_{b_0, b_1 \in \{-1, 1\}}$ est une matrice 2×2 ,

alors

$$(A^N)_{b_0, b_N} = \sum_{b_1 \in \{-1, 1\}} \sum_{b_2 \in \{-1, 1\}} \dots \sum_{b_{N-1} \in \{-1, 1\}} A_{b_0, b_1} A_{b_1, b_2} \dots A_{b_{N-1}, b_N}$$

$$\text{Tr}(A^N) = \sum_{b_0} \sum_{b_1} \dots \sum_{b_{N-1}} A_{b_0, b_1} A_{b_1, b_2} \dots A_{b_{N-1}, b_0}$$

$$\text{Tr}(A^N) = \sum_{b \in \{-1, 1\}^N} A_{b_0, b_1} A_{b_1, b_2} \dots A_{b_{N-1}, b_0}$$

$$\textcircled{5} \quad \sum_b p(b) = 1$$

$$\Leftrightarrow Z = \sum_{b \in \{-1, 1\}^N} e^{-\beta E(b)}$$

$$= \sum_{b_0 \in \{-1, 1\}} \sum_{b_1 \in \{-1, 1\}} \dots \sum_{b_{N-1} \in \{-1, 1\}} e^{\beta (b_0 b_1 + b_1 b_2 + \dots + b_{N-1} b_N)} e^{+\beta B (b_0 + b_1 + \dots + b_{N-1})}$$

$$= \sum_{b_0} \sum_{b_1} \dots \sum_{b_{N-1}} e^{\beta (b_0 b_1 + B b_0)} e^{\beta (b_1 b_2 + B b_1)} \dots e^{\beta (b_{N-1} b_N + B b_{N-1})}$$

$$= \text{Tr}(A^N)$$

avec $A_{b_0 b_1} = e^{\beta (b_0 b_1 + B b_0)}$

donc

$$A = \begin{matrix} & \begin{matrix} (b_1 = -1) & (b_1 = +1) \end{matrix} \\ \begin{matrix} (b_0 = -1) \\ (b_0 = +1) \end{matrix} & \begin{pmatrix} e^{\beta(1-B)} & e^{\beta(-1-B)} \\ e^{\beta(-1+B)} & e^{\beta(1+B)} \end{pmatrix} \end{matrix}$$

