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PREQUANTUM CHAOS: RESONANCES OF THE PREQUANTUM CAT MAP

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ABSTRACT. Prequantum dynamics was introduced in the 70s by Kostant, Souriau and Kirillov as an intermediate between classical and quantum dynamics. In common with the classical dynamics, prequantum dynamics transports functions on phase space, but adds some phases which are important in quantum interference effects. In the case of hyperbolic dynamical systems, it is believed that the study of the prequantum dynamics will give a better understanding of the quantum interference effects for large time, and of their statistical properties. We consider a linear hyperbolic map M in $\mathrm{SL}(2,\mathbb{Z})$ which generates a chaotic dynamical system on the torus. The dynamics is lifted to a prequantum fiber bundle. This gives a unitary prequantum (partially hyperbolic) map. We calculate its resonances and show that they are related to the quantum eigenvalues. A remarkable consequence is that quantum dynamics emerges from long-term behavior of prequantum dynamics. We present trace formulas, and discuss perspectives of this approach in the nonlinear case.

CONTENTS

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1. INTRODUCTION

Quantum chaos is the study of wave dynamics (quantum dynamics) and its spectral properties, in the limit of small wavelength, when in this limit, the corresponding classical dynamical system is chaotic [19]. This limit is also denoted by $\hbar \to 0$, and called the semiclassical limit. The usual mathematical models to study quantum chaos are models of hyperbolic dynamics, because there, the classical chaotic features are important and quite well understood (mixing, exponential decay of correlations, central limit theorem for observables, etc.) [7, 21]. On the quantum side, semiclassical formulas like the Gutzwiller trace formula (resp. the Van-Vleck formula) give descriptions of the quantum spectrum (resp. the description of the wave evolution) in the semiclassical limit, in terms of sum of complex amplitudes along different classical trajectories. One important problem in quantum chaos is that these semiclassical formulas are mathematically proved only for moderate time (versus $\hbar \to 0$), whereas some numerical experiments suggest that they could be valid for much larger time, like $t \simeq 1/\hbar^{\alpha}$, $\alpha > 0$ [31, 10][1], and a lot of work in the physics literature of quantum chaos is based on this last hypothesis ([11] for example). The main difficulty to prove this hypothesis is related to the fact that the number of classical trajectories which enter in the semiclassical formulas increases exponentially fast with time, like $e^{\lambda t}$, where λ is the Lyapunov exponent, and this makes it difficult to control the error terms.

For large time the structure described by the classical orbits in phase space is much finer than the Planck cells.[2] The validity of the semiclassical formulas could be due to some average effects in the sum of the huge number of complex amplitudes, at the scale of the Planck cells. One goal is to justify and understand this averaging process.

It is known that classical hyperbolic dynamical systems have trace formulas which are exact, even in *nonlinear* cases, [4, page 97], [15]. These trace formulas give the trace of the so-called regularized *transfer operator*, in terms of periodic orbits. The eigenvalues of the regularized transfer operator are called Ruelle–Pollicott resonances and are useful to describe convergence towards equilibrium and the decay of time-correlation functions in hyperbolic dynamical systems. A

[1] In [13], we show the validity of semiclassical formulas for time large as $t \simeq C \log(1/\hbar)$, for any $C > 0$, for a quantized hyperbolic nonlinear map on the torus.

[2] Planck cells are the "best resolution" of phase space made by quantum mechanics at the scale $\hbar$. The limitation is due to the uncertainty principle.

remarkable result in this theory, and which could be useful to exploit in quantum chaos, is the exactness of these trace formulas. As these formulas involve a sum over classical orbits, they can be interpreted as an averaging process over these orbits. We hope to be able to extend this formalism of classical dynamical systems to the semiclassical setting, in order to better control the averaging process between complex amplitudes for large time, and possibly to suggest an appropriate statistical approach for quantum chaos.

To follow this program we have to find a classical transfer operator whose trace formula is the semiclassical trace formula, and then be able to compare (in the operator norm) this transfer operator with the quantum evolution operator, in order to prove the validity of the semiclassical trace formulas for the quantum dynamics. This paper is a first step towards this objective. We propose here such an operator, and perform its study for a particular hyperbolic dynamical system, namely a linear hyperbolic map on the torus. However the objective is not yet reached because *linear* hyperbolic maps are very particular and the semiclassical trace formulas are already exact. The aforementioned problem is therefore not fully present in this paper, but it is the main motivation for this work, and we think that this analysis can be extended to the nonlinear case and will then reveal its interest.

The transfer operator we propose is the prequantum evolution operator. The prequantum dynamics is a natural dynamics at the border between classical and quantum dynamics. Similarly to the classical dynamics, prequantum dynamics transports functions on phase space (more precisely sections of a bundle), but introduces some complex phases which are determined by the actions of the classical trajectories. These phases are known to govern interference phenomena which are characteristic of wave dynamics and quantum dynamics. However, the difference from quantum dynamics is that there is no uncertainty principle in prequantum dynamics, and this simplifies its study in an essential way. The uncertainty principle (which is mathematically introduced by the choice of a complex polarization, or a complex structure on phase space, see Section 3.6), introduces a cutoff in phase space at the scale of the Planck constant $\hbar$. One consequence of the absence of this cutoff in the prequantum setting is that the prequantum formulas are exact. Another consequence is that the prequantum Hilbert space is much larger than the quantum one, and the hyperbolicity hypothesis on the dynamics implies that the prequantum wave functions escape towards finer and finer scales. This escape of the prequantum wave function from macroscopic scales towards microscopic scales for large time is described by a discrete set of "prequantum resonances". Another way to say this is that the prequantum resonances describe the time decay of correlations between smooth prequantum functions. The biggest prequantum resonance(s) (i.e., those with greatest modulus) dominate for long time and describe the part of the prequantum wave functions which remain at the macroscopic scale (i.e., at a scale larger than the Planck cells $\hbar$). We therefore expect a general relation between these outer prequantum resonances and the quantum eigenvalues which describe the quantum wave evolution.

The role of the prequantum dynamics and the corresponding trace formulas for quantum dynamics has already been suggested by many authors [27, 10, 30], in particular V. Guillemin in [18, page 504].

In this paper, starting from a linear hyperbolic map on the torus, we show how to define the hyperbolic prequantum map on the torus and establish a relation between the discrete resonance spectrum of the prequantum map and the discrete spectrum of the quantum map, see Theorem 1 on page 259. In the conclusion, we discuss some perspectives.

2. STATEMENT OF THE RESULTS

In this section we state the main result of this paper, and discuss some consequences. In the next sections, we will give precise definitions and recall the basics of the prequantum dynamics.

2.1. **Prequantum resonances and quantum eigenvalues.**

Let $M\colon \mathbb{T}^2 \to \mathbb{T}^2$ be a *hyperbolic linear map* on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, i.e., $M \in \mathrm{SL}(2,\mathbb{Z})$, $\mathrm{Tr} M > 2$. This map is Anosov (uniformly hyperbolic), with strong chaotic properties, such as ergodicity and mixing, see [21, p. 154].

The prequantum line bundle L is a Hermitian complex line bundle over $\mathbb{T}^2$, with constant curvature $\Theta = i2\pi N\omega$, where $\omega = dq \wedge dp$ is the symplectic two-form on $\mathbb{T}^2$ and $N \in \mathbb{N}^*$ is the Chern index of the line bundle. N is related to $\hbar$ by $N = 1/(2\pi\hbar)$. The *prequantum Hilbert space* is the space $\tilde{\mathcal{H}}_N := L^2(L)$ of L^2 sections of L. Note that $\tilde{\mathcal{H}}_N$ is infinite-dimensional. The prequantum dynamics is a lift of the map M to the bundle L that preserves the connection. This prequantum dynamics induces a transport of sections, and defines a unitary operator acting on $\tilde{\mathcal{H}}_N$ called *prequantum map* $\tilde{M}$. (In the following sections, this operator will be denoted by $\tilde{M}_N$).

The *quantum Hilbert space* $\mathcal{H}_N$ is the space of antiholomorphic sections of L (after the introduction of a complex structure on $\mathbb{T}^2$). Contrary to the prequantum case, $\mathcal{H}_N$ is finite-dimensional, and $\dim \mathcal{H}_N = N$ from the Riemann–Roch Theorem. The *quantum map* $\hat{M}$ is obtained by Weyl quantization of M. It is a unitary operator acting on $\mathcal{H}_N$ [20, 22, 9]. The *quantum spectrum* is the set of the eigenvalues of $\hat{M}$ denoted by $\exp(i\varphi_k)$, $k = 1,\dots,N$.

Classical resonances. We first review the concept of time correlation functions and Ruelle–Pollicott resonances for the classical map M. These concepts give a fruitful approach to the study of chaotic properties of classical dynamics, such as mixing or central limit theorem for observables, etc., see [4]. Let $\varphi,\phi \in L^2(\mathbb{T}^2) \cap C^\infty(\mathbb{T}^2)$, and define the transfer operator $M_{\text{class.}}$ acting on such functions by $(M_{class}\varphi)(x) := \varphi(M^{-1}x)$, $x \in \mathbb{T}^2$. For $t \in \mathbb{N}$, the *classical time correlation function* is defined by:

$$C_{\phi,\varphi}(t) := \langle \phi | M^t_{class} \varphi \rangle,$$

where the scalar product takes place in $L^2(\mathbb{T}^2)$. Using the Fourier decomposition of ϕ, φ, it is easy to show that $C_{\phi,\varphi}(t)$ decreases with t faster than any exponential (see [4, p. 226]). That is, for any $\kappa > 0$:

$$C_{\phi,\varphi}(t) = \langle\phi|1\rangle\langle 1|\varphi\rangle + o\left(e^{-\kappa t}\right), \tag{1}$$

where $|1\rangle$ stands for the constant function 1, and $\langle 1|\varphi\rangle = \int_{\mathbb{T}^2} \varphi(x)\,dx$. Eq.(1) reveals the mixing property of the classical map M. In order to study quantitatively the decay of $C_{\phi,\varphi}(t)$, we introduce its Fourier transform:

$$\tilde{C}_{\phi,\varphi}(\omega) := \sum_{t\in\mathbb{N}} e^{it\omega} C_{\phi,\varphi}(t).$$

The *classical resonances of Ruelle–Pollicott* are $e^{i\omega}$ such that ω is a pole of the meromorphic extension of $\tilde{C}_{\phi,\varphi}(\omega)$, $\omega \in \mathbb{C}$. They control the decay of $C_{\phi,\varphi}(t)$. In our case, there is a simple pole $e^{i\omega} = 1$, corresponding to the mixing property, see Figure 1 (a). The superexponential decay implies that there are no other resonances. For a *nonlinear* hyperbolic map we expect to observe other resonances $e^{i\omega}$, with $\left|e^{i\omega}\right| < 1$, see e.g., [15].

Prequantum resonances. We proceed similarly for prequantum dynamics. Given two smooth sections $\tilde{\varphi}, \tilde{\phi} \in L^2(L) \cap L^\infty(L)$, we define their *prequantum time-correlation function* by

$$C_{\tilde{\phi},\tilde{\varphi}}(t) := \langle\tilde{\phi}|\tilde{M}^t|\tilde{\varphi}\rangle, \quad t \in \mathbb{N}$$

We wish to study the decay of $C_{\tilde{\phi},\tilde{\varphi}}(t)$. The *prequantum resonances of Ruelle–Pollicott* are defined as $e^{i\omega}$ such that ω is a pole of the meromorphic extension of the Fourier transform of $C_{\tilde{\phi},\tilde{\varphi}}(t)$. These resonances govern the decay of $C_{\tilde{\phi},\tilde{\varphi}}(t)$. The main result of this paper is the following theorem, illustrated by Figure 1.

THEOREM 1. *Let $\tilde{M}$ be the prequantum map. There exists an operator $\tilde{B}$, such that*

$$\tilde{R} = \tilde{B}\tilde{M}\tilde{B}^{-1}$$

is defined on a dense domain of $L^2(L)$, and such that $\tilde{R}$ extends uniquely to a trace class operator in $L^2(L)$. The eigenvalues of $\tilde{R}$ are the prequantum resonances *and are given by*

$$r_{n,k} = \exp\left(i\varphi_k - \lambda_n\right), \qquad k = 1\ldots N, \quad n \in \mathbb{N} \tag{2}$$

with $\exp\left(i\varphi_k\right)$ *being the eigenvalues of the quantum map $\hat{M}$ (*quantum eigenvalues*), and $\lambda_n = \lambda\left(n + \frac{1}{2}\right)$, with λ being the Lyapunov exponent (i.e.,* $\exp(\pm\lambda)$ *are the eigenvalues of M).*

Figure 1 shows

(a) Ruelle–Pollicott resonances of the *classical map M*. The isolated value 1 traduces mixing property of the map. The absence of resonances traduces superexponential decay of time correlation functions (See [4, page 225], or [15] for a simple description of the classical resonances as eigenvalues of a trace class operator).

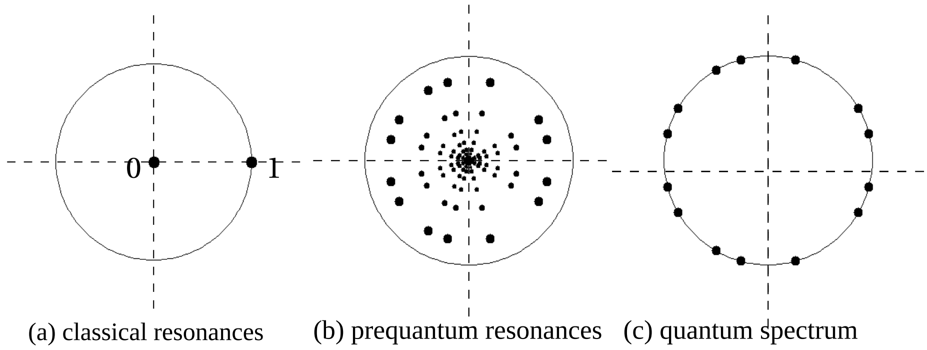

FIGURE 1. Spectra for the linear cat map with $N = 1/(2\pi\hbar) = 14$.

(b) Resonances $r_{n,k}$ of the *prequantum map* $\tilde{M}$, calculated in this paper. $r_{n,k} = \exp\left(i\varphi_k - \lambda(n+1/2)\right)$, $k = 1\dots N$, $n \in \mathbb{N}$. There are N resonances on each circle of radius $e^{-\lambda/2}e^{-\lambda n}$, $n \in \mathbb{N}$.
(c) Eigenvalues of the *quantum map* $\hat{M}$: $\exp\left(i\varphi_k\right)$, $k = 1\dots N$.

REMARK. • It is easy to see that the prequantum resonances are the eigenvalues of $\tilde{R}$. Indeed, if $\tilde{\varphi}, \tilde{\phi} \in \tilde{\mathscr{H}}_N = L^2(L)$ are sections which belong to the domains of $\tilde{B}, \tilde{B}^{-1}$ respectively, then the time-correlation function $C_{\tilde{\phi},\tilde{\varphi}}(t) := \langle\tilde{\phi}|\tilde{M}^t|\tilde{\varphi}\rangle$, $t \in \mathbb{N}$, can be expressed using the trace class operator $\tilde{R}$ as

$$C_{\tilde{\phi},\tilde{\varphi}}(t) := \langle\tilde{\phi}|\tilde{M}^t|\tilde{\varphi}\rangle = \left(\langle\tilde{\phi}|\tilde{B}^{-1}\right)\tilde{R}^t\left(\tilde{B}|\tilde{\varphi}\rangle\right).$$

Using a spectral decomposition of $\tilde{R}$, we deduce that the discrete spectrum of $\tilde{R}$ gives the explicit exponential decay of $C_{\tilde{\phi},\tilde{\varphi}}(t)$, and more precisely that the eigenvalues of $\tilde{R}$ are the prequantum resonances as defined above.

- The way we obtain the resonances of $\tilde{M}$ by conjugation with a nonunitary operator $\tilde{B}$ is well-known in quantum mechanics and is called the "complex scaling method" [8]. It is usually used in order to obtain the "quantum resonances of open quantum systems". Remind that $\tilde{M}$ is a unitary operator. It will appear in the paper, that it has a continuous spectrum on the unit circle.

Sketch of the proof. The proof of Theorem 1 will be obtained in Section 4.2 page 280. The main steps in the proof is to show that the prequantum Hilbert space is unitarily equivalent to a tensor product $\tilde{\mathscr{H}}_N \equiv \mathscr{H}_N \otimes L^2(\mathbb{R})$ involving the quantum Hilbert space $\mathscr{H}_N$ and an $L^2(\mathbb{R})$ space (this is Eq. (51) page 276), and then that the prequantum operator writes $\tilde{M} \equiv \hat{M} \otimes \exp\left(-i\hat{N}/\hbar\right)$, where $\hat{M}$ is the quantum map acting on $\mathscr{H}_N$ and $\hat{N} = \mathrm{Op}_{\mathrm{Weyl}}\left(\lambda qp\right)$ acting on $L^2(\mathbb{R})$ is the Weyl quantization of a hyperbolic fixed point dynamics. It is well-known that $\exp\left(-i\hat{N}/\hbar\right)$ has a continuous spectrum but a discrete set of resonances $\exp(-\lambda(n+1/2))$, $n \in \mathbb{N}$. So Eq.(2) follows.

2.2. **Dynamical appearance of the quantum space.** For large time t, the N external prequantum resonances on the circle of radius $\exp(-\lambda/2)$ will dominate, and with a suitable rescaling, $C_{\tilde{\phi},\tilde{\varphi}}(t)$ behaves for large time like quantum correlation functions, i.e., *matrix elements of the quantum propagator*. More precisely:

PROPOSITION 2. *if $\tilde{\phi},\tilde{\varphi} \in \tilde{\mathscr{H}}_N$ are prequantum wave functions, let us define $\phi = \hat{\Pi}\tilde{B}^{-1}\tilde{\phi}$, $\varphi = \hat{\Pi}\tilde{B}\tilde{\varphi}$, where $\hat{\Pi} = \tilde{\mathscr{H}}_N \to \mathscr{H}_N$ is the orthogonal projector called the Toeplitz projector (this requires $\tilde{\phi},\tilde{\varphi}$ to have suitable regularity so that they belong to the corresponding domains). Then for large time t*

$$\langle\tilde{\phi}|\tilde{M}^t|\tilde{\varphi}\rangle = \langle\phi|\hat{M}^t|\varphi\rangle e^{-\lambda t/2}\left(1+\mathcal{O}\left(e^{-\lambda t}\right)\right)$$

This means that quantum dynamics emerges as the long-term behavior of prequantum dynamics.

The proof is given in Section 4.3 on page 281.

Let us comment on Theorem 1 and Proposition 2. It is remarkable that the exterior circle of prequantum resonances is identified with the quantum eigenvalues. So the (generalized) eigenspace associated with these resonances is equivalent to the quantum space. This unitary isomorphism appears explicitly in the proof of the theorem. In some sense, and this is what Proposition 2 shows, the quantum space appears dynamically under the prequantum dynamics, and corresponds to "long lived" states. In this way the quantum dynamics appears here without any quantization procedure, but by the prequantum dynamics itself (which is itself a natural extension of the classical dynamics as a lift to a line bundle).

2.3. **Trace formulas.** As usual with transfer operators, trace formulas express the trace of a regularized transfer operator in terms of periodic orbits. The prequantum unitary operator $\tilde{M}$ is not trace class, so the trace formula expresses the trace of $\tilde{R}^t$ which is trace class. What is particular to prequantum dynamics (compared to classical dynamics), is the appearance of complex phases, related to the classical actions of the periodic orbits.

PROPOSITION 3. *For $t \in \mathbb{N}^*$, the trace formula for the prequantum dynamics expresses the trace of $\tilde{R}^t$ in terms of periodic points of M on $\mathbb{T}^2$ of period t:*

$$Tr\left(\tilde{R}^t\right) = \sum_{x\equiv M^t x\,[\mathbb{Z}^2]} \frac{1}{\left|det\left(1-M^t\right)\right|} e^{iA_{x,t}/\hbar} \tag{3}$$

where $A_{x,t} = \oint \frac{1}{2}\left(qdp - pdq\right) + Hdt$ is the classical action of the periodic orbit starting from $x = \left(q,p\right)$, and $\left|det\left(1-M^t\right)\right|^{-1} = \left(e^{\lambda t/2} - e^{-\lambda t/2}\right)^{-2}$ is related to its instability. More explicitly, for a periodic point characterized by $x = \left(q,p\right) \in \mathbb{R}^2$ and $M^t x = x + n$, $n \in \mathbb{Z}^2$, we have $A_{x,t} = \frac{1}{2} n \wedge x$.

The proof of Proposition 3 is given in Section 4.4, and follows the usual procedure to obtain a trace formula for transfer operators ([4, page 103] or [15]). The idea is to use the fact that the prequantum dynamics is a lift of the classical transport with additional phases, and therefore use the Schwartz kernel of $\tilde{M}$. Formally we write:

(4)
$$\mathrm{Tr}^{\flat}\left(\tilde{M}^t\right) = \int_{\mathbb{T}^2} dx \langle x|\tilde{M}^t|x\rangle = \int_{\mathbb{T}^2} dx \delta\left(M^t x - x\right) e^{i\frac{A_{x,t}}{\hbar}} = \sum_{x=M^t x} \frac{1}{\left|\det\left(1-M^t\right)\right|} e^{i\frac{A_{x,t}}{\hbar}}$$

This short calculation is made rigorous in the proof of Proposition 3 page 281, using a suitable regularization.

Relation with the quantum trace formula.

COROLLARY 4. *From Eq.(2), we deduce a relation between traces of operators. For $t \in \mathbb{Z}$,*

$$Tr\left(\hat{M}^t\right) = \sqrt{\left|det\left(1-M^t\right)\right|}\, Tr\left(\tilde{R}^t\right), \tag{5}$$

and from Eq.(3),

$$Tr\left(\hat{M}^t\right) = \sum_{x=M^t x} \frac{1}{\sqrt{\left|det\left(1-M^t\right)\right|}} e^{iA_{x,t}/\hbar}. \tag{6}$$

Proof. We have $\mathrm{Tr}\left(\hat{M}^t\right) = \sum_{k=1}^N e^{i\varphi_k}$, $\mathrm{Tr}\left(\tilde{R}^t\right) = \sum_{k=1}^N \sum_{n\geq 0} e^{i\varphi_k - \lambda_n}$ and $\sum_{n\geq 0} e^{-\lambda_n t} = \sum_{n\geq 0} e^{-\lambda\left(n+\frac{1}{2}\right)t} = \left(e^{\lambda t/2} - e^{-\lambda t/2}\right)^{-1}$, and finally $\sqrt{\left|\det\left(1-M^t\right)\right|} = \left(e^{\lambda t/2} - e^{-\lambda t/2}\right)$. □

REMARK.
- Formula Eq.(6) can be proved directly, see e.g., [22].
- It is important to realize that the trace formula for the quantum operator Eq.(6) is exact in our case, because we consider a *linear* hyperbolic map M. For a nonlinear map we expect that the trace formula for the prequantum map would still be exact, whereas there is no longer an exact trace formula for the quantum operator. What is known are semiclassical trace formulas that give $\mathrm{Tr}\left(\hat{M}^t\right)$ in the limit $N \to \infty$, but for relatively short time: $t = \mathcal{O}\left(\log N\right)$, see [13]. We give more comments on these trace formulas in the conclusion of this paper.

3. PREQUANTUM DYNAMICS ON $\mathbb{R}^2$

In this section we recall the basics of prequantization on the euclidean phase space $\mathbb{R}^2$. We will need this material in the next section. This is well-known, see [35], or [5] for an introduction to geometric quantization on more general phase spaces, i.e., Kähler manifolds.

3.1. **Hamiltonian dynamics.**

We first start with a classical Hamiltonian flow. We consider the phase space $\mathbb{R}^2$ and write $x = (q,p) \in \mathbb{R}^2$. The symplectic two-form is $\omega = dq \wedge dp$. A real-valued *Hamiltonian function* $H \in C^\infty\left(\mathbb{R}^2\right)$ defines a Hamiltonian vector field X_H by $\omega\left(X_H, \cdot\right) = dH$ and given explicitly by

$$X_H = \left(\frac{\partial H}{\partial p}\right)\frac{\partial}{\partial q} - \left(\frac{\partial H}{\partial q}\right)\frac{\partial}{\partial p}. \tag{7}$$

The Poisson bracket of $f,g \in C^{\infty}(\mathbb{R}^2)$ is $\{f,g\} = \omega(X_f, X_g) = X_g(f) = -X_f(g)$. The vector field X_H generates a Hamiltonian flow $\phi_t \colon \mathbb{R}^2 \to \mathbb{R}^2$, $t \in \mathbb{R}$. Explicitly, $(q(t), p(t)) = \phi_t(q(0), p(0))$, if $\frac{dq}{dt} = \frac{\partial H}{\partial p}, \frac{dp}{dt} = -\frac{\partial H}{\partial q}$. The flow transports functions: the action of ϕ_t on $f \in C^{\infty}(\mathbb{R}^2)$ is defined by $f_t := (f \circ \phi_{-t}) \in C^{\infty}(\mathbb{R}^2)$. The corresponding evolution equation is $\frac{df_t}{dt} = \{H, f_t\} = -X_H(f_t)$. In order to explain the introduction of the prequantum operator below, we rewrite this last equation as

$$\frac{df_t}{dt} = -\frac{i}{\hbar}(-i\hbar X_H) f_t \tag{8}$$

where $\hbar > 0$. A complex-valued function $f \in C^{\infty}(\mathbb{R}^2)$ can be seen as a section of the trivial bundle $\mathbb{R}^2 \times \mathbb{C}$ over $\mathbb{R}^2$. Prequantum dynamics, which we will define now, is a generalization of the transport of f_t but for sections of a nonflat bundle over $\mathbb{R}^2$.

3.2. **The prequantum line bundle.** We introduce $\hbar > 0$, called the "Planck constant" and consider a *Hermitian complex line bundle* L over $\mathbb{R}^2$, with a Hermitian connection[3] D. Each fiber L_x over $x \in \mathbb{R}^2$ is isomorphic to $\mathbb{C}$. A C^{∞} section s of L is a C^{∞} map $\mathbb{R}^2 \ni x \mapsto s(x) \in L_x$. We write $s \in A^0(L)$. The covariant derivative D is an operator $D \colon A^0(L) \to A^1(L)$ that acts on C^{∞} sections of L and gives a L-valued 1-form, see Figure 2. We require

1. *Leibniz's rule:* if $s \in A^0(L)$ is a section of L, and $f \in C^{\infty}(\mathbb{R}^2)$ a function, then $D(f.s) = df \otimes s + f.D(s)$.
2. If $h_x(\cdot,\cdot)$ denotes the Hermitian metric in the fiber L_x, the connection D should be compatible with h: $d(h(s_1, s_2)) = h(Ds_1, s_2) + h(s_1, Ds_2)$. In other words, if the section s follows the connection in direction X, i.e., $D_X s = 0$, then $h(s,s)$ is constant in this direction, i.e., $X(h(s,s)) = 0$.
3. The curvature two-form of the connection is
$$\Theta = \frac{i}{\hbar}\omega$$
where $\omega = dq \wedge dp$ is the symplectic two-form. This means that the holonomy of a closed loop surrounding a surface $\mathscr{S} \subset \mathbb{R}^2$ is $\exp\left(i\int_{\mathscr{S}} \omega/\hbar\right) = \exp(i2\pi(\mathscr{A}/h))$, where $\mathscr{A}/h$ is interpreted as the number of quanta $h = 2\pi\hbar$ contained in the area $\mathscr{A} = \int_{\mathscr{S}} \omega$, see Figure 3.

A section of reference. As the base space $\mathbb{R}^2$ is contractible, we can choose a unitary global section r of L, i.e., such that $|r(x)| = \sqrt{h_x(r(x), r(x))} = 1$, for every $x \in \mathbb{R}^2$. The section r is called the *reference section* and gives a trivialization of the bundle L. We write its covariant derivative $Dr = \theta r$, where θ is a 1-form on $\mathbb{R}^2$. The requirements on D above[4] impose that $\theta = \frac{i}{\hbar}\eta$ with a real one-form η

[3] For a general introduction to Hermitian line bundles, see [17, p. 71–77] or [34, p. 67, 77]

[4] The fact that θ is purely imaginary reflects the fact that the connection is compatible with the Hermitian metric. Indeed, $h(r,r) = 1$, which gives $0 = h(Dr, r) + h(r, Dr) \Leftrightarrow 0 = \mathrm{Re}(h(r, \theta r)) = \mathrm{Re}(\theta h(r,r)) = \mathrm{Re}(\theta)$. One requires that $\Theta = d\theta = i\omega/\hbar \Leftrightarrow d\eta = \omega$.

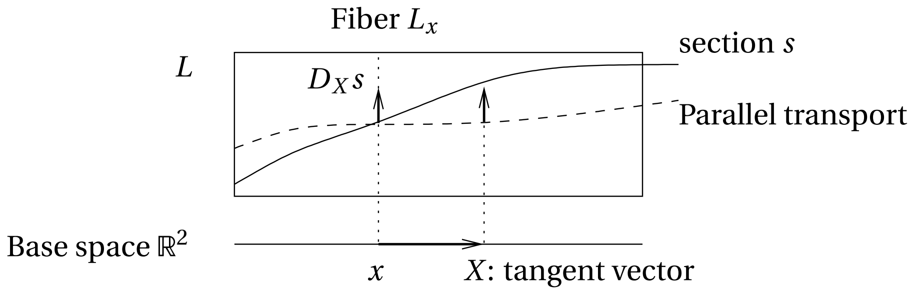

FIGURE 2. The covariant derivative of a section s with respect to a tangent vector X, is $D_X s \in L_x$ and characterizes the infinitesimal departure of the section s from the parallel transport in the direction of X.

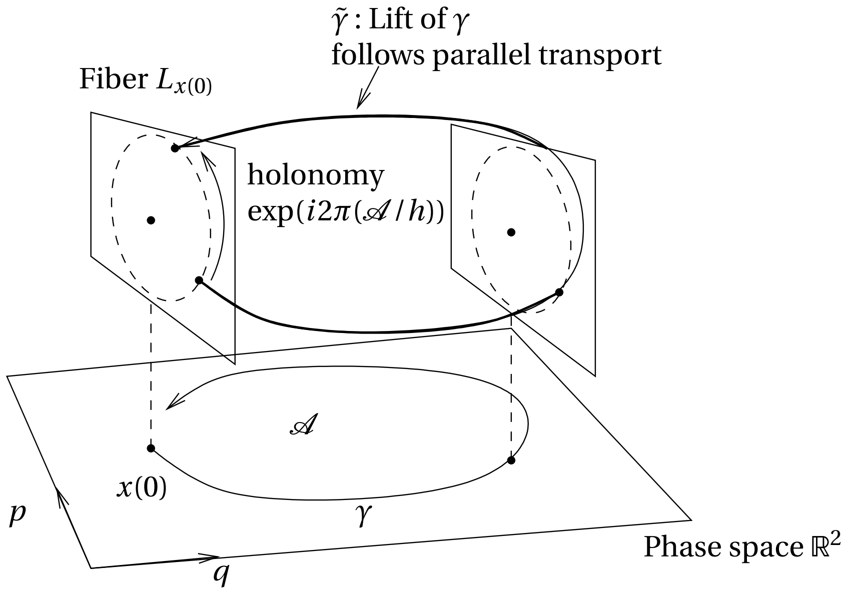

FIGURE 3. A closed path γ is lifted in the line bundle following the parallel transport. The holonomy of the lifted path $\tilde{\gamma}$ is equal to the phase $\exp(i2\pi\mathscr{A}/h)$ where $\mathscr{A}$ is the area of the closed path also called the classical action of γ. $\mathscr{A}/h$ is called the number of quanta enclosed in γ. These phases are responsible for interference effects in quantum dynamics (wave dynamics).

such that $d\eta = \omega$. In order to simplify some expressions below, the section r is chosen such that[5]

$$\eta := \frac{1}{2}\left(q\,dp - p\,dq\right), \tag{9}$$

With respect to the reference section r, any section $s \in A^0(L)$ is represented by a complex-valued function ψ on $\mathbb{R}^2$ defined by:

$$s(x) = \psi(x)\, r(x), \qquad \psi(x) \in \mathbb{C}, \quad x \in \mathbb{R}^2$$

and $|s(x)| = \sqrt{h_x(s(x), s(x))} = |\psi(x)| \sqrt{h_x(r(x), r(x))} = |\psi(x)|$.

The space of interest for us, called the *prequantum Hilbert space*, denoted by $L^2(L)$, is the space of sections of L with finite L^2 norm:

$$\begin{aligned} L^2(L) &:= \left\{ s,\ \|s\|^2 = \int_{\mathbb{R}^2} dx\, |s(x)|^2 < \infty \right\} \\ &\cong L^2\left(\mathbb{R}^2\right) = \left\{ \psi, \int_{\mathbb{R}^2} dx\, |\psi(x)|^2 < \infty \right\}, \text{ with } s = \psi r, \end{aligned} \tag{10}$$

where the last *unitary isomorphism* is obtained by the identification $s \equiv \psi$ given by Eq.(10). We will use this unitary isomorphism all along the paper and work most of time with the space $L^2\left(\mathbb{R}^2\right)$.

REMARK. If $\|s\| = 1$, the function $\mathrm{Hus}_s(x) = |s(x)|^2 = |\psi(x)|^2$ is a probability measure on phase space $\mathbb{R}^2$ (i.e., $\int_{\mathbb{R}^2} \mathrm{Hus}_s(x)\, dx = \|s\|^2 = 1$), and is called *Husimi distribution* of the section s in the physics literature [6, 16].

3.3. **The prequantum operator.** The *prequantum operator of Kostant–Souriau–Kirillov* acts on the Hilbert space $L^2(L)$, Eq. (10), and is defined by

$$\mathbf{P}_H := -i\hbar D_{X_H} + H, \tag{11}$$

where D is the covariant derivative, X_H is the Hamiltonian vector field in Eq.(7), and H denotes multiplication of a section by the function H. If H is a real function and X_H is complete, then $\mathbf{P}_H$ is a self-adjoint operator (see [35, page 162]).

Writing $s = \psi r$ as in Eq.(10), we use Leibniz's rule to write

$$D_{X_H}(s) = D_{X_H}(\psi r) = d\psi(X_H)\, r + D_{X_H}(r) = \left(X_H(\psi) + \frac{i}{\hbar}\eta(X_H)\,\psi \right) r \tag{12}$$

and obtain that

$$\mathbf{P}_H(s) = \left(-i\hbar X_H \psi + \eta(X_H)\,\psi + H\psi\right) r = \left(P_H \psi\right) r$$

so $\mathbf{P}_H$ is isomorphic to the differential operator

$$P_H = -i\hbar X_H + \left(\eta(X_H) + H\right), \tag{13}$$

which acts on $L^2\left(\mathbb{R}^2\right)$. The last two terms in Eq.(11) are the multiplication operator by the function $\eta(X_H) + H = -\frac{1}{2}\left(q\left(\frac{\partial H}{\partial q}\right) + p\left(\frac{\partial H}{\partial p}\right)\right) + H$. The role of the differential operator P_H (respect. $\mathbf{P}_H$) is to generate the "prequantum dynamics",

[5] The geometric meaning of η, also called the symmetric gauge, is that the reference section r follows the parallel transport along radial lines issued from the origin $x = 0$. Indeed $\eta = \frac{1}{2}\left(q\,dp - p\,dq\right) \equiv \frac{1}{2} x \wedge dx$, so if $X \in T_x\mathbb{R}^2 \equiv \mathbb{R}^2$ is such that $x \wedge X = 0$, then $D_X r = \frac{i}{\hbar}\eta(X)\, r = 0$.

i.e., the evolution of $\psi(t) \in L^2(\mathbb{R}^2)$ (respect. $s(t) \in L^2(L)$) by the "*prequantum Schrödinger equation*"

$$\frac{d\psi(t)}{dt} = -\frac{i}{\hbar} P_H \psi(t), \qquad \frac{ds(t)}{dt} = -\frac{i}{\hbar} \mathbf{P}_H s(t) \tag{14}$$

whose solution is $\psi(t) = \tilde{U}_t \psi(0)$ (respect. $s(t) = \tilde{\mathbf{U}}_t s(0)$), with the unitary operator in $L^2(\mathbb{R}^2)$:

$$\tilde{U}_t := \exp\left(-\frac{i}{\hbar} P_H t\right), \qquad \tilde{\mathbf{U}}_t := \exp\left(-\frac{i}{\hbar} \mathbf{P}_H t\right) \tag{15}$$

It can be shown that the term H in Eq.(11) is necessary so that $\tilde{\mathbf{U}}_t$ preserves the connection (see [35, page 163]).

The Geometric and Dynamical phases. In this paragraph we interpret the terms which enter in the expression of P_H, Eq.(13). The reader can skip it and go directly to Section 3.4. According to Eq.(8), the first term $(-i\hbar X_H)$ is just responsible for the transport of the function ψ along the Hamiltonian flow. The second term $\eta(X_H)$ comes from the covariant derivative in Eq.(12), and without the third term H, it would mean that the transported section $s(t)$ follows parallel transport over each trajectory $x(t)$. The third term H gives a *departure from the parallel transport.* The last two terms together change the value of the function $\psi(t)$ at point $x = (q, p)$ by the amount:

$$\left(\frac{d\psi}{dt}\right)_{(2)} \equiv \left(-\frac{i}{\hbar}\right)\left(\eta(X_H) + H\right)\psi \equiv \left(-\frac{i}{\hbar}\right)\left(\frac{1}{2}\left(q\frac{dp}{dt} - p\frac{dq}{dt}\right) + H\right)\psi.$$

We recognize the infinitesimal action of the trajectory, see [2]. As it is purely imaginary, it changes the phase of the function $\psi(t)$. The first term related to the parallel transport over the trajectory is called the "geometric phase" in physics literature, whereas the second term which depends explicitly on H is called the "dynamical phase"[29].

In order to be more precise, let $x(t) = \phi_t(x(0))$, $t \in \mathbb{R}$, be a trajectory on base space $\mathbb{R}^2$, and $p(0) \in L_{x(0)}$ a point in the fiber over the point $x(0)$. Let us denote $p_\parallel(t)$ the lifted path over $x(t)$ which starts from $p(0)$ and follows parallel transport. Then the prequantum dynamics is the unique lifted path over $x(t)$ given by $p(t) = e^{\frac{-i}{\hbar}\int_0^t H(x(s))ds} p_\parallel(t)$, i.e., with a departure from the parallel transport given by the dynamical phase. From that point of view, prequantum dynamics is a flow in the fiber bundle L, which will be denoted by $p(t) = \tilde{\phi}_t p(0)$. The unitary operator $\tilde{\mathbf{U}}_t$ defined in Eq.(15), can be expressed by $\left(\tilde{\mathbf{U}}_t s\right)(x(t)) = \tilde{\phi}_t(s(x(0)))$.

If $p(0) = r_{x(0)}$, then $p(t)$ is explicitly given with respect to the reference section $r_{x(t)} \in L_{x(t)}$ by:

$$p(t) = e^{-\frac{i}{\hbar}\int_\gamma dF} r_{x(t)} \tag{16}$$

where $\gamma\colon x(0) \to x(t)$ is the classical trajectory on the phase space $\mathbb{R}^2$ and dF is the one-form on the extended phase space $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$:

$$dF = \left(\eta(X_H) + H\right)dt = \frac{1}{2}\left(q\,dp - p\,dq\right) + H\,dt \tag{17}$$

which is the sum of the geometrical phase plus the dynamical phase. See Figure 4. In other words, the solution $\psi(t)$ of Eq.(14), is given in terms of the classical flow by:

$$\left(\tilde{U}_t\psi\right)(x(t)) = e^{-\frac{i}{\hbar}\int_\gamma dF}\psi(x(0)). \tag{18}$$

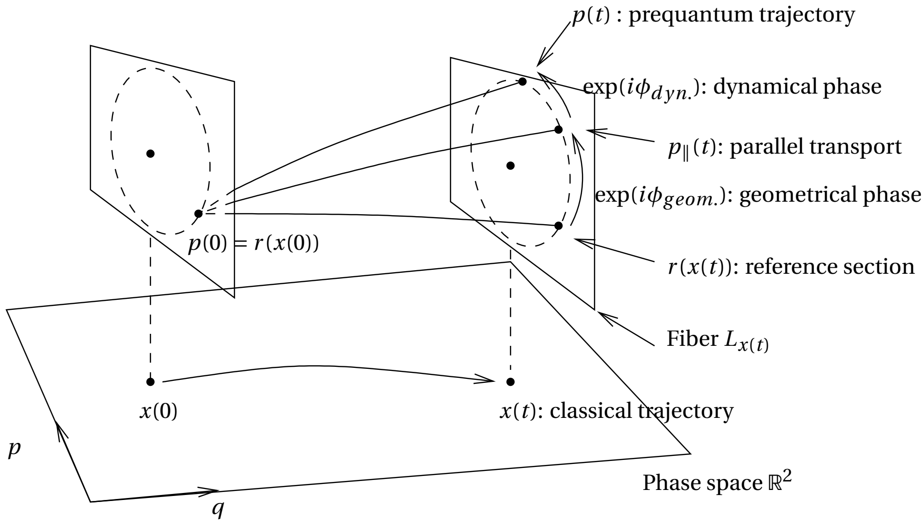

FIGURE 4. The prequantum dynamics is a lift of the classical dynamics $x(t)$ in phase space, where the lifted path $p(t)$ follows the parallel transport $p_{\parallel}(t)$ with an additional phase $\phi_{dyn} = \frac{1}{\hbar}\int_0^t H(x(s))\,ds$ called the dynamical phase. With respect to the reference section $r(x(t))$, the parallel transport is given by $p_{\parallel}(x(t)) = e^{i\phi_{geom}}r(x(t))$, where $\phi_{geom} = -\frac{1}{2\hbar}\int\left(q\,dp - p\,dq\right)$ is called the geometrical phase.

Correspondence principle. An important interest in the prequantum operators comes from the following proposition (see [35, page 157]).

PROPOSITION 5. *If $f,g \in C^\infty\left(\mathbb{R}^2\right)$ then*

$$[P_f, P_g] = i\hbar P_{\{f,g\}}. \tag{19}$$

In other words, $f \in \left(C^\infty\left(\mathbb{R}^2\right), \{\cdot,\cdot\}\right) \mapsto P_f \in \left(L\left(\mathcal{H}_p\right), [\cdot,\cdot]\right)$ is a *Lie algebra homomorphism.* In particular, it gives the following basic commutation relation of quantum mechanics between position and momentum, called the *"correspondence principle"*[6]: $[P_q, P_p] = i\hbar P_{\{q,p\}} = i\hbar P_1 = i\hbar \hat{\mathrm{Id}}$.

Proof. If $f, g \in C^\infty(M)$ then $[X_f, X_g] = -X_{\{f,g\}}$. If β is a one-form, and X, Y two vector fields then $X(\beta(Y)) - Y(\beta(X)) = d\beta(X,Y) + \beta([X,Y])$ (see e.g., [36, p. 207]). With these two relations we deduce:

$$\begin{aligned}
[P_f, P_g] &= (-i\hbar)^2 [X_f, X_g] - i\hbar [X_f, \eta(X_g) + g] - i\hbar [\eta(X_f) + f, X_g] \\
&= \hbar^2 X_{\{f,g\}} - i\hbar X_f(\eta(X_g)) + i\hbar\{f,g\} + i\hbar X_g(\eta(X_f)) - i\hbar\{g,f\} \\
&= \hbar^2 X_{\{f,g\}} + 2i\hbar\{f,g\} - i\hbar(d\eta(X_f, X_g) + \eta([X_f, X_g])) \\
&= \hbar^2 X_{\{f,g\}} + 2i\hbar\{f,g\} - i\hbar\omega(X_f, X_g) - i\hbar\eta([X_f, X_g]) \\
&= i\hbar\left(-i\hbar X_{\{f,g\}} + 2\{f,g\} - \{f,g\} + \eta\left(X_{\{f,g\}}\right)\right) = i\hbar P_{\{f,g\}}
\end{aligned}$$

□

3.4. **Canonical basis of operators in $L^2(\mathbb{R}^2)$.** In this section we show that the Hilbert space $L^2(\mathbb{R}^2)$ (of prequantum sections, Eq.(10)) is an irreducible space for a convenient Weyl–Heisenberg algebra of operators constructed with the covariant derivative. This will give a decomposition of the space $L^2(\mathbb{R}^2)$ useful for later use.

We have chosen coordinates $(q,p) \in \mathbb{R}^2$ on phase space. Consider the covariant derivative operators respectively in the directions $\partial/\partial p$ and $\partial/\partial q$. We denote them by:

$$\hat{Q}_2 := -i\hbar D_{\frac{\partial}{\partial p}}, \qquad \hat{P}_2 := -i\hbar D_{\frac{\partial}{\partial q}}$$

With the unitary isomorphism Eq.(10), we identify these operators with operators in $L^2(\mathbb{R}^2)$. Using Eq.(12), and Eq.(9), this gives $\hat{Q}_2 \equiv \left(-i\hbar\frac{\partial}{\partial p} + \eta\left(\frac{\partial}{\partial p}\right)\right) = \left(-i\hbar\frac{\partial}{\partial p} + \frac{1}{2}q\right)$. Similarly for $\hat{P}_2$. We obtain:

$$\hat{Q}_2 \equiv \left(-i\hbar\frac{\partial}{\partial p}\right) + \frac{1}{2}q, \qquad \hat{P}_2 \equiv \left(-i\hbar\frac{\partial}{\partial q}\right) - \frac{1}{2}p. \tag{20}$$

Using the well-known commutation relation $\left[q, \left(-i\hbar\frac{\partial}{\partial q}\right)\right] = i\hbar\hat{\mathrm{Id}}$ (similarly with p), we deduce that $(\hat{Q}_2, \hat{P}_2, \hat{\mathrm{Id}})$ form a Weyl–Heisenberg algebra:

$$[\hat{Q}_2, \hat{P}_2] = \hat{\mathrm{Id}}.$$

In order to complete this algebra, define

$$\hat{Q}_1 := \mathbf{P}_q, \qquad \hat{P}_1 := \mathbf{P}_p \tag{21}$$

[6]Note that $P_{f=1} = \hat{\mathrm{Id}}$ is obtained thanks to the third term in (13). $f \to (-X_f)$ is also a Lie algebra homomorphism (a more simple one), but $X_{f=1} = 0 \neq \hat{\mathrm{Id}}$.

to be the prequantum operator for functions q and p respectively. As before, the corresponding self-adjoint operators in $L^2\left(\mathbb{R}^2\right)$ are explicitly obtained from Eq.(13):

$$\hat{Q}_1 \equiv -\left(-i\hbar\frac{\partial}{\partial p}\right) + \frac{1}{2}q, \qquad \hat{P}_1 \equiv \left(-i\hbar\frac{\partial}{\partial q}\right) + \frac{1}{2}p. \tag{22}$$

We directly check (or use Eq.(19)) that $\left[\hat{Q}_1, \hat{P}_1\right] = i\hbar\hat{\mathrm{Id}}$. But also

$$\left[\hat{Q}_i, \hat{P}_j\right] = i\hbar\hat{\mathrm{Id}}\delta_{ij}, \qquad \left[\hat{Q}_i, \hat{Q}_j\right] = 0, \quad \left[\hat{P}_i, \hat{P}_j\right] = 0.$$

So $\left(\hat{Q}_1, \hat{P}_1, \hat{Q}_2, \hat{P}_2, \hat{\mathrm{Id}}\right)$ form a basis of the Weyl–Heisenberg algebra with "two degrees of freedom" in $L^2\left(\mathbb{R}^2\right)$. In fact, we have obtained a new basis, from the original basis $(q, \left(-i\hbar\frac{\partial}{\partial q}\right), p, \left(-i\hbar\frac{\partial}{\partial p}\right), \hat{\mathrm{Id}})$ by a metaplectic transformation [16]. We summarize:

PROPOSITION 6. *The space $L^2\left(\mathbb{R}^2\right)$ is an irreducible representation space for the Weyl–Heisenberg algebra of operators $\left(\hat{Q}_1, \hat{P}_1, \hat{Q}_2, \hat{P}_2, \hat{Id}\right)$. As a consequence we have a unitary isomorphism:*

$$L^2\left(\mathbb{R}^2\right) \cong L^2\left(\mathbb{R}_{(1)}\right) \otimes L^2\left(\mathbb{R}_{(2)}\right) \tag{23}$$

where $L^2\left(\mathbb{R}_{(1)}\right)$ (resp. $L^2\left(\mathbb{R}_{(2)}\right)$) denotes the Hilbert space of L^2 functions of one variable $f\left(Q_1\right), Q_1 \in \mathbb{R}$ (resp. $f\left(Q_2\right), Q_2 \in \mathbb{R}$), in which $\hat{Q}_1$ acts as $\left(\hat{Q}_1 f\right)\left(Q_1\right) = Q_1 f\left(Q_1\right)$ and $\left(\hat{P}_1 f\right)\left(Q_1\right) = -i\hbar\frac{df}{dQ_1}\left(Q_1\right)$ (resp. for $f\left(Q_2\right)$). In other words, the decomposition Eq.(23), means that $\psi\left(q,p\right) \in L^2\left(\mathbb{R}^2\right)$ is transformed into a function $\Psi\left(Q_1, Q_2\right) \in L^2\left(\mathbb{R}_{(1)}\right) \otimes L^2\left(\mathbb{R}_{(2)}\right)$, see Eq.(37) below for an explicit formula.

We will see that the decomposition of the prequantum Hilbert space Eq.(23) plays a major role for our understanding of the prequantum dynamics.

3.5. **Case of a linear Hamiltonian function.** Consider the special case where H is a linear function on $\mathbb{R}^2$, with $v = \left(v_q, v_p\right) \in \mathbb{R}^2$:

$$H\left(q,p\right) = v_q p - v_p q \tag{24}$$

then $X_H = v_q\frac{\partial}{\partial q} + v_p\frac{\partial}{\partial p}$. The Hamiltonian flow after time 1 is a translation on $\mathbb{R}^2$ by the vector v, and we denote it by T_v:

$$T_v(x) := x + v. \tag{25}$$

From definition Eq.(21) and linearity of Eq.(11), we deduce that

$$P_H = v_q\hat{P}_1 - v_p\hat{Q}_1. \tag{26}$$

The unitary operator generated by P_H after time 1 will be written:

$$\tilde{T}_v := \exp\left(-\frac{i}{\hbar}P_H\right) = \exp\left(-\frac{i}{\hbar}\left(v_q\hat{P}_1 - v_p\hat{Q}_1\right)\right). \tag{27}$$

It is the prequantum lift of the classical translation Eq.(25).

REMARK. The prequantum operator P_H, depends only on the operators $\hat{Q}_1, \hat{P}_1$ and not on $\hat{Q}_2, \hat{P}_2$. Therefore with respect to the decomposition Eq.(23), operators P_H and $\tilde{T}_v$ act trivially[7] in the space $L^2\left(\mathbb{R}_{(2)}\right)$, i.e., can be written as

$$\tilde{T}_v = \tilde{T}_v^{(1)} \otimes \hat{\mathrm{Id}}^{(2)} \tag{28}$$

The operator P_H restricted to the space $L^2\left(\mathbb{R}_{(1)}\right)$ is identical to the Weyl-quantized operator $\mathrm{Op}_{\mathrm{Weyl}}\left(H\left(Q_1, P_1\right)\right) = v_q \hat{P}_1 - v_p \hat{Q}_1$, see [16].

PROPOSITION 7. *The prequantum translation operators satisfy the algebraic relation of the* Weyl–Heisenberg group*: for any $v, v' \in \mathbb{R}^2$,*

$$\tilde{T}_v \tilde{T}_{v'} = e^{-iS/\hbar} \tilde{T}_{v+v'}, \tag{29}$$

with $S = \frac{1}{2} v \wedge v' = \frac{1}{2}\left(v_1 v_2' - v_2 v_1'\right)$.

Proof. There are two ways to see that. The first one (more algebraic) is to use the explicit expression Eq.(26) of P_H in terms of the operators $\hat{Q}_1, \hat{P}_1$ and use $\left[\hat{Q}_1, \hat{P}_1\right] = i\hbar \hat{Id}$ (Weyl–Heisenberg algebra) as well as the Baker–Campbell–Hausdorff relation $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$ for any operators which satisfy $[A,B] = C.\hat{\mathrm{Id}}$, $C \in \mathbb{C}$.

The second (more geometrical) one is to consider the initial point $p = r_x \in L_x$ in the fiber over $x \in \mathbb{R}^2$. We want to compute the phase F obtained after a lift over the closed triangular path $x \to \left(x + v'\right) \to \left(\left(x + v'\right) + v\right) \to x$ in the plane $\mathbb{R}^2$:

$$\tilde{T}_{v+v'}^{-1} \tilde{T}_v \tilde{T}_{v'}\left(p\right) = e^{-\frac{i}{\hbar}F} p.$$

For a unique translation of v, starting at x, Eq.(24) and Eq.(17) give the phase

$$F_{\mathrm{trans.}} = \frac{1}{2} v \wedge x. \tag{30}$$

So for the closed triangular path:

$$F = \frac{1}{2}\left(v' \wedge x\right) + \frac{1}{2}\left(v \wedge \left(x + v'\right)\right) + \frac{1}{2}\left(-\left(v + v'\right) \wedge \left(x + v + v'\right)\right) = \frac{1}{2} v \wedge v'.$$

□

3.6. **The quantum Hilbert space.** The usual Hilbert space of quantum mechanics which corresponds to the phase space $\left(q, p\right) \in \mathbb{R}^2$, is the space of functions $\psi\left(q\right) \in L^2\left(\mathbb{R}\right)$ [26]. The prequantum Hilbert space $L^2\left(\mathbb{R}^2\right)$, Eq.(10), is obviously too large. The usual procedure to construct the quantum Hilbert space from the prequantum one in geometric quantization is to add a complex structure on the phase space $\mathbb{R}^2$, called a complex polarization, which induces a holomorphic structure on the line bundle L, and then to consider the subspace of antiholomorphic sections of L, (see [35, 5]). We will show below that this indeed gives the “standard” Hilbert space of quantum wave functions $\psi\left(q\right)$.

[7]We will see in Section 3.6, that this is related to the fact that translations on $\mathbb{R}^2$ preserve the complex structure of $\mathbb{C} \equiv \mathbb{R}^2$.

We consider the canonical complex structure J on phase space $(q,p)\in\mathbb{R}^2$ defined by $J\left(\frac{\partial}{\partial q}\right)=\frac{\partial}{\partial p}$. Then $x=(q,p)\in\mathbb{R}^2$ is identified with $z\in\mathbb{C}$ by[8]:

$$z=\frac{1}{\sqrt{2\hbar}}(q+ip). \tag{31}$$

The *quantum Hilbert space* is defined to be the space of *antiholomorphic sections*:

$$\mathscr{H}:=\left\{\text{section } s\in L^2(L)\ /\ D_{X^+}s=0,\ \text{for all } X^+\in T^{1,0}(\mathbb{C})\right\}, \tag{32}$$

where the space of tangent vectors of type $(1,0)$ (holomorphic tangent vectors) at point $x\in\mathbb{R}^2$ is spanned by $X^+=\frac{\partial}{\partial q}-i\frac{\partial}{\partial p}=\sqrt{\frac{2}{\hbar}}\frac{\partial}{\partial z}$.

Characterization of the quantum Hilbert space $\mathscr{H}$. Let us define the usual "annihilation" and "creation" operators $a_2,a_2^\dagger$ by:

$$a_2:=\frac{1}{\sqrt{2\hbar}}\left(\hat{Q}_2+i\hat{P}_2\right),\quad a_2^\dagger:=\frac{1}{\sqrt{2\hbar}}\left(\hat{Q}_2-i\hat{P}_2\right)$$

The three operators $\left(a_2,a_2^\dagger,\hat{Id}\right)$, with the relation $\left[a_2,a_2^\dagger\right]=\hat{Id}$, form a Cartan basis for the Weyl–Heisenberg algebra of operators acting in the space $L^2\left(\mathbb{R}_{(2)}\right)$, which enters in the decomposition Eq.(23). Note also that the introduction of this basis of operators is natural after the choice of the complex structure Eq.(31). Similarly the operators $\left(a_1,a_1^\dagger\right)$ can be constructed with respect to the space $L^2\left(\mathbb{R}_{(1)}\right)$, but we will not need them. We recall that there is an orthonormal basis[9] of $L^2\left(\mathbb{R}_{(2)}\right)$ related to the "Harmonic Oscillator", with vectors denoted by $|n_2\rangle\in L^2\left(\mathbb{R}_{(2)}\right)$, $n_2\in\mathbb{N}$ and defined by

$$|0_2\rangle\in\mathrm{Ker}(a_2)\quad\text{(one-dimensional space)}$$

$$a_2|n_2\rangle=\sqrt{n_2}|n_2-1\rangle,\qquad a_2^\dagger|n_2\rangle=\sqrt{n_2+1}|n_2+1\rangle,\qquad n_2\in\mathbb{N} \tag{33}$$

$$\left(a_2^\dagger a_2\right)|n_2\rangle=n_2|n_2\rangle$$

PROPOSITION 8. *With the unitary isomorphism Eq.(10), a section $s\in\mathscr{H}$ (Eq. (32)) is identified with a function $\psi\in L^2\left(\mathbb{R}^2\right)$ such that $a_2\psi=0$, but also with the Bargmann space of antiholomorphic functions with weight $e^{-z\bar{z}/2}$* [6, 16]*:*

$$\mathscr{H}\cong\left\{\psi\in L^2\left(\mathbb{R}^2\right),\quad \psi\in Ker(a_2)\right\} \tag{34}$$

$$\cong\left\{\psi\in L^2\left(\mathbb{R}^2\right)/\ \psi(q,p)=e^{-z\bar{z}/2}\varphi(\bar{z}),\ \varphi(\bar{z})\ \textit{antiholomorphic}\right\}:\ \textit{Bargmann space}$$

---

[8] The factor $1/\sqrt{2\hbar}$ is just a matter of choice.

[9] This orthonormal basis has a nice physical meaning: for a free particle in configuration space $\mathbb{R}^2$, with a constant magnetic field $B=(2\pi\hbar)^{-1}\omega$, the Hamiltonian is $\hat{H}=\frac{1}{2}\left(-i\hbar\partial/\partial q-\frac{1}{2}p\right)^2+\frac{1}{2}\left(-i\hbar\partial/\partial p+\frac{1}{2}q\right)^2=\frac{1}{2}\hat{P}_2^2+\frac{1}{2}\hat{Q}_2^2=a_2^\dagger a_2+\frac{1}{2}$, whose eigenspaces are $L^2\left(\mathbb{R}_{(1)}\right)\otimes(\mathbb{C}|n_2\rangle)$ and eigenvalues $n_2+\frac{1}{2}$ called Landau levels.

With Eq.(33) and the unitary isomorphism Eq.(23), we get unitary isomorphisms[10]*:*

$$\mathcal{H} \cong L^2\left(\mathbb{R}_{(1)}\right) \otimes (\mathbb{C}|0_2\rangle) \cong L^2\left(\mathbb{R}_{(1)}\right) \tag{36}$$

where $(\mathbb{C}|0_2\rangle)$ *denotes the one-dimensional space Span*$(|0_2\rangle)$, *and the second isomorphism is related to the choice of a vector*[11] $|0_2\rangle \in Ker(a_2)$.

Proof. If $s = \psi r$, and $X^+ = \frac{\partial}{\partial q} - i\frac{\partial}{\partial p} = \sqrt{\frac{2}{\hbar}}\frac{\partial}{\partial z}$, then

$$-i\hbar D_{X^+} s = -i\hbar D_{\frac{\partial}{\partial q}} s - i(-i\hbar) D_{\frac{\partial}{\partial p}} s = \left(\left(\hat{P}_2 - i\hat{Q}_2\right)\psi\right) r = -i\sqrt{2\hbar}\left(a_2\psi\right) r$$

so $D_{X^+} s = 0 \Leftrightarrow a_2\psi = 0 \Leftrightarrow \psi \in \mathrm{Ker}(a_2)$. We also write $D_{X^+} s = -i\sqrt{2\hbar}\left(\frac{\partial\psi}{\partial z} + \frac{1}{2}\overline{z}\psi\right) r$, and $D_{X^+} s = 0 \Leftrightarrow \frac{\partial\psi}{\partial z} = -\frac{1}{2}\overline{z}\psi \Leftrightarrow \psi = e^{-z\overline{z}/2}\varphi\left(\overline{z}\right)$, with an antiholomorphic function $\varphi\left(\overline{z}\right)$. □

Correspondence with the usual Quantum Hilbert space $L^2(\mathbb{R})$. We can make the connection between the space $\mathcal{H}$ and the usual space of quantum wave functions more explicit. In "standard quantum mechanics" also called "position representation", the quantum Hilbert space associated with the phase space $(q, p) \in \mathbb{R}^2 \equiv T^*\mathbb{R}$ consists of wave functions $\varphi(q) \in L^2(\mathbb{R})$. In this section, we show that this space $L^2(\mathbb{R})$ coincides with the space $L^2\left(\mathbb{R}_{(1)}\right)$ used in Eq.(36). For that purpose we have to show that the map $\varphi \in L^2\left(\mathbb{R}_{(1)}\right) \to \psi \in \mathcal{H} \subset L^2\left(\mathbb{R}^2\right)$ coincides with the *Bargmann Transform* [6] of φ.

PROPOSITION 9. *If* $\varphi \in L^2\left(\mathbb{R}_{(1)}\right)$, *the isomorphism* $\mathcal{H} \cong L^2\left(\mathbb{R}_{(1)}\right)$ *in Eq.(36) is given by* $\varphi \in L^2\left(\mathbb{R}_{(1)}\right) \mapsto \psi \in \mathcal{H} \subset L^2\left(\mathbb{R}^2\right)$, *with*

$$\psi(q,p) = \frac{1}{(\pi\hbar)^{1/4}} e^{iqp/(2\hbar)} \int_{\mathbb{R}} dQ_1 \varphi(Q_1)\, e^{-iQ_1 p/\hbar} e^{-(Q_1-q)^2/(2\hbar)}.$$

We recognize the Bargmann transform [6] *of* φ.

Proof. From Eq.(22),Eq.(20), we have an explicit relation between the representation of a function ψ in (q,p) variables or (Q_1, Q_2) variables:

$$\psi(q,p) = \int dQ_1 dQ_2 \langle qp|Q_1Q_2\rangle \Psi(Q_1,Q_2) \tag{37}$$

---

[10]We can introduce an orthogonal projector in the prequantum space onto the quantum space, called the *Toeplitz projector:*

$$\hat{\Pi}: L^2(L) \to \mathcal{H}.$$

With the identifications given by the unitary isomorphisms Eq.(23) and Eq.(36), $\hat{\Pi}$ is the projector in the space $L^2\left(\mathbb{R}_{(1)}\right) \otimes L^2\left(\mathbb{R}_{(2)}\right)$ onto the linear subspace $L^2\left(\mathbb{R}_{(1)}\right) \otimes (\mathbb{C}|0_2\rangle)$, and can be written

$$\hat{\Pi} \equiv \hat{\mathrm{Id}}_{(1)} \otimes (|0_2\rangle\langle 0_2|) \tag{35}$$

This projector is used in geometric quantization to defined Toeplitz quantization rules, see [5].

[11]In geometrical terms, the complex structure J is associated to the one-dimensional space $\mathbb{C}|0_2\rangle$. More generally, the space of all possible homogeneous complex structures on $\mathbb{R}^2$ (which is the hyperbolic half plane $\mathbb{H}$) is identified with the so called squeezed coherent states, which are the orbit of the space $(\mathbb{C}|0_2\rangle)$ under the action of the metaplectic group $\mathrm{Mp}(2,\mathbb{R})$ (generated by quadratic functions of $\hat{Q}_2, \hat{P}_2$).

with

$$\langle qp|Q_1Q_2\rangle := \delta\left(Q_1+Q_2-q\right)e^{i\frac{1}{2}(Q_2-Q_1)p/\hbar}$$

(which comes from $\langle p_0|\xi_p\rangle = e^{i\xi_p p_0/\hbar}$, $\langle q_0|q\rangle = \delta\left(q_0-q\right)$ and $q = Q_1+Q_2$, $\xi_p = \frac{1}{2}\left(Q_2-Q_1\right)$). Now if $\psi \in \mathscr{H}$, then from Eq.(36), $\Psi\left(Q_1,Q_2\right) = \varphi\left(Q_1\right)\varphi_0\left(Q_2\right)$, where $\varphi_0\left(Q_2\right) = \langle Q_2|0_2\rangle = (\pi\hbar)^{-1/4}\exp\left(-Q_2^2/(2\hbar)\right)$. This gives

$$\begin{aligned}\psi\left(q,p\right) &= \int dQ_1 dQ_2 \delta\left(Q_1+Q_2-q\right)e^{i\frac{1}{2}(Q_2-Q_1)p/\hbar}\varphi\left(Q_1\right)\frac{1}{(\pi\hbar)^{1/4}}\exp\left(-\frac{Q_2^2}{2\hbar}\right)\\ &= \frac{1}{(\pi\hbar)^{1/4}}e^{iqp/(2\hbar)}\int dQ_1 e^{-iQ_1p/\hbar}\varphi\left(Q_1\right)\exp\left(-\frac{\left(q-Q_1\right)^2}{2\hbar}\right)\end{aligned}$$

□

3.7. **The case of a quadratic Hamiltonian function.** We consider now the special case where the Hamiltonian $H\left(q,p\right)$ is a quadratic function:

$$H\left(q,p\right) = \frac{1}{2}\alpha q^2 + \frac{1}{2}\beta p^2 + \gamma qp, \qquad \alpha,\beta,\gamma \in \mathbb{R}. \tag{38}$$

Let us denote by $M \in \mathrm{SL}(2,\mathbb{R})$ the flow on $\mathbb{R}^2$ generated by the quadratic Hamiltonian H after time 1 (M is a linear symplectic map).

PROPOSITION 10. *With respect to the decomposition Eq.(23), the prequantum operator is*

$$P_H = P_H^{(1)} \otimes Id_{(2)} + Id_{(1)} \otimes P_H^{(2)} \tag{39}$$

with

$$P_H^{(1)} := \frac{1}{2}\alpha \hat{Q}_1^2 + \frac{1}{2}\beta\hat{P}_1^2 + \gamma\left(\frac{1}{2}\hat{Q}_1\hat{P}_1 + \frac{1}{2}\hat{P}_1\hat{Q}_1\right) = Op^{(1)}_{Weyl}(H), \tag{40}$$

which acts on $L^2\left(\mathbb{R}_{(1)}\right)$*, and*

$$P_H^{(2)} := -\frac{1}{2}\alpha \hat{Q}_2^2 - \frac{1}{2}\beta\hat{P}_2^2 + \gamma\left(\frac{1}{2}\hat{Q}_2\hat{P}_2 + \frac{1}{2}\hat{P}_2\hat{Q}_2\right) = Op^{(2)}_{Weyl}\left(H_{(2)}\right),$$

which acts on $L^2\left(\mathbb{R}_{(2)}\right)$*. Here,* $Op^{(i)}_{Weyl}$*,* $i = 1,2$*, means usual Weyl (symmetric) quantization of quadratic symbols, with, respectively,* (Q_1,P_1) *or* (Q_2,P_2)*. The function*

$$H_{(2)}\left(q,p\right) := -\frac{1}{2}\alpha q^2 - \frac{1}{2}\beta p^2 + \gamma qp \tag{41}$$

can be written as $H_{(2)} = -H\circ\mathscr{T}$ *where* $\mathscr{T}\left(q,p\right) = \left(q,-p\right)$ *is the "time reversal" operation.*

Proof. The Hamiltonian vector field is $X_H = \left(\gamma q + \beta p\right)\frac{\partial}{\partial q} - \left(\alpha q + \gamma p\right)\frac{\partial}{\partial p}$. We compute then

$$\eta\left(X_H\right) + H = 0 \tag{42}$$

so $P_H = -i\hbar X_H = \left(\gamma q + \beta p\right)\left(-i\hbar\frac{\partial}{\partial q}\right) - \left(\alpha q + \gamma p\right)\left(-i\hbar\frac{\partial}{\partial p}\right)$. Note that this means that the prequantum transport by P_H is equivalent to the Hamiltonian transport

Eq.(8). Using Eq.(20) and Eq.(22), we deduce the expression of P_H in terms of the operators $\left(\hat{Q}_i, \hat{P}_i\right)$. □

REMARK.
- The separation of terms in Eq.(39), has the following direct consequence on the prequantum dynamics. Let

$$\tilde{M}_{(1),t} := \exp\left(-\frac{i}{\hbar} P_H^{(1)} t\right), \quad \tilde{M}_{(2),t} := \exp\left(-\frac{i}{\hbar} P_H^{(2)} t\right)$$

  be the unitary operators acting on $L^2\left(\mathbb{R}_{(1)}\right)$ and $L^2\left(\mathbb{R}_{(2)}\right)$ respectively, and generated by $P_H^{(1)}$ and $P_H^{(2)}$ respectively. Then the total unitary operator in $L^2\left(\mathbb{R}^2\right)$ (the prequantum propagator) decomposes as a tensor product:

$$\tilde{M}_t := \exp\left(-\frac{i}{\hbar} P_H t\right) = \tilde{M}_{(1),t} \otimes \tilde{M}_{(2),t}. \tag{43}$$

  We will see that this tensor product is the main phenomenon which explains that the spectrum of prequantum resonances is a product of two spectra in Eq.(2).
- Note that the prequantum evolution does not preserve the quantum Hilbert space $\mathcal{H} \cong L^2\left(\mathbb{R}_{(1)}\right) \otimes (\mathbb{C}|0_2\rangle)$, except if $|0_2\rangle$ is an eigenvector of $P_H^{(2)}$, i.e., if $H = \frac{1}{2}\alpha\left(q^2 + p^2\right)$ is the Harmonic oscillator. The geometrical meaning is that the linear symplectic map $M \in \mathrm{SL}(2,\mathbb{R})$ does not preserve the complex structure J except if $M \in \mathrm{U}(1)$ is a rotation.
- *Spectrum of the prequantum Harmonic oscillator:* With $\alpha = \beta = 1$ and $\gamma = 0$ in Eq.(38), we obtain $H = \frac{1}{2}\left(q^2 + p^2\right)$. From Eq.(39), we observe that P_H is the sum of two “quantum Harmonic oscillators in 1:(-1) resonances”, i.e., $P_H = \frac{1}{2}\left(\hat{Q}_1^2 + \hat{P}_1^2\right) - \frac{1}{2}\left(\hat{Q}_2^2 + \hat{P}_2^2\right)$. We deduce that its spectrum $\sigma(P_H)$ is the set of eigenvalues $\lambda_{n_1,n_2} = \hbar\left(n_1 + \frac{1}{2}\right) - \hbar\left(n_2 + \frac{1}{2}\right) = \hbar(n_1 - n_2)$, with $n_1, n_2 \in \mathbb{N}$. So $\sigma(P_H) = \hbar\mathbb{Z}$, with infinite multiplicity[12].

LEMMA 11. *For any $v \in \mathbb{R}^2$ one trivially has $M T_v = T_{Mv} M$. This conjugation relation persists at the prequantum level:*

$$\tilde{M}\tilde{T}_v = \tilde{T}_{Mv}\tilde{M}, \tag{44}$$

where $\tilde{T}_v$ is defined by Eq.(26), and $\tilde{M} = \exp\left(-\frac{i}{\hbar}P_H\right)$.

Proof. For any point $x \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$, the linear relation $M(x+v) = M(x) + M(v)$ gives $MT_v = T_{Mv}M$. Consider the initial point $p = r_x \in L_x$ in the fiber over $x \in \mathbb{R}^2$. We want to compute the phase F obtained on the lifted path over the piecewise closed path $x = T_v^{-1} M^{-1} T_{Mv} M(x)$, defined by

$$\tilde{T}_v^{-1}\tilde{M}^{-1}\tilde{T}_{Mv}\tilde{M}(p) = e^{-\frac{i}{\hbar}F} p. \tag{45}$$

For a path generated by the quadratic Hamiltonian H, Eq.(17), Eq.(42) gives that the phase is $F = 0$. So only the translations contribute to the phase. From Eq.(30)

[12] More generally it could be interesting to compute the spectrum of a prequantum operator if the classical Hamiltonian flow is integrable.

and Eq.(45), we obtain:

$$F = \frac{1}{2}(Mv \wedge Mx) - \frac{1}{2} v \wedge (x+v) = 0 \tag{46}$$

using also the fact that M preserves area. □

4. LINEAR CAT MAP ON THE TORUS $\mathbb{T}^2$

After the necessary presentation of prequantization on phase space $\mathbb{R}^2$, we can now pass to the quotient $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. In this section we recall the definition of the hyperbolic cat map on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and present its prequantization in the same way its quantization is usually obtained (see e.g., [20, 1, 14]).

We start from a hyperbolic map

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{47}$$

on $\mathbb{R}^2$, i.e., with integer coefficients such that $AD - BC = 1$ and $\mathrm{Tr}(M) = A + D > 2$. A simple example is the "cat map" $M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$[3]. For any $x \in \mathbb{R}^2, n \in \mathbb{Z}^2$, $M(x+n) = M(x) + M(n) \equiv M(x) \pmod 1$ so M induces a map on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ also denoted by M, which is fully chaotic.

4.1. **Prequantum Hilbert space of the torus.**

In this paragraph, we explicitly construct the prequantum Hilbert space $\tilde{\mathcal{H}}_N$ associated to the torus phase space $\mathbb{T}^2$, and the prequantum map $\tilde{M} \in \mathrm{End}(\tilde{\mathcal{H}}_N)$ acting on it (respectively the quantum map $\hat{M} \in \mathrm{End}(\mathcal{H}_N)$ acting on the quantum Hilbert space $\mathcal{H}_N$).

Prequantum and Quantum Hilbert space for the torus $\mathbb{T}^2$ phase space. The integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ is generated by the two vectors $(1,0)$ and $(0,1)$. We consider the corresponding prequantum translation operators $\tilde{T}_1 := \tilde{T}_{(1,0)}$ and $\tilde{T}_2 := \tilde{T}_{(0,1)}$, defined by Eq.(27), which satisfy $\tilde{T}_1\tilde{T}_2 = e^{-i/\hbar}\tilde{T}_2\tilde{T}_1$ as a result of Eq.(29). So for special values of $\hbar$ given by:

$$N = \frac{1}{2\pi\hbar} \in \mathbb{N}^\star,$$

one has the property $[\tilde{T}_1, \tilde{T}_2] = 0$. We assume this relation from now on.

We have seen in Eq.(28) that each operator has a trivial action in the space $L^2(\mathbb{R}_{(2)})$ entering the decomposition Eq.(23). So we will first consider their action in the space $L^2\left(\mathbb{R}_{(1)}\right)$. Let us define the space of "periodic distributions"[13]:

$$\mathscr{H}_{(1),N} := \left\{\psi \in \mathscr{S}'\left(\mathbb{R}_{(1)}\right) \text{ such that } \tilde{T}_1\psi = \psi,\ \tilde{T}_2\psi = \psi\right\}. \tag{48}$$

Characterization of the space $\mathscr{H}_{(1),N}$.

LEMMA 12. *$\dim\mathscr{H}_{(1),N} = N$. An explicit orthonormal basis of $\mathscr{H}_{(1),N}$ is given by distributions $\left(\varphi_n\right)_{n=0\dots N-1}$ made of a Dirac comb:*

$$\varphi_n\left(Q_1\right) = \frac{1}{\sqrt{N}}\sum_{k\in\mathbb{Z}}\delta\left(Q_1 - \left(\frac{n}{N}+k\right)\right), \qquad n = 0,\dots,N-1 \tag{49}$$

Proof. First observe that the operator $\tilde{T}_1 = \tilde{T}_{(1,0)} = \exp\left(-\frac{i}{\hbar}\hat{P}_1\right) = \exp\left(-\frac{\partial}{\partial Q_1}\right)$ on $L^2\left(\mathbb{R}_{(1)}\right)$ translates functions by one unit: $\left(\tilde{T}_1\psi\right)\left(Q_1\right) = \psi\left(Q_1 - 1\right)$, and similarly the operator $\tilde{T}_2 = \tilde{T}_{(0,1)} = \exp\left(-\frac{i}{\hbar}\left(-\hat{Q}_1\right)\right)$ translates the $\hbar$–Fourier Transform by one unit: $\left(\tilde{T}_2\hat{\psi}\right)\left(P_1\right) = \hat{\psi}\left(P_1 - 1\right)$, with $\hat{\psi}\left(P_1\right) := \frac{1}{\sqrt{2\pi\hbar}}\int_{\mathbb{R}}\psi\left(Q_1\right)e^{-iP_1Q_1/\hbar}$. So the space $\mathscr{H}_{(1),N}$ consists of distributions $\psi\left(Q_1\right)$ which are periodic with period one, and such that the Fourier transform is also periodic with period one. As a result $\psi\left(Q_1\right) = \frac{1}{\sqrt{N}}\sum_{n\in\mathbb{Z}}\psi_n\delta\left(Q_1 - nh\right)$ with $h = \frac{1}{N} = 2\pi\hbar$, and with components $\psi_n \in \mathbb{C}$ which satisfy the periodicity relation $\psi_{n+N} = \psi_n$. So there are only N independent components, and $\psi = \sum_{n=0}^{N-1}\psi_n\varphi_n$. □

Similarly to Eq.(48), let us define the *prequantum Hilbert space of the torus* by:

$$\tilde{\mathscr{H}}_N := \left\{\text{sections } s \in \Gamma^\infty(L) \text{ such that } \tilde{T}_1 s = s,\ \tilde{T}_2 s = s,\ \int_{[0,1]^2}|s(x)|^2 < \infty\right\} \tag{50}$$

With the unitary isomorphism Eq.(23), and with Eq.(48), we can write[14 15]:

$$\tilde{\mathscr{H}}_N \equiv \mathscr{H}_{(1),N} \otimes L^2\left(\mathbb{R}_{(2)}\right). \tag{51}$$

[13] We could have given a more general presentation with a decomposition of $L^2\left(\mathbb{R}_{(1)}\right)$ into common eigenspaces of the operators $\tilde{T}_1, \tilde{T}_2$:

$$\begin{aligned} L^2\left(\mathbb{R}_{(1)}\right) &= \int_{[0,2\pi]^2}^{\oplus}\mathscr{H}_{(1),N,\theta}\frac{d^2\theta}{(2\pi)^2}, \\ \mathscr{H}_{(1),N,\theta} &:= \left\{\psi_{(1)} \in \mathscr{S}'\left(\mathbb{R}_{(1)}\right) \text{ such that } \tilde{T}_1\psi_{(1)} = e^{i\theta_1}\psi_{(1)},\ \tilde{T}_2\psi_{(1)} = e^{i\theta_2}\psi_{(1)}\right\} \end{aligned}$$

with $\theta = (\theta_1, \theta_2) \in [0,2\pi]^2$. In this paper, we only consider the space $\mathscr{H}_{(1),N} = \mathscr{H}_{(1),N,\theta=0}$ which is sufficient for our purpose, and avoids more complicated notations. See [14, Section 3.2], where this more general presentation is done.

[14] Note that this isomorphism gives an explicit orthonormal basis of the prequantum Hilbert space $\tilde{\mathscr{H}}_N$ of L^2 sections of the Hermitian line bundle L over $\mathbb{T}^2$, which is not obvious a priori. Namely $\phi_{n,m} = \varphi_n \otimes \psi_m$ where $\varphi_n, n = 1 \to N$, Eq.(49), is an o.n. basis of $\mathscr{H}_{(1),N}$ and $\psi_m, m \in \mathbb{N}$ is an orthonormal basis of $L^2\left(\mathbb{R}_{(2)}\right)$ (for example the eigenstates of the Harmonic oscillator given in Eq.(33)). This basis has in fact a well-known physical meaning: each space $\mathscr{H}_{(1),N} \otimes \left(\mathbb{C}\psi_m\right)$ is the eigenspace for the Hamiltonian of a free particle moving on the torus $\mathbb{T}^2$ with a constant magnetic field $B = N\omega$. The corresponding eigenvalues are called the Landau levels.

[15] The tensor product decomposition Eq.(51) which is an important step in order to obtain Theorem 1 can be considered as a simple (and surely well-known) result of pure representation theory of the Heisenberg group. More precisely, let $H_{\mathbb{R}}$ be the Heisenberg group and $H_{\mathbb{Z}}$ be the integral Heisenberg group. Then Eq.(51) concerns the decomposition of $L^2\left(H_{\mathbb{R}} \setminus H_{\mathbb{Z}}\right)$ under the action of $H_{\mathbb{R}}$ (whose Lie algebra is represented in this paper by the operators $\hat{Q}_2, \hat{P}_2, Id$).

The definition Eq.(50) is a space of sections of $L \to \mathbb{R}^2$ that are periodic with respect to some action of $\mathbb{Z}^2$. The space $\tilde{\mathcal{H}}_N$ can be identified with the space of L^2 sections of a nontrivial line bundle $L \to \mathbb{T}^2$ over the torus, with Chern index N. With respect to the trivialization r the space $\tilde{\mathcal{H}}_N$ consists of quasiperiodic functions:

$$\tilde{\mathcal{H}}_N \equiv \Big\{\psi \text{ s.t. } \psi(x+n) = \psi(x)\, e^{-i2\pi\frac{N}{2}n\wedge x} e^{-i2\pi\frac{N}{2}n_1n_2}, \forall x \in \mathbb{R}^2, \forall n \in \mathbb{Z}^2 \\ \text{and} \int_{[0,1]^2} |\psi(x)|^2\, dx < \infty\Big\}. \tag{52}$$

Proof. $\tilde{T}_n = \tilde{T}_{(n_1,0)+(0,n_2)} = e^{i2\pi\frac{N}{2}n_1n_2}\tilde{T}_{(n_1,0)}\tilde{T}_{(0,n_2)} = e^{i2\pi\frac{N}{2}n_1n_2}\tilde{T}_1^{n_1}\tilde{T}_2^{n_2}$ by Eq.(29). Then with Eq.(50), Eq.(30) and Eq.(18) $s = \psi r \in \tilde{\mathcal{H}}_N \Leftrightarrow \{\tilde{T}_n s = e^{i2\pi\frac{N}{2}n_1n_2} s$ for any $n \in \mathbb{Z}^2\} \Leftrightarrow \psi(x)\, e^{-i2\pi\frac{N}{2}n\wedge x} = e^{i2\pi\frac{N}{2}n_1n_2}\psi(x+n)$. □

In the same manner, let us define the *quantum Hilbert space of the torus* by:

$$\mathcal{H}_N := \{s \in \tilde{\mathcal{H}}_N \text{ such that } s \text{ is antiholomorphic}\}. \tag{53}$$

From Eq.(36) we have:

$$\mathcal{H}_N \equiv \mathcal{H}_{(1),N} \otimes (\mathbb{C}|0_2\rangle) \equiv \mathcal{H}_{(1),N}.$$

Note that there is a "perfect decoupling" between the antiholomorphic condition which concerns the $L^2(\mathbb{R}_{(2)})$ part of the decomposition Eq.(23), and the torus-periodicity which concerns the $L^2(\mathbb{R}_{(1)})$ part.

The prequantum cat map and the quantum cat map. In order to obtain the prequantum map or quantum map corresponding to $M\colon \mathbb{T}^2 \to \mathbb{T}^2$ given in Eq.(47), we have first to describe M as a Hamiltonian flow[16]. The hyperbolic linear map $M\colon \mathbb{R}^2 \to \mathbb{R}^2$, $M \in \mathrm{SL}(2,\mathbb{Z})$, can be realized as a time-1 map of a flow on $\mathbb{R}^2$ phase space generated by a *hyperbolic quadratic Hamiltonian function*:

$$H(q,p) = \frac{1}{2}\alpha q^2 + \frac{1}{2}\beta p^2 + \gamma qp. \tag{54}$$

From Hamiltonian equations $dq(t)/dt = \partial_p H = \gamma q + \beta p$, $dp(t)/dt = -\partial_q H = -\alpha q - \gamma p$, we deduce that the constants $\alpha, \beta, \gamma \in \mathbb{R}$ are obtained by solving $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \exp\begin{pmatrix} \gamma & \beta \\ -\alpha & -\gamma \end{pmatrix}$. The *Lyapunov exponent* is given by $\lambda = \sqrt{\gamma^2 - \alpha\beta} = \log\left(\frac{T+\sqrt{T^2-4}}{2}\right)$, with $T = \mathrm{Tr}(M) = A + D$, and gives the two eigenvalues $e^{\pm\lambda}$ of M.

In Section 3.7, Eq.(43), we have considered such quadratic Hamiltonian functions and obtained that the prequantum map $\tilde{M} = \exp\left(-\frac{i}{\hbar}P_H\right)$ which is a unitary operator acting on $L^2(\mathbb{R}^2) \equiv L^2(\mathbb{R}_{(1)}) \otimes L^2(\mathbb{R}_{(2)})$, decomposes as $\tilde{M} = \tilde{M}_{(1)} \otimes \tilde{M}_{(2)}$.

[16] The reason is essentially that a map itself has not all the information necessary to define the prequantum or quantum map in a unique way. In particular the "classical action" of the trajectories are not defined *a priori*. If the map is obtained from a Poincaré section or a stroboscopic section of a Hamiltonian flow, then there is less arbitrariness to (pre)quantizing it.

LEMMA 13. *If N is even, the prequantum map $\tilde{M}$ in $L^2(\mathbb{R}^2)$ defines in a natural way unitary endomorphisms associated with the torus phase space:*

$$\hat{M}_{(1),N}: \mathscr{H}_{(1),N} \to \mathscr{H}_{(1),N}: \text{ the quantum catmap}$$
$$\tilde{M}_N \equiv \hat{M}_{(1),N} \otimes \hat{M}_{(2)}: \tilde{\mathscr{H}}_N \to \tilde{\mathscr{H}}_N: \text{ the prequantum catmap}$$

Proof. Recall that the passage from the prequantum space $L^2(\mathbb{R}^2) \equiv L^2(\mathbb{R}_{(1)}) \otimes L^2(\mathbb{R}_{(2)})$ to the torus prequantum space concerns only the $L^2(\mathbb{R}_{(1)})$ part. Let us define a projector from the space $L^2(\mathbb{R}_{(1)})$ onto the space $\tilde{\mathscr{H}}_{(1),N}$ by:

$$\tilde{\mathscr{P}}_{(1)} := \sum_{(n_1,n_2)\in\mathbb{Z}^2} \tilde{T}_1^{n_1}\tilde{T}_2^{n_2} = \sum_{(n_1,n_2)\in\mathbb{Z}^2} \tilde{T}_n, \tag{55}$$

(we have used $\tilde{T}_n = \tilde{T}_1^{n_1}\tilde{T}_2^{n_2}$, from Eq.(29), and the hypothesis that N is even). The domain of $\tilde{\mathscr{P}}_{(1)}$ consists of fast decreasing sections. We extend $\tilde{\mathscr{P}}_{(1)}$ on the whole prequantum space $L^2(\mathbb{R}^2) \equiv L^2(\mathbb{R}_{(1)}) \otimes L^2(\mathbb{R}_{(2)})$ by $\tilde{\mathscr{P}} = \tilde{\mathscr{P}}_{(1)} \otimes \hat{Id}_{(2)}$. Using Eq.(55) and Eq.(44), we have

$$\tilde{M}\tilde{\mathscr{P}} = \sum_{n\in\mathbb{Z}^2} \tilde{M}\tilde{T}_n = \sum_{n\in\mathbb{Z}^2} \tilde{T}_{Mn}\tilde{M} = \sum_{n\in\mathbb{Z}^2} \tilde{T}_n\tilde{M} = \tilde{\mathscr{P}}\tilde{M}$$

using that M is one-to-one on $\mathbb{Z}^2$. In particular $\tilde{M}_{(1)}\tilde{\mathscr{P}}_{(1)} = \tilde{\mathscr{P}}_{(1)}\tilde{M}_{(1)}$. This gives a commutative diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}_{(1)}) & \xrightarrow{\tilde{M}_{(1)}} & L^2(\mathbb{R}_{(1)}) \\ \downarrow \tilde{\mathscr{P}}_{(1)} & & \downarrow \tilde{\mathscr{P}}_{(1)} \\ \mathscr{H}_{(1),N} & \xrightarrow{\tilde{M}_{(1)}} & \mathscr{H}_{(1),N}, \end{array}$$

which means that $\tilde{M}_{(1)}$ induces a map denoted $\tilde{M}_{(1),N}: \mathscr{H}_{(1),N} \to \mathscr{H}_{(1),N}$ (the quantum map), and similarly that $\tilde{M}$ induces a map denoted by $\tilde{M}_N: \tilde{\mathscr{H}}_N \to \tilde{\mathscr{H}}_N$ (the prequantum map). The fact that $\tilde{M}_{(1),N}$ is the "usual" quantum map is because its generator is obtained by Weyl quantization in Eq.(40). □

4.2. **Prequantum resonances.**

Spectrum of the quantum map. The spectrum of the quantum cat map, i.e., the unitary operator $\tilde{M}_{N,(1)}$ in the N-dimensional space $\mathscr{H}_{(1),N}$, is well-studied in the literature [22, 23, 24, 25]. Let

$$\tilde{M}_{N,(1)}|\psi_{(1),k}\rangle = e^{i\varphi_k}|\psi_{(1),k}\rangle, \quad k = 1 \to N \tag{56}$$

be the eigenvectors and eigenvalues of $\tilde{M}_{N,(1)}$. (See Figure 1 on page 260). The prequantum map is the unitary map $\tilde{M}_N = \tilde{M}_{N,(1)} \otimes \tilde{M}_{(2)}$ acting on the infinite-dimensional space $\mathscr{H}_{(1),N} \otimes L^2(\mathbb{R}_{(2)})$. The unitary operator $\tilde{M}_{(2)} = \exp\left(-\frac{i}{\hbar}P_H^{(2)}\right)$ is generated by $P_H^{(2)} = \mathrm{Op}_{\mathrm{Weyl}}(H_{(2)})$, with the *hyperbolic* quadratic Hamiltonian $H_{(2)}$ given by Eq.(41). $P_H^{(2)}$ has a continuous spectrum with multiplicity two, therefore $\tilde{M}_{(2)}$ has a continuous spectrum on the unit circle. The spectrum of $\tilde{M}_N$ is then obtained by a product from the spectra of $\tilde{M}_{N,(1)}$ and $\tilde{M}_{(2)}$. The aim of this section is to show that $\tilde{M}_{(2)}$ and therefore $\tilde{M}_N$, have nevertheless a well-defined discrete spectrum of resonances.

Normal form of the operator $\tilde{M}_{(2)}$. We consider the operator $\tilde{M}_{(2)} = \exp\left(-\frac{i}{\hbar}P_H^{(2)}\right)$ with $P_H^{(2)} = \mathrm{Op}_{\mathrm{Weyl}}\left(H_{(2)}\right) = -\frac{1}{2}\alpha\hat{Q}_2^2 - \frac{1}{2}\beta\hat{P}_2^2 + \frac{\gamma}{2}\left(\hat{Q}_2\hat{P}_2 + \hat{P}_2\hat{Q}_2\right)$ acting on the space $L^2\left(\mathbb{R}_{(2)}\right)$. The classical symbol $H_{(2)}\left(q,p\right) = -\frac{1}{2}\alpha q^2 - \frac{1}{2}\beta p^2 + \gamma qp$ is a *hyperbolic* quadratic function on $\mathbb{R}^2$. Therefore, there exists a linear symplectic transformation $D \in SL(2,\mathbb{R})$ which transforms $H_{(2)}$ into the hyperbolic normal form:

$$N = H_{(2)} \circ D, \qquad N\left(q,p\right) = \lambda qp$$

with the Lyapunov exponent $\lambda = \sqrt{\gamma^2 - \alpha\beta}$ (this last quantity is the unique symplectic invariant of the function $H_{(2)}$).

At the operator level, there is a similar result: there is exists a metaplectic operator (unitary operator in $L^2\left(\mathbb{R}_{(2)}\right)$), given by $\hat{D} = \exp\left(-i\mathrm{Op}_{\mathrm{Weyl}}(d)/\hbar\right)$ (with $d\left(q,p\right)$ a quadratic form which generates D) such that:

$$\hat{N} = \hat{D}P_H^{(2)}\hat{D}^{-1} = \mathrm{Op}_{\mathrm{Weyl}}(N) = \frac{\lambda}{2}\left(\hat{Q}_2\hat{P}_2 + \hat{P}_2\hat{Q}_2\right). \tag{57}$$

As a result, $\tilde{M}_{(2)} = \exp\left(-\frac{i}{\hbar}P_H^{(2)}\right) = \hat{D}^{-1}\exp\left(-\frac{i}{\hbar}\hat{N}\right)\hat{D}$ is conjugate to the normal form, so we can consider the operator $\exp\left(-\frac{i}{\hbar}\hat{N}\right)$ or $\hat{N}$ itself, which is simpler to handle.

Quantum resonances of the quantum hyperbolic fixed point. "Quantum resonances" of $\hat{N} = \mathrm{Op}_{\mathrm{Weyl}}\left(\lambda qp\right)$ are well-known. Note that with a canonical transform, $N\left(q,p\right) = \lambda qp$ is transformed to the inverted potential barrier: $H(x,\xi) = \frac{1}{2}\xi^2 - \frac{1}{2}\lambda^2x^2$. We recall here how to define and obtain these resonances by the complex scaling method [8]. Consider first the classical flow on $\left(\mathbb{R}^2, dq\wedge dp\right)$ generated by the hyperbolic Hamiltonian function $N\left(q,p\right) = \lambda qp$. The point $(0,0)$ is a hyperbolic fixed point, with an unstable direction $\left\{p=0\right\}$, and a stable direction $\left\{q=0\right\}$. Let us introduce the quadratic "escape function":

$$f_\alpha\left(q,p\right) = \frac{\alpha}{2}\left(p^2 - q^2\right), \qquad \alpha > 0$$

and define

$$\hat{f}_\alpha := \mathrm{Op}_{\mathrm{Weyl}}\left(f_\alpha\right), \quad \hat{A}_\alpha := \exp\left(\hat{f}_\alpha\right).$$

For $|\alpha| < \pi/2$, the domains $D_A := \mathrm{dom}\left(\hat{A}_\alpha\right)$ and $C_A := \mathrm{dom}\left(\hat{A}_\alpha^{-1}\right)$ are dense in $L^2\left(\mathbb{R}_{(2)}\right)$. One can explicitly check that they contain Gaussian wave functions. The choice of the escape function f_α is related to the property that it decreases along the flow of N: let $X_N = \lambda\left(q\frac{\partial}{\partial q} - p\frac{\partial}{\partial p}\right)$ be the Hamiltonian vector field associated to N, then $X_N\left(f_\alpha\right) = -\alpha\lambda\left(q^2+p^2\right) < 0$ if $q,p \neq 0$.

LEMMA 14. *For $|\alpha| < \pi/2$, let $\hat{K}_\alpha := \frac{i}{\hbar}\hat{A}_\alpha\hat{N}\hat{A}_\alpha^{-1}$. Then $\hat{K}_\alpha$ is defined on a dense domain in $L^2\left(\mathbb{R}_{(2)}\right)$, and*

$$\hat{K}_\alpha = \frac{1}{\hbar}\lambda\sin(2\alpha)\,Op_{Weyl}\left(\frac{1}{2}\left(q^2+p^2\right)\right) + \frac{i}{\hbar}\lambda\cos(2\alpha)\,Op_{Weyl}\left(qp\right).$$

In particular for $\alpha = \frac{\pi}{4}$,

$$\hat{K}_{\pi/4} = \lambda \frac{1}{2\hbar}\left(\hat{q}^2 + \hat{p}^2\right) \tag{58}$$

is the quantum Harmonic oscillator with discrete spectrum $\lambda_n = \lambda\left(n + \frac{1}{2}\right)$, $n \in \mathbb{N}$.

We keep now the simple choice $\alpha = \pi/4$, and write $\hat{K} := \hat{K}_{\pi/4}$, $\hat{A} := \hat{A}_{\pi/4}$.

Proof. The proof requires some standard calculation with the complexified metaplectic group, whose Lie algebra $sp(2,\mathbb{R})^{\mathbb{C}} = sl(2,\mathbb{C})$ is generated by the three operators $\mathrm{Op}_{\mathrm{Weyl}}(qp)$, $\mathrm{Op}_{\mathrm{Weyl}}\left(\frac{1}{2}\left(p^2 - q^2\right)\right)$, $\mathrm{Op}_{\mathrm{Weyl}}\left(\frac{1}{2}\left(p^2 + q^2\right)\right)$, see [16, Chapter 4], or [33, p. 896]. □

COROLLARY 15. *Let* $\hat{B} := \hat{A}\hat{D}$. *By a (nonunitary) conjugation,* $\tilde{M}_{(2)} = \exp\left(-\frac{i}{\hbar}P_H^{(2)}\right)$ *is transformed on a dense domain into a trace class operator:*

$$\hat{R} := \hat{B}\tilde{M}_{(2)}\hat{B}^{-1} = \exp\left(-\hat{K}\right) \tag{59}$$

with eigenvalues

$$\exp(-\lambda_n), \qquad \lambda_n := \lambda\left(n + \frac{1}{2}\right), \qquad n \in \mathbb{N}$$

REMARK.
- We would obtain the same result with any choice of $0 < \alpha < \pi/2$.
- Because $\hat{R}$ is defined on a dense domain, and is a bounded operator, it extends in a unique way to $L^2\left(\mathbb{R}_{(2)}\right)$. The eigenvalues $\exp(-\lambda_n)$ are called the "quantum resonances" of the unitary operator $\tilde{M}_{(2)}$. The meaning of the operator $\hat{R}$ and its eigenvalues, appears in the study of the decay of time-correlation functions. If $\varphi \in D_C = \mathrm{dom}\left(\hat{B}\right)$, and $\phi \in C_C = \mathrm{dom}\left(\hat{B}^{-1}\right)$ are suitable functions, then $C_{\phi,\varphi}(t) := \langle\phi|\tilde{M}^t_{(2)}|\varphi\rangle$, $t \in \mathbb{N}$, can be expressed using $\hat{R}$ as
$$C_{\phi,\varphi}(t) = \left(\langle\phi|\hat{B}^{-1}\right)\hat{R}^t\left(\hat{B}|\varphi\rangle\right)$$
Then, the spectrum of $\hat{R}$ gives the explicit exponential decay of the time-correlation function $C_{\phi,\varphi}(t)$. The decay comes from the simple fact that there is an unstable fixed point at the origin, and therefore the wave function $\varphi_t = \tilde{M}^t_{(2)}\varphi$ spreads along the unstable direction. This is general in physics and mathematics [37].

Resonances of the prequantum operator. The conjugation operator $\hat{B} = \hat{A}\hat{D}$ has been defined on $L^2\left(\mathbb{R}_{(2)}\right)$ and can be extended to the prequantum space $\tilde{\mathscr{H}}_N \equiv \mathscr{H}_{(1),N} \otimes L^2\left(\mathbb{R}_{(2)}\right)$ by $\tilde{B} := \hat{\mathrm{Id}}_{(1)} \otimes \hat{B}$. We use it to conjugate the prequantum map $\tilde{M}_N = \tilde{M}_{(1),N} \otimes \tilde{M}_{(2)}$ and deduce from Eq.(56) and Eq.(59):

THEOREM 16. *The conjugated operator*

$$\tilde{R} := \tilde{B}\tilde{M}_N\tilde{B}^{-1} = \tilde{M}_{N,(1)} \otimes \hat{R} \tag{60}$$

is a trace class operator in the prequantum space $\tilde{\mathscr{H}}_N \equiv \mathscr{H}_{(1),N} \otimes L^2\left(\mathbb{R}_{(2)}\right)$, *with eigenvalues:*

$$r_{n,k} := \exp\left(i\varphi_k - \lambda_n\right), \quad \lambda_n := \lambda\left(n + \frac{1}{2}\right),\ n \in \mathbb{N}, \quad \varphi_k \in [0, 2\pi],\ k \in [1, N].$$

The eigenvalues $r_{n,k}$ are called the *resonances of the prequantum map*. This gives Theorem 1 on page 259, the main result of this paper.

4.3. **Relation between prequantum time-correlation functions and quantum evolution of wave functions.** Let $\varphi,\phi \in \tilde{\mathscr{H}}_N$ be prequantum wave functions, that belong respectively to the domains of $\tilde{B}$ and $\tilde{B}^{-1}$. Let us define $\tilde{\phi} = \hat{\Pi}\tilde{B}^{-1}\phi$, $\tilde{\varphi} = \hat{\Pi}\tilde{B}\varphi$, where $\hat{\Pi} = \hat{I}_1 \otimes |0_2\rangle\langle 0_2| : \tilde{\mathscr{H}}_N \to \mathscr{H}_{(1),N}$ is the orthogonal Toeplitz projector. Then we have $\langle\phi|\tilde{M}_N^t|\varphi\rangle = \langle\phi|\tilde{M}_{(1),N}^t \otimes \tilde{M}_{(2)}^t|\varphi\rangle$, but $\tilde{M}_{(2)}^t = \tilde{B}^{-1}\tilde{R}^t\tilde{B}$ and $\tilde{R}^t = \sum_{n_2\in\mathbb{N}} |n_2\rangle\langle n_2| \exp\left(-\lambda\left(n_2+\frac{1}{2}\right)t\right)$.

We deduce that $\langle\phi|\tilde{M}_N^t|\varphi\rangle = \langle\phi|\tilde{M}_{(1),N}^t \otimes \left(\tilde{B}^{-1}|0_2\rangle\langle 0_2|\tilde{B}\right)|\varphi\rangle e^{-\lambda t/2}\left(1+\mathscr{O}\left(e^{-\lambda t}\right)\right)$, hence

$$\langle\phi|\tilde{M}_N^t|\varphi\rangle = \langle\tilde{\phi}|\tilde{M}_{(1),N}^t|\tilde{\varphi}\rangle e^{-\lambda t/2}\left(1+\mathscr{O}\left(e^{-\lambda t}\right)\right)$$

This gives Proposition 2 on page 261. Let us remark that $|0_2\rangle$ does not belong to the domains of $\tilde{B}$ or $\tilde{B}^{-1}$, but $\tilde{B}|0_2\rangle$ can be interpreted as a distribution, so $\langle 0_2|\tilde{B}|\varphi\rangle$ makes sense even if φ does not belong to the domain of $\tilde{B}$.

4.4. **Proof of the trace formula.** We prove here Proposition 3 on page 261. We just follow the calculation of Eq.(4), but with a suitable regularization, and show that it gives $\mathrm{Tr}\left(\tilde{R}^t\right)$. We follow a calculation similar to one in [15]. This proof does not use crucially the hypothesis that M is a linear map, so it would work for nonlinear prequantum hyperbolic map as well.

Let us introduce a cutoff operator in the space $L^2\left(\mathbb{R}_{(2)}\right)$ defined in Eq. (23):

$$P_\nu := \exp\left(-\nu\frac{1}{2}\left(\hat{P}_2^2+\hat{Q}_2^2\right)\right), \quad \nu > 0.$$

This operator is diagonal in the basis $|n_2\rangle$ of the Harmonic oscillator, and truncates high values of n_2. We choose here a metaplectic operator for future convenience. The operator P_ν is trace class, and converges strongly to the identity for $\nu \to 0$. Consequently, $\langle Q_2'|P_\nu|Q_2\rangle \to \delta\left(Q_2'-Q_2\right)$ for $\nu \to 0$ and uniformly with respect to $Q_2 \in K \subset \mathbb{R}_{(2)}$ in a compact set. We extend this operator in $\tilde{\mathscr{H}}_N = \mathscr{H}_{(1),N} \otimes L^2\left(\mathbb{R}_{(2)}\right)$ by $Id_{(1)} \otimes P_\nu$ and denote it again by P_ν. Using Eq.(37) it is possible to show that $\langle x'|P_\nu|x\rangle \underset{\nu\to 0}{\longrightarrow} \delta\left(x-x'\right)$ uniformly with respect to $x \in K \subset \mathbb{R}^2$ in a compact set.

LEMMA 17. $\left(\tilde{M}_N^t P_\nu\right)$ *is a trace class operator in* $\tilde{\mathscr{H}}_N$ *for any* $t>0$, $\nu>0$, *and*

$$Tr\left(\tilde{M}_N^t P_\nu\right) \underset{\nu\to 0}{\longrightarrow} \sum_{x\equiv M^t x[1]} \frac{1}{\left|det\left(1-M^t\right)\right|} e^{iA_{x,t}/\hbar},$$

where the sum is over points $x \in [0,1[^2$ *such that* $M^t x = x+n$, *with* $n \in \mathbb{Z}^2$, *i.e., periodic points on* $\mathbb{T}^2$. $A_{x,t} = \frac{1}{2} n \wedge x$ *is the "classical action" of the periodic point* x.

Proof. First $\tilde{M}_N^t P_\nu$ is trace class because it is a product of a unitary and trace class operator. Using Eq.(18) for the prequantum evolution and Dirac notations, we write

$$\left(\tilde{M}^t\psi\right)(x) = \langle x|\tilde{M}^t|\psi\rangle = \psi\left(M^{-t}x\right)e^{-iF_{M^{-t}x,t}/\hbar} = \psi\left(M^{-t}x\right)$$

because since M is linear, we have shown in Eq.(42) that the phase is $F_{x,t} = 0$. Then with $|\psi_{x,v}\rangle := P_v|x\rangle$, the operator $\tilde{\mathscr{P}}$ defined in Eq.(55), and using $\tilde{T}_n|x\rangle = e^{-i\frac{1}{2\hbar}n\wedge x}|x+n\rangle$,

$$\begin{aligned}\operatorname{Tr}\left(\tilde{M}_N^t P_v\right) &= \int_{]0,1[^2} \langle x|\tilde{\mathscr{P}}\tilde{M}^t P_v|x\rangle dx = \int_{]0,1[^2} \langle x|\tilde{\mathscr{P}}\tilde{M}^t|\psi_{x,v}\rangle dx \\ &= \sum_{n\in\mathbb{Z}^2}\int_{]0,1[^2} e^{i\frac{1}{2\hbar}n\wedge x}\langle x+n|\tilde{M}^t|\psi_{x,v}\rangle dx \\ &= \sum_{n\in\mathbb{Z}^2}\int_{]0,1[^2} e^{i\frac{1}{2\hbar}n\wedge x}\psi_{x,v}\left(M^{-t}(x+n)\right)dx\end{aligned}$$

We have seen that $\psi_{x,v}\left(x'\right) = \langle x'|P_v|x\rangle \underset{v\to 0}{\longrightarrow} \delta\left(x-x'\right)$ uniformly with respect to x in a compact set, so

$$\begin{aligned}\operatorname{Tr}\left(\tilde{M}_N^t P_v\right) \underset{v\to 0}{\longrightarrow} &\sum_{n\in\mathbb{Z}^2}\int_{]0,1[^2}\delta\left(x-M^{-t}(x+n)\right)e^{i\frac{1}{2\hbar}n\wedge x}dx \\ &= \sum_{x\equiv M^t x\,[1]}\frac{1}{\left|\det\left(1-M^t\right)\right|}e^{i\frac{1}{2\hbar}n\wedge x}.\end{aligned}$$

We have used a change of variable $x \to y = x - M^{-t}x - M^{-t}n$, where a periodic point $x\in]0,1[^2$ is specified by $M^t x = x+n$, $n\in\mathbb{Z}^2$. □

LEMMA 18. *$Tr\left(\tilde{M}^t P_v\right) \underset{v\to 0}{\longrightarrow} Tr\left(\tilde{R}^t\right)$.*

Proof. We have $\tilde{M}^t P_v = \tilde{B}^{-1}\tilde{R}^t\tilde{B}P_v$, so $\operatorname{Tr}\left(\tilde{M}^t P_v\right) = \operatorname{Tr}\left(\tilde{R}^t\tilde{B}P_v\tilde{B}^{-1}\right)$. This involves a product of metaplectic operators, and using a representation of $SL(2,\mathbb{C})$, we explicitly check that $\tilde{R}^t\tilde{B}P_v\tilde{B}^{-1}$ converges to $\tilde{R}^t$ as $v\to 0$. (Notice that for nonlinear maps, this last argument would have failed, and the proof would have been longer). □

With Lemma 17 and Lemma 18 taken together, we conclude the proof of Proposition 3.

5. CONCLUSION

In this paper we defined the prequantum map associated to a linear hyperbolic map on the torus $\mathbb{T}^2$, and showed that it has well-defined resonances. These resonances form a discrete spectrum and can be explicitly expressed in terms of the eigenvalues of the unitary quantum map. In Section 2, we discussed the interpretation of this spectrum of resonances in terms of decay of time correlation functions, and compared them with the matrix elements of the quantum map after time t. We have also compared the trace formula for the quantum propagator and for the prequantum one (the sum over its resonances) after a large time.

We would like first to make a general remark about prequantum dynamics. Prequantization has been well-known for many years, and it is known to be a beautiful theory from a geometrical point of view. Many works have studied the geometrical aspects and shown how to define prequantization in general cases, for example Hodge manifolds. From a mathematical perspective in dynamical systems, prequantization is directly defined from the Hamiltonian flow, so that it is natural to investigate its properties, for example, its spectrum. Nevertheless, it seems that few works have yet investigated its dynamical properties and its spectrum. This paper goes in this direction, and we would like to emphasize that the prequantum spectrum is not only interesting by itself, but may rather be a useful approach for semiclassical analysis, especially for quantum hyperbolic dynamics, i.e., "quantum chaos".

It is natural to ask if such results have been investigated for the geodesic flow on negatively curved manifolds. In fact, in the case of cotangent phase spaces, the prequantum bundle is trivial, and the prequantum operator can be expressed as a classical transfer operator with a suitable weighted function. Such an operator is well-studied and it is known that the spectrum of classical resonances for the geodesic flow on constant negative curvature is related to the spectrum of the Laplacian which plays the role of the quantum operator (the relation can be obtained using the Selberg zeta function [32], or by a group theory approach in [28]).

Some interesting questions arise naturally in the framework of prequantum chaos, similar to questions which exist in quantum chaos, namely concerning the "semiclassical limit" $N = 1/(2\pi\hbar) \to \infty$, where the curvature of the prequantum bundle goes to infinity. If properly defined, one could investigate the problem of "prequantum ergodicity" or "unique prequantum ergodicity". For example, in [14], the existence of scarred quantum eigenfunctions has been obtained, i.e., non-equidistributed eigenfunctions over the torus in the limit $N \to \infty$. Because of the explicit relation between quantum eigenfunctions and prequantum resonances we have obtained, this could lead to "prequantum scarred distributions" (but this needs some correct definition). Let us remark that the Ehrenfest time $t_E := \frac{1}{\lambda} \log N$ is known to play an important role as a characteristic time scale in quantum chaos [12]. Its usual interpretation is the time after which a detail of the size of $\hbar$, i.e., the minimum size in phase space allowed by the quantum uncertainty principle, called the Planck cell, is exponentially amplified towards finite size: $\hbar e^{\lambda t_E} \simeq 1$. In prequantum dynamics, there is no uncertainty principle any more because the dynamics evolves smooth sections over phase space. But the prequantum bundle has a curvature $\Theta = \frac{i}{\hbar}\omega$, and there is still the notion of Planck cell on the torus as the elementary surface over which the curvature integral is one. Therefore the Ehrenfest time may still play an important role for prequantum dynamics, at least in the semiclassical limit $N \to \infty$.

Perspectives in the nonlinear case. For a linear map M, we have shown that there is an exact correspondence between the spectrum of prequantum resonances and the quantum spectrum. In a future work we plan to study nonlinear prequantum hyperbolic map on the torus, and expect to obtain similar results[17] (with possibly introducing some weight function $\varphi = \lambda/2$ in the transfer operator, where λ is the local expanding rate). We expect then that there still exists an exact prequantum trace formula for $\mathrm{Tr}\left(\tilde{R}_\varphi^t\right)$ in terms of periodic orbits, similar to Eq.(3). We hope to be able to compare the prequantum operator $\tilde{R}_\varphi$ with the quantum operator $\hat{M}$, and possibly their spectra as we did in Eq.(2), at least in the limit $N \to \infty$, and then deduce validity of the semiclassical Gutzwiller trace formula and other semiclassical formula for long times. Some interesting questions would appear then, such as: does the random matrix theory apply to the outlying prequantum spectra?

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