Un ruban qui glisse

(1) $E_c = \frac{1}{2} m (\dot{z})^2$

$$dU(z) = - dm \cdot g z = -\left(\frac{dm}{dz}\right) g z \, dz$$

$$= -\left(\frac{m}{L}\right) g z \, dz$$

$$\rightarrow \quad U(z) = \int_0^z dU = \int_0^z -\left(\frac{m}{L}\right) g \, z' dz' = -\left(\frac{m}{L}\right) g \left(\frac{z^2}{2}\right)$$

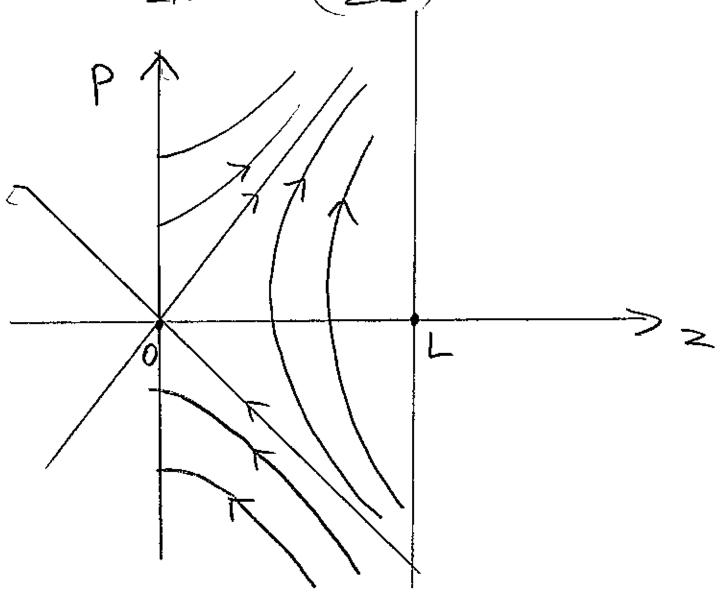
: pour $0 \leq z \leq L$

(2) $L(z, \dot{z}) = E_c - U = \frac{1}{2} m (\dot{z})^2 + \left(\frac{m}{L}\right) g \frac{z^2}{2}$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m \dot{z}$$

(3) $H(z, p_z) = p_z \dot{z} - L = \frac{p_z^2}{2m} - \left(\frac{m g}{2L}\right) z^2$: pour $0 \leq z \leq L$

$$E = H(z, p_z) = \frac{p_z^2}{2m} - \left(\frac{m g}{2L}\right) z^2 = \text{cste} \; : \; \text{hyperboles}$$



: valable pour $0 \leq z \leq L$, $p_z \in \mathbb{R}$

(4) • pour $z \leq 0$, le ruban est sur la table,

$$U(z) = U(0) = 0$$

donc $H(z, p_z) = \dfrac{p_z^2}{2m}$: lignes de niveaux = droites

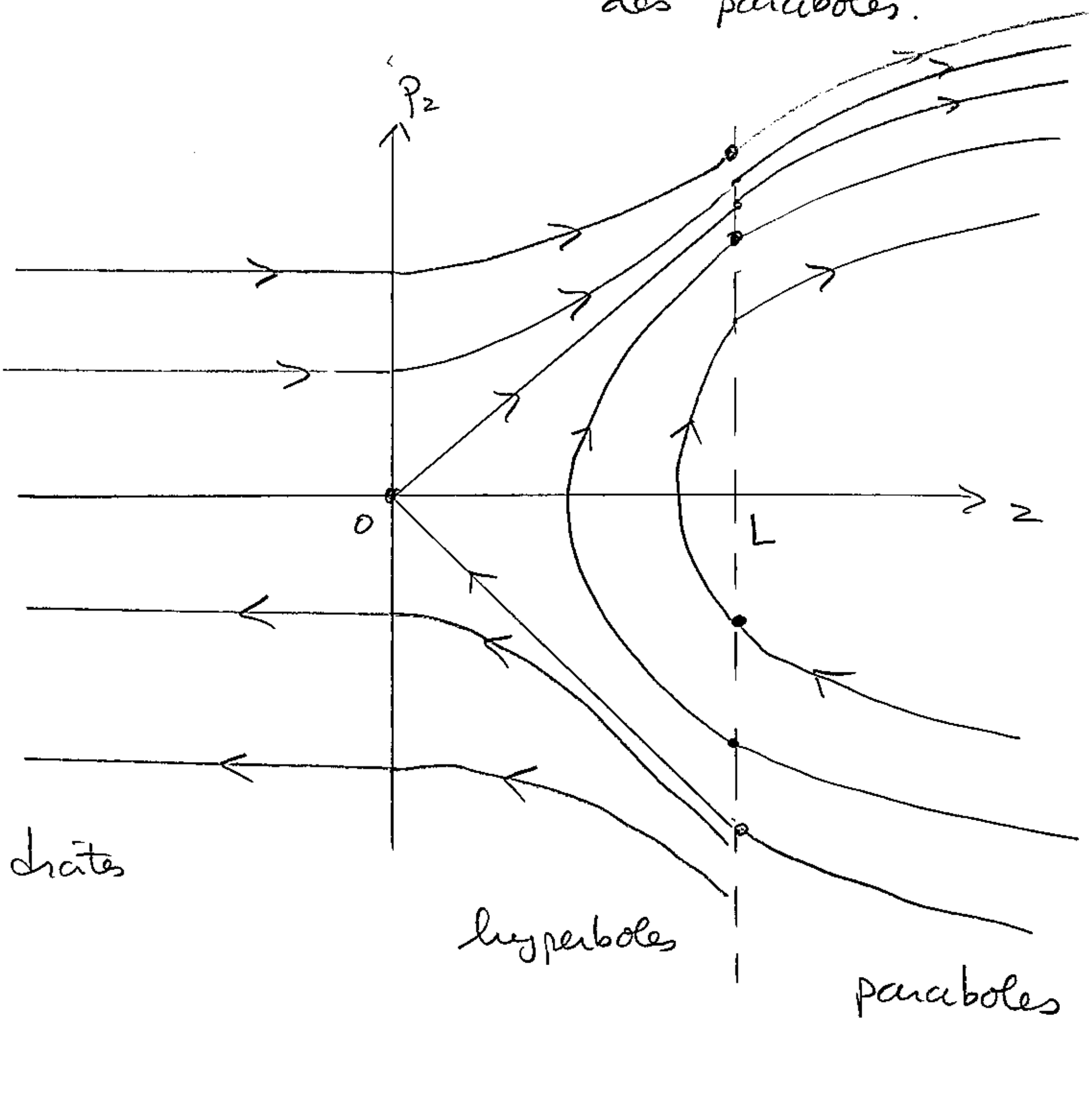
• pour $z \geqslant L$, $U(z) = \displaystyle\int_{z-L}^{z} -\left(\frac{m}{L}\right) g\, z' dz' = -\frac{mg}{L}\left[\frac{z'^2}{2}\right]_{z-L}^{z}$

$$= -\left(\frac{mg}{L}\right)\left(\frac{z^2}{2} - \frac{(z-L)^2}{2}\right)$$

$$= -mg\left(\frac{2z-L}{2}\right)$$

↑ position du centre

les lignes de Niveaux de $H(z, p_z) = \dfrac{p_z^2}{2m} - mg\left(\dfrac{2z-L}{2}\right)$ sont des paraboles.



5 Equation de Hamilton pour $0 \leq z \leq L$

$$\begin{cases} \dot{p}_z = -\dfrac{\partial H}{\partial z} = \left(\dfrac{mg}{L}\right) z \\[2ex] \dot{z} = +\dfrac{\partial H}{\partial p_z} = \dfrac{p_z}{m} \end{cases}$$: linéarisation

d'après le cours, $\lambda = \left(\dfrac{-U''(0)}{m}\right)^{1/2} = \left(\dfrac{g}{L}\right)^{1/2}$

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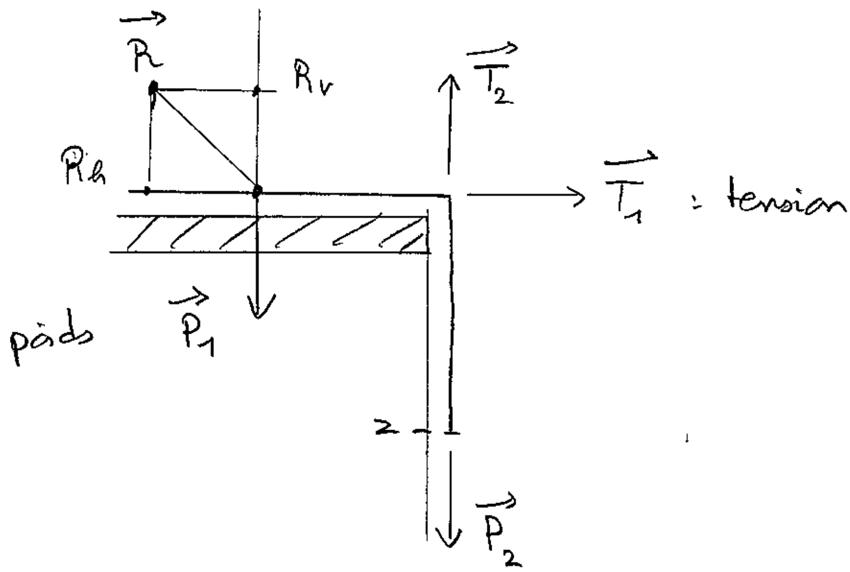


on suppose $0 \leq z \leq L$

- Bilan sur la partie horizontale immobile :

$$\vec{R} + \vec{P_1} + \vec{T_1} = 0$$

$$\Longleftrightarrow \begin{cases} R_v + P_1 = 0 \\ R_h + T_1 = 0 \end{cases} \Longleftrightarrow \begin{cases} |R_v| = mg\left(\dfrac{L-z}{L}\right) \\ |R_H| = |T_1| \end{cases}$$ ← proportion horizontale

- Partie verticale :

$$T_2 + P_2 = 0$$

$$\Longleftrightarrow \quad |T_2| = mg\left(\frac{z}{L}\right)$$ ← proportion verticale

or la tension $|T_1| = |T_2|$

donc $|R_H| = mg\left(\dfrac{z}{L}\right)$

(7) Le ruban reste immobile si

$$|R_H| < f\,|R_v|$$

$$\Leftrightarrow \quad mg\frac{z}{L} < f\,mg\left(\frac{L-z}{L}\right)$$

$$\Leftrightarrow \quad z < f(L-z)$$

$$\Leftrightarrow \quad z < \left(\frac{f}{1+f}\right)\cdot L = z_0$$

(8) Equation de Newton : si $|R_h| = f\,|R_v|$, pour $z \geqslant z_0$

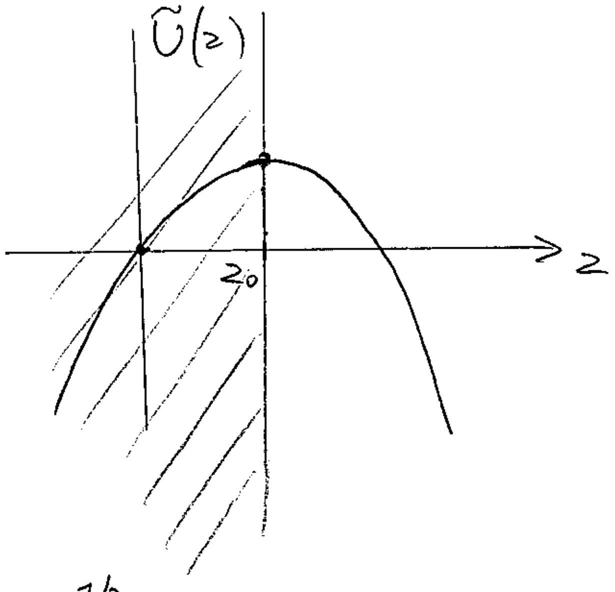
$$m\ddot{z} = \text{forces externes} = P_z + R_h = P_z - f\cdot R_v$$

$$= mg\left(\frac{z}{L}\right) - f\,mg\left(\frac{L-z}{L}\right)$$

$$= \frac{mg}{L}\left((1+f)z - fL\right)$$

$$= -\frac{d\tilde{U}(z)}{dz}$$

avec $\tilde{U}(z) = \frac{mg}{L}\left(-(1+f)\frac{z^2}{2} + fLz\right)$



$$\lambda = \left(\frac{-\tilde{U}''}{m}\right)^{1/2}$$

$$= \left[\frac{g}{L}(1+f)\right]^{1/2}$$: l'instabilité augmente avec f.