Modes normaux du double pendule

1) a)
$$\begin{cases} x_1 = l_1 \sin\theta_1 \simeq l_1 \theta_1 \\ z_1 = -l\cos\theta_1 \simeq -l + \frac{l}{2}\theta_1^2 \end{cases}$$

$$\begin{cases} x_2 = l\sin\theta_1 + l\sin\theta_2 \simeq l(\theta_1+\theta_2) \\ z_2 = -l\cos\theta_1 - l\cos\theta_2 \simeq -2l + \frac{l}{2}(\theta_1^2+\theta_2^2) \end{cases}$$

b) vitesse : $\dot{x}_1 \simeq l\,\dot{\theta}_1$, $\dot{z}_1 = l\,\theta_1\dot{\theta}_1$

$\dot{x}_2 \simeq l(\dot{\theta}_1+\dot{\theta}_2)$ $\quad \dot{z}_2 = l(\theta_1\dot{\theta}_1 + \theta_2\dot{\theta}_2)$

$$E_c = \frac{1}{2}m_1(\dot{x}_1^2+\dot{z}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{z}_2^2)$$

$$E_c \simeq \frac{1}{2}m_1 l^2(\dot{\theta}_1)^2 + \frac{1}{2}m_2 l^2(\dot{\theta}_1+\dot{\theta}_2)^2 \quad : \text{à l'ordre 2}$$

$$= \frac{1}{2}m_1 l^2(\dot{\theta})^2 + \frac{1}{2}m_2 l^2(\dot{\varphi})^2$$

$$U = m_1 g z_1 + m_2 g z_2 \simeq -m_1 g l + m_1 g\frac{l}{2}\theta_1^2 - m_2 g 2l + m_2 g\frac{l}{2}(\theta_1^2+\theta_2^2)$$

$$= \underbrace{-gl(m_1+2m_2)}_{\text{cste}} + m_1 g\frac{l}{2}\theta^2 + m_2 g\frac{l}{2}(\theta^2+(\varphi-\theta)^2)$$

2 a) $L = E_c - U$

$$\begin{cases} p_\theta = \dfrac{\partial L}{\partial \dot\theta} = m_1 \ell^2 \dot\theta \\ p_\varphi = \dfrac{\partial L}{\partial \dot\varphi} = m_2 \ell^2 \dot\varphi \end{cases}$$

$$H = p_\theta \dot\theta + p_\varphi \dot\varphi - L$$

$$= \frac{p_\theta^2}{2m_1 \ell^2} + \frac{p_\varphi^2}{2m_2\ell^2} + U ,$$

b) $\mu = \dfrac{m_2}{m_1}$. $\quad \varphi = \dfrac{1}{\sqrt{\mu}} \psi , \quad p_\varphi = \sqrt{\mu}\, p_\psi$

$$p_\varphi^2 = \frac{m_2}{m_1} p_\psi^2$$

donc

$$H(\theta, p_\theta, \psi, p_\psi) = \frac{1}{2m_1 \ell^2} \left(p_\theta^2 + p_\psi^2 \right) + U$$

$$U = m_1 g \frac{\ell}{2} \theta^2 + m_2 g \frac{\ell}{2} \left(\theta^2 + \left(\frac{1}{\sqrt{\mu}} \psi - \theta \right)^2 \right)$$

$$= \frac{g \ell m_1 \sqrt{1+\mu}}{2} \left[\frac{1}{\sqrt{1+\mu}} \theta^2 + \frac{\mu}{\sqrt{1+\mu}} \left(\theta^2 + \frac{1}{\mu} \psi^2 + \theta^2 - \frac{2}{\sqrt{\mu}} \theta\psi \right) \right]$$

$$= \frac{g\ell m_1 \sqrt{1+\mu}}{2} \left[\left(\frac{1+2\mu}{\sqrt{1+\mu}} \right) \theta^2 + \frac{1}{\sqrt{1+\mu}} \psi^2 - \frac{2\sqrt{\mu}}{\sqrt{1+\mu}} \theta\psi \right]$$

donc

$$\begin{cases} A_{11} = \dfrac{1+2\mu}{\sqrt{1+\mu}} \\ A_{12} = -\dfrac{\sqrt{\mu}}{\sqrt{1+\mu}} = A_{21} \\ A_{22} = \dfrac{1}{\sqrt{1+\mu}} \end{cases}$$

3) On a :

$$p_\theta^2 + p_\psi^2 = p_x^2 + p_y^2 \quad \text{car } \mathbb{P} \text{ est orthogonale}$$

et $\mathbb{A} = \mathbb{P}^{-1}\, \mathbb{D}\, \mathbb{P}$ avec $\mathbb{D} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$

$\mathbb{P}^{-1} = \mathbb{P}^T$

$$(\theta, \psi) \cdot \mathbb{A} \cdot \begin{pmatrix} \theta \\ \psi \end{pmatrix} = (\theta, \psi) \cdot \mathbb{P}^T . \mathbb{D} . \mathbb{P} \begin{pmatrix} \theta \\ \psi \end{pmatrix}$$

$$= \left(\mathbb{P} \begin{pmatrix} \theta \\ \psi \end{pmatrix} \right)^T \mathbb{D} \left(\mathbb{P} \begin{pmatrix} \theta \\ \psi \end{pmatrix} \right) = (x, y)\, \mathbb{D} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \lambda_+ x^2 + \lambda_- y^2$$

donc

$$U = \frac{g \ell m_1 \sqrt{1+\mu}}{2} \left(\lambda_+ x^2 + \lambda_- y^2 \right).$$

$$H(x, p_x, y, p_y) = H_+(x, p_x) + H_-(y, p_y)$$

avec

$$\begin{cases} H_+(x, p_x) = \dfrac{1}{2 m_1 \ell^2} p_x^2 + \dfrac{g \ell m_1 \sqrt{1+\mu}\, \lambda_+}{2} x^2 \\[2ex] H_-(y, p_y) = \dfrac{1}{2 m_1 \ell^2} p_y^2 + \dfrac{g \ell m_1 \sqrt{1+\mu}\, \lambda_-}{2} y^2 \end{cases}$$

4) Donc

$$\Omega_+ = \left(\frac{g \ell m_1 \sqrt{1+\mu}\, \lambda_+}{m_1 \ell^2} \right)^{1/2} = \sqrt{\frac{g}{\ell}} \cdot \left(\sqrt{1+\mu}\, \lambda_+ \right)^{1/2}$$

$$\Omega_- = \left(\frac{g \ell m_1 \sqrt{1+\mu}\, \lambda_-}{m_1 \ell^2} \right)^{1/2} = \sqrt{\frac{g}{\ell}} \left(\sqrt{1+\mu}\, \lambda_- \right)^{1/2}$$

- pour $\mu \to 0$, $\lambda_\pm \to 1$

  donc $\Omega_\pm \to \sqrt{\frac{g}{\ell}}$

- pour $\mu = 1$, $\lambda_+ = \sqrt{2} + 1$, $\lambda_- = \sqrt{2} - 1$

  donc $\begin{cases} \Omega_+ = \sqrt{\frac{g}{\ell}} \left(2+\sqrt{2}\right)^{1/2} \simeq 1,84 \cdot \sqrt{\frac{g}{\ell}} \\ \Omega_- = \sqrt{\frac{g}{\ell}} \left(2-\sqrt{2}\right)^{1/2} \simeq 0,76 \cdot \sqrt{\frac{g}{\ell}} \end{cases}$

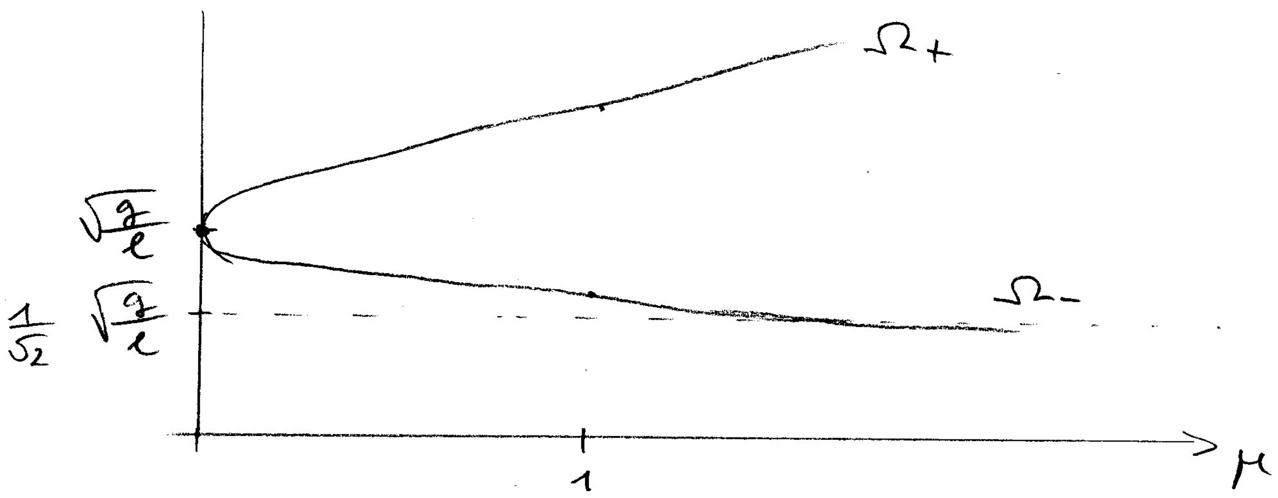
- pour $\mu \to \infty$, $\lambda_+ \sim 2\sqrt{\mu}$

  donc $\Omega_+ \sim \sqrt{\frac{g}{\ell}} \sqrt{2\mu} \longrightarrow +\infty$

  et $\sqrt{1+\mu} \cdot \lambda_- = 1+\mu - \sqrt{\mu+\mu^2} = 1+\mu-\mu\sqrt{1+\frac{1}{\mu}}$

  $$\sim 1+\mu-\mu\left(1+\frac{1}{2\mu}\right) \sim \frac{1}{2}$$

  donc $\Omega_- \longrightarrow \sqrt{\frac{g}{\ell}} \frac{1}{\sqrt{2}} \simeq 0,71 \sqrt{\frac{g}{\ell}}$



5) Le mode 1 est lorsque $y = 0$, $p_y = 0$

or $0 = y = V_-^{(1)} \theta + V_-^{(2)} \psi$

$$\Leftrightarrow \quad \lambda_- \theta + \psi = 0$$

$$\Leftrightarrow \quad \lambda_- \theta_1 + \sqrt{\mu}\,(\theta_1 + \theta_2) = 0$$

$$\Leftrightarrow \quad \theta_2 = -\frac{1}{\sqrt{\mu}}\left(\lambda_- + \sqrt{\mu}\right)\theta_1$$

$$\Leftrightarrow \quad \theta_2(t) = -\sqrt{1+\frac{1}{\mu}}\;\theta_1(t)$$: "opposition de phase"

Le mode 2 est lorsque $x = 0$, $p_x = 0$

or $0 = x = V_+^{(1)} \theta + V_+^{(2)} \psi$

$$\Leftrightarrow \quad \lambda_+ \theta - \psi = 0$$

$$\Leftrightarrow \quad \lambda_+ \theta_1 - \sqrt{\mu}\,(\theta_1 + \theta_2) = 0$$

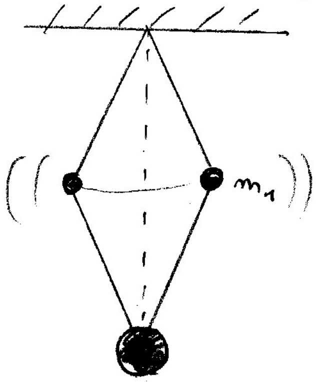
$$\Leftrightarrow \quad \theta_2 = \frac{1}{\sqrt{\mu}}\left(\lambda_+ - \sqrt{\mu}\right)\theta_1$$

$$\Leftrightarrow \quad \theta_2(t) = \sqrt{1+\frac{1}{\mu}}\;\theta_1(t)$$: "en phase"

• si $m_1 \ll m_2 \Leftrightarrow \mu \to \infty$, $\sqrt{1+\frac{1}{\mu}} \to 1$

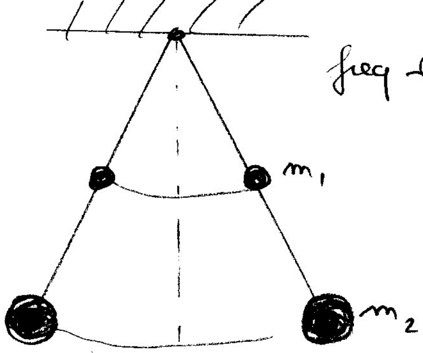
mode 1 : $\theta_2 = -\theta_1$

freq $\Omega_+ \to +\infty$



mode 2 : $\theta_2 = \theta_1$

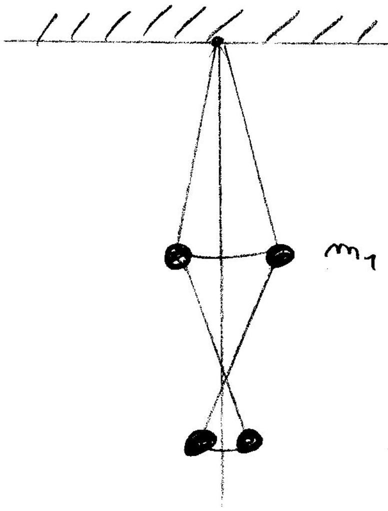
freq $\Omega_- = \frac{1}{\sqrt{2}}\sqrt{\frac{g}{\ell}}$



• si $m_1 = m_2 \iff \mu = 1, \quad \sqrt{1+\frac{1}{\mu}} = \sqrt{2}$

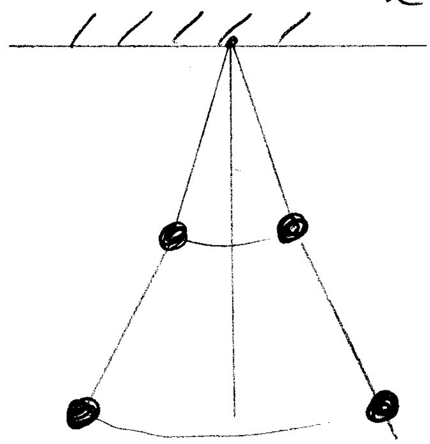
mode 1 $\quad \theta_2 = -\sqrt{2}\,\theta_1$

freq $\Omega_+ \simeq 1,84 \cdot \sqrt{\frac{g}{\ell}}$

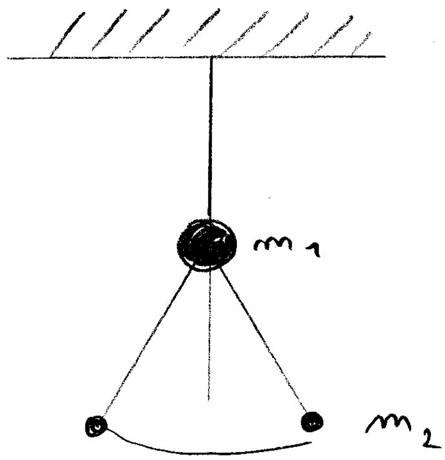


mode 2 $\quad \theta_2 = \sqrt{2}\,\theta_1$

$\Omega_- \simeq 0,76 \cdot \sqrt{\frac{g}{\ell}}$



• si $m_1 \gg m_2 \iff \mu \to 0, \quad \sqrt{1+\frac{1}{\mu}} \to \infty, \quad \theta_1 = 0$



$\Omega_\pm = \sqrt{\frac{g}{\ell}}$