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Ruelle–Pollicott resonances for real analytic hyperbolic maps

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Abstract

We study two simple real analytic uniformly hyperbolic dynamical systems: expanding maps on the circle S^1 and hyperbolic maps on the torus $\mathbb{T}^2$. We show that the Ruelle–Pollicott resonances which describe time correlation functions of the chaotic dynamics can be obtained as the eigenvalues of a trace class operator in Hilbert space $L^2(S^1)$ or $L^2(\mathbb{T}^2)$, respectively. The trace class operator is obtained by conjugation of the Ruelle transfer operator in a similar way to how quantum resonances are obtained in open quantum systems. We comment on this analogy.

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1. Introduction

In this paper, we study the Ruelle–Pollicott resonances of two simple models which are uniformly hyperbolic: expanding maps on the circle S^1 and hyperbolic maps on the torus $\mathbb{T}^2$. These models are the most simple examples of chaotic dynamical systems, which exhibit strong chaotic properties, such as ergodicity, mixing, decay of correlations and central limit theorem for observables (see [3, 9, 14]). Expansivity or hyperbolicity makes every trajectory unstable and are therefore important hypotheses responsible for these chaotic properties. One of these properties, the 'decay of time correlation', is fundamental to the establishment of other chaotic properties. Decay of time correlations can be studied by means of the spectral analysis of 'Ruelle transfer operators', which transport functions on S^1 or $\mathbb{T}^2$ according to the dynamics [7,20]. The simplest Ruelle transfer operator is the 'pull-back operator' or 'Koopman operator' defined by $(\hat{M}\varphi)(x) \overset{\text{def}}{=} \varphi(M(x))$, where $M : S^1 \to S^1$ is the map (respectively $M : \mathbb{T}^2 \to \mathbb{T}^2$) and $\varphi \in C^0(S^1)$. The time correlation of two functions $\phi, \varphi \in C^0(S^1)$ is defined by $C_{\phi,\varphi}(t) \overset{\text{def}}{=} \langle \phi | \hat{M}^t | \varphi \rangle = \int_{S^1} \phi(x)(\varphi \circ M^t(x))\mathrm{d}x$. The main effect of the hyperbolic

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dynamics is to transform an initial function φ into a function $\hat{M}^t\varphi$ with finer and finer structures, as time t evolves. In other words, the information on the initial function φ is sent towards 'microscopic scales' or equivalently at infinity in the Fourier space. On the 'macroscopic scale' (i.e. if $\hat{M}^t\varphi$ is tested on a regular function ϕ), there remains only a constant function, i.e. the function 1 times a weight $\mu_{\mathrm{SRB}}(\varphi)$ where μ_{SRB} is called the Sinai–Ruelle–Bowen (SRB) measure. The number $\mu_{\mathrm{SRB}}(\varphi) \in \mathbb{C}$ is the only information on φ which remains on the macroscopic scale after a long time evolution. With this point of view, the decay of time correlation functions $C_{\phi,\varphi}(t)$ is due to the escape of the function $\hat{M}^t\varphi$ towards infinity in the Fourier space, implying a decay of the small Fourier components. This suggests the study of the decay of correlations using a 'window of observation' in the Fourier space, centred on small Fourier components (in the unstable direction). This is the role of the operator $\hat{A}$ below. This situation is very similar to open quantum systems, where the decay of the quantum wave function in a compact domain of space is due to the escape of the wave function towards infinity. In such situations people study the decay by a 'complex scaling method' which consists of conjugating the dynamical operator in such a way that the wave function is toned down at infinity ([6], chapter 8). Then the 'quantum resonances' which appear are suitable for describing the decay. Here we will follow this general approach.

We will suppose in both models that the map M is *real analytic*. We will show that the time correlation functions can be obtained from an 'effective dynamical operator' $\hat{R}$ obtained from $\hat{M}$ by a conjugation $\hat{R} = \hat{A}\hat{M}\hat{A}^{-1}$, where $\hat{A}$ tones down the high Fourier modes in the unstable direction of the dynamics. The main result is to show that $\hat{R}$ is a trace class operator in the Hilbert space $L^2(S^1)$ (respectively $L^2(\mathbb{T}^2)$). The effective long time dynamics is obtained by $\hat{R}^t = \hat{A}\hat{M}^t\hat{A}^{-1}$ with $t \in \mathbb{N}$, for which the spectral properties of $\hat{R}$ are important. The eigenvalues of $\hat{R}$ are called the 'Ruelle–Pollicott resonances'. This approach, with a conjugation, is the one which is usually followed in complex scaling methods [6]. An alternative but equivalent approach would be to keep the operator $\hat{M}$ but to work with another norm instead of the L^2 one, namely with $(\phi, \psi)_A \stackrel{\mathrm{def}}{=} (\hat{A}\phi, \hat{A}\psi)_{L^2}$. This last formulation is preferred in [4, 5, 11, 15]. In this approach, the operator $\hat{A}$ is seen as an isomorphism between two Hilbert spaces, and one gets $(\phi, \hat{M}\psi)_A = (\phi', \hat{R}\psi')_{L^2}$ with $\phi' = \hat{A}\phi$, $\psi' = \hat{A}\psi$.

The correlation spectrum for analytic maps has already been studied through trace class operators but with different approaches: the case of expanding maps has been studied by Ruelle in [20]. The case of analytic hyperbolic maps has been studied by Rugh in [21]. Our approach which consists of working in the Fourier space (or more technically conjugating $\hat{M}$ by a pseudo-differential operator) has already been investigated for hyperbolic diffeomorphisms by Baladi and Tsujii [4]. In their paper, they consider a much broader class of dynamical systems (they do not suppose analyticity) and they show quasi-compacity for the transfer operator in a Banach space of distributions. Although our results are more restrictive, we believe that they have their own meaning because of their technical simplicity (we obtain a trace class operator in a Hilbert space and the proof is quite simple). A technical difference between the two approaches appears for example with the choice of the operator $\hat{A}$. We have to choose an operator which has an exponential expression in the Fourier basis, whereas in [4] the operator $\hat{A}$ has an algebraic dependence on the Fourier modes ($\hat{A}$ is a power of the Laplacian). This difference is crucial to obtain some of the results presented in this paper.

Our techniques do not allow us to treat any hyperbolic maps on the torus but maps which are closed enough to the linear hyperbolic map[3]. More precisely, as the proof in section 4.2 will show, the technical assumption is that the unstable and stable tangent directions are uniformly contained in constant cones defined by the linear map. This restriction of our method, due to

[3] We discuss some possible extensions in the conclusion.

the very simple expression of the operator $\hat{A}$, prevents us on the other hand from making a partition of the unity as in [4].

With a different approach, results close to those obtained by Baladi and coworkers were obtained by Liverani and coworkers in [5, 11, 15]. The connection between dynamical determinants and Ruelle resonances is established there in great generality. See [4] for historical remarks and comparison between the different approaches.

The paper is organized as follows. In section 2, we define the operator $\hat{R}$ which governs the decay of the correlation function. We state the theorems which say that $\hat{R}$ is a trace class operator in both cases S^1 and $\mathbb{T}^2$. These results are proved in sections 3 and 4, respectively. Our method is very convenient for numerical analysis since the truncation of the high Fourier modes produces a small error. In section 3.3, we present numerical illustrations of some aspects of the SRB measure and the Ruelle–Pollicott (RP) resonances. In section 2.1.4, we show the equivalence of our approach with a more common approach known as 'randomly perturbed dynamics' or 'noisy models' [5, 8]. We use this equivalence to show the (well-known) trace formula in terms of periodic orbits [3]. We mention that in [11] a powerful method is developed in the C^r case and allows the authors to show spectral stability for a wider class of deterministic and random perturbations.

2. Statement of the results

2.1. Expanding map on the circle

Let $M : \mathbb{R} \to \mathbb{R}$ be the map defined by[4]

$$M(x) = 2x + f(x),$$

where f is *real analytic* and periodic: $f(x+1) = f(x), \forall x \in \mathbb{R}$. We suppose moreover that

$$f'_{\min} \stackrel{\text{def}}{=} \min_{x\in\mathbb{R}} \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) > -1,$$

so that $M'_x = 2 + \mathrm{d}f/\mathrm{d}x > 1$ (M is called strictly expanding). A simple example used later for numerical illustrations is

$$f(x) = \frac{\delta}{2\pi}\sin(2\pi x), \qquad |\delta| < 1. \tag{1}$$

For all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$, $M(x+n) \equiv M(x) \bmod 1$; hence M defines an expanding map on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ (also denoted by M). M is not invertible, but it is the simplest model exhibiting chaotic dynamics (see [14]): expansivity is responsible for the high sensitivity to initial conditions, mixing and positive entropy of the dynamics.

The pull-back operator $\hat{M}$ on $L^2(S^1)$ is the (non-unitary) bounded operator defined by[5]

$$(\hat{M}\varphi)(x) \stackrel{\text{def}}{=} \varphi(Mx). \tag{2}$$

For any $n \in \mathbb{Z}$, we denote[6] by $|n\rangle \in L^2(S^1)$ the Fourier mode $\varphi_n(x) = \exp(\mathrm{i}2\pi nx)$ and define the operator $\hat{A}$ by

$$\hat{A}|n\rangle \stackrel{\text{def}}{=} \exp(-a|n|)|n\rangle, \qquad n \in \mathbb{Z}, \quad \text{with } a > 0. \tag{3}$$

4 More generally, we could have supposed that $M(x) = dx + f(x)$ with $d \in \mathbb{N}$, $d \geqslant 2$. This does not change the results we obtain.

5 We can also consider a more general class of operators, called 'Ruelle transfer operators': $(\hat{M}_g\varphi)(x) \stackrel{\text{def}}{=} e^{g(x)}\varphi(Mx)$, with complex valued function g. The results obtained in this paper extend to this case provided g is real analytic.

6 Throughout the paper, we use **Dirac notations** for vectors in Hilbert space $\mathcal{H} = L^2(S^1)$: $\varphi \in \mathcal{H}$ is written as $|\varphi\rangle$. Its dual metric is written as $\langle\varphi|$. A scalar product is written as $\langle\phi|\varphi\rangle = \int_{S^1} \overline{\phi(x)}\varphi(x)\mathrm{d}x$. If $\hat{M}$ is an operator, we write $\langle\phi|\hat{M}|\varphi\rangle \stackrel{\text{def}}{=} \langle\phi|\hat{M}\varphi\rangle = \langle\hat{M}^*\phi|\varphi\rangle$ (where $\hat{M}^*$ is the adjoint operator). Finally, $|\phi\rangle\langle\varphi| \stackrel{\text{def}}{=} |\phi\rangle \otimes \langle\varphi|$.

$\hat{A}$ is diagonal in the Fourier basis. The image $C_A = \hat{A}(L^2(S^1))$ is a set of very regular functions (analytic in a complex neighbourhood of S^1 of radius a). The operator $\hat{A}$ is used to define[7] the operator

$$\hat{R} \stackrel{\text{def}}{=} \hat{A}\hat{M}\hat{A}^{-1}, \tag{4}$$

with domain C_A dense in $L^2(S^1)$.

In this paper, we will prove the following theorem.

Theorem 1. *There exists $a > 0$ entering in equation (3), such that matrix elements of $\hat{R}$ decrease exponentially: $|\langle n'|\hat{R}|n\rangle| < \exp(-c(|n'|+|n|))$, with $c > 0$. In particular $\hat{R}$ extends to a trace class operator in Hilbert space $L^2(S^1)$.*

As the proof will show, the result holds for any $a \in]0, a_0]$, with some $a_0 > 0$, and c depends on a.

For general results on trace class operators, see [18,19] or ([10] chapter 4). The eigenvalues of $\hat{R}$ are called the RP resonances of the pull-back operator $\hat{M}$.

2.1.1. Dynamical correlation functions. The operator $\hat{R}$ is useful for studying time-correlation functions. Indeed, if $|\phi\rangle \in C_A$ is a regular 'test function', and $|\varphi\rangle \in L^2(S^1)$, then for $t \in \mathbb{N}$, $C_{\phi,\varphi}(t) \stackrel{\text{def}}{=} \langle\phi|\hat{M}^t|\varphi\rangle$ can be expressed using $\hat{R}$ as

$$C_{\phi,\varphi}(t) \stackrel{\text{def}}{=} \langle\phi|\hat{M}^t|\varphi\rangle = (\langle\phi|\hat{A}^{-1})\hat{R}^t(\hat{A}|\varphi\rangle). \tag{5}$$

In 'physical terms' it means that the operator $\hat{R}$ is a nice effective operator to express the dynamics of $\hat{M}$ in $L^2(S^1)$ provided it is tested on the regular function space C_A.

2.1.2. Finite rank approximation. There is a direct consequence of theorem 1 which is useful for the numerical computation of RP resonances. Let $\hat{M}_N$ be the matrix of the operator $\hat{M}$ expressed in the Fourier basis and truncated to the N first Fourier modes (i.e. $|n| \leqslant N$, the matrix $\hat{M}_N$ has thus a size $(2N + 1) \times (2N + 1)$).

Corollary 2. *The eigenvalues of $\hat{M}_N$ converge towards the RP resonances, when $N \to \infty$.*

Proof. If $\hat{R}_N$ is the matrix of the operator $\hat{R}$ restricted to $\mathcal{H}_N = \text{Span}\{|n\rangle, |n| \leqslant N\} \equiv \mathbb{C}^{2N+1}$ then the eigenvalues of $\hat{R}_N$ converge to the RP resonances as $N \to \infty$ because $\hat{R}_N$ converges to $\hat{R}$ in operator norm. But in $\mathcal{H}_N \equiv \mathbb{C}^{N+1}$, $\hat{R}_N$ is conjugate to $\hat{M}_N$ by the invertible and diagonal matrix $\hat{A}_N = \text{Diag}(\exp(-a|n|), n = -N \to N)$; $\hat{R}_N$ and $\hat{M}_N$ have the same spectrum. □

2.1.3. Exponential concentration of Ruelle–Pollicott resonances near zero. Since $\hat{R}$ is a compact operator (and moreover a trace class operator), its eigenvalues converge to zero. From theorem 1, matrix elements $\langle n'|\hat{R}|n\rangle$ decrease exponentially fast for large $|n|, |n'|$. A quite direct consequence of this is the exponential concentration of the eigenvalues of $\hat{R}$ near zero.

Theorem 3. *Let $\lambda_i \in \mathbb{C}$, $i = 0, 1, \ldots$, be the non-zero eigenvalues of $\hat{R}$ (the RP resonances), such that $|\lambda_{i+1}| \leqslant |\lambda_i|$, and counting multiplicity. Let $l_i = \log|\lambda_i|$ and $C_1 = (2(1 + \mathrm{e}^{-c}))/(1 - \mathrm{e}^{-c})^2$. For any $i \geqslant 0$,*

$$l_i \leqslant -\frac{c}{4}\mathrm{i} + \log C_1. \tag{6}$$

The constant c is given in theorem 1 and the proof of this theorem is shown in section 3.3. (The best estimate is for the largest possible value of c, which is neither very explicit in the proof nor very sharp.)

[7] Equivalently $\hat{A}$ can be seen as a change of norm on $L^2(S^1)$.

2.1.4. Relation with randomly perturbed operators or noisy models. A different approach for obtaining RP resonances of transfer operators is to add a small 'noise' or 'random perturbation' to the dynamical operator $\hat{M}$ at each step of evolution. These models are often used for theoretical or numerical calculations. In [8] Baladi and Young show that for expanding maps the randomly perturbed operator is compact and that its eigenvalues are the RP resonances in the limit when the perturbation vanishes. A similar result is shown by Blank *et al* in [5] for Anosov maps. Also see [17]. In this section we consider such 'noisy operators' and show with a very simple proof that they have the same eigenvalues as $\hat{R}$ (i.e. the RP resonances) when the perturbation vanishes. The same result holds for Anosov maps on $\mathbb{T}^2$ considered in the next section.

Let $\Delta \equiv -\mathrm{d}^2/\mathrm{d}x^2$ be the Laplacian operator on S^1, and for $\varepsilon > 0$, let

$$\hat{\mathcal{D}}_\varepsilon \stackrel{\text{def}}{=} \mathrm{e}^{-\varepsilon\Delta/(2\pi)^2}$$

be the heat operator. This operator is diagonal in the Fourier basis:

$$\hat{\mathcal{D}}_\varepsilon |n\rangle = \mathrm{e}^{-\varepsilon n^2} |n\rangle. \tag{7}$$

The main effect of $\hat{\mathcal{D}}_\varepsilon$ is to truncate the high Fourier components. In the 'real space' $x \in S^1$, the operator $\hat{\mathcal{D}}_\varepsilon$ convolutes with a Gaussian distribution of size $\sim \sqrt{\varepsilon}$, so $\hat{\mathcal{D}}_\varepsilon$ as the same effect as a Gaussian noise.

Let us define the 'noisy perturbed operator' by

$$\hat{M}_\varepsilon \stackrel{\text{def}}{=} \hat{M}\hat{\mathcal{D}}_\varepsilon. \tag{8}$$

$\hat{M}_\varepsilon$ is a trace class operator because it is the product of $\hat{\mathcal{D}}_\varepsilon$ which is trace class with $\hat{M}$ which is bounded (cf [18] p 207).

Theorem 4. *The eigenvalues of the noisy perturbed operator $\hat{M}_\varepsilon$ converge to the RP resonances, when $\varepsilon \to 0$.*

Proof. Let us define the operator

$$\hat{R}_\varepsilon \stackrel{\text{def}}{=} \hat{R}\hat{\mathcal{D}}_\varepsilon = \hat{A}\hat{M}\hat{A}^{-1}\hat{\mathcal{D}}_\varepsilon = \hat{A}\hat{M}\hat{\mathcal{D}}_\varepsilon\hat{A}^{-1} = \hat{A}\hat{M}_\varepsilon\hat{A}^{-1},$$

where we have used the fact that $\hat{A}$ and $\hat{\mathcal{D}}_\varepsilon$ commute. Let $\hat{R}_{\varepsilon,N}$ (respectively $\hat{M}_{\varepsilon,N}$) the matrix of the operator $\hat{R}_\varepsilon$ (respectively $\hat{M}_\varepsilon$) be expressed in the Fourier basis and truncated to the first N Fourier modes. We have $\hat{R}_{\varepsilon,N} = \hat{A}\hat{M}_{\varepsilon,N}\hat{A}^{-1}$, so the matrices $\hat{R}_{\varepsilon,N}$ and $\hat{M}_{\varepsilon,N}$ have the same spectrum. But $\hat{R}_{\varepsilon,N}$ converges to $\hat{R}_\varepsilon$ in operator norm for $N \to \infty$ (respectively $\hat{M}_{\varepsilon,N} \to \hat{M}_\varepsilon$ for $N \to \infty$). We deduce that $\hat{R}_\varepsilon$ and $\hat{M}_\varepsilon$ have the same spectrum. Now $\hat{R}_\varepsilon$ converges to $\hat{R}$ in operator norm for $\varepsilon \to 0$, which shows that eigenvalues of $\hat{R}_\varepsilon$ converge to eigenvalues of $\hat{R}$. □

2.1.5. Trace formula. An important feature of Ruelle transfer operators is the existence of exact trace formulae in terms of periodic orbits ([3] p 100). We recall here the trace formula for the operator $\hat{R}$ defined by equation (4).

Proposition 5. *For any $t \geqslant 1$,*

$$\mathrm{Tr}(\hat{R}^t) = \sum_{x \in \mathrm{Fix}(M^t)} \frac{1}{|D_x M^t - 1|},$$

where $\mathrm{Fix}(M^t)$ *denotes the set of fixed points of M^t and $D_x M^t(x) = \mathrm{d}M^t/\mathrm{d}x(x)$.*

Proof. We consider first the trace of the operator $\hat{M}_{t,\varepsilon} \stackrel{\text{def}}{=} \hat{M}^t \hat{D}_\varepsilon$ (similarly to equation (8)). From ([10] theorem 8.1 p 70),

$$\mathrm{Tr}(\hat{M}_{t,\varepsilon}) = \int_0^1 \mathrm{d}x \langle x|\hat{M}_{t,\varepsilon}|x\rangle = \int_0^1 \mathrm{d}x \delta_\varepsilon(M^t(x) - x),$$

where $\langle x'|\hat{M}_{t,\varepsilon}|x\rangle$ denotes the Schwartz kernel of the operator $\hat{M}_{t,\varepsilon}$ on $L^2(S^1)$ and where $\delta_\varepsilon = \hat{D}_\varepsilon \delta$ denotes the regularized Dirac distribution at $x = 0$ (i.e. δ_ε is a periodic Gaussian function with width $\sim \sqrt{\varepsilon}$). With the ($(2^t - 1)$-valued) change in variable $y = M^t(x) - x$, we obtain

$$\mathrm{Tr}(\hat{M}_{t,\varepsilon}) = \int_0^{2^t-1} \mathrm{d}y \frac{1}{|D_x M^t - 1|} \delta_\varepsilon(y),$$

so $\mathrm{Tr}(\hat{M}_{t,\varepsilon}) \to \sum_{x \in \mathrm{Fix}(M^t)} (1/|D_x M^t - 1|)$ for $\varepsilon \to 0$. On the other hand, with $\hat{R}_{t,\varepsilon} \stackrel{\text{def}}{=} \hat{R}^t \hat{\mathcal{D}}_\varepsilon$, we show (as in the proof of theorem 4) that $\mathrm{Tr}(\hat{M}_{t,\varepsilon}) = \mathrm{Tr}(\hat{R}_{t,\varepsilon})$ and that $\mathrm{Tr}(\hat{R}_{t,\varepsilon} - \hat{R}^t) \to 0$, for $\varepsilon \to 0$. This gives the result. □

2.2. *Hyperbolic map on the torus*

With the same approach, one can study a nonlinear hyperbolic map on the torus (expressed as a linear map with a small perturbation). Let $M_0 \in SL(2, \mathbb{Z})$ be a hyperbolic matrix (i.e. $\mathrm{Tr} M_0 > 2$) and $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a real analytic periodic function:

$$f(x + n) = f(x), \qquad \forall x \in \mathbb{R}^2, \quad \forall n \in \mathbb{Z}^2.$$

Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be defined as M_0 perturbed by f:

$$M(x) = M_0(x) + f(x).$$

Then $M(x + n) \equiv M(x)[\mathbb{Z}^2]$; hence, M induces a map on $\mathbb{T}^2$ also denoted by M. Structural stability asserts that the map M on $\mathbb{T}^2$ is Anosov (uniformly hyperbolic) whenever $\|f\|_{C^1}$ is small enough ([2] p 122)([14] p 89)[8].

The pull-back operator $\hat{M}$ on $L^2(\mathbb{T}^2)$ is the bounded operator defined by

$$(\hat{M}\varphi)(x) \stackrel{\text{def}}{=} \varphi(Mx). \tag{9}$$

Note that the operator $\hat{M}$ is not unitary except if M preserves the area on $\mathbb{T}^2$.

Example. Choose $M_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and

$$f(x) = \left(0, \frac{\delta}{2\pi} \sin(2\pi(2x_1 + x_2))\right). \tag{10}$$

In this example, M preserves[9] area $\mathrm{d}x_1\,\mathrm{d}x_2$.

For each $n = (n_1, n_2) \in \mathbb{Z}^2$, denote by $|n\rangle \in L^2(\mathbb{T}^2)$ the Fourier mode $\varphi_n(x) = \exp(\mathrm{i}2\pi(n \cdot x))$. Let $u, s \in \mathbb{R}^2$ be unstable/stable eigenvectors of the transposed matrix M_0^{t}, i.e. $M_0^{\mathrm{t}} u = \mathrm{e}^{\lambda_0} u$ and $M_0^{\mathrm{t}} s = \mathrm{e}^{-\lambda_0} s$, with $\lambda_0 > 0$. A vector $v = (v_1, v_2) \in \mathbb{R}^2$ is written $\tilde{v} \equiv (v_u, v_s)$ with respect to the basis (u, s). In particular $n \in \mathbb{Z}^2$ is mapped to $\tilde{n} = (n_u, n_s)$. Define the operator $\hat{A}$ by

$$\hat{A}|n\rangle \stackrel{\text{def}}{=} \exp(-a|n_u| + a|n_s|)|n\rangle, \qquad n \in \mathbb{Z}^2, \quad \text{with } a > 0. \tag{11}$$

[8] In our case f is supposed to be real analytic. So $\|f\|_{C^1}$ is controlled by $\|f\|_{C^0}$.

[9] Because in this example, M can be written as $M = M_1 M_0$, where M_1 is an Hamiltonian time 1 flow, generated by Hamiltonian function $H_1(x_1, x_2) = (\delta/(2\pi)^2) \cos(2\pi x_1)$ on $\mathbb{T}^2$.

It is diagonal in the Fourier basis. $\hat{A}$ is defined on a domain $D_A \stackrel{\text{def}}{=} \text{Dom}(\hat{A}) = \{|\varphi\rangle = \sum_n \varphi_n |n\rangle,\ s.t.\ \sum_n |\varphi_n|^2 e^{2a|n_s|-2a|n_u|} < \infty,\ \sum_n |\varphi_n|^2 < \infty\}$ dense in $L^2(\mathbb{T}^2)$ and consists of functions with exponentially fast decreasing Fourier components (i.e very regular) in the stable direction. Similarly,

$$C_A \stackrel{\text{def}}{=} \text{Dom}(\hat{A}^{-1}) = \{|\phi\rangle = \sum_n \phi_n |n\rangle,\ s.t.\ \sum_n |\phi_n|^2 e^{-2a|n_s|+2a|n_u|} < \infty,\ \sum_n |\phi_n|^2 < \infty\} \subset L^2(\mathbb{T}^2)$$

consists of functions with exponentially fast decreasing Fourier components in the unstable direction. One checks that $C_A = \hat{A}(D_A)$, $D_A = \hat{A}^{-1}(C_A)$.

Define

$$\hat{R} \stackrel{\text{def}}{=} \hat{A}\hat{M}\hat{A}^{-1} \tag{12}$$

on a suitable domain included in C_A (one has $\text{Dom}(\hat{R}) = C_A$ if $\hat{M}(D_A) \subset D_A$).

Theorem 6. *For a C^1-small enough perturbation f, i.e. $\|f\|_{C^1} < \varepsilon$ with $\varepsilon > 0$, there exists $a > 0$ such that $\hat{R}$ is defined on the domain $\text{Dom}(\hat{R}) = C_A$ and its matrix elements decrease exponentially: $|\langle n'|\hat{R}|n\rangle| < \exp(-c(|n'_1|+|n'_2|+|n_1|+|n_2|))$, with $c > 0$. Therefore, $\hat{R}$ extends to a trace class operator in Hilbert space $L^2(\mathbb{T}^2)$.*

As the proof will show, the result holds for any $a \in]0, a_0]$, with some $a_0 > 0$, and c depends on a. The operator $\hat{R}$ is useful for studying time-correlation functions: if $|\phi\rangle \in C_A$ is a regular 'test function', and $|\varphi\rangle \in D_A$, then

$$C_{\phi,\varphi}(t) \stackrel{\text{def}}{=} \langle\phi|\hat{M}^t|\varphi\rangle = (\langle\phi|\hat{A}^{-1})\hat{R}^t(\hat{A}|\varphi\rangle).$$

The different corollaries and applications we mentioned for expanding maps work as well for hyperbolic maps on the torus (with only minor modifications): (1) finite rank approximation, (2) exponential concentration of RP resonances near zero, (3) relation with randomly perturbed operators and (4) trace formula in terms of periodic orbits.

2.2.1. Exponential concentration of Ruelle–Pollicott resonances near zero.

Theorem 7. *Let $\lambda_n \in \mathbb{C}$, $n = 0, 1, \ldots$, be the non-zero eigenvalues of $\hat{R}$ (the RP resonances), such that $|\lambda_{n+1}| \leqslant |\lambda_n|$, and counting multiplicity. Let $l_n = \log|\lambda_n|$. There exists $C_1 > 0$ such that for any $n \geqslant 0$*

$$l_n \leqslant -\frac{c}{3}\sqrt{n}\frac{1}{(1+1/n)} + \log C_1, \tag{13}$$

where the constant c is given in theorem 1, and the proof of the theorem is shown in section 4.2.1. The main difference with (6) is the appearance of $n^{1/2}$, where the power is $1/d$ with $d = \dim(\mathbb{T}^2) = 2$. Let us note that such an asymptotic behaviour of eigenvalues is also met in quantum mechanics in the spectrum $(E_n)_n$ of the harmonic oscillator on $\mathbb{R}^d$. From the semi-classical Weyl law $E_n \simeq \text{Cste}\, n^{1/d}$ (see [13], chapter 16). This suggests that (13) could be obtained or interpreted within a semi-classical approach, with a Weyl asymptotic.

2.2.2. Relation with randomly perturbed operators. The analysis made in section 2.1.4 can be repeated with no change except for the definition of the heat operator:

$$\hat{\mathcal{D}}_\varepsilon|n\rangle = e^{-\varepsilon(n_1^2+n_2^2)}|n\rangle,$$

which is used to define the randomly perturbed operator:

$$\hat{M}_\varepsilon \stackrel{\text{def}}{=} \hat{M}\hat{\mathcal{D}}_\varepsilon.$$

We have the following theorem.

Theorem 8. *The eigenvalues of the noisy perturbed operator $\hat{M}_\varepsilon$ converge towards the RP resonances, as $\varepsilon \to 0$.*

(The same proof as in section 2.1.4.)

2.2.3. Trace formula. We have the following trace formula for $\hat{R}^t$ in terms of periodic orbits.

Proposition 9. *For any $t \geqslant 1$,*

$$\mathrm{Tr}(\hat{R}^t) = \sum_{x\in \mathrm{Fix}(M^t)} \frac{1}{|\det(D_x M^t - Id)|},$$

where $\mathrm{Fix}(M^t)$ denotes the set of fixed points of M^t and $D_x M^t(x)$ is the differential at point $x \in \mathbb{T}^2$.

The proof follows the same lines as those for proposition 5.

3. Expanding map on the circle

In this section, we prove theorem 1.

3.1. Matrix elements of the operator $\hat{M}$

Denote $|n\rangle$ the Fourier mode $\varphi_n(x) = \exp(\mathrm{i}2\pi nx)$, with $n \in \mathbb{Z}$, $x \in S^1$. The set $(|n\rangle)_{n\in\mathbb{Z}}$ form an orthonormal basis of $L^2(S^1)$ and matrix elements of $\hat{M}$ in this basis are explicitly given by

$$\langle n'|\hat{M}|n\rangle = \int_0^1 \mathrm{d}x \exp(\mathrm{i}2\pi((2n-n')x + nf(x))). \tag{14}$$

3.1.1. Remarks.

- For a vanishing perturbation $f = 0$, then

  $$\langle n'|\hat{M}_0|n\rangle = \delta_{2n=n'}, \tag{15}$$

  i.e. in the plane (n', n) matrix elements are zero except on the 'line' $n' = 2n$. For a non-zero perturbation f, we will show that matrix elements are very small outside the cone containing this line.
- Since f is real, we have the symmetry

  $$\langle -n'|\hat{M}|-n\rangle = \overline{\langle n'|\hat{M}|n\rangle},$$

  and if $n = 0$, we have

  $$\langle n'|\hat{M}|0\rangle = \delta_{n'=0}.$$

  Therefore, we only have to study matrix elements with $n > 0$.

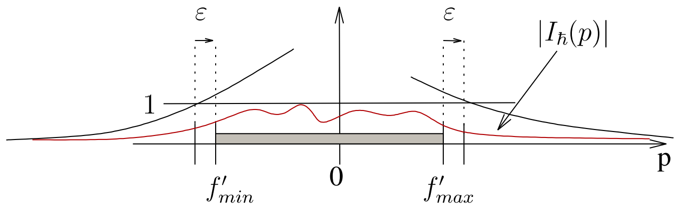

Figure 1. Upper bounds for the function $|I_\hbar(p)|$.

3.1.2. Localization property of matrix elements. For simplicity of the presentation, we borrow notations from semi-classical analysis (see [16]). For $n > 0$, let us make the change in variables $(n, n') \Leftrightarrow (h, p)$, with

$$h = \frac{1}{n}, \qquad p = \frac{1}{n}(n' - 2n) = n'h - 2, \tag{16}$$

and define $\hbar = h/(2\pi)$. A matrix element can be written as the oscillating integral:

$$I_\hbar(p) \stackrel{\text{def}}{=} \langle n'|\hat{M}|n\rangle = \int_0^1 \mathrm{d}x \exp(\mathrm{i}(f(x) - px)/\hbar).$$

From the 'non-stationary phase theorem' below, this integral is 'very small' for values of p outside the interval $[f'_{\min}, f'_{\max}]$, with

$$f'_{\min} \stackrel{\text{def}}{=} \min_x \frac{\mathrm{d}f}{\mathrm{d}x}, \qquad f'_{\max} \stackrel{\text{def}}{=} \max_x \frac{\mathrm{d}f}{\mathrm{d}x}.$$

Theorem 10 ('Non-stationary phase'). *Assume that $f(x)$ is a periodic function and can be continued analytically in some strip $|\mathrm{Im}(x)| < Y$. Assume $p/(2\pi\hbar) \in \mathbb{Z}$ and $I_\hbar(p) = \int_0^1 \mathrm{d}x \exp(\mathrm{i}(f(x) - px)/\hbar)$. For any $\varepsilon > 0$, there exists a constant $C > 0$, such that for any $p, \hbar > 0$ (see figure 1),*

$$|I_\hbar(p)| \leqslant \min(1, \mathrm{e}^{-C(f'_{\min} - p - \varepsilon)/\hbar}, \mathrm{e}^{-C(p - f'_{\max} - \varepsilon)/\hbar}). \tag{17}$$

Proof. Write $z = x + \mathrm{i}y$ and $f(z) = a(z) + \mathrm{i}b(z)$, with a, b real-valued functions. Analyticity of f gives $\partial_y b = \partial_x a$. For $y = 0$ and $x \in [0, 1]$, one has $b(x, 0) = 0$ and $(\partial_y b)(x, 0) = (\partial_x a)_{y=0} \geqslant f'_{\min}$. Therefore, $\forall \varepsilon > 0, \exists y_0 > 0$ $y_0 < Y$ such that $b(x, y_0) > (f'_{\min} - \varepsilon)y_0$ for all x. So for $z = x + \mathrm{i}y_0$,

$$|\exp(\mathrm{i}(f(z) - pz)/\hbar)| = \exp(-(b(x, y_0) - py_0)/\hbar) < \exp(-(f'_{\min} - \varepsilon - p)y_0/\hbar).$$

Since f is analytic, we can deform the integration path to $z = x + \mathrm{i}y_0$, $x = 0 \to 1$, y_0 fixed. This gives

$$|I_\hbar(p)| < \exp(-(f'_{\min} - \varepsilon - p)y_0/\hbar).$$

A similar argument for the integral $\overline{I}_\hbar(p) = \int_0^1 \mathrm{d}x \exp(\mathrm{i}(-f(x) + px)/\hbar)$ provides

$$|I_\hbar(p)| < \exp(-(p - f'_{\max} - \varepsilon)y'_0/\hbar).$$

We finally choose $C = \min(y_0, y'_0)$. □

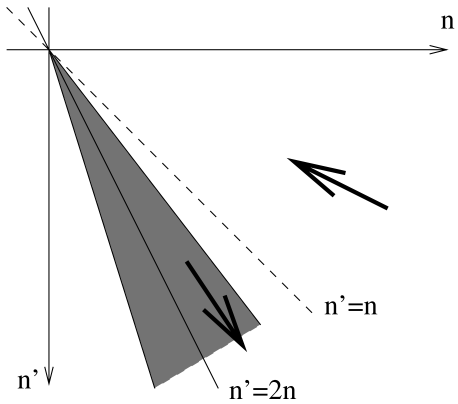

Figure 2. Matrix elements $\langle n'|\hat{M}|n\rangle$ are small outside the grey cone, for $|n| \to \infty$. For $f = 0$, this cone reduces to the line $n' = 2n$. These matrix elements can be interpreted as transition amplitudes for the dynamics $n \to n'$. For long times (many iterations), the dynamics goes to infinity in Fourier space on the sector $|n'| > |n|$ and tends to small Fourier components on the sector $|n'| < |n|$ (see the black arrows). Important matrix elements of $\hat{M}$ are localized on the sector $|n'| > |n|$ only and thus generate an escape towards infinity in Fourier space, responsible for chaos, as discussed in the introduction.

3.1.3. Remarks.

- For $p \in [f'_{\min}, f'_{\max}]$, the stationary phase formula gives the asymptotic value of $I_\hbar(p)$ when $\hbar \to 0$. It says that the asymptotic value of the matrix element $\langle n'|\hat{M}|n\rangle \equiv I_\hbar(p)$ of operator $\hat{M}$ depends only on the point x' in the integral, such that $p = (\mathrm{d}f/\mathrm{d}x)(x')$, or equivalently $n' = (2+(\mathrm{d}f/\mathrm{d}x)(x'))n$. We now comment on a semi-classical interpretation of this result. The map M^{-1} acting in S^1 is two-valued. Let $\tilde{M}^{-1}$ denote its lifted action on the cotangent space T^*S^1. If $x = M(x') = 2x' + f(x')$ then $\partial/\partial x' = (2 + (\mathrm{d}f/\mathrm{d}x)(x'))\partial/\partial x$. So, if (x, k) denote coordinates on T^*S^1, and $(x', k') = \tilde{M}^{-1}(x, k)$, then $x = M(x')$ and $k' = (2+(\mathrm{d}f/\mathrm{d}x)(x'))k$. So in a sense, which needs to be specified, $\hat{M}$ acting in $L^2(S^1)$ is a 'semi-classical quantization' of the map $\tilde{M}^{-1}$ acting in the symplectic space T^*S^1. This 'semi-classical aspect' of hyperbolic dynamics will be investigated in a future work.
- If we come back to variables n, n', this last theorem shows that matrix elements $\langle n'|\hat{M}|n\rangle$ in the plane (n', n) are exponentially small outside the cone defined by (for $n > 0$)

  $$(f'_{\min} + 2 - \varepsilon)n < n' < (f'_{\max} + 2 + \varepsilon)n.$$

  Expansivity hypothesis of M gives $f'_{\min} + 2 > 1$, so for ε small enough this cone does not contain the diagonal $n' = n$ (cf figure 2).
- With example (1), we can explicitly express matrix elements in term of Bessel functions of the first kind (see [1], section 9.1.21, p 360):

  $$\langle n'|\hat{M}|n\rangle = (-1)^{(2n-n')} J_{(2n-n')}(\delta n).$$

  This gives

  $$|I_\hbar(p)| = \left| J_{(-p/h)}\left(\frac{\delta}{h}\right)\right|.$$

Asymptotic results for Bessel functions (see [1], sections 9.3.1 and 9.3.2, p 365) give

$$\log|I_h(p)| \sim -\frac{p}{h}\left(\log\left(\frac{2p}{e\delta}\right)\right), \qquad \text{for } h \text{ fixed, and } p \to \infty,$$
$$\sim -\frac{p}{h}(\alpha - \tanh\alpha), \qquad \text{with } \cosh\alpha = \frac{p}{\delta} \geqslant 1 \text{ fixed, and } h \to 0,$$

which are sharper than the upper bound in (17).

3.2. *Proof of theorem 1*

3.2.1. Idea of the proof and remarks. In the *linear case*, with a vanishing perturbation $f = 0$, the result is obvious. Indeed from equation (15), matrix elements of $\hat{R}$ lie on the line $n' = 2n$ and decrease similarly to $\langle n'|\hat{R}|n\rangle = \delta_{n'=2n}\mathrm{e}^{a(|n|-|n'|)} = \delta_{n'=2n}\mathrm{e}^{-a|n|}$ (the spectrum is $\sigma(\hat{R}) = \{1\} \cup \{0\}$, with 1 as a simple eigenvalue). Note that choosing the operator $\hat{A}$ with the algebraic form $\hat{A}|n\rangle = 1/|n|^{\alpha}|n\rangle$, $n \in \mathbb{Z}$, would not give decreasing matrix elements: $\langle n'|\hat{R}|n\rangle = \delta_{n'=2n}(|n|^a/|n'|^a) = \delta_{n'=2n}(1/2^a)$. On the other hand, the choice $\hat{A}|n\rangle = \mathrm{e}^{-|n|^a}|n\rangle$ with $0 < a < 1$ or $\hat{A}|n\rangle = \mathrm{e}^{-a\log^2(1+|n|)}|n\rangle$ would be suitable as well.

In the *nonlinear case*, with $f \neq 0$, we have shown that $|\langle n'|\hat{M}|n\rangle|$ decreases fast outside the cone in the (n', n) plane. The conjugation with $\hat{A}$ gives the multiplicative factor $e^{+a(|n|-|n'|)}$, which decreases in the sector $|n'| > |n|$ but increases in the sector $|n'| < |n|$. The expansivity hypothesis insures that the cone is strictly included in the first sector. Moreover, provided $a > 0$ is small enough, the decrease in $|\langle n'|\hat{M}|n\rangle|$ dominates the increase in $\mathrm{e}^{+a(|n|-|n'|)}$ in the sector $|n'| < |n|$. As a final result we obtain that $|\langle n'|\hat{R}|n\rangle|$ decreases exponentially fast for $|n|, |n'| \to \infty$.

3.2.2. Exponential decrease of matrix elements. From (3) and (4) it follows that for any $n > 0$:

$$\langle n'|\hat{R}|n\rangle = \mathrm{e}^{+a(n-|n'|)}\langle n'|\hat{M}|n\rangle.$$

Instead of $n', n \in \mathbb{Z}$, we prefer to use 're-normalized' indices for $n > 0$, defined by

$$h = \frac{1}{n}, \qquad \nu' = n'h = p + 2,$$

where h, p were already defined in (16).

We can write $\mathrm{e}^{+a(n-|n'|)} = \mathrm{e}^{(1/h)A(\nu')}$, with $A(\nu') \stackrel{\text{def}}{=} a(1 - |\nu'|)$, and equation (17) gives the upper bound

$$|\langle n'|\hat{M}|n\rangle| < e^{\frac{1}{h}B(\nu')}, \qquad B(\nu') \stackrel{\text{def}}{=} \min(0, 2\pi C(\nu' - b_{\min}), 2\pi C(b_{\max} - \nu'))$$

with

$$b_{\min} = 2 + f'_{\min} - \varepsilon, \qquad b_{\max} = 2 + f'_{\max} + \varepsilon.$$

Because M is expanding, we can choose $\varepsilon > 0$ such that $\min_x(M'_x) = 2 + f'_{\min} > 1 + \varepsilon$; hence $b_m > 1$. For large $|\nu'|$, the functions $A(\nu')$ and $B(\nu')$ and have respective slope a and $2\pi C$. Choosing $a < 2\pi C$ implies that a maximum of $F(\nu') = A(\nu') + B(\nu')$ is reached for $\nu' = b_m$ (see figure 3).

We need an upper bound $|\langle n'|\hat{R}|n\rangle| < \exp(-c(|n| + |n'|)) = \exp(-(c/h)(1 + |\nu'|))$; therefore, we now look for a constant $c > 0$ such that $F(\nu') = A(\nu') + B(\nu') < -c(1 + |\nu'|)$, for any $\nu' \in \mathbb{R}$. This requires $c < a$ and $F(b_{\min}) \leqslant -c(1 + b_{\min}) \Leftrightarrow c < a(b_{\min} - 1)/(b_{\min} + 1) < a$. Consequently, we choose $c < a(b_{\min} - 1)/(b_{\min} + 1)$, and this proves the exponential estimates of theorem 1.

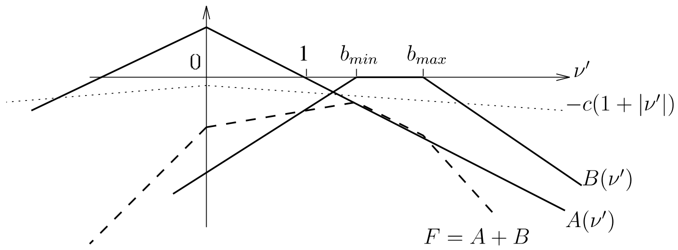

Figure 3. Representation of functions $F(\nu') = A(\nu') + B(\nu')$ for the upper bounds $|\langle n'|\hat{R}|n\rangle| < \exp((1/h)F(\nu')) < \exp(-(c/h)(1+|\nu'|))$.

3.2.3. Trace class operator. First from $|\langle n'|\hat{R}|n\rangle| < \exp(-c(|n|+|n'|))$, $\hat{R}$ is a Hilbert–Schmidt operator, therefore bounded. Its domain C_A is dense in $L^2(S^1)$. From a classical result, $\hat{R}$ extends in a unique way to a bounded operator in $L^2(S^1)$. Now let $\hat{B}$ be the operator diagonal in the Fourier basis, defined by $\hat{B}|n\rangle = 1/|n|^\alpha|n\rangle$, with $\alpha > 1/2$. $\hat{B}$ is a Hilbert–Schmidt operator and $\hat{C} \stackrel{\text{def}}{=} \hat{R}\hat{B}^{-1}$ is also an Hilbert–Schmidt operator, so $\hat{R} = \hat{C}\hat{B}$ being a product of two Hilbert–Schmidt operators is a trace class operator (cf [10], lemma 7.2, p 67).

3.3. Exponential accumulation of Ruelle–Pollicott resonances near zero

In this section, we prove theorem 3.

Let $\hat{R}$ be the trace class operator obtained in theorem 1, with the estimation $|\langle n'|\hat{R}|n\rangle| < \exp(-c(|n|+|n'|))$ on its matrix elements, with $c > 0$. We first deduce an estimation on the singular values of $\hat{R}$.

Lemma 1. *Let μ_j, $j = 0, 1, \ldots$, be the non-zero singular values of $\hat{R}$ (namely the eigenvalues of the self-adjoint operator $\sqrt{\hat{R}^*\hat{R}}$), such that $\mu_{j+1} \leqslant \mu_j$, repeated as many times as the value of their multiplicity. Then*

$$\mu_j \leqslant C_1 \mathrm{e}^{-cj/2} \tag{18}$$

with $C_1 = (2(1+\mathrm{e}^{-c}))/((1-\mathrm{e}^{-c})^2)$.

Proof. We borrow an argument from [12] (in the proof of proposition 3.2). From the min–max theorem,

$$\mu_j = \min_{V \subset L^2(S^1),\, \mathrm{codim}\, V = j} \; \max_{v \in V,\, \|v\|=1} \|\hat{R}v\|.$$

Consider the Fourier basis $|n\rangle$, $n \in \mathbb{Z}$ and $V_l = \mathrm{span}(|n\rangle)_{|n|>l}$, hence $\mathrm{codim}\, V_l = 2l+1$. If $|v\rangle \in V_l$, we compute

$$\begin{aligned}
\|\hat{R}|v\rangle|\| &= \left\| \sum_{n' \in \mathbb{Z}, |n|>l} |n'\rangle\langle n'|\hat{R}|n\rangle\langle n|v\rangle \right\| \leqslant \sum_{n' \in \mathbb{Z}, |n|>l} |\langle n'|\hat{R}|n\rangle\langle n|v\rangle| \\
&\leqslant \|v\| \sum_{n' \in \mathbb{Z}, |n|>l} \exp(-c(|n|+|n'|)) = \|v\| \left(\sum_{|n|>l} \mathrm{e}^{-c|n|} \right) \sum_{n' \in \mathbb{Z},} \mathrm{e}^{-c|n'|} \\
&= \|v\| 2S_{l+1}(1+2S_1) = \|v\| \mathrm{e}^{-c(l+1)} 2S_0(1+2S_1),
\end{aligned}$$

with $S_j \stackrel{\text{def}}{=} \sum_{n\geqslant j} \mathrm{e}^{-cn} = \mathrm{e}^{-cj}/1-\mathrm{e}^{-c} = \mathrm{e}^{-cj} S_0$. We deduce that $\mu_{2l+1} \leqslant \mathrm{e}^{-c(l+1)} 2S_0(1+2S_1)$; hence for j odd, $\mu_j \leqslant \mathrm{e}^{-cj/2}\mathrm{e}^{-c/2}2S_0(1+2S_1) < C_1\mathrm{e}^{-cj/2}$ with $C_1 = 2S_0(1+2S_1) = 2(1+\mathrm{e}^{-c})/(1-\mathrm{e}^{-c})^2$. For j even, $\mu_j \leqslant \mu_{j-1} \leqslant \mathrm{e}^{-cj/2}2S_0(1+2S_1) = C_1\mathrm{e}^{-cj/2}$. □

There is a fundamental relation between eigenvalues $(\lambda_j)_j$ of $\hat{R}$ (sorted such that $|\lambda_{j+1}| \leqslant |\lambda_j|$ and repeated as many times as the value of their multiplicity) and singular values $(\mu_j)_j$ (cf [10], theorem 3.1, p 52):

$$\prod_{j=0}^{n} |\lambda_j| \leqslant \prod_{j=0}^{n} \mu_j, \qquad n \geqslant 0. \tag{19}$$

For non-zero eigenvalues, define

$$l_j = \log|\lambda_j|, \qquad m_j = \log\mu_j.$$

(These sequences tend to $-\infty$ as $j \to \infty$.) Then the above inequality reads $\sum_{j=0}^{n} l_j \leqslant \sum_{j=0}^{n} m_j$. Equation (18) gives $m_j \leqslant \log C_1 - cj/2$. We deduce that $\sum_{j=0}^{n} l_j \leqslant (n+1)\log C_1 - (c/2)(n(n+1))/2$; hence

$$\frac{1}{(n+1)}\sum_{j=0}^{n} l_j \leqslant \log C_1 - \frac{c}{4}n.$$

But $l_n \leqslant l_j$ for $j \leqslant n$, so $l_n \leqslant 1/(n+1)\sum_{j=0}^{n} l_j \leqslant \log C_1 - (c/4)n$, which proves theorem 3.

3.4. Numerical illustrations: Sinai–Ruelle–Bowen measure and Ruelle–Pollicott resonances

In order to illustrate the previous result, we discuss here some well-known aspects of the SRB measure and RP resonances of example (1), obtained by numerical diagonalization of operator $\hat{R}$ (in the Fourier basis).

3.4.1. The Sinai–Ruelle–Bowen measure. The zero Fourier mode (constant function) $|v_0\rangle = |n=0\rangle$ is an eigenvector of $\hat{M}$ (and thus $\hat{R}$) with eigenvalue $\lambda_0 = 1$. It is known that expanding maps such as equation (2) are mixing [14], which implies as we will see that $\lambda_0 = 1$ is an isolated eigenvalue of multiplicity 1 and all other eigenvalues of $\hat{R}$ are $|\lambda_i| < 1$, $i = 1, 2, \ldots$. Let $|w_0\rangle \in L^2(S^1)$ be the dual eigenvector, i.e. $\langle v_0|w_0\rangle = 1$ and $\hat{R}^*|w_0\rangle = |w_0\rangle \Leftrightarrow \langle w_0|\hat{R} = \langle w_0|$. So in operator norm,

$$\hat{R}^t \equiv |v_0\rangle\langle w_0| + \mathcal{O}(|\lambda_1|^t), \qquad |\lambda_1| < 1.$$

If $|\varphi\rangle \in L^2(S^1)$, and $|\phi\rangle \in C_A$, equation (5) gives an exponential decay of correlation for large t:

$$\begin{aligned} C_{\phi,\varphi}(t) = \langle\phi|\hat{M}^t|\varphi\rangle &= \langle\phi|\hat{A}^{-1}|v_0\rangle\langle w_0|\hat{A}|\varphi\rangle + \mathcal{O}(|\lambda_1|^t), \qquad |\lambda_1| < 1 \\ &= \langle\phi|v_0\rangle\langle\mu_{\mathrm{SRB}}|\varphi\rangle + \mathcal{O}(|\lambda_1|^t), \end{aligned}$$

where $|\mu_{\mathrm{SRB}}\rangle \stackrel{\text{def}}{=} \hat{A}|w_0\rangle \in C_A$ is called the SRB measure. Its density is a regular (real analytic function) on S^1, (cf figure 4). The physical meaning of the last equation is that for large t the function $\hat{M}^t|\varphi\rangle$ behaves (as seen from test functions, i.e. from a macroscopic point of view) like the constant function $|v_0\rangle$ times $\langle\mu_{\mathrm{SRB}}|\varphi\rangle$. This is mixing property. Another interpretation of μ_{SRB} is that for almost all $x_0 \in S^1$,

$$\langle\mu_{\mathrm{SRB}}| \equiv \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\langle\delta_{M^t x_0}|,$$

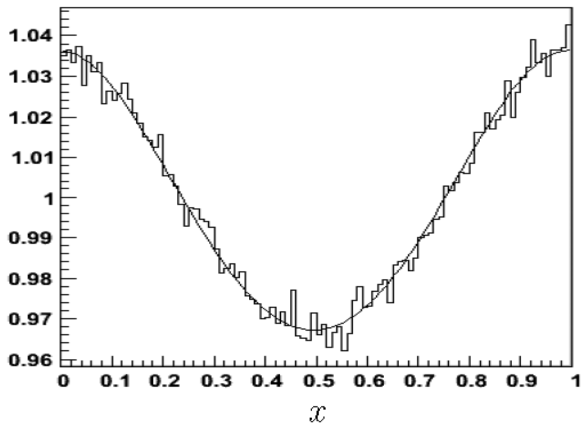

Figure 4. SRB measure: the solid line is the density $\mu_{\mathrm{SRB}}(x)$ computed from a numerical diagonalization of $\hat{R}$ (in the Fourier basis), for the perturbation given in (1), with $\delta = 0.4$. The histogram is constructed from a trajectory of length $T = 10^7$ starting from a random initial point x_0.

(equality of measures) where δ_x is the Dirac measure at $x \in S^1$. The right-hand side is called the 'physical measure' because it is constructed from a typical trajectory ([7] p 640, [3] p 73), (cf figure 4).

3.4.2. The Ruelle–Pollicott resonances. Suppose for simplicity the first N eigenvalues of $\hat{R}$ are *simple*, $\hat{R}|v_i\rangle = \lambda_i|v_i\rangle$, $i = 0 \to (N-1)$, with $\lambda_0 = 1$, $|\lambda_i| < 1$, $|\lambda_{i+1}| \leqslant |\lambda_i|$ and $|\lambda_N| < |\lambda_{N-1}|$. Let us write $|w_i\rangle$ the dual vectors, i.e. $\langle w_i|v_j\rangle = \delta_{i,j}$ and $\hat{R}^*|w_i\rangle = \overline{\lambda}_i|w_i\rangle \Leftrightarrow \langle w_i|\hat{R} = \lambda_i\langle w_i|$. Then

$$\hat{R}^t \equiv |v_0\rangle\langle w_0| + \sum_{i=1}^{N-1} \lambda_i^t |v_i\rangle\langle w_i| + \mathcal{O}(|\lambda_N|^t), \qquad |\lambda_N| < 1,$$

shows that the RP resonances λ_i govern the asymptotic behaviour of the correlation functions (5) and the convergence towards equilibrium:

$$C_{\phi,\varphi}(t) = \langle\phi|\hat{M}^t|\varphi\rangle = \langle\phi|v_0\rangle\langle\mu_{\mathrm{SRB}}|\varphi\rangle + \sum_{i=1}^{N-1} \lambda_i^t \langle\phi|\hat{A}^{-1}|v_i\rangle\langle w_i|\hat{A}|\varphi\rangle + \mathcal{O}(|\lambda_N|^t).$$

Note that in this last expression $|v_i\rangle, |w_i\rangle \in L^2(S^1)$; hence, $\hat{A}|w_i\rangle \in C_A$ is a regular function, but $\hat{A}^{-1}|v_i\rangle$ may not belong to $L^2(S^1)$. We have to interpret $\hat{A}^{-1}|v_i\rangle$ as a linear form on the space C_A. Vectors $|v_i\rangle, |w_i\rangle$ depend on the choice of operator $\hat{A}$, but eigenvalues λ_i and distributions $\hat{A}|w_i\rangle$, $\hat{A}^{-1}|v_i\rangle$ do not.

Let us write $\lambda_i = \rho_i e^{i\theta_i}$, $\rho_i > 0$, hence $\log\lambda_i = \log(\rho_i) + i\theta_i$. Figure 5 shows the first nine RP resonances $\log(\lambda_i)$, obtained by a numerical diagonalization of $\hat{R}$ in the Fourier basis. In particular, we note that there are two symmetric clusters of eigenvalues. λ_4 and λ_5 are very close to each other: $\log\rho_4 = -5.271$, $\theta_4 = 1.101$ and $\log\rho_5 = -5.285$, $\theta_4 = 1.119$. We have no explanation for this. In semi-classical analysis such clusters occur because of the 'tunnelling effect'. It would be nice to find such a semi-classical interpretation here. Note that (3) predicts that the values $\log(\rho_i)$ tend at least linearly to $-\infty$, as $i \to \infty$.

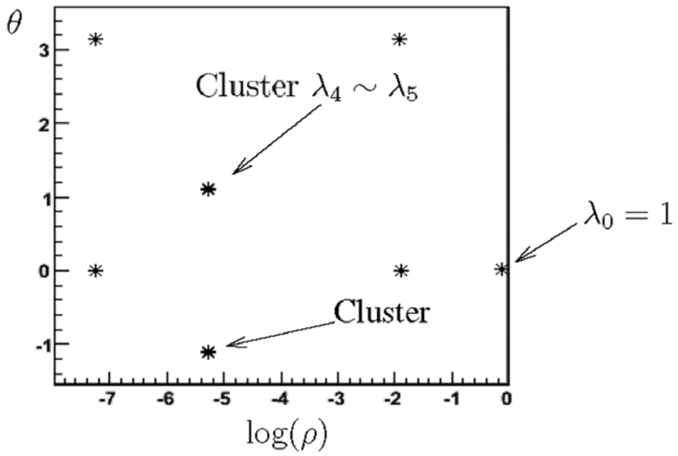

Figure 5. The first nine RP resonances $\lambda_i = \rho_i e^{i\theta_i}$ in log scale, for example in equation (1), with $\delta = 0.4$. We note two clusters of nearby eigenvalues.

4. Hyperbolic map on the torus

We follow essentially the same lines as in the case of an expanding map on the circle, in order to prove theorem 6.

4.1. Matrix elements of the operator $\hat{M}$

Consider the operator $\hat{M}$ defined in equation (9). Let $|n\rangle$ denote the Fourier mode on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, defined by $\varphi_n(x) = \exp(i2\pi n \cdot x)$, with $n \in \mathbb{Z}^2$, $x \in \mathbb{T}^2$. The set $(|n\rangle)_{n\in\mathbb{Z}^2}$ forms a orthonormal basis of $L^2(\mathbb{T}^2)$, and matrix elements of $\hat{M}$ in this basis are explicitly given by

$$\begin{aligned}\langle n'|\hat{M}|n\rangle &= \int_{\mathbb{T}^2} dx \exp(-i2\pi n' \cdot x) \exp(i2\pi n \cdot (M_0(x) + f(x))) \\ &= \int_{\mathbb{T}^2} dx \exp(i2\pi((M_0^t(n) - n') \cdot x + n \cdot f(x))),\end{aligned}$$

with transposed matrix M_0^t.

4.1.1. Remarks.

- For a vanishing perturbation $f = 0$, then

$$\langle n'|\hat{M}_0|n\rangle = \delta_{n'=M_0^t n}. \qquad (20)$$

  For a non-vanishing perturbation f, we will now show that in the plane $n' = (n'_1, n'_2)$ matrix elements are very small outside some of the domains surrounding the point $n' = M_0^t n$.
- Since f is real, we have the symmetry

$$\langle -n'|\hat{M}| - n\rangle = \overline{\langle n'|\hat{M}|n\rangle},$$

  and if $n = 0$, we have

$$\langle n'|\hat{M}|0\rangle = \delta_{n'=0}.$$

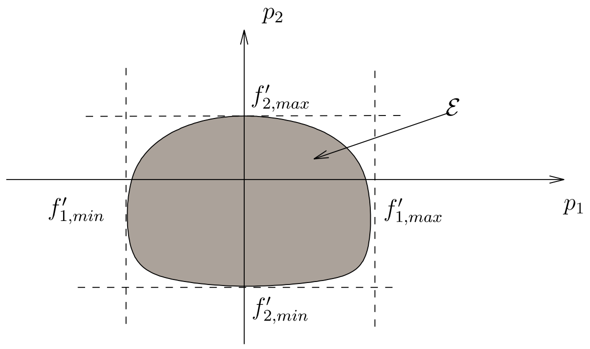

Figure 6. Picture of the domain $\mathcal{E}$.

- With the example given by equation (10), one can explicitly compute the matrix elements in terms of the Bessel functions of the first kind (cf [1] section 9.1.21, p 360):

$$\langle n'|\hat{M}|n\rangle = (-1)^{N_1}\delta_{N_2=0}\, J_{N_1}(\delta n_2),$$

$$\text{with } N = (n - (M_0^{-1})^{\mathrm{t}}(n')) = \begin{cases} N_1 = n_1 - n'_1 + n'_2, \\ N_2 = n_2 + n'_1 - 2n'_2. \end{cases}$$

4.1.2. Localization property of the matrix elements. Let us note the following changes in variables $(n, n') \Leftrightarrow (h, \nu, p)$, for $n \neq 0$,

$$\nu = \frac{n}{|n|} \in S^1, \qquad h = \frac{1}{|n|} > 0, \qquad p = h(n' - M_0^{\mathrm{t}}(n)) \in \mathbb{R}^2, \tag{21}$$

with $|n| = \sqrt{n_1^2 + n_2^2}$ and S^1 the unit circle in Fourier space $\mathbb{R}^2$. Define $\hbar = h/(2\pi)$. Any matrix element can be written as the oscillating integral:

$$I_{\hbar,\nu}(p) \stackrel{\text{def}}{=} \langle n'|\hat{M}|n\rangle = \int_{\mathbb{T}^2} \mathrm{d}x \exp(\mathrm{i}(\nu \cdot f(x) - p \cdot x)/\hbar).$$

Theorem 11 ('Non-stationary phase'). *Let $f : \mathbb{T}^2 \to \mathbb{R}^2$ be real analytic, $\hbar > 0$, $\nu \in U(1)$ and $p/(2\pi\hbar) \in \mathbb{Z}^2$. Then $\nu \cdot (D_x f) \in \mathbb{R}^2$. Consider the compact domain $\mathcal{E} \stackrel{\text{def}}{=} \{\nu \cdot (D_x f)\ s.t.\ x \in \mathbb{T}^2, \nu \in S^1\} \subset \mathbb{R}^2$ which contains 0. We denote by $[f'_{1,\min}, f'_{1,\max}]$ and $[f'_{2,\min}, f'_{2,\max}]$ the projections of $\mathcal{E}$ the axes p_1 and p_2, respectively (see figure 6). For any $\varepsilon > 0$, there exists $C > 0$, such that, for any $p = (p_1, p_2)$ with $p_1 < f'_{1,\min} - \varepsilon$, any $\hbar > 0$ and any $\nu \in S^1$, one has*

$$|I_{\hbar,\nu}(p)| \leqslant \mathrm{e}^{-C(f'_{1,\min} - p_1 - \varepsilon)/\hbar}. \tag{22}$$

Similarly, we have exponential upper bounds for the other three half-planes $p_1 > f'_{1,\max} + \varepsilon$, $p_2 < f'_{2,\min} - \varepsilon$ and $p_2 > f'_{2,\max} + \varepsilon$ with the same constant C. Moreover we always have the general bound $|I_{\hbar,\nu}(p)| < 1$.

Proof. Let $\varepsilon > 0$ and write $I_{\hbar,\nu}(p) = \int_0^1 \mathrm{d}x_2\, \mathrm{e}^{-\mathrm{i}p_2 x_2/\hbar} \mathcal{I}_1(x_2)$ with $\mathcal{I}_1(x_2) = \int_0^1 \mathrm{d}x_1\, \mathrm{e}^{\mathrm{i}(\nu \cdot f(x_1,x_2) - p_1 x_1)/\hbar}$. For ν, x_2 fixed, let $\tilde{f}(x_1) = \nu \cdot f(x_1, x_2)$. Then it follows

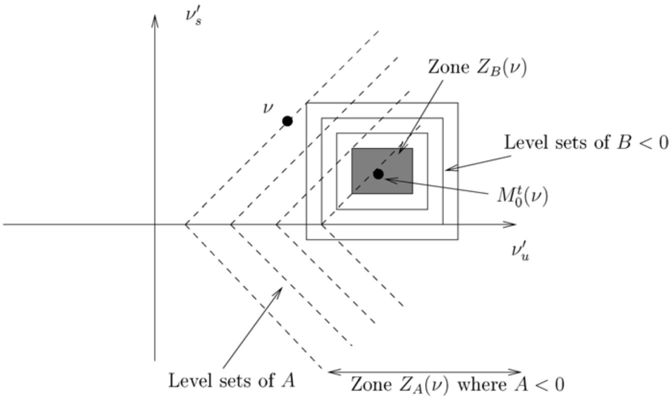

Figure 7. Schematic representation of level sets of functions $A(\nu')$ and $B(\nu')$, with respect to the frame of unstable/stable directions. Here $|\nu| = 1$ and ν is sent to $M_0^{\mathrm{t}}(\nu)$ by the dynamics.

from (17) that

$$|\mathcal{I}_1(x_2)| < \mathrm{e}^{-C_1(\tilde{f}'_{\min} - p_1 - \varepsilon)/\hbar},$$

where C_1 and $\tilde{f}'_{\min}$ depend on ν and x_2. Let $f'_{1,\min} = \min_{\nu,x_2}(\nu \cdot (\partial f(x_1, x_2)/\partial x_1))$ and $C = \min_{\nu,x_2}(C_1) > 0$. Then, for $p_1 < (f'_{1,\min} - \varepsilon)$, one has $|I_{\hbar,\nu}(p)| < |\mathcal{I}_1(x_2)| < \mathrm{e}^{-C(f'_{1,\min} - p_1 - \varepsilon)/\hbar}$. Similarly, we define $f'_{1,\max}$, $f'_{2,\min}$ and $f'_{2,\max}$ and obtain similar estimates. □

Remark. We have shown that the integral $I_{\hbar,\nu}(p)$ is 'small' outside a rectangle containing $p = 0$ in the plane $\mathbb{R}^2$. This rectangle shrinks to 0 when the perturbation f is C_1 small. Coming back to variables (n', n), this means that for n fixed the matrix elements of $\langle n'|\hat{M}|n\rangle$ are 'small' except in a domain surrounding the point $n' = M_0^{\mathrm{t}} n$.

4.2. Proof of theorem 6

Instead of the variables $n, n' \in \mathbb{Z}^2$, for $n \neq 0$ we prefer to use,

$$h = \frac{1}{|n|} > 0, \qquad \nu = \frac{n}{|n|} \in S^1, \qquad \nu' = \frac{n'}{|n|} \in \mathbb{R}^2.$$

From (21), we have $p = \nu' - M_0^{\mathrm{t}}\nu$. From (11) and (12), we have, for $n \neq 0$,

$$R_{n',n} \stackrel{\mathrm{def}}{=} \langle \hat{A}n'|\hat{M}|\hat{A}^{-1}n\rangle = \exp\left(\frac{1}{h}A(\nu')\right)\langle n'|\hat{M}|n\rangle$$

with

$$A(\nu') = a(|\nu_u| - |\nu_s| - |\nu'_u| + |\nu'_s|),$$

where n, h, ν are considered as fixed in the discussion. Note that we do not as yet use the notation $\langle n'|\hat{R}|n\rangle$, but $R_{n',n}$ instead, because we do not know yet if $|n\rangle$ belongs to the domain of $\hat{R}$. This will be proved below.

Then equation (22) gives the upper bound: $|\langle n'|\hat{M}|n\rangle| < \exp((1/h)B(\nu'))$, where the function $B(\nu')$ is equal to 0 on a rectangle domain denoted by $Z_B(\nu)$, containing the point $M_0^{\mathrm{t}}\nu$. The function $B(\nu')$ decreases linearly outside this rectangle, with a slope $2\pi C$ which does not depend on ν and h (see figure 7). The size of the domain $Z_B(\nu)$ goes to 0, whenever $\|f\|_{C^1} \to 0$.

We deduce that $|R_{n',n}| < \exp((1/h)F(\nu'))$, with $F(\nu') = A(\nu') + B(\nu')$. The function $A(\nu')$ is zero for $\nu' = \nu$, and it is negative and decreases with a constant slope a on a domain denoted by $Z_A(\nu)$ (see figure 7). At the point $\nu' = M_0^t(\nu) = (e^{\lambda_0}\nu_u, e^{-\lambda_0}\nu_s)$, the value of $A(\nu') = -a(|\nu_u|(e^{\lambda_0}-1)+|\nu_s|(1-e^{-\lambda_0})) < \mathcal{A} < 0$ is strictly negative, uniformly with respect to $\nu \in S^1$. Therefore, the domain $Z_B(\nu)$ is strictly included in $Z_A(\nu)$ if the perturbation f is small enough in C^1 norm.

If we choose a such that A increases more slowly than B decreases (i.e. $a < c'2\pi C$ where $c' > 0$) then there exists $c > 0$ such that

$$F(\nu') < -c(|\nu_1| + |\nu_2| + |\nu_1'| + |\nu_2'|).$$

This gives $|R_{n',n}| < \exp(-c(|n_1| + |n_2| + |n_1'| + |n_2'|))$.

Let us first deduce that the operator $\hat{R} = \hat{A}\hat{M}\hat{A}^{-1}$ is defined on the domain C_A. If $|\phi\rangle \in C_A$,

$$|\phi\rangle \in \mathrm{Dom}(\hat{R}) \Leftrightarrow \hat{M}\hat{A}^{-1}|\phi\rangle \in D_A \Leftrightarrow \sum_{n'} |\langle \hat{A}n'|\hat{M}\hat{A}^{-1}|\phi\rangle|^2 < \infty$$

$$\Leftrightarrow \sum_{n'} \left| \sum_n \langle \hat{A}n'|\hat{M}|\hat{A}^{-1}n\rangle\langle n|\phi\rangle \right|^2 < \infty \Leftrightarrow \sum_{n'} \left| \sum_n R_{n',n}\langle n|\phi\rangle \right|^2 < \infty.$$

Let us now show that the last estimate is actually fulfilled. If $|\phi\rangle \in L^2(\mathbb{T}^2)$

$$\sum_{n'} \left| \sum_n R_{n',n}\langle n|\phi\rangle \right|^2 \leqslant \sum_{n'} \left(\sum_n |R_{n',n}||\langle n|\phi\rangle| \right)^2$$

$$\leqslant \|\phi\|^2 \sum_{n'} e^{-2c(|n_1'|+|n_2'|)} \left(\sum_n e^{-c(|n_1|+|n_2|)} \right)^2 < \infty.$$

Therefore, the operator $\hat{R} = \hat{A}\hat{M}\hat{A}^{-1}$ is defined on the domain C_A and its matrix elements are obviously $\langle n'|\hat{R}|n\rangle = R_{n',n}$. With the same arguments as those used earlier for the expanding map on S^1, we deduce that $\hat{R}$ extends to a trace class operator on $L^2(\mathbb{T}^2)$.

4.2.1. Proof of the exponential concentration of Ruelle–Pollicott resonances The proof of theorem 7 is very similar to the proof we gave in section 3.3. Here we use the same notations and emphasize the differences.

Let $\hat{R}$ be the trace class operator obtained in theorem 6, with the estimation $|\langle n'|\hat{R}|n\rangle| < \exp(-c(|n_1| + |n_2| + |n_1'| + |n_2'|))$ on its matrix elements, with $c > 0$.

Lemma 2. *Let μ_j, $j = 0, 1, \ldots$, be the non-zero singular values of $\hat{R}$, such that $\mu_{j+1} \leqslant \mu_j$, repeated as many times as the value of their multiplicity. Then*

$$\mu_j \leqslant C_1 e^{-(c/2)\sqrt{j}} \tag{23}$$

with $C_1 > 0$.

Proof. From the min–max theorem,

$$\mu_j = \min_{V \subset L^2(\mathbb{T}^2),\ \mathrm{codim}\ V = j} \ \max_{v \in V,\ \|v\|=1} \|\hat{R}v\|.$$

Consider the Fourier basis $|n\rangle, n = (n_1, n_2) \in \mathbb{Z}^2$ and $V_l = \mathrm{span}(|n\rangle)_{\max(|n_1|,|n_2|)>l}$; hence $\mathrm{codim}\, V_l = (2l+1)^2$. If $|v\rangle \in V_l$, we compute

$$\|\hat{R}|v\rangle|\| = \left\| \sum_{n'\in\mathbb{Z},|n|>l} |n'\rangle\langle n'|\hat{R}|n\rangle\langle n|v\rangle \right\|$$
$$\leqslant \|v\| \left(\sum_{n'\in\mathbb{Z}^2} \exp(-c(|n'_1| + |n'_2|)) \right) \left(\sum_{n/\max(|n_1|,|n_2|)>l} \exp(-c(|n_1| + |n_2|)) \right)$$
$$\leqslant \|v\| C e^{-cl}, \qquad C > 0.$$

We deduce that $\mu_j \leqslant C_1 \mathrm{e}^{-c\sqrt{j}/2}$, with $C_1 > 0$. □

For non-zero eigenvalues, define

$$l_j = \log|\lambda_j|, \qquad m_j = \log\mu_j.$$

(These sequences tend to $-\infty$ as $j \to \infty$.) Inequality (19) reads $\sum_{j=0}^{n} l_j \leqslant \sum_{j=0}^{n} m_j$. Equation (23) gives $m_j \leqslant \log C_1 - c\sqrt{j}/2$. We deduce that $\sum_{j=0}^{n} l_j \leqslant (n+1)\log C_1 - (c/2)\sum_{j=0}^{n}\sqrt{j}$. But $\sum_{j=0}^{n}\sqrt{j} \geqslant \int_0^n \sqrt{x}\,\mathrm{d}x = \frac{2}{3}n^{3/2}$. Hence, $1/(n+1)\sum_{j=0}^{n} l_j \leqslant \log C_1 - (c/3)(n^{3/2}/(n+1))$. But $l_n \leqslant l_j$ for $j \leqslant n$, so $l_n \leqslant 1/(n+1)\sum_{j=0}^{n} l_j \leqslant \log C_1 - (c/3)(n^{1/2}/(1+1/n)$, which proves theorem 7.

5. Conclusions

For specific models of chaotic dynamics, namely real analytic expanding maps on the circle S^1 and real analytic hyperbolic maps on the torus $\mathbb{T}^2$, we have shown that the decay of time correlation functions can be described by a trace class operator in $L^2(S^1)$ (respectively $L^2(\mathbb{T}^2)$). We have followed an approach similar to the well-known 'complex scaling method', to study the decay of quantum states in open quantum systems. As explained in the introduction, this approach has been already used by Baladi and Tsujii [4] for hyperbolic diffeomorphisms in a more general context, but our methods differ sightly and allowed us to obtain different results. To make a more precise comparison, our operator $\hat{A}$ defined in equation (11) and the conjugation in equation (12) correspond to their definition of anisotropic norms. But they use the powers of the Fourier components whereas we use their exponential. This exponential is important for us to obtain a trace class operator $\hat{R}$, as explained in section 3.2.1.

In this paper, some 'semi-classical aspects' of hyperbolic dynamics have appeared many times: (i) in theorem 10 concerning the localization of the matrix elements in Fourier space and the remark which follows, (ii) in a remark on the semi-classical Weyl law after equation (13) and (iii) in figure 5 where a cluster of eigenvalues suggests some semi-classical tunnelling effect. The role of semi-classical parameter is played by the inverse of the distance in Fourier space: $\hbar \equiv 1/|n|$. A direction of research would be to make this semi-classical theory more precise.

We would like to comment on some limitations to our results and possible extensions of them. First we have assumed that the dynamics is given by a real analytic map. This is a severe limitation because in hyperbolic dynamical system theory one has to use Hölder potential functions [3, 14]. We have presented here the results in their simplest form. In particular, for expanding maps on the torus, we have shown that $\hat{R}$ is trace class for any expanding map, but for hyperbolic maps on the torus we have assumed that the nonlinear perturbation f is weak enough. In geometric terms we have supposed that $\|f\|_{C^1}$ is weak enough so that the unstable and stable foliations are, respectively, contained in fixed cones (the cones adapted to the linear map, and which enter in the definition of $\hat{A}$ equation (11)). It could be possible to generalize in this direction and treat in this way *any* uniform analytic hyperbolic map on the torus, using a local choice of cones. As in [4], this could be possible using pseudo-differential operators instead of $\hat{A}$. Then the localization property of the matrix elements in theorem 10 would be replaced by a 'microlocal' version in the cotangent space T^*S^1 (respectively $T^*\mathbb{T}^2$). Some other directions of research could be to take advantage of the relative simplicity of this approach to investigate non-uniform hyperbolic dynamics or other kinds of dynamical systems which exhibit some chaotic behaviour, where many questions still remain open.

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