Lecture notes,
Spectrum, traces and zeta functions in hyperbolic dynamics

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Abstract

These are lectures notes for the school 23-27 April 2018 at the University Cheikh Anta Diop in Dakar, Sénégal. For different models of deterministic hyperbolic dynamics or probabilistic dynamics on graph, we present the Ruelle spectrum that describes the time behavior of time correlation functions and Trace formula and dynamical zeta functions that relate the spectrum to periodic orbits of the dynamics. We also present some recent results, in particular obtained with J. Sjöstrand and M. Tsujii, some applications as counting periodic orbits and perspectives.

Contents

Acknowledgement 0.1. I thank the organizers of the conference in particular Masseye Gaye from Dakar.

1 Introduction

Remark 1.1. This file is still "under construction". This is not considered as a achieved version. You can find a more recent version on this page. On this pdf file, you can click on the colored words, they contain an hyperlink to wikipedia or other multimedia contents.

Some references: For more informations on this subject we recommend the lectures notes in Grenoble 2013 and Warsaw 2015, by Mark Pollicott on his web page. About Ruelle resonances, there are lectures notes [14].

1.1 Objectives

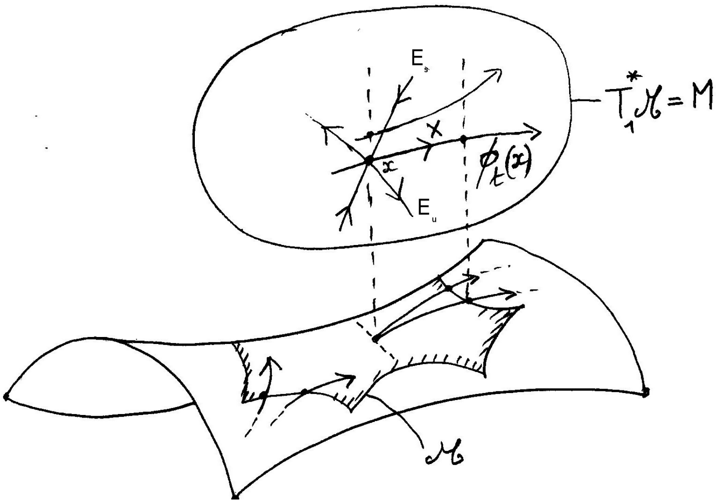
The objective of these lecture notes is to define the Ruelle discrete spectrum for uniformly hyperbolic dynamical systems and explain some **relations between this discrete spectrum and periodic orbits** of the dynamics. At the end of the lectures we will explain the Theorems 1.3 and 1.7 presented in this introduction.

Some precise definitions and examples will be given in the lectures, but for this introduction, an **Anosov flow** $\phi^t : M \to M$ on a manifold M, with $t \in \mathbb{R}$, is such that at every point $m \in M$ there is an invariant decomposition

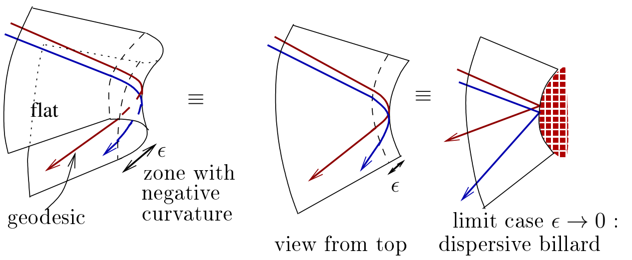
$$T_m M = E_u(m) \oplus E_s(m) \oplus E_0(m)$$

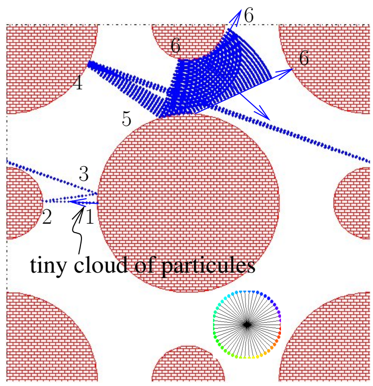
where $E_0(m)$ is the one-dimensional direction of the flow, and $E_u(m), E_s(m)$ are directions such that for $t \geq 0$ the differential of the flow $D\phi^t(m) : E_u(m) \to E_u(\phi^t(m))$ is expanding and $D\phi^t(m) : E_s(m) \to E_s(\phi^t(m))$ is contracting. These conditions called **"sensitivity to initial conditions"** will generates "**chaos**" (confusion, unpredictability).

A typical but very special example is the **geodesic flow** on a (strictly) negatively curved Riemannian manifold $(\mathcal{M}, g)$. In this case $M = (T^*\mathcal{M})_1$ is the unit cotangent bundle. On the following picture, $d = 1$, $\mathcal{M}$ is a surface with negative Gauss curvature:



Remark 1.2. **Sinai dispersive billard** is a limit case of **Anosov geodesic flow,** where the negative curvature is concentrated on the boundaries:





- See **movie** "Anosov linkage" by **Mickael Kourganoff, Jos Leys** (2015). It is shown that the dynamics of the free linkage system (2nd image) is equivalent to the geodesic flow on the surface of 3rd image, and seen from above on the 1st image. If the mass of the small blue ball goes to zero then the geodesic flow on the first image converges to the dynamics in a dispersive Sinai billard.
- See **movie** showing the chaotic dynamics in a dispersive Sinai billard.

Theorem 1.3 (Giulietti, Liverani, Pollicott 2012 [18]). *Suppose that X is a smooth vector field on a closed manifold M that generates an* ***Anosov flow*** $\phi^t = e^{tX} : M \to M$*. Let $V \in C^\infty(M;\mathbb{C})$ an arbitrary smooth function on M, called potential. We define the* ***"spectral determinant zeta function"*** *for $z \in \mathbb{C}, \mathrm{Re}(z) \gg 1$,*

$$d(z) := \exp\left(-\sum_{\gamma}\sum_{n\geq 1}\frac{e^{nV_\gamma}.e^{-zn|\gamma|}}{n\left|\det\left(\mathrm{Id}-D_{/E_u\oplus E_s}\phi^{n|\gamma|}(m)\right)\right|}\right) \tag{1.1}$$

where the sum is over ***primitive periodic orbits*** *γ with period $|\gamma|$, and $V_\gamma := \int_0^{|\gamma|} V(\phi^s(m))\,ds$ is the Birkhoff sum of V along γ, $m \in \gamma$ is an arbitrary point on γ. Then $d(z)$ has an* ***holomorphic extension*** *on $\mathbb{C}$. Its zeroes are the* ***discrete Ruelle eigenvalues of*** *$A := X + V$ in specific anisotropic Sobolev spaces and where the vector field X is considered as a first order differential operator.*

Remark 1.4. Theorem 1.3 has been obtained soon after by Dyatlov-Zworski in [8] using the microlocal approach developped in [12].

Remark 1.5. There is a natural and important generalization of Eq.(1.1) where the vector field operator X acts naturally on sections of differential forms $C^\infty\left(M;\Lambda^k(M)\right)$ of any degree k or action on more general bundles over M.

Typical applications of Theorem 1.3 is **counting periodic orbits**, i.e. evaluating $\pi(T) := \sharp\{\gamma, \quad |\gamma| \leq T\} = \sum_{\gamma,|\gamma|\leq T} 1$ for $T \gg 1$, with the following result[1]

Corollary 1.6. *[18](with pinching hypothesis) there exists $\delta > 0$ s.t.*

$$\pi(T) = \mathrm{Ei}(h_{top}T) + O\left(e^{(h_{top}-\delta)T}\right) \underset{T\to\infty}{\sim} \frac{e^{h_{top}T}}{h_{top}T}$$

with $\mathrm{Ei}(x) := \int_{x_0}^{x}\frac{e^y}{y}dy$ and h_{top} is the dominant eigenvalue of $A = X + \mathrm{div}X_{/E_u(x)}$ called ***topological entropy****.*

For the next Theorem, we consider the more special case of **geodesic flow** on a (strictly) negatively curved Riemannian manifold $(\mathcal{M}, g)$, or **contact Anosov flow**. The curvature is possibly non constant. Theorem 1.3 applies as a special case and the next Theorem gives a more precise description of the spectrum. See Figure 1.1.

[1] Observe that $\left|\det\left(1-D_{/E_u\oplus E_s}\phi^{n|\gamma|}(m)\right)\right|^{-1} \underset{n|\gamma|\gg 1}{\simeq} \det\left(D\phi^{n|\gamma|}_{/E_u}\right)^{-1}$ i.e. expanding dominates. The choice of potential $V = \mathrm{div}X_{/E_u}$ gives $e^{nV_\gamma} = \det\left(D\phi^{n|\gamma|}_{/E_u}\right)$ and $e^{nV_\gamma}\left|\det\left(\mathrm{Id}-D_{/E_u\oplus E_s}\phi^{n|\gamma|}(m)\right)\right|^{-1} \sim 1$ and is favorable to count periodic orbits using (1.1).

Theorem 1.7 (Faure, Tsujii 2013 [16]). *Suppose that $\phi^t = e^{tX}$ is the* ***geodesic flow*** *on $M = (T^*\mathcal{M})_1$ where $(\mathcal{M}, g)$ is a (strictly)* ***negatively curved Riemanian manifold****. We define the* ***"semi-classical zeta function"*** *for $z \in \mathbb{C}, \mathrm{Re}(z) \gg 1$,*

$$d_{sc}(z) := \exp\left(-\sum_{\gamma}\sum_{n\geq 1}\frac{e^{-zn|\gamma|}}{n\left|\det\left(\mathrm{Id} - D_{/E_u\oplus E_s}\phi^{n|\gamma|}(m)\right)\right|^{1/2}}\right) \tag{1.2}$$

where the sum is over ***primitive periodic orbits*** *γ with period $|\gamma|$. Then $d_{sc}(z)$ has a* ***meromorphic extension*** *on $\mathbb{C}$. $\exists\gamma_1^+ < 0$, $\forall\epsilon > 0$, $\exists\omega_\epsilon > 0$ such that on $D_\epsilon := \left\{z \in \mathbb{C}, s.t. \mathrm{Re}(z) > \gamma_1^+ + \epsilon, \quad |\mathrm{Im}(z)| > \omega_\epsilon\right\}$, $d_{sc}(z)$ is* ***holomorphic*** *with the zeroes contained in the vertical band $B_0 := \{z \in \mathbb{C}, s.t. |Re(z)| \leq \epsilon\}$ with density*

$$\sharp\{\textit{zeroes } z \in \mathbb{C}, s.t. |Re(z)| \leq \epsilon \textit{ and } \omega \leq \mathrm{Im}(z) \leq \omega + 2\pi\} \sim \mathrm{Vol}(M)\frac{\omega^d}{(2\pi)^d}.$$

In other words the ***zeroes accumulate on the imaginary axis*** *with a density given by the* ***Weyl law****.*
The zeroes of $d_{sc}(z)$ on D_ϵ coincide with the ***discrete Ruelle eigenvalues*** *of $A = X + \frac{1}{2}\mathrm{div}\left(X_{/E_u}\right)$ in specific anisotropic Sobolev spaces.*

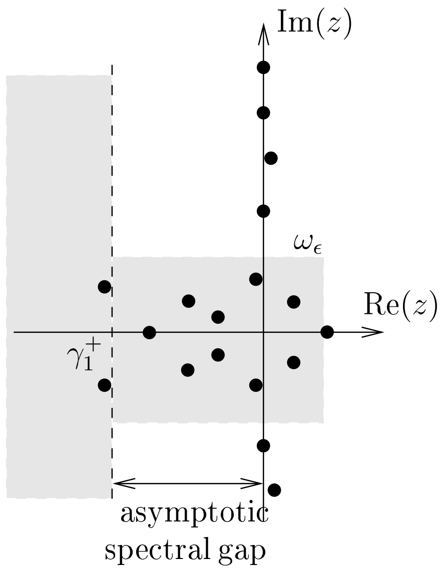


Figure 1.1: Zeroes of $d_{sc}(z)$ accumulate of the imaginary axis.

The motivation for studying $d_{sc}(z)$ comes from the Gutzwiller semiclassical trace formula in quantum chaos. Also in the case of a **compact surface with constant negative**

curvature $\mathcal{M} = \Gamma\backslash \mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2$, the function $d_{sc}(z)$ coincide with the **Selberg zeta function (1956)**

$$\zeta_{\text{Selberg}}(z) := \prod_{\gamma}\prod_{m\geq 0}\left(1-e^{-(z+m)|\gamma|}\right)$$

up to a shift 1/2:

$$d_{sc}(z) = \zeta_{\text{Selberg}}\left(z+\frac{1}{2}\right) \tag{1.3}$$

Remark 1.8. Recall that $\zeta_{\text{Selberg}}(z)$ has zeroes on the vertical line $\mathrm{Re}(z) = 1/2$ related to the eigenvalues of the operator $\Delta = d^*d$ on $\mathcal{M}$. Compare figure 1.2 with Figure 1.1. Somehow, the function $d_{sc}(z)$, in (1.2) is a generalization of the Selberg zeta function to non constant negative curvature manifolds. There is a noticeable resemblance between ζ_{Selberg} and the famous Riemann zeta function where the product is over the prime numbers:

$$\zeta_{\text{Riemann}}(z) := \prod_{p}\left(1-e^{-z\log p}\right)^{-1} = \sum_{n\geq 1}\frac{1}{n^z}$$

In this analogy, $\log p$ is related to the period $|\gamma|$.

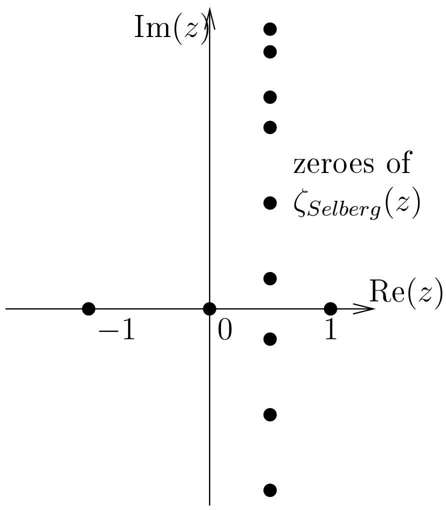


Figure 1.2: Zeroes of the holomorphic function $\zeta_{\text{Selberg}}(z) := \prod_{\gamma}\prod_{m\geq 0}\left(1-e^{-(z+m)|\gamma|}\right)$ are on $-\mathbb{N}$ and $\frac{1}{2} \pm i\sqrt{\mu_j - \frac{1}{4}}$ where μ_j are the eigenvalues of the operator $\Delta = d^*d$ on the surface $\mathcal{M}$.

Proof of (1.3). On a surface $\mathcal{M} = \Gamma\backslash \mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2$ the expanding and contracting rate are constant and equal to one, hence $D_{/E_u\oplus E_s}\phi^{n|\gamma|}(m) \equiv \begin{pmatrix} e^{|\gamma|n} & 0 \\ 0 & e^{-|\gamma|n}\end{pmatrix}$. we also use, with

$x = e^{-|\gamma|n}$ that

$$\left|\det\left(\mathrm{Id} - D_{/E_u \oplus E_s}\phi^{n|\gamma|}(m)\right)\right|^{-1/2} = \left|\det\begin{pmatrix} 1-x^{-1} & 0 \\ 0 & 1-x \end{pmatrix}\right|^{-1/2} = \left(\left(x^{-1}-1\right)(1-x)\right)^{1/2}$$
$$= x^{1/2}(1-x)^{-1} = x^{1/2}\sum_{m\geq 0} x^m.$$

For $|X| < 1$, we have the series $\ln(1-X) = -\sum_{n\geq 1}\frac{1}{n}X^n$. This gives

$$d_{sc}(z) \underset{(1.2)}{=} \exp\left(-\sum_{\gamma}\sum_{n\geq 1}\sum_{m\geq 0}\frac{1}{n}e^{-n|\gamma|\left(z+\frac{1}{2}+m\right)}\right) = \exp\left(-\sum_{\gamma}\sum_{m\geq 0}\ln\left(1-e^{-|\gamma|\left(z+\frac{1}{2}+m\right)}\right)\right)$$
$$= \prod_{\gamma}\prod_{m\geq 0}\left(1-e^{-|\gamma|\left(z+\frac{1}{2}+m\right)}\right) = \zeta_{\mathrm{Selberg}}\left(z+\frac{1}{2}\right)$$

$\square$

Remark 1.9. The most difficult part in the proof of Theorem 1.3 and 1.7 is the spectral aspects of the operator X: using semiclassical analysis, showing that X has discrete spectrum in specific anisotropic Sobolev spaces and finally that the zeta functions are no more than (or related to) spectral determinant functions hence holomorphic on $\mathbb{C}$.

1.2 Outline of these lecture notes

We will consider different models, starting from a simple complex $N \times N$ matrix $\mathcal{L}$ and ending with the geodesic flow transfer operator $\mathcal{L}^t = e^{tX}$ on a negatively curved manifold. Some important formula that we will encounter in every models are the "Atiyah-Bott trace formula" and "dynamical zeta functions". In each step (each Section) we introduce and explain some new ingredient.

In Section 2 we consider a $N \times N$ matrix $\mathcal{L}$. In that case the discrete Ruelle spectrum $(z_j)_j$ are the eigenvalues of the matrix. We will associate a graph G to the matrix $\mathcal{L}$ and show some relations between the periodic trajectories on the graph G and the spectrum of the matrix $\mathcal{L}$, namely the "Atiyah-Bott trace formula" and "dynamical zeta functions" usually called the "Bowen-Lanford zeta function".

In Section 3 the matrix $\mathcal{L}$ is replaced by an operator $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ acting on $l^2(\mathbb{Z})$ where $\mathcal{L}_0$ is the simple right shift operator, i.e. with matrix elements $(\mathcal{L}_0)_{j+1,j} = 1$ and $(\mathcal{L}_0)_{k,j} = 0$ otherwise and $\mathcal{L}_1$ is a finite rank perturbation (a finite matrix). We can also associate a dynamics on a graph G. We will show that the operator $\mathcal{L}$ has some discrete spectrum $\left(z_j^+\right)_j$ in some specific anisotropic Sobolev space $\mathcal{H}_W(\mathbb{Z})$. We will explain how $\mathcal{H}_W(\mathbb{Z})$ is constructed from the dynamics on the graph G. This spectrum is called the future Ruelle spectrum[2] of $\mathcal{L}$. Then we will show some relations between this Ruelle spectrum

[2]In this model, there is also a "past Ruelle spectrum" $\left(z_j^-\right)_j$ associated to the past dynamics on the graph G^- associated to the inverse operator $\mathcal{L}^{-1}$.

and periodic orbits on the graph G using again the "Atiyah-Bott trace formula" and the "dynamical zeta functions".

In Section 4 we consider a linear vector field on $\mathbb{R}$ given by $X = \lambda x \frac{d}{dx}$, with $\lambda > 0$. Its Ruelle spectrum (i.e. eigenvalues of $-X$) seems quite obvious: this is the discrete eigenvalues $z_k = -k\lambda$, $k \in \mathbb{N}$ with eigenfunction $\varphi_k(x) = x^k$. This spectrum is obviously not in $L^2(\mathbb{R})$ but in a specific anisotropic Sobolev space $\mathcal{H}_W(\mathbb{R})$ that we will construct in detail. In this simple model the periodic orbit is only the fixed point $x = 0$.

In Section 5 we consider a smooth Anosov flow on a compact manifold M. This flow is generated from a vector field X. A typical example is the geodesic flow on a negatively curved closed manifold $\mathcal{M}$ and $M = (T^*\mathcal{M})_1$ is the unit cotangent bundle. Then we will show that the vector field $X = \sum_j X_j \frac{\partial}{\partial x_j}$ (as a differential operator) has some intrinsic Ruelle spectrum in some specific anisotropic Sobolev space $\mathcal{H}_W(M)$. For this model we will explain the "Atiyah-Bott trace formula" and the recent result of [18] that relates the "dynamical zeta functions" to the Ruelle spectrum of X.

2 Spectrum, traces and zeta functions for a matrix

In this Section, let $N \geq 2$. We consider $\mathcal{L}$ a $N \times N$ **complex matrix** with matrix elements $\mathcal{L}_{j,i} \in \mathbb{C}$, $i, j = 1 \dots N$.

Example 2.1. Let $N = 2$ and $\mathcal{L} = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$.

2.1 Spectrum of $\mathcal{L}$

The matrix $\mathcal{L}$ can put in Jordan normal form

$$\mathcal{L} = PJP^{-1}$$

with a matrix $J = (J_{j,k})_{j,k}$ having $J_{j,j} = z_j \in \mathbb{C}$ on the diagonal, called **eigenvalues** of $\mathcal{L}$, and others elements vanish $J_{j,k} = 0$ for $j \neq k$ except possibly $J_{j,j+1} = 1$ (in the case $z_{j+1} = z_j$). We will assume $|z_j| \geq |z_{j+1}|, \forall j$.

The **spectral determinant** of $\mathcal{L}$ is, for $z \in \mathbb{C}$,

$$\begin{aligned} d(z) &:= \det(z\mathrm{Id} - \mathcal{L}) \\ &= \det(z\mathrm{Id} - J) = \prod_{j=1}^{N} (z - z_j) \end{aligned} \tag{2.1}$$

Hence $d(z)$ is a polynomial of degree N. For $n \geq 1$, we have $\mathcal{L}^n = PJ^nP^{-1}$ and

$$\mathrm{Tr}(\mathcal{L}^n) = \mathrm{Tr}(J^n) = \sum_{j=1}^{N} z_j^n$$

Example 2.2. Let $N = 2$ and $\mathcal{L} = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$. Then

$$d(z) = \det \begin{pmatrix} z-1 & -i \\ -i & z \end{pmatrix} = z^2 - z + 1 = (z - z_1)(z - z_2)$$

with

$$z_1 = \frac{1}{2}\left(1 + i\sqrt{3}\right), \quad z_2 = \frac{1}{2}\left(1 - i\sqrt{3}\right).$$

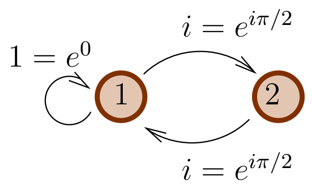
2.2 Associated graph and periodic orbits

For $i, j \in \{1, \dots N\}$, if $\mathcal{L}_{j,i} \neq 0$ we denote

$$\mathcal{L}_{j,i} = e^{V_{j,i}},$$

with $V_{j,i} \in \mathbb{C}$ called "**potential function**". Notice that $\mathrm{Im}(V_{j,i})$ is defined modulo 2π. Let $G = (V, E)$ be the finite graph with vertices $V = \{1, \dots N\}$ and oriented edges $i \to j$ if $\mathcal{L}_{j,i} = e^{V_{j,i}} \neq 0$.

Example 2.3. $N = 2$. Let $\mathcal{L} = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$. The graph with two vertices is



Definition 2.4. A **marked periodic orbit** of period $n \geq 1$ starting at vertex i_0 is a sequence $w = (i_0, i_1, i_2, \dots i_{n-1})$ such that $\mathcal{L}_{i_{k+1}, i_k} \neq 0$ for $k = 0, \dots n-1$ and with $i_n = i_0$. $\mathcal{W}_n$ is the set of marked periodic orbits w of period n and $\mathcal{W} := \bigcup_{n \geq 1} \mathcal{W}_n$. The period is $|w| := n$. The sum

$$V_w := V_{i_n i_{n-1}} + \dots V_{i_2 i_1} + V_{i_1 i_0} = \sum_{k=0}^{n-1} V_{i_{k+1} i_k} \tag{2.2}$$

is called the **Birkhoff sum** of V along the periodic orbit w.
A **primitive periodic orbit** of period $m \geq 1$ on the graph is a sequence of vertices $\gamma = (i_0, i_1, i_2, \dots i_{m-1})$ with $\mathcal{L}_{i_{k+1}, i_k} \neq 0$ for $k = 0, \dots m-1$ and with $i_m = i_0$ but $i_j \neq i_0$ if $0 < j < m$. We identify γ with its circular permuttation. Γ is the set of primitive periodic orbits γ and $|\gamma| := m$ is the period. We denote

$$V_\gamma := \sum_{k=0}^{m-1} V_{i_{k+1} i_k}.$$

In other words a primitive periodic has no marked point and does not contain a smaller periodic orbit. It is a "geometric closed path on the graph" with no repetition.

Example 2.5. Let $\mathcal{L} = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$. From the graph we observe that marked periodic orbits are $w = (1)$, $w = (1,1)$, $w = (1,2)$, $w = (2,1)$, $w = (1,1,1)$, $w = (1,1,2)$, $w = (1,2,1)$, $w = (2,1,1)$, $w = (1,1,1,1)$, $w = (1,2,1,2)$, $w = (2,1,2,1)$,$w = (1,1,1,2)$,$w = (1,1,2,1)$,$w = (1,2,1,1)$,$w = (2,1,1,1)$, etc. The number of marked periodic orbits is

$$\mathcal{N}(1) = 1, \quad \mathcal{N}(2) = 3, \quad \mathcal{N}(3) = 4, \quad \mathcal{N}(4) = 7, \ldots$$

Primitive periodic orbits are are $\gamma = (1)$, $\gamma = (1,2)$, $\gamma = (1,1,2)$, $\gamma = (1,1,1,2)$, $\gamma = (1,1,1,1,2)$, $\gamma = (1,1,2,1,2)$. etc. Hence the number of primitive orbits is

$$\pi(1) = 1, \quad \pi(2) = 1, \quad \pi(3) = 1, \quad \pi(4) = 1, \quad \pi(5) = 2, \ldots$$

We will obtain formula for these numbers at the end of the Section.

Remark 2.6. From a primitive orbit γ, if $i_0 \in \gamma$ is a given vertex and $k \geq 1$, then the repeated sequence $w = \Big(\underbrace{\gamma, \gamma, \ldots \gamma}_{k}\Big) \in \mathcal{W}_n$ gives a marked periodic orbit of period $n = k|\gamma|$. Moreover $e^{V_w} = e^{kV_\gamma}$.

The following formula that relates the spectrum $(z_j)_j$ of $\mathcal{L}$ to marked periodic orbits $w \in \mathcal{W}_n$ and primitive periodic orbits $\gamma \in \Gamma$ on the graph.

Lemma 2.7. ***"Trace formula for matrices".*** *For $n \geq 1$, we have*

$$\mathrm{Tr}(\mathcal{L}^n) = \sum_{w \in \mathcal{W}_n} e^{V_w} = \sum_{\gamma \in \Gamma} |\gamma| \sum_{k \geq 1} e^{kV_\gamma} \delta_{n=k|\gamma|} = \sum_{j=1}^{N} z_j^n \tag{2.3}$$

Proof. We have

$$\mathrm{Tr}(\mathcal{L}^n) = \sum_{i_0, i_1, \ldots, i_{n-1}, i_n = i_0} \mathcal{L}_{i_n i_{n-1}} \mathcal{L}_{i_{n-1} i_{n-2}} \ldots \mathcal{L}_{i_1 i_0} = \sum_{i_0, i_1, \ldots, i_{n-1}, i_n = i_0} e^{V_{i_n, i_{n-1}}} \ldots e^{V_{i_1, i_0}}$$

$$= \sum_{w \in \mathcal{W}_n} e^{V_w} = \sum_{\gamma \in \Gamma} |\gamma| \sum_{k \geq 1} e^{kV_\gamma} \delta_{n=k|\gamma|} = \sum_{j=1}^{N} z_j^n.$$

□

From (2.3) we can express an other way the relation between the spectrum $(z_j)_j$ of $\mathcal{L}$ and marked or primitive periodic orbits of the graph.

Proposition 2.8. *The "**Bowen-Lanford zeta function for matrices**" (1968) is defined by*

$$\zeta_{BL}(z) := \exp\left(-\sum_{n\geq 1}\frac{1}{nz^n}\sum_{w\in\mathcal{W}_n} e^{V_w}\right)$$
$$= \prod_{\gamma\in\Gamma}\left(1 - e^{\cancel{|\gamma|(V_\gamma - \ln z)}}\right). \qquad e^{V_\gamma - |\gamma| \ln z}$$

For $|z| > |z_1|$, the sum and the product are convergent hence $\zeta_{BL}(z)$ is holomorphic and non zero. $\zeta_{BL}(z)$ has a holomorphic extension on $\mathbb{C}\setminus\{0\}$ given by

$$\zeta_{BL}(z) = z^{-N} d(z).$$

Where $d(z)$ is given by (2.1). Hence the zeroes of $\zeta_{BL}(z)$ on $\mathbb{C}\setminus\{0\}$ are the eigenvalues of $\mathcal{L}$.

Proof. We have to express the spectral determinant $d(z)$ in terms of $\mathrm{Tr}(\mathcal{L}^n)$. We will use the series $\log(1-x) = -\sum_{n\geq 1}\frac{1}{n}x^n$ if $|x| < 1$. If $z \neq 0$, we write

$$d(z) = \prod_{j=1}^{N}(z - z_j) = z^N \prod_{j=1}^{N}\left(1 - \frac{z_j}{z}\right)$$
$$= z^N \exp\left(\sum_j \log\left(1 - \frac{z_j}{z}\right)\right)$$
$$= z^N \exp\left(-\sum_j\sum_{n\geq 1}\frac{1}{nz^n}z_j^n\right) = z^N \exp\left(-\sum_{n\geq 1}\frac{1}{nz^n}\mathrm{Tr}(\mathcal{L}^n)\right)$$
$$= z^N \exp\left(-\sum_{n\geq 1}\frac{1}{nz^n}\sum_{w\in\mathcal{W}_n} e^{V_w}\right)$$

where the series is convergent if $\left|\frac{z_j}{z}\right| < 1, \forall j$, i.e. if $|z| > |z_1|$. We have

$$-\sum_{n\geq 1}\frac{1}{nz^n}\sum_{w\in\mathcal{W}_n} e^{V_w} = -\sum_{n\geq 1}\frac{1}{nz^n}\sum_{\gamma\in\Gamma}\ \sum_{k\geq 1, s.t. n=k|\gamma|}\ \sum_{i\in\gamma} e^{\cancel{n}V_\gamma} \quad k$$
$$= -\sum_{\gamma\in\Gamma}\sum_{k\geq 1}\frac{1}{k|\gamma| z^{k|\gamma|}}|\gamma| e^{k\cancel{|\gamma|}V_\gamma}$$
$$= -\sum_{\gamma\in\Gamma}\sum_{k\geq 1}\frac{\left(e^{V_\gamma - \ln z}\right)^{k|\gamma|}}{k} = \sum_{\gamma\in\Gamma}\log\left(1 - e^{|\gamma|\left(\frac{V_\gamma}{|\gamma|} - \ln z\right)}\right)$$
$$= \log\prod_{\gamma\in\Gamma}\left(1 - e^{|\gamma|\left(\frac{V_\gamma}{|\gamma|} - \ln z\right)}\right)$$

hence

$$d(z) = z^N \prod_{\gamma \in \Gamma} \left(1 - e^{|\gamma| \left(\frac{V_\gamma}{|\gamma|} - \ln z\right)}\right)$$

□

2.3 Special case of an adjacency matrix. Counting periodic orbits.

We suppose that $\mathcal{L}$ is an adjacency matrix, i.e. $\forall i, j, \mathcal{L}_{j,i} = 0$ or $\mathcal{L}_{j,i} = 1 = e^0$. Hence, this corresponds to the potential function $V_{j,i} = 0$.

Example 2.9. *Let* $\mathcal{L} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. *Then*

$$d(z) = \det \begin{pmatrix} z-1 & -1 \\ -1 & z \end{pmatrix} = z^2 - z - 1 = (z - z_1)(z - z_2)$$

with

$$z_1 = \frac{1}{2}\left(1 + \sqrt{5}\right) = 1.62\ldots, \qquad z_2 = \frac{1}{2}\left(1 - \sqrt{5}\right) = -0.62\ldots.$$

Let us assume that the matrix $\mathcal{L}$ is **mixing** (or primitive or a-periodic), i.e.:

$$\exists n > 0, \forall i, j, \quad \left(\mathcal{L}^n\right)_{j,i} > 0.$$

Then from Perron-Frobenius Theorem (for $N \geq 2$), the eigenvalue $z_1 > 1$ is positive, simple and dominant:

$$\forall j > 1, \quad \frac{|z_j|}{z_1} \leq \kappa < 1.$$

where $\kappa = \frac{|z_2|}{|z_1|} < 1$ is called the **spectral gap**.

Proposition 2.10. *The* ***number of marked periodic orbits*** *of period n is*

$$\mathcal{N}(n) := \sharp \mathcal{W}_n = \mathrm{Tr}\mathcal{L}^n = \sum_{j=1}^{n} z_j^n = z_1^n \left(1 + \sum_{j>1} \left(\frac{z_j}{z_1}\right)^n\right),$$
$$\underset{n\gg 1}{=} e^{nh_{\mathrm{top}}}\left(1 + O\left(\kappa^n\right)\right)$$

with

$$h_{\mathrm{top}} := \ln(z_1) > 0$$

called the ***topological entropy.*** *The* ***number of primitive periodic orbits*** *is*

$$\pi(n) := \sharp\{\gamma \in \Gamma, |\gamma| \leq n\}$$

We have

$$\mathcal{N}(n) = \sum_{d\backslash n} \frac{n}{d}\pi\left(\frac{n}{d}\right)$$

where the sum is over the divisors d of n and conversely

$$\pi(n) = \frac{1}{n}\sum_{d\backslash n} \mu(d)\,\mathcal{N}\left(\frac{n}{d}\right)$$
$$\underset{n\gg 1}{=} \frac{e^{nh_{\mathrm{top}}}}{n}\left(1 + O\left(\kappa^n\right) + O\left(e^{-\frac{n}{2}h_{\mathrm{top}}}\right)\right)$$

with the Möebius function $\mu(d) \in \{-1, 0, 1\}$ defined by $\mu(1) = 1$ and $\sum_{d\backslash n} \mu(d) = 0$ if $n \geq 2$.

Let us compute the first values of μ. We have $\mu(1) = 1$. We have $\mu(1)+\mu(2) = 0$ hence $\mu(2) = -1$. We have $\mu(1) + \mu(3) = 0$ hence $\mu(3) = -1$. We have $\mu(1) + \mu(2) + \mu(4) = 0$ hence $\mu(4) = 0$. Etc. Here are the first values of $\mu(n)$:

n	1	2	3	4	5	6	7	...
$\mu(n)$	1	-1	-1	0	-1	1	-1	...

Remark 2.11. As the proof below shows, the spectrum $(z_j)_j$ is directly related to the marked periodic orbits and we need some arithmetic to deduce information about the primitive periodic orbits.

Proof. The function $\mathcal{N}(n)$ is analogous to the "**Riemann counting function**" for counting the prime numbers. We have

$$\mathcal{N}(n) \underset{(2.3)}{=} \sum_{\gamma\in\Gamma} |\gamma| \sum_{k\geq 1} \delta_{n=k|\gamma|} = n\pi(n) + \frac{n}{2}\pi\left(\frac{n}{2}\right) + \frac{n}{3}\pi\left(\frac{n}{3}\right) + \dots$$
$$= \sum_{d\backslash n} \frac{n}{d}\pi\left(\frac{n}{d}\right)$$

Let $P(n) := \frac{1}{n}\sum_{d\backslash n}\mu(d)\mathcal{N}\left(\frac{n}{d}\right)$. We want to show that $P(n) = \pi(n)$. We have

$$\begin{aligned}
\sum_{d_1\backslash n}\frac{n}{d_1}P\left(\frac{n}{d_1}\right) &= \sum_{d_1\backslash n}\left(\frac{n}{d_1}\right)\left(\frac{d_1}{n}\right)\sum_{d_2\backslash(n/d_1)}\mu(d_2)\mathcal{N}\left(\frac{n}{d_2d_1}\right)\\
&= \sum_{m\backslash n}\sum_{d_2\backslash m, d_1=\frac{m}{d_2}}\mu(d_2)\mathcal{N}\left(\frac{n}{d_2d_1}\right)\\
&= \sum_{m\backslash n}\mathcal{N}\left(\frac{n}{m}\right)\sum_{d_2\backslash m}\mu(d_2) = \sum_{m\backslash n}\mathcal{N}\left(\frac{n}{m}\right)\delta_{m=1} = \mathcal{N}(n)
\end{aligned}$$

Hence $P(n) = \pi(n)$. We deduce that for $n \gg 1$,

$$\begin{aligned}
\pi(n) &= \frac{e^{nh_{\mathrm{top}}}}{n}\left(1 + O(\kappa^n)\right) + O\left(\frac{e^{\frac{n}{2}h_{\mathrm{top}}}}{n}\right)\\
&= \frac{e^{nh_{\mathrm{top}}}}{n}\left(1 + O(\kappa^n) + O\left(e^{-\frac{n}{2}h_{\mathrm{top}}}\right)\right)
\end{aligned}$$

□

Example 2.12. Let $\mathcal{L} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. *We have* $\mathcal{L}^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $\mathcal{L}^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$, $\mathcal{L}^4 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$. The number of marked periodic orbits is then

$$\begin{gathered}
\mathcal{N}(1) = \mathrm{Tr}(\mathcal{L}) = 1, \quad \mathcal{N}(2) = \mathrm{Tr}\left(\mathcal{L}^2\right) = 3, \quad \mathcal{N}(3) = \mathrm{Tr}\left(\mathcal{L}^3\right) = 4,\\
\mathcal{N}(4) = \mathrm{Tr}\left(\mathcal{L}^4\right) = 7, \quad \mathcal{N}(5) = \mathrm{Tr}\left(\mathcal{L}^5\right) = 11, \ldots
\end{gathered}$$

We deduce the number of primitive periodic orbits

$$\begin{aligned}
&\pi(1) = \mu(1)\mathcal{N}(1) = 1, \quad \pi(2) = \frac{1}{2}\left(\mu(1)\mathcal{N}(2) + \mu(2)\mathcal{N}\left(\frac{2}{2}\right)\right) = \frac{1}{2}(3-1) = 1,\\
&\pi(3) = \frac{1}{3}\left(\mu(1)\mathcal{N}(3) + \mu(3)\mathcal{N}\left(\frac{3}{3}\right)\right) = \frac{1}{3}(4-1) = 1,
\end{aligned}$$

$$\begin{aligned}
\pi(4) &= \frac{1}{4}\left(\mu(1)\mathcal{N}(4) + \mu(2)\mathcal{N}\left(\frac{4}{2}\right) + \mu(4)\mathcal{N}\left(\frac{4}{4}\right)\right) = \frac{1}{4}(7-3+0) = 1,\\
\pi(5) &= \frac{1}{5}\left(\mu(1)\mathcal{N}(5) + \mu(5)\mathcal{N}\left(\frac{5}{5}\right)\right) = \frac{1}{5}(11-1) = 2, \ldots
\end{aligned}$$

as we found before.

The matrix $\mathcal{L}$ is mixing and $h_{\mathrm{top}} = \log(z_1) = 048\ldots > 0$. Hence $\mathcal{N}(n) \sim z_1^n = (1.62\ldots)^n$ grows exponentially fast. The number of primitive periodic orbits is

$$\pi_n \sim \frac{e^{nh_{\mathrm{top}}}}{n} = \frac{z_1^n}{n} = \frac{1}{n}(1.62\ldots)^n$$

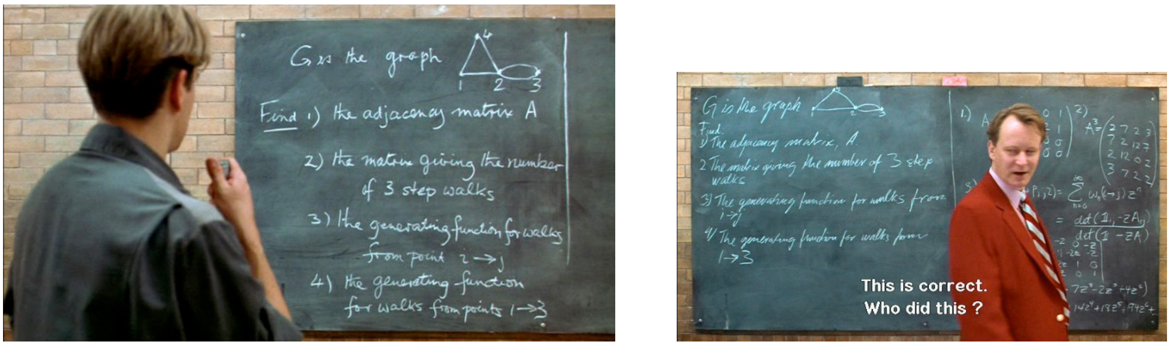
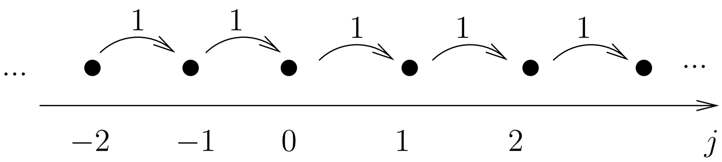


Figure 2.1: Extracts from the movie "Good will hunting" 1997. A cleaning man solved the problem in the corridors of MIT.

3 Finite rank perturbation of the shift operator

3.1 Ruelle spectrum of the perturbed shift operator

We start with the graph of the "right shift map" on $\mathbb{Z}$:



We associate the infinite matrix $\mathcal{L}_0 = (\mathcal{L}_0)_{i,j\in\mathbb{Z}}$ with zero elements except for $(\mathcal{L}_0)_{j+1,j} = 1$ for every $j \in \mathbb{Z}$. Then we consider another infinite matrix $\mathcal{L}_1 = (\mathcal{L}_1)_{i,j\in\mathbb{Z}}$ with finite rank, i.e. there exists $N \geq 0$ such that $(\mathcal{L}_1)_{i,j} \neq 0 \Rightarrow 1 \leq i, j \leq N$. Finally we consider the sum

$$\mathcal{L} := \mathcal{L}_0 + \mathcal{L}_1.$$

Proposition 3.1. *We denote the spectrum of the $N \times N$ matrix $(\mathcal{L}_1)_{i,j=1\ldots N}$ by $\mathcal{L}_1 u_k^+ = z_k^+ u_k^+$ with $k \in \{1, \ldots N\}$, eigenvalues $z_k^+ \in \mathbb{C}$ and eigenvectors $u_k^+ \in \mathbb{C}^N$, then for each k such that $z_k^+ \neq 0$ we associate a formal eigenvector of $\mathcal{L}$:*

$$U_k^+ = \left(\underbrace{\ldots 0}_{-\infty\to 0}, \underbrace{u_k^+}_{1\to N}, \underbrace{v^+}_{N+1\to+\infty} \right) \in \mathbb{C}^{\mathbb{Z}}$$

where $v^+ = \left(v_j^+\right)_{j\geq N+1}$ has components $v_j^+ = \frac{1}{\left(z_k^+\right)^{j-N}} \left(u_k^+\right)_N$. Then

$$\mathcal{L}U_k^+ = z_k^+ U_k^+$$

If $\left|z_k^+\right| > 1$ then $U_k^+ \in l^2(\mathbb{Z})$.

Example 3.2. Let $w_0, w_1 \in \mathbb{C}\backslash 0$. Let $\mathcal{L}_1 = (\mathcal{L}_1)_{i,j\in\mathbb{Z}}$ with zero elements except for $(\mathcal{L}_1)_{0,0} = w_0 \in \mathbb{C}\backslash\{0\}$, $(\mathcal{L}_1)_{2,0} = -w_1^{-1} \in \mathbb{C}$. The sum $\mathcal{L} := \mathcal{L}_0 + \mathcal{L}_1$ is represented by the following graph.

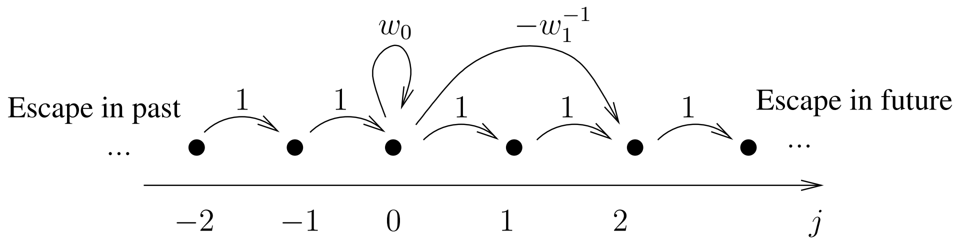


Figure 3.1: Graph associated to the operator $\mathcal{L}$ in (3.1).

We have

$$\mathcal{L} := \begin{pmatrix} \ddots & & & & \\ \ddots & 0 & & & \\ & 1 & w_0 & & \\ & & 1 & 0 & \\ 0 & & -w_1^{-1} & 1 & \ddots \end{pmatrix}, \qquad \mathcal{L}^{-1} := \begin{pmatrix} \ddots & 1 & & 0 & \\ & 0 & 1 & -w_0 & \\ & & 0 & 1 & \\ 0 & & 0 & w_1^{-1} & \ddots \\ & & & & \ddots \end{pmatrix} \tag{3.1}$$

The matrix $\mathcal{L}$ has the following eigenvectors U and V:

- $\mathcal{L}U = w_0 U$ with vector $U = \left(\dots, 0, U_0, \frac{1}{w_0}U_0, \frac{1}{w_0^j}\left(1 - \frac{w_0}{w_1}\right)U_0, \dots\right) \in \mathbb{C}^{\mathbb{Z}}$, i.e. zero components $U_j = 0$ for $j < 0$, $U_0 \in \mathbb{C}$, $U_1 = \frac{1}{w_0}U_0$, $U_j = \frac{1}{w_0^j}\left(1 - \frac{w_0}{w_1}\right)U_0$ for $j \geq 2$. Hence $U \in l^2(\mathbb{Z})$ if and only if $|w_0| > 1$.
- $\mathcal{L}V = w_1 V$ with vector $V = \left(\dots, w_1^{1-j}\left(1 - \frac{w_0}{w_1}\right)V_1, w_1 V_1, V_1, 0, \dots\right) \in \mathbb{C}^{\mathbb{Z}}$, i.e. components $V_1 \in \mathbb{C}$, $V_0 = w_1 V_1$, $V_j = w_1^{1-j}\left(1 - \frac{w_0}{w_1}\right)V_1$ for $j \leq -1$ and zero components $V_j = 0$ for $j \geq 2$. Hence $V \in l^2(\mathbb{Z})$ if and only if $|w_1| < 1$.

Definition 3.3. Let $r \in \mathbb{R}$ and the function $W : j \in \mathbb{Z} \to W(j) = \exp(-rj)$ called an **escape function** or **Lyapunov function** because $\frac{W(j+1)}{W(j)} = e^{-r} < 1$ if $r > 0$, i.e. W decays along the trajectories of the right shift map. We define

$$\mathcal{H}_W(\mathbb{Z}) := \mathrm{Diag}(W)^{-1}\left(l^2(\mathbb{Z})\right)$$

called **anisotropic Sobolev space** with **weight** $W(j) = e^{-rj}$. This means that the norm of a vector $u \in \mathcal{H}_W(\mathbb{Z})$ is

$$\|u\|^2_{\mathcal{H}_W(\mathbb{Z})} := \|\mathrm{Diag}(W)\,u\|^2_{l^2(\mathbb{Z})} = \sum_{j\in\mathbb{Z}} \left|e^{-rj}u_j\right|^2 \tag{3.2}$$

Remark 3.4. In other words we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}_W(\mathbb{Z}) & \xrightarrow{\mathcal{L}} & \mathcal{H}_W(\mathbb{Z}) \\ {\scriptstyle \mathrm{Diag}(W)}\downarrow & & {\scriptstyle \mathrm{Diag}(W)}\downarrow \\ l^2(\mathbb{Z}) & \xrightarrow{\mathcal{L}_W:=\mathrm{Diag}(W)\circ\mathcal{L}\circ\mathrm{Diag}(W)^{-1}} & l^2(\mathbb{Z}) \end{array}$$

where $\mathrm{Diag}(W) : \mathcal{H}_W(\mathbb{Z}) \to l^2(\mathbb{Z})$ is an isometry (by definition) and

$$\mathcal{L}_W := \mathrm{Diag}(W) \circ \mathcal{L} \circ \mathrm{Diag}\left(W^{-1}\right).$$

Observe that $\mathcal{L}_{0,W} := \mathrm{Diag}(W) \circ \mathcal{L}_0 \circ \mathrm{Diag}(W^{-1})$ has non zero matrix elements only on a line at $(\mathcal{L}_{0,W})_{j+1,j} = \frac{W(j+1)}{W(j)} = e^{-r}$. From Shur test, the operator $\mathcal{L}_{0,W} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ has essential spectrum on the circle of radius e^{-r}. Equivalently the operator $\mathcal{L}_0 : \mathcal{H}_W(\mathbb{Z}) \to \mathcal{H}_W(\mathbb{Z})$ has essential spectrum on the circle of radius e^{-r}. We refer to [4, chap. 1][20, p.51] for the spectrum of Toeplitz operators.

Proposition 3.5. *The operator $\mathcal{L} : \mathcal{H}_W(\mathbb{Z}) \to \mathcal{H}_W(\mathbb{Z})$ has essential spectrum on the circle of radius e^{-r} and **discrete spectrum** outside given by eigenvalues $\left(z_k^+\right)_k$ for which $\left|z_k^+\right| > e^{-r}$. The eigenvector is $U_k^+ \in \mathcal{H}_W(\mathbb{Z})$. More generally eigenspaces associated to those eigenvalues z_k^+ belong to $\mathcal{H}_W(\mathbb{Z})$ and have support on the positive half-line $[0, 1, \ldots + \infty[$. They do not depend on $\mathcal{H}_W(\mathbb{Z})$ or r. The eigenvalues $\left(z_k^+\right)_k$ are called the **future Ruelle spectrum** of $\mathcal{L}$.*

For the example 3.2, w_0 is an eigenvalue with eigenvector $U \in \mathcal{H}_W(\mathbb{Z})$ if and only $|w_0| > e^{-r}$. w_1 is an eigenvalue with eigenvector $V \in \mathcal{H}_W(\mathbb{Z})$ if and only $|w_1| < e^{-r}$. See Figure 3.2.

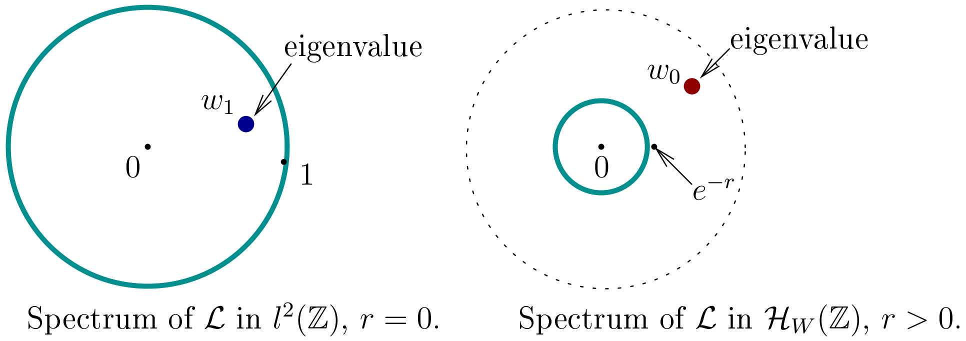


Figure 3.2: In this picture we suppose $|w_0| = |w_1| < 1$. The green circle of radius e^{-r} is the essential spectrum of $\mathcal{L}$ in the space $\mathcal{H}_W(\mathbb{Z})$ that depends on $r \in \mathbb{R}$. As $r \to +\infty$ this circle shrinks to zero and we reveal the intrinsic "future discrete spectrum" of $\mathcal{L}$, in red, here this is eigenvalue w_0, as soon as $e^{-r} < |w_0|$. As $r \to -\infty$ this circle goes to infinity and we reveal the intrinsic "past discrete spectrum" of $\mathcal{L}$, in blue, here this is eigenvalue w_1, as soon as $|w_1| < e^{-r}$.

3.2 Flat trace and periodic orbits

Remark 3.6. We have $\sum_j (\mathcal{L}_0)_{j,j} = 0$ but the operator $\mathcal{L}_0$ is not trace class because $\mathcal{L}_0^*\mathcal{L}_0 = \mathrm{Id}$, $\mathrm{Tr}(\mathcal{L}_0^*\mathcal{L}_0) = \infty$. For $n \geq 1$, the operator $\mathcal{L}^n = (\mathcal{L}_0 + \mathcal{L}_1)^n$ is not trace class neither. However $\sum_j (\mathcal{L}^n)_{j,j}$ is finite.

As before, from the operator $\mathcal{L}$ we associate a graph G with vertices $j \in \mathbb{Z}$ and oriented edges $i \to j$ if $\mathcal{L}_{j,i} = e^{V_{j,i}} \neq 0$, with $V_{j,i} \in \mathbb{C}$. See Figure 3.1 for example. As in Definition 2.4, we introduce the notations $\mathcal{W}_n$ for marked periodic orbit of period n and Γ for primitive periodic orbits. For every marked periodic orbit $w \in \mathcal{W}_n$ on the graph G we associate the Birkhoff sum V_w, (2.2).

Proposition 3.7. *"Trace formula for perturbed shift operator".* *For $n \geq 1$, we define the "**flat trace**":*

$$\mathrm{Tr}^\flat \mathcal{L}^n := \sum_j (\mathcal{L}^n)_{j,j}$$

that is finite and we have

$$\mathrm{Tr}^\flat (\mathcal{L}^n) = \sum_{w \in \mathcal{W}_n} e^{V_w} = \sum_{j=1}^{N} \left(z_j^+\right)^n$$

where the second sum is over marked periodic orbits in the graph G and last sum is over future Ruelle resonances.

The strategy of the following proof follows the idea used in the papers [2],[15, 16]. This strategy can be used for Anosov diffeomorphisms and Anosov flows later. However a simpler proof could be obtained for Lemma 3.7, by taking a more efficient escape function.

Proof. For every $n \geq 1$, we have $\mathrm{Tr}^\flat \mathcal{L}_0^n = 0$. If A is a bounded operator and B is trace class operator then AB, BA are trace class. Here $\mathcal{L}_1$ is finite rank hence trace class and $\mathcal{L}_0$ is bounded, so we deduce by induction that $\mathcal{L}^n = \mathcal{L}_0^n + C$ with C trace class. This gives $\mathrm{Tr}^\flat \mathcal{L}^n = \mathrm{Tr} C$. The proof of $\mathrm{Tr}^\flat (\mathcal{L}^n) = \sum_{w \in \mathcal{W}_n} e^{V_w}$ has been given for Lemma 2.7. We will prove $\mathrm{Tr}^\flat (\mathcal{L}^n) = \sum_{j=1}^N \left(z_j^+\right)^n$. Take $r > 0$ large enough so that $e^{-r} < \left|z_j^+\right|$ for every j such that $z_j^+ \neq 0$, i.e. so that every Ruelle resonance is outside the essential spectrum on the circle of radius e^{-r}. By taking a Riesz contour integral on the circle of radius $e^{-r+\epsilon}$ with arbitrary small $\epsilon > 0$, we obtain a spectral decomposition

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{R}$$

with $[\mathcal{L}_2, \mathcal{R}] = 0$ hence $\mathcal{L}^n = \mathcal{L}_2^n + \mathcal{R}^n$. Also we have using Shur test for the operator $\mathcal{R}$,

$$\exists C > 0, \forall n \geq 1, \|\mathcal{R}^n\| \leq C e^{n(-r+\epsilon)}$$

with C independent on r and $\mathrm{Tr}(\mathcal{L}_2^n) = \sum_{j=1}^N \left(z_j^+\right)^n$. We will show that

$$\left|\mathrm{Tr}^\flat (\mathcal{R}^n)\right| \leq C e^{n(-r+\epsilon)}. \tag{3.3}$$

Then we deduce that
$$\left|\mathrm{Tr}^{\flat}\left(\mathcal{L}^{n}\right)-\mathrm{Tr}\left(\mathcal{L}_{2}^{n}\right)\right|\leq Ce^{n(-r+\epsilon)},$$
and making $r\to+\infty$ gives $\mathrm{Tr}^{\flat}\left(\mathcal{L}^{n}\right)=\mathrm{Tr}\left(\mathcal{L}_{2}^{n}\right)=\sum_{j=1}^{N}\left(z_{j}^{+}\right)^{n}$. This finish the proof of Lemma 3.7.

To prove (3.3), write
$$\mathcal{R}=\mathcal{L}-\mathcal{L}_{2}=\mathcal{L}_{0}+\left(\mathcal{L}_{1}-\mathcal{L}_{2}\right)=A+B$$
with $A=\mathcal{L}_{0}$ and $B=\mathcal{L}_{1}-\mathcal{L}_{2}$ and we use Lemma 3.8 below. □

Lemma 3.8. *[2]Suppose that $R=A+B$ with operators A,B such that $\exists C>0,r\in\mathbb{R},\forall n\geq1,\mathrm{Tr}^{\flat}A^{n}=0$, $\left\Vert R^{n}\right\Vert \leq Ce^{-nr}$, $\left\Vert B\right\Vert_{\mathrm{Tr}}\leq C$, $\left\Vert A^{n}\right\Vert \leq Ce^{-nr}$. Then*
$$\left|\mathrm{Tr}^{\flat}\left(R^{n}\right)\right|\leq nCe^{-nr}.$$

Proof. First let us show by induction that for every $n\geq1$,
$$R^{n}=A^{n}+\sum_{k=0}^{n-1}A^{k}BR^{n-k-1}.$$
Indeed for $n=1$, $R=A+A^{0}BR^{0}=A+B$. And
$$\begin{aligned}R^{n+1}=R^{n}R=A^{n}\left(A+B\right)+\sum_{k=0}^{n-1}A^{k}BR^{n-k}\\ =A^{n+1}+A^{n}BR^{0}+\sum_{k=0}^{n-1}A^{k}BR^{n+1-k-1}\\ =A^{n+1}+\sum_{k=0}^{(n+1)-1}A^{k}BR^{(n+1)-k-1}\end{aligned}$$
Then
$$\left|\mathrm{Tr}^{\flat}\left(R^{n}\right)\right|\leq\sum_{k=0}^{n-1}\left\Vert A^{k}\right\Vert \left\Vert B\right\Vert_{\mathrm{Tr}}\left\Vert R^{n-k-1}\right\Vert \leq nCe^{-nr}.$$
□

Proposition 3.9. *We define the "**zeta function for perturbed shift operator**":*
$$\begin{aligned}\zeta\left(z\right):=\exp\left(-\sum_{n\geq1}\frac{1}{nz^{n}}\sum_{w\in\mathcal{W}_{n}}e^{V_{w}}\right)\\ =\prod_{\gamma\in\Gamma}\left(1-e^{|\gamma|(V_{\gamma}-\ln z)}\right)\end{aligned}$$
$\zeta\left(z\right)$ is holomorphic, non zero for $|z|>\left|z_{1}^{+}\right|$ and admits a holomorphic extension on $\mathbb{C}\backslash\left\{0\right\}$. The zeroes of $\zeta\left(z\right)$ on $\mathbb{C}\backslash\left\{0\right\}$ are the future Ruelle resonances $\left(z_{j}^{+}\right)_{j}$ of $\mathcal{L}$.

4 Hyperbolic fixed point on $\mathbb{R}$ (or $\mathbb{R}^n$)

In previous Sections we have considered a finite or infinite matrix $\mathcal{L}$. In this Section the operator $\mathcal{L}$ will be associated to a deterministic hyperbolic flow. However this will be the most simple example of deterministic hyperbolic flow and we will see how to extend the previous results to this case. This will prepare the strategy to handle general hyperbolic (or Anosov) flows later in Section 5.

4.1 Vector field and flow

Let $\lambda > 0$ and consider the linear vector field X on $\mathbb{R}$ with coordinate $x \in \mathbb{R}$ given by

$$X = \lambda x \frac{d.}{dx}$$

The flow map at time $t \in \mathbb{R}$ is

$$\phi^t : \begin{cases} \mathbb{R} & \rightarrow \mathbb{R} \\ x & \rightarrow \phi^t(x) = e^{\lambda t} x \end{cases}$$

since (by definition) for any function $u \in C^\infty(\mathbb{R}), \forall t \in \mathbb{R}, \forall x \in \mathbb{R}$, $\frac{du\left(\phi^t(x)\right)}{dt} = (Xu)\left(\phi^t(x)\right)$.

4.2 The transfer operator

The one-parameter group of **transfer operators** with parameter $t \in \mathbb{R}$ is

$$\mathcal{L}^t = e^{-tX} : \begin{cases} \mathcal{S}(\mathbb{R}) & \rightarrow \mathcal{S}(\mathbb{R}) \\ u(x) & \rightarrow \left(e^{-tX}u\right)(x) = \underbrace{u\left(\phi^{-t}(x)\right)}_{\text{transport}} \end{cases} \tag{4.1}$$

The choice e^{-tX} instead of e^{tX} is maybe not very natural but corresponds to the "push foward". See figure 4.1.

Remark 4.1. In particular the L^2-adjoint $\left(\mathcal{L}^t\right)^* \delta_x = \delta_{\phi^{-t}(x)}$ transports Dirac measures. So we loose no information with studying the transfer operator $\mathcal{L}^t$ instead of individual trajectoires $\phi^t(x)$.

For $t > 0$ and two functions $u, v \in \mathcal{S}(\mathbb{R})$, let us consider the matrix element of the operator $\mathcal{L}^t$:

$$\begin{aligned} \langle v|\mathcal{L}^t u\rangle_{L^2(\mathbb{R})} &= \int \overline{v(x)} u\left(e^{-\lambda t}x\right) dx \\ &\underset{\text{Taylor expansion}}{=} \int \overline{v(x)} \sum_{k\geq 0} \frac{1}{k!} u^{(k)}(0) \left(e^{-\lambda t}\right)^k x^k dx \\ &= \sum_{k\geq 0} e^{-k\lambda t} \langle v|x^k\rangle_{L^2} \langle \frac{1}{k!}\delta^{(k)}|u\rangle_{L^2} \end{aligned} \tag{4.2}$$

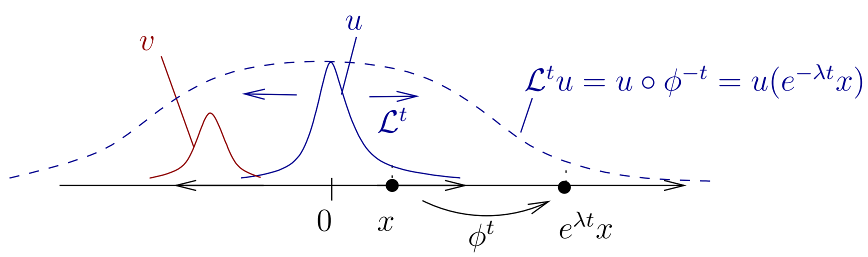


Figure 4.1: Illustration of the correlation function $\langle v|\mathcal{L}^t u\rangle_{L^2(\mathbb{R})}$: the evolved function $\mathcal{L}^t u$ is tested against function v.

where $\delta^{(k)}$ denotes the k-th derivative of the Dirac distribution. We have $\left(\frac{d^k x^l}{dx^k}\right)(0) = 0$ if $k \neq l$ and $= k!$ if $k = l$. Hence for $k, l \geq 0$

$$\langle \frac{1}{k!}\delta^{(k)}|x^l\rangle_{L^2} = \delta_{k=l}. \tag{4.3}$$

Let[3]

$$\Pi_k := |x^k\rangle\langle \frac{1}{k!}\delta^{(k)}|.\rangle_{L^2} \tag{4.4}$$

be a rank one operator. Then (4.3) implies that

$$\Pi_k \circ \Pi_l = \delta_{k=l}.\Pi_k$$

i.e. $(\Pi_k)_k$ is a family of rank one projectors and the Taylor expansion (4.2) writes:

$$\langle v|\mathcal{L}^t u\rangle_{L^2(\mathbb{R})} = \sum_{k\geq 0} e^{-k\lambda t}\langle v|\Pi_k u\rangle = \sum_{k\geq 0} e^{z_k t}\langle v|\Pi_k u\rangle \tag{4.5}$$

with

$$z_k = -k\lambda, \quad k \in \mathbb{N}.$$

Remark 4.2. Since $\mathcal{L}^t = e^{t(-X)}$, this relation suggests that the **numbers** $z_k = -k\lambda \in \mathbb{C}$ **are eigenvalues of the operator (the vector field)** $-X$ and that Π_k are the **associated rank one spectral projectors**, i.e. x^k are **eigenfunctions**.However this cannot be true in the Hilbert space $L^2(\mathbb{R})$ because $x^k \notin L^2(\mathbb{R})$ and $\frac{1}{k!}\delta^{(k)} \notin L^2(\mathbb{R})^* = L^2(\mathbb{R})$. The purpose of this Section is to construct a Hilbert space $\mathcal{H}_W(\mathbb{R})$ so that this is true, i.e. in which $-X$ has discrete spectrum given by $(z_k)_{k\in\mathbb{N}}$ in $\mathcal{H}_W(\mathbb{R})$. In fact in the space $L^2(\mathbb{R})$ the spectrum of $-X = -\lambda x\frac{d}{dx}$ is essential (continuous) spectrum on the line $\mathrm{Re}(z) = \frac{\lambda}{2}$. This is because $-X = -i\lambda H + \frac{\lambda}{2}$ with selfadjoint operator $H := \frac{1}{2}\left(x\left(-i\frac{d}{dx}\right) + \left(-i\frac{d}{dx}\right)x\right)$ and H has continuous spectrum on $\mathbb{R}$.

[3] $|x^k\rangle\langle\frac{1}{k!}\delta^{(k)}|$ is a notation (called "Dirac notation" in physics) for the rank one operator $x^k\langle\frac{1}{k!}\delta^{(k)}|.\rangle_{L^2}$.

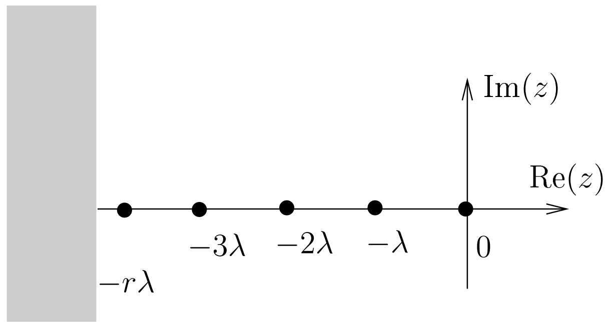


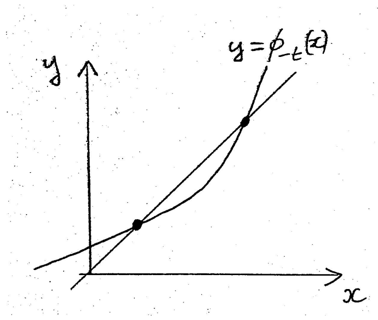
Figure 4.2: Spectrum of the vector field $(-X) : \mathcal{H}_W(\mathbb{R}) \to \mathcal{H}_W(\mathbb{R})$ is given by $z_k = -k\lambda, \quad k \in \mathbb{N}$ on $\mathrm{Re}(z) > -\lambda r$.

4.3 The flat trace

Let δ denotes the Dirac measure on $\mathbb{R}$ defined by $\forall u \in \mathcal{S}(\mathbb{R}), \delta(u) = u(0)$. We also write $\delta(u) = \int \delta(x) u(x) dx = u(0)$. We write the transfer operator for $u \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}\left(\mathcal{L}^t u\right)(x) &\underset{(4.1)}{=} \left(u\left(\phi^{-t}(x)\right)\right) = \int \delta\left(y - \phi^{-t}(x)\right) u(y)\, dy \\ &= \int K_t(x, y)\, u(y)\, dy\end{aligned}$$

with the distributional Schwartz kernel given by $K_t(x, y) = \delta(y - \phi^{-t}(x))$ (it is supported by the graph of the flow $y = \phi^{-t}(x)$).



Notice that for $t > 0$, the operator $\mathcal{L}^t : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is not trace class, however we can define its "flat trace" (as for a matrix) by the integral over the diagonal.

Theorem 4.3. ***"Atiyah-Bott trace formula for expanding flow on $\mathbb{R}$".*** *For $t > 0$, we define the flat trace of the operator $\mathcal{L}^t$ by*

$$\mathrm{Tr}^\flat\left(\mathcal{L}^t\right) := \int K_t(x, x)\, dx.$$

We get

$$\mathrm{Tr}^\flat\left(\mathcal{L}^t\right) = \frac{1}{|1 - e^{-\lambda t}|} = \sum_{k\geq 0} e^{z_k t}, \tag{4.6}$$

with

$$z_k = -k\lambda, \quad k \in \mathbb{N}.$$

Remark 4.4. From remark 4.2, the eigenvalues of $\mathcal{L}^t = e^{t(-X)}$ are $(e^{z_k t})$ hence (4.6) is what we expect naively (as for a matrix). The question (solved below) is again to find some specifi Sobolev space $\mathcal{H}_W(\mathbb{R})$ in which $(e^{z_k t})_k$ are eigenvalues of $\mathcal{L}^t$ for $t > 0$.

Proof. We have $K_t(x, x) = \delta\left(x - \phi^{-t}(x)\right) = \delta(f(x))$ with $f(x) = x - \phi^{-t}(x) = \left(1 - e^{-\lambda t}\right) x$. We do the change of variable $y = f(x)$ and use the series $\frac{1}{1-X} = \sum_{k\geq 0} X^k$ that gives

$$\begin{aligned}\mathrm{Tr}^\flat\left(\mathcal{L}^t\right) &:= \int_{\mathbb{R}} \delta(f(x))\, dx = \frac{1}{|f'(0)|} \int_{\mathbb{R}} \delta(y)\, dy = \frac{1}{|1 - e^{-\lambda t}|} \\ &= \sum_{k\geq 0} e^{-k\lambda t} = \sum_{k\geq 0} e^{z_k t}\end{aligned}$$

□

4.4 Microlocal analysis with wave-packets. Ruelle discrete spectrum

4.4.1 Wave packet transform

Let $(x, \xi) \in \mathbb{R}^2$. The following function of $y \in \mathbb{R}$

$$\varphi_{x,\xi}(y) := a e^{i\xi y} e^{-\frac{1}{2}|y-x|^2}$$

is called a **wave-packet** at position $(x, \xi) \in T^*\mathbb{R}$ (indeed $\xi : y \to \xi.y \in \mathbb{R}$ is seen as a linear form on $\mathbb{R}$ hence a cotangent vector). We take $a := \frac{1}{(2\pi)^{1/2}\pi^{1/4}}$ so that $\|\varphi_{x,\xi}\|_{L^2(\mathbb{R})} = 1$ (using Gaussian integral).

Remark 4.5. $|\varphi_{x,\xi}(y)| = a e^{-\frac{1}{2}|x-y|^2}$ is negligle if y is far from x. Taking the Fourier tranform $(\mathcal{F}\varphi_{x,\xi})(\xi') := \frac{1}{\sqrt{2\pi}} \int e^{-i\xi' y} \varphi_{x,\xi}(y)\, dy = a e^{-ix\xi'} e^{-\frac{1}{2}|\xi'-\xi|^2}$ we see that $|\mathcal{F}\varphi_{x,\xi}|(\xi') = a e^{-\frac{1}{2}|\xi'-\xi|^2}$ is negligible if ξ' is far from ξ.

To this family of wave-packets we associate the following metric on $T^*\mathbb{R}$:

$$g = dx^2 + d\xi^2$$

that is compatible with the symplectic form $\omega = dx \wedge d\xi$. g is called the **metric on phase space**.

Definition 4.6. The **wave-packet transform** is the operator

$$\mathcal{T} : \begin{cases} \mathcal{S}(\mathbb{R}) & \to \mathcal{S}(T^*\mathbb{R}) \\ u & \to \langle \varphi_{x,\xi} | u \rangle_{L^2(\mathbb{R})} \end{cases}$$

We consider the Hilbert spaces $L^2(\mathbb{R}, dx)$ and $L^2\left(T^*\mathbb{R}, \frac{dxd\xi}{2\pi}\right)$. Observe that L^2-adjoint operator is

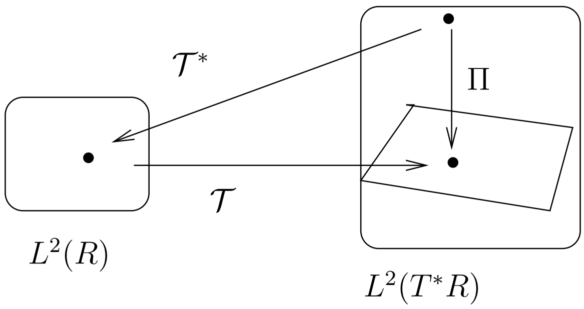
$$\mathcal{T}^* : \begin{cases} \mathcal{S}(T^*\mathbb{R}) & \to \mathcal{S}(\mathbb{R}) \\ v(x,\xi) & \to u(y) = \int v(x,\psi)\, \varphi_{x,\xi}(y) \frac{dxd\xi}{2\pi} \end{cases}$$

We have the important following formula

Lemma 4.7 ("**Resolution of identity**").

$$\mathcal{T}^* \circ \mathcal{T} = \mathrm{Id}$$

Proof. Compute the Schartz kernel of $\mathcal{T}^*\mathcal{T}$ using Gaussian integral. $\square$



Remark 4.8. $\mathcal{T} : L^2(\mathbb{R}) \to \mathrm{Im}(\mathcal{T}) \subset L^2(T^*\mathbb{R})$ is an isomorphism. Hence $\mathcal{T}$ **"lifts the analysis form $\mathbb{R}$ to $T^*\mathbb{R}$"**. The operator $\Pi = \mathcal{T} \circ \mathcal{T}^* : L^2(T^*\mathbb{R}) \to \mathrm{Im}(\mathcal{T})$ is an orthogonal projector because $\Pi^2 = \Pi$ and $\Pi^* = \Pi$. We have the "reconstruction formula" $\forall u \in C^\infty(M), \quad u(y) = (\mathcal{T}^*\mathcal{T}u)(y) = \int_{T^*\mathbb{R}} \varphi_{x,\xi}(y) \langle \varphi_{x,\xi} | u \rangle \frac{dxd\xi}{2\pi}$.

4.4.2 Propagation of singularities

We want to "analysis the operator $\mathcal{L}^t$ on the cotangent space $T^*\mathbb{R}$". We consider the commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{L}^t = e^{-tX}} & L^2(\mathbb{R}) \\ \mathcal{T}\big\downarrow & & \mathcal{T}\big\downarrow \\ L^2(T^*\mathbb{R}) & \xrightarrow{\tilde{\mathcal{L}}^t := \mathcal{T} \circ \mathcal{L}^t \circ \mathcal{T}^*} & L^2(T^*\mathbb{R}) \end{array} \tag{4.7}$$

We have a second important Lemma that concerns the Schwartz kernel of the lifted operator $\tilde{\mathcal{L}}^t$:

Lemma 4.9 (**"Micro-locality of the transfer operator"**). $\forall t \geq 0, \forall N > 0, \exists C_{N,t} > 0, \forall \rho, \rho' \in T^*\mathbb{R}$,

$$\left| \langle \delta_{\rho'} | \tilde{\mathcal{L}}^t \delta_\rho \rangle_{L^2(T^*\mathbb{R})} \right| = \left| \langle \varphi_{\rho'} | \mathcal{L}^t \varphi_\rho \rangle_{L^2(\mathbb{R})} \right| \leq C_{N,t} \left\langle \mathrm{dist}_g \left(\rho', \tilde{\phi}^t (\rho) \right) \right\rangle^{-N}$$

with

$$\tilde{\phi}^t : (x, \xi) \to \left(e^{\lambda t} x, e^{-\lambda t} \xi \right)$$

is the canonical lift of ϕ^t on $T^*\mathbb{R}$.

This Lemma means that the Schwartz kernel of $\tilde{\mathcal{L}}^t$ is negligible outside the graph of the lifted flow $\tilde{\phi}^t : T^*\mathbb{R} \to T^*\mathbb{R}$. In particular, if $t = 0$, it shows that wave packets are almost orthogonal to each other if they are far from each other with respect to the metric g on phase space $T^*\mathbb{R}$.

Proof. □

5 Anosov flow

Uniformly hyperbolic dynamics (Anosov or Axiom A) have "sensitivity to initial conditions" and manifest "deterministic chaotic behavior", e.g. mixing, statistical properties etc. In the 70', David Ruelle, Rufus Bowen and others have introduced a functional and spectral approach in order to study these dynamics which consists in describing the evolution not of individual trajectories but of functions, and observing the convergence towards equilibrium in the sense of distribution. This approach has progressed and these last years, it has been shown by V. Baladi, C. Liverani, M. Tsujii and others that this evolution operator ("transfer operator") has a discrete spectrum, called "Ruelle-Pollicott resonances" which describes the effective convergence and fluctuations towards equilibrium.

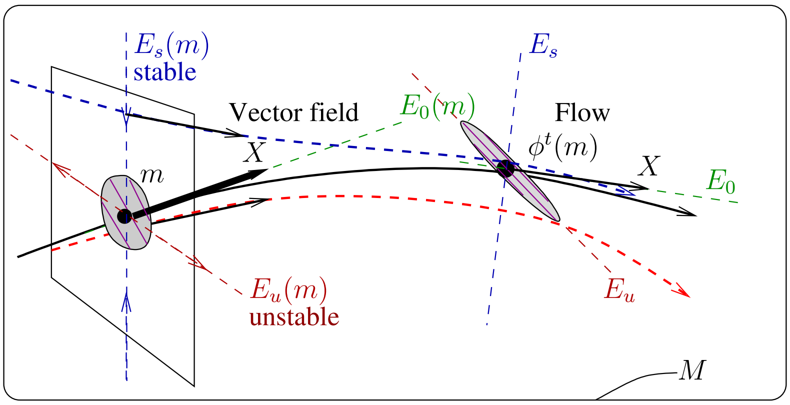
Due to hyperbolicity, the chaotic dynamics sends the information towards small scales (high Fourier modes) and technically it is convenient to use "microlocal analysis" which permits to treat fast oscillating functions. More precisely it is appropriate to consider the dynamics lifted in the cotangent space T^*M of the initial manifold M (this is an Hamiltonian flow). We observe that at fixed energy, this lifted dynamics has a relatively compact non-wandering set called the trapped set and that this lifted dynamics on T^*M scatters on this trapped set. Then the existence and properties of the Ruelle-Pollicott spectrum enters in a more general theory of semiclassical analysis developed in the 80' by B. Helffer and J. Sjöstrand called "quantum scattering on phase space".

5.1 Definitions

Let M be a closed manifold. Let X be a C^∞ **vector field** on M. The **flow** is $\phi^t = e^{tX}$ with $t \in \mathbb{R}$.

Definition 5.1. X is **Anosov** if there exists a continuous splitting, $\forall m \in M,\ T_m M = E_u(m) \oplus E_s(m) \oplus \underbrace{E_0(m)}_{\mathbb{R}X}$,

$$\exists C > 0, \lambda > 0, \forall t \geq 0, m \in M, \quad \left\| D\phi^t_{/E_s(m)} \right\|_g \leq C e^{-\lambda t}, \quad \left\| D\phi^{-t}_{/E_u(m)} \right\|_g \leq C e^{-\lambda t},$$



this **"sensitivity to initial conditions"** will generates "**chaos**" (confusion, unpredictability).

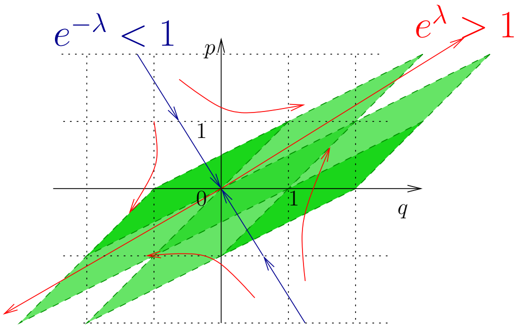
Proposition 5.2. *Anosov property is **stable** under any (small C^1) perturbation of X. The maps $m \to E_u(m), E_s(m), E_u(m) \oplus E_s(m)$ are **Hölder-continuous** with some respective exponents $0 < \beta_u, \beta_s, \beta_0 < 1$.*

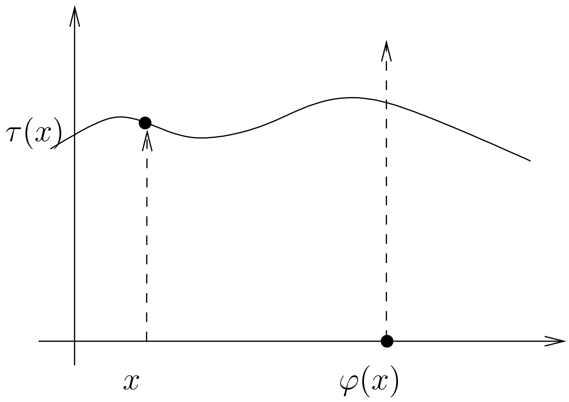
5.2 Examples

5.2.1 Suspension of a Anosov diffeomorphism

Suppose $\varphi : N \to N$ is an Anosov diffeomorphism. For example the "cat map"

$$\varphi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2_{q,p} = (\mathbb{R}/\mathbb{Z})^2 \to \mathbb{T}^2_{q,p}$$

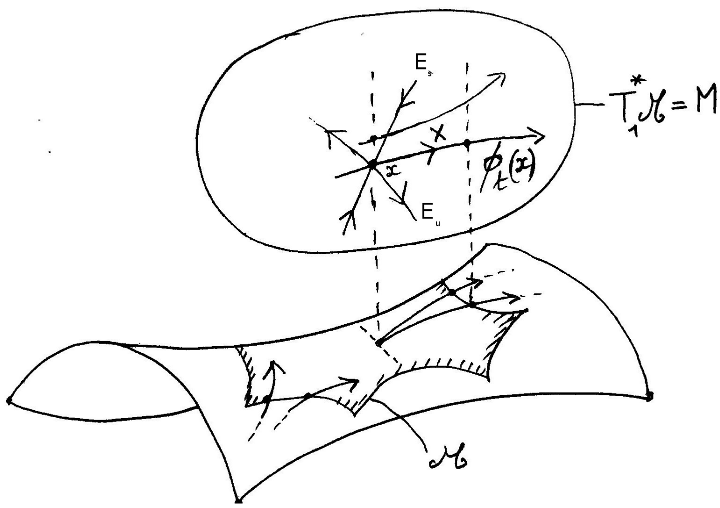




Suppose $\tau : N \to \mathbb{R}^+$ (called roof function). Let $M := (N \times \mathbb{R}) / \sim$ with the relation $\forall x \in N, \forall z \in \mathbb{R}, (x, z + \tau(x)) \sim (\varphi(x), z)$. Then $X = \frac{\partial}{\partial z}$ is an **Anosov vector field on** M**.**

5.2.2 Special example (contact)

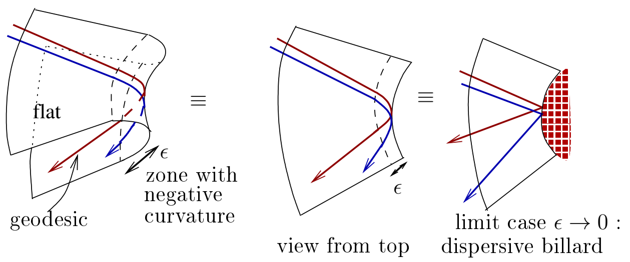
Let $(\mathcal{M}, g)$ a Riemannian manifold of negative curvature. $\dim \mathcal{M} = d + 1$. The **Geodesic flow** on $M = (T^*\mathcal{M})_1$ is **Anosov**. $\dim M = 2d + 1$. $\dim E_u = \dim E_s = d$. A more special example is a surface of constant curvature. In that case, $\mathcal{M} = \Gamma \setminus \underbrace{(\mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2)}_{\mathbb{H}^2}$, $M = (T^*\mathcal{M})_1 = \Gamma \backslash \mathrm{SL}_2(\mathbb{R})$. More details below. Possible generalization to $\Gamma \backslash SO_{1,n}/SO_{n-1}$, $n \geq 3$. On the following picture, $d = 1$, $\mathcal{M}$ is a surface with negative Gauss curvature:

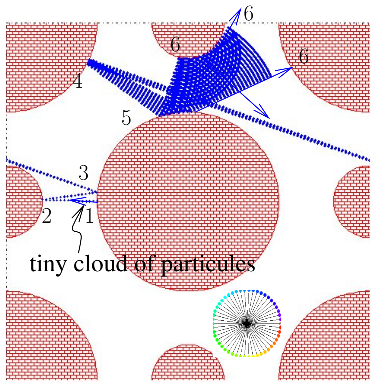


Remark 5.3. Geodesic flow is a very special example because it preserves the canonical Liouville one form $\mathcal{A} = \sum_j \xi_j dx^j$ on M hence $E_u \oplus E_s = \mathrm{Ker}(\mathcal{A})$ is **smooth** and **contact** (i.e. a maximally non integrable distribution). Equivalently $E_0^* = (E_u \oplus E_s)^\perp \subset T^*M$ is smooth and symplectic.

5.2.3 Extreme special example (Sinai billard)

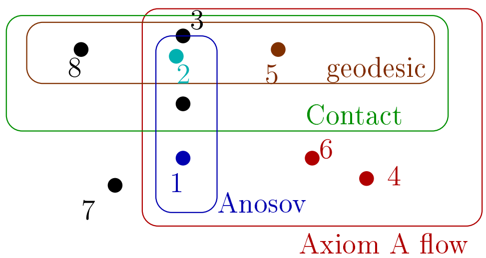
Sinai dispersive billardis a limit case of **Anosov geodesic flow,** where the negative curvature is concentrated on the boundaries:





- See **movie** "Anosov linkage" by **Mickael Kourganoff, Jos Leys** (2015).
- See **movie** showing the chaotic dynamics in a dispersive Sinai billard.

5.2.4 Other examples (and non examples) of hyperbolic dynamics



1: suspension of Anosov diffeom.
2: geodesic flow in $\kappa < 0$
3: Sinai billard

4: Horse shoe flow
5: geodesic flow on Schottky surface.
6: Morse Smale flow

7: Lorenz flow
8: general geodesic flow

Remark 5.4. Microlocal analysis explained below has been be extended to more general hyperbolic dynamics than Anosov: Axiom A (Dyatlov-Guillarmou 14 [9]).

5.3 Transfer operators

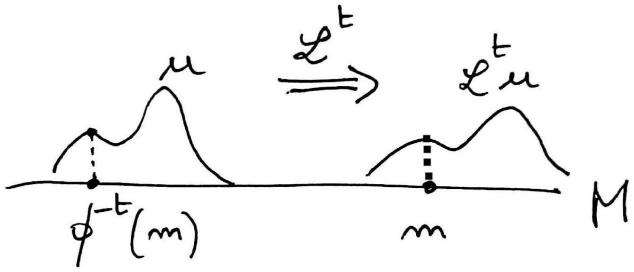
Definition 5.5. Let X an Anosov vector field on M. Let $V \in C^\infty(M;\mathbb{C})$ ("potential"). The first order differential operator on $C^\infty(M)$

$$A := -X + V = -\sum_{j=1}^{\dim M} X_j(x) \frac{\partial}{\partial x^j} + V \tag{5.1}$$

generates the one-parameter group of **transfer operators** $\mathcal{L}^t = e^{tA}$**,** $t \in \mathbb{R}$**,**

$$\mathcal{L}^t : u \in C^\infty(M) \to e^{tA}u = \underbrace{e^{V_{[-t,0]}(m)}}_{\text{amplitude}} \cdot \underbrace{\left(u \circ \phi^{-t}\right)}_{\text{transport}} \in C^\infty(M) \tag{5.2}$$

with the **Birkhoff sum** $V_{[-t,0]}(m) := \int_{-t}^{0} V(\phi^s(m))\, ds$.



Remark 5.6. In particular the L^2-adjoint $(\mathcal{L}^t)^* \delta_m = e^{V_{[-t,0]}(m)} \delta_{\phi^{-t}(m)}$ transports Dirac measures. So we loose no information with studying the transfer operator $\mathcal{L}^t$ instead of individual trajectoires $\phi^t(m)$.

Remark 5.7. More generally we can consider any (lifted) action of the flow on Sections $C^\infty(M;E)$ of a vector bundle $E \to M$.

5.4 Ruelle spectrum in two examples

Ruelle spectrum of resonances will be the discrete spectrum of the operator A in some specific anisotropic Sobolev space $\mathcal{H}_W(M)$. The spectrum and its eigenspace do not depend on the space. Before giving the general definition of Ruelle spectrum for hyperbolic dynamics we describe here the spectrum for two special examples.

5.4.1 Suspension of an Anosov diffeomorphism

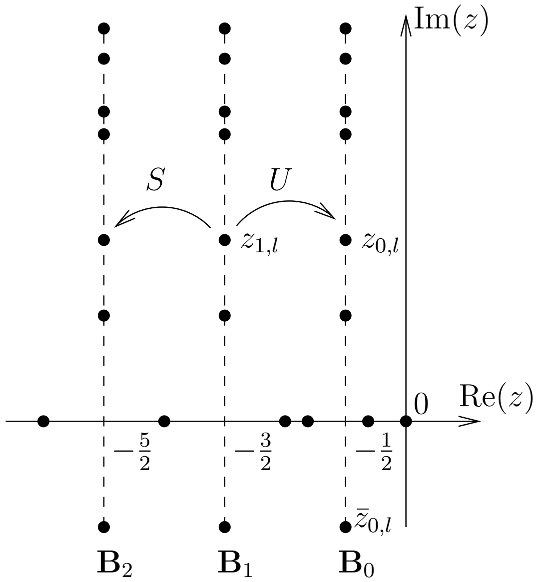
We have defined the suspension of an Anosov diffeomorphism in Section 5.2.1 For the very special case of constant roof function $\tau = \text{cste} = 1$, the cat map φ (or any hyperbolic matrix in $\mathrm{SL}_n(\mathbb{Z})$) and $V = 0$, the Ruelle spectrum is $z_k = i2\pi k$ with $k \in \mathbb{Z}$. The associated eigenfunctions are $\varphi_k(x,z) = \exp(i2\pi kz)$.

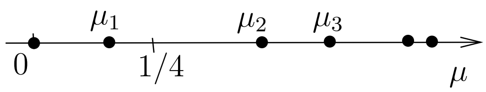
5.4.2 Anosov geodesic flow on hyperbolic surface

In the special case of a closed hyperbolic surface $\mathcal{M} = \Gamma\backslash \mathrm{SL}_2\mathbb{R}/\mathrm{SO}_2\mathbb{R}$ where $\Gamma \subset \mathrm{SL}_2\mathbb{R}$ is a discrete co-compact subgroup one computes from representation theory (Dyatlov-F-Guillarmou 14[7]) that the Ruelle spectrum is

$$z_{k,l} = -\frac{1}{2} - k \pm i\sqrt{\mu_l - \frac{1}{4}} \in \mathbb{C} \tag{5.3}$$

with $k \in \mathbb{N}$ and $0 \leq \mu_0 \leq \mu_1 \leq \mu_2 \ldots$ are the discrete eigenvalues of the Laplacian operator Δ on $\mathcal{M}$, and also $z_n = -n$ with $n \in \mathbb{N}$. This gives that the Ruelle spectrum is made by identical copies from the spectrum of the Laplacian operator and on vertical lines $\mathrm{Re}\,(z) = -\frac{1}{2} - k$:





We will see a generalization of this band structure for geodesic flow on negative non constant curvature manifold, in the end of the Lecture.

Sketch of proof of (5.3). The Lie algebra of $\mathrm{SL}_2\mathbb{R}$ has a basis of three vectors X, U, S

$$X = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

that satisfy

$$[U, X] = U, \quad [S, X] = -S, \quad [S, U] = 2X.$$

The vector field X generates the geodesic flow, and U, S generate respectively the unstable, stable horocycle flow. In other words, U, S span E_u, E_s respectively. Hence the generator is $A = -X$.

Let us suppose that $(-X)\,u = zu$ with a Ruelle resonance $\mathrm{Re}\,(z) \leq 0$. Then we deduce a whole family of Ruelle resonances by:

$$(-X)\,(Uu) = (-UX + U)\,u = (z+1)\,(Uu)\,,$$

$$(-X)\,(Su) = (-SX - S)\,u = (z-1)\,(Su)$$

Since the spectrum is on the half plane $\mathrm{Re}\,(z) \leq 0$, we must have $\exists k$ s.t. $U^{k+1}u = 0, U^k u \neq 0$ and we say that $u \in \mathbf{B}_k$ "band k". If $u \in \mathbf{B}_0$, i.e. $Uu = 0$ with $u \neq 0$ then

$$\underbrace{\triangle}_{\text{Casimir}}\ u := \left(-X^2 - \frac{1}{2}SU - \frac{1}{2}US\right)u = \left(-X^2 + X - SU\right)u = \left(-(-X)^2 - (-X) - SU\right)u$$
$$= -z\,(z+1)\,u = \mu u$$

hence by averaging, $\langle u \rangle_{SO_2} \in C^\infty(\mathcal{M})$ is an eigenfunction of the Laplacien operator $\Delta \equiv -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ on the surface $\mathcal{M}$. Hence $\mu \in \mathbb{R}^+$ and

$$z = -\frac{1}{2} \pm i\sqrt{\mu - \frac{1}{4}}$$

□

Remark 5.8. The smallest eigenvalue of the Laplacian $\mu_1 > 0$ corresponds to the second eigenvalue λ_1 responsible for the exponential rate of mixing.

5.5 Microlocal analysis with wave-packets. Ruelle discrete spectrum

The objective is to **understand the spectrum of the generator** $A := -X + V = -\sum_{j=1}^{\dim M} X_j\,(x)\,\frac{\partial}{\partial x^j} + V$ and show that A has intrinsic **discrete Ruelle spectrum** in some specific anisotropic Sobolev spaces.

- Technically $P = iA$ is a differential operator with principal symbol $\sigma_P\,(m, \Xi) = X\,(\Xi)$ on T^*M. The transfer operator $\mathcal{L}^t = e^{tA} = e^{-itP}$ is a Fourier integral operator. We will see that hyperbolicity assumption implies that $\mathcal{L}^t$ transports functions to high frequency regime. This suggests to use microlocal analysis developed by Hörmander and other people form the 70'.

- We will use **wave-packet transform** and quantization, also called "FBI, wavelet, Bargmann, Anti-Wick, Wick, Toeplitz, Coherent-states" quantization. "Wave-packet calculus" is equivalent and less usual than the usual Weyl quantization and pseudo-differential-operator (PDO) calculus but is more convenient for Hölder regularity of the present situation.

- We will observe **"quantum scattering on a compact trapped set"** in T^*M. From Helffer-Sjöstrand like analysis (86), we will obtain a **discrete spectrum of "Ruelle resonances" in suitable anisotropic Sobolev spaces** as in the toy model of Section 3

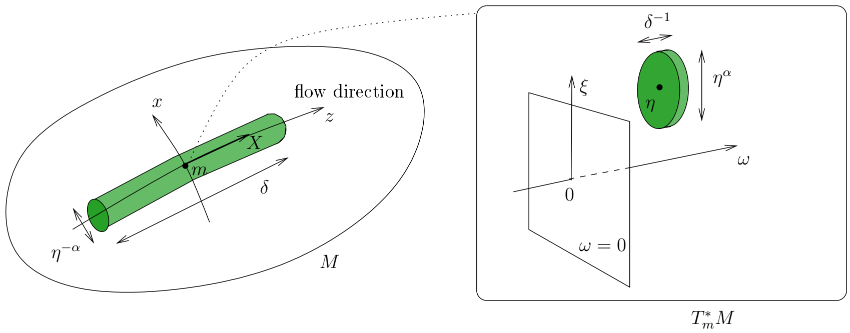
- This microlocal approach with PDO for hyperbolic dynamics has been introduced in [11, 12]. The microlocal approach with wave packets has been introduced in [17].

5.5.1 Wave packets

Consider local flow box coordinates on M: $y = (x, z) \in \mathbb{R}^n \times \mathbb{R}$ s.t. $X = \frac{\partial}{\partial z}$ and dual coordinates $\eta = (\xi, \omega) \in \mathbb{R}^n \times \mathbb{R}$ on $T_y^* M$. Let $\frac{1}{2} \leq \alpha < 1$ and $0 < \delta \ll 1$. A **wave packet function** is:

$$\varphi_{(y,\eta)}(y') \underset{|\eta|\gg 1}{\approx} a \exp\left(i\eta.y' - \left|\frac{x'-x}{\langle\eta\rangle^{-\alpha}}\right|^2 - \left|\frac{z'-z}{\delta}\right|^2\right), \qquad \left\|\varphi_{(y,\eta)}\right\|_{L^2(M)} \underset{|\eta|\gg 1}{\approx} 1$$

Notice that this wave packet function is negligible outside a domain (in green) of size $\Delta x \sim \eta^{-\alpha}$, $\Delta z \sim \delta$. After Fourier transform, we also observe that $\Delta\xi \sim \eta^{\alpha}$ and $\Delta\omega \sim \delta^{-1}$:



We associated the **metric** g **on** T^*M, compatible with the canonical symplectic form $\Omega = dy \wedge d\eta$:

$$g_{y,\eta} = \left(\frac{dx}{\langle\eta\rangle^{-\alpha}}\right)^2 + \left(\frac{d\xi}{\langle\eta\rangle^{\alpha}}\right)^2 + \left(\frac{dz}{\delta}\right)^2 + \left(\frac{d\omega}{\delta^{-1}}\right)^2$$

On the picture, the green domain is a unit ball for this metric.

Remark 5.9. $\alpha \geq \frac{1}{2} \Leftrightarrow g$ remains equivalent uniformly$/\eta$ after change of flow box coordinates.

5.5.2 Wave packet transform (or FBI, wavelet, Bargmann, Anti-Wick... transform)

Remark 5.10. We will abuse with notations because we forget charts and partitions of unity.

Definition 5.11. Let

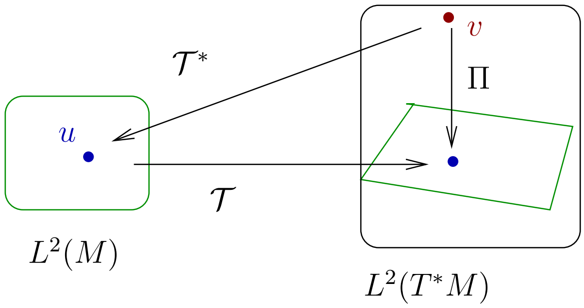
$$\mathcal{T}:\begin{cases} C^{\infty}(M) & \to \mathcal{S}(T^{*}M)\\ u(y') & \to (\mathcal{T}u)(y,\eta) := \langle \varphi_{y,\eta}, u\rangle_{L^{2}(M)}\end{cases}$$

called **wave packet transform**.

Here is the first fundamental lemma for microlocal analysis:

Lemma 5.12 ("**Resolution of identity**")**.**

$$\mathcal{T}^{*}\circ\mathcal{T} = \mathrm{Id}$$



Remark 5.13. $\mathcal{T}: L^{2}(M)\to \mathrm{Im}(\mathcal{T})\subset L^{2}(T^{*}M)$ is an isomorphism. Hence $\mathcal{T}$ **"lift the analysis to $T^{*}M$".** The operator $\Pi = \mathcal{T}\circ\mathcal{T}^{*}: L^{2}(T^{*}M)\to \mathrm{Im}(\mathcal{T})$ is an orthogonal projector . We have the reconstruction formula $\forall u\in C^{\infty}(M)$, $\quad u(y') = \int_{T^{*}M}\varphi_{y,\eta}(y')\langle\varphi_{y,\eta},u\rangle\frac{dyd\eta}{(2\pi)^{n+1}}$.

5.5.3 Propagation of singularities

We want to "analysis the operator $\mathcal{L}^{t}$ on the cotangent space $T^{*}M$". We consider the commutative diagram

$$\begin{array}{ccc} L^{2}(M) & \xrightarrow{\mathcal{L}^{t}=e^{tA}} & L^{2}(M)\\ {\scriptstyle\mathcal{T}}\big\downarrow & & {\scriptstyle\mathcal{T}}\big\downarrow\\ L^{2}(T^{*}M) & \xrightarrow{\tilde{\mathcal{L}}^{t}:=\mathcal{T}\circ\mathcal{L}^{t}\circ\mathcal{T}^{*}} & L^{2}(T^{*}M)\end{array} \tag{5.4}$$

We have a second fundamental Lemma in microlocal analysis that concerns the Schwartz kernel of the lifted operator $\tilde{\mathcal{L}}^{t}$:

Lemma 5.14 ("**Micro-locality of the transfer operator**")**.** $\forall t\geq 0, \forall N>0, \exists C_{N,t}>0, \forall\rho,\rho'\in T^{*}M$,

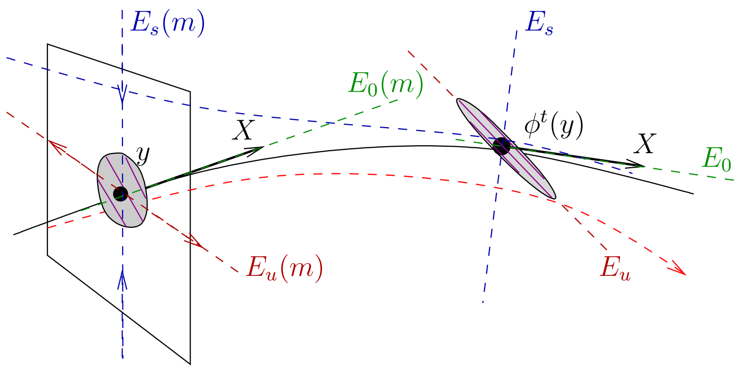
$$\left|\langle\delta_{\rho'},\tilde{\mathcal{L}}^{t}\delta_{\rho}\rangle_{L^{2}(T^{*}M)}\right| = \left|\langle\varphi_{\rho'},\mathcal{L}^{t}\varphi_{\rho}\rangle_{L^{2}(M)}\right|\leq C_{N,t}\left\langle \mathrm{dist}_{g}\left(\rho',\tilde{\phi}^{t}(\rho)\right)\right\rangle^{-N}$$

with $\tilde{\phi}^{t}: T^{*}M\to T^{*}M$ canonical lift of ϕ^{t}.

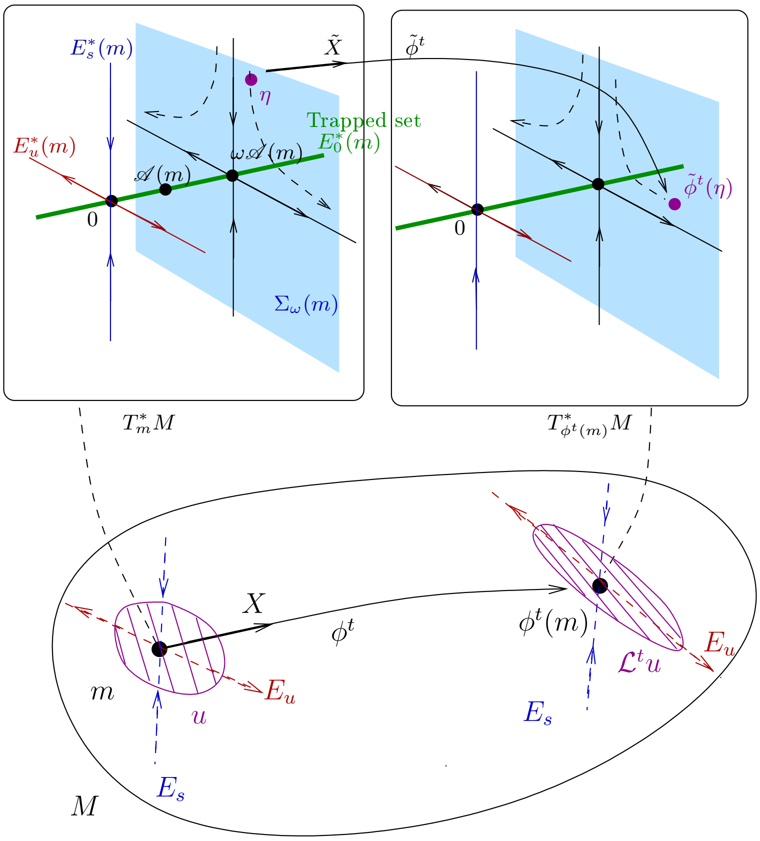
This Lemma means that the Schwartz kernel of $\tilde{\mathcal{L}}^t$ is negligible outside the graph of the lifted flow $\tilde{\phi}^t : T^*M \to T^*M$. In particular, $t = 0$, shows that wave packets are almost orthogonal to each other.

5.5.4 Observations of the Hamiltonian flow $\tilde{\phi}^t$ in T^*M

From the previous Lemma 5.14 it is necessary to understand the behaviour of the the lifted flow $\tilde{\phi}^t : T^*M \to T^*M$. In this Section we do some observations. Let $(y, \eta) \in T^*M$ (represented by lines on the picture).



- In $S^*M := T^*M/\mathbb{R}^+$, for $t \to +\infty$, $\left[\tilde{\phi}^t(m, \eta)\right]$ approaches $E_u^* := (E_u \oplus E_0)^\perp$. For $t \to -\infty$, $\left[\tilde{\phi}^t(m, \eta)\right]$ approaches $E_s^* := (E_s \oplus E_0)^\perp$.
- The trapped set (i.e. the non wandering set) is $E_0^* := (E_u \oplus E_s)^\perp \subset T^*M$.
- $\tilde{\phi}^t$ is generated by the Hamiltonian $\sigma_{iA}(y, \eta) = X(\eta)$ hence preserves the energy $\omega = X(\eta)$ (i.e. frequency along X).



Let $\mathscr{A}$ be the one form s.t. $\mathscr{A}(X) = 1$, $\quad \mathrm{Ker}(\mathscr{A}) = E_u \oplus E_s$. The trapped set is $E_0^* = (E_u \oplus E_s)^\perp = \{\omega\mathscr{A}(m)\,;\ \omega \in \mathbb{R}, m \in M\} \subset T^*M$. $E_0^*(m)$ is β_0-Hölder continuous.

5.5.5 Escape function W in T^*M

Following ideas of Combes 70', Helffer-Sjöstrand 86. Let $\rho = (y, \eta) \in T^*M$. Decompose the cotangent vector $\eta = \underbrace{\eta_u}_{\in E_u^*} + \underbrace{\eta_s}_{\in E_s^*} + \underbrace{\omega\mathcal{A}}_{\in E_0^*}$. With time evolution, $\eta_u(t)$ increases and $\eta_s(t)$ decreases. From this obervation we will construct an escape function W, that should satisfy other "technical properties".

Let $R > 0$ and $h_\gamma(\rho) = h_0 \left\langle \|\eta_u + \eta_s\|_g \right\rangle^{-\gamma}$, $1 - \frac{\alpha \min(\beta_u, \beta_s)}{1-\alpha} \leq \gamma < 1$, $\frac{1}{1+\beta_0} \leq \alpha \leq \frac{1}{1+\min(\beta_u, \beta_s)}$, $h_0 > 0$. Define the **escape function** $W \in C(T^*M; \mathbb{R}^+)$:

$$W(\rho) := \frac{\left\langle h_\gamma(\rho) \|\eta_s\|_g \right\rangle^R}{\left\langle h_\gamma(\rho) \|\eta_u\|_g \right\rangle^R}$$

Proposition 5.15. *If $R \geq 0$, W* ***decays outside a parabolic neighborhood of the trapped set:***

$$\forall t \geq 0, \qquad \frac{1}{C} e^{-(\lambda_{\max} r) t} \leq \frac{W\left(\tilde{\phi}^t(\rho)\right)}{W(\rho)} \leq \begin{cases} C \\ C e^{-(\lambda r) t} & \text{if } \|\eta_u + \eta_s\|_{g(\rho)} > C_t \end{cases}$$

*with $r = R(1-\gamma)\left(1-\alpha^{\perp}\right)$ (the order of W). W is h_γ-**temperate**:*

$$\frac{W(\rho')}{W(\rho)} \leq C \langle h_\gamma(\rho) \operatorname{dist}_g(\rho', \rho) \rangle^{N_0}$$

h_γ-temperate property means that the function W is not growing faster than polynomialy and has bounded variations at distances smaller than h_γ^{-1}.

5.5.6 Anisotropic Sobolev space $\mathcal{H}_W(M)$

Definition 5.16. For $u, v \in C^\infty(M)$, let

$$\langle u, v \rangle_{\mathcal{H}_W} := \langle \mathcal{M}_W \mathcal{T} u, \mathcal{M}_W \mathcal{T} v \rangle_{L^2(T^*M)},$$

and define the **Anisotropic Sobolev space** $\mathcal{H}_W(M)$ by completion:

$$\mathcal{H}_W(M) := \overline{\{u \in C^\infty(M)\}}^{\|\cdot\|_{\mathcal{H}_W}}$$

In other words, we have isometries

$$\mathcal{H}_W(M) \xrightarrow{\mathcal{T}} L^2\left(T^*M; W^2\right) \xrightarrow{\mathcal{M}_W} L^2(T^*M)$$

and inclusion in standard Sobolev space of constant orders:

$$H^r(M) \subset \mathcal{H}_W(M) \subset H^{-r}(M).$$

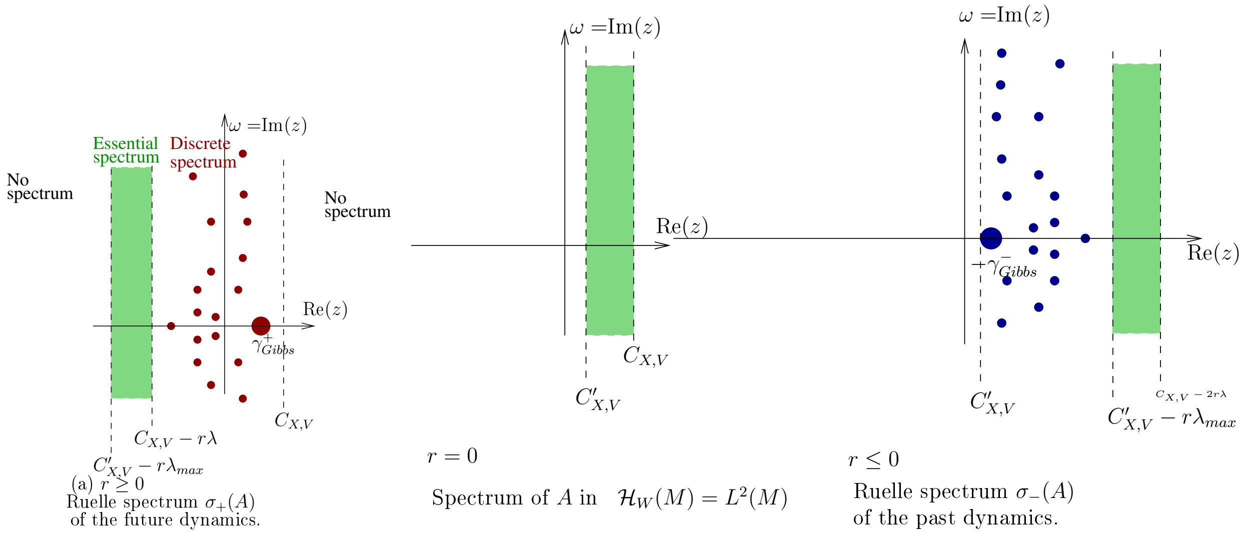
5.5.7 Ruelle Pollicott discrete spectrum of resonances

First versions of the next Theorem for diffeomorphisms have been obtained by Blank, Keller, Liverani 2002[3], Gouëzel, Liverani 2005[19], Baladi, Tsujii 2005,2008 [1, 2]. For Flows: Butterley-Liverani 2007[5], F.-Sjöstrand 2011 using PDO[12]. Dyatlov-Guillarmou 2014 for Axiom A flows[9]. Dang-Rivière 2016 for Morse-Smale flows[6]. F-Tsujii 2017 using wave-packet transform [17].

Theorem 5.17 (F.-Tsujii 17[17]). *For any $r \in \mathbb{R}$, W can be designed such that $\mathcal{L}^t = e^{tA} : \mathcal{H}_W(M) \to \mathcal{H}_W(M), t \in \mathbb{R}$ is a **strongly continuous group** with estimates: for $r \geq 0, t \geq 0$, $\|\mathcal{L}^t\|_{\mathcal{H}_W} \leq Ce^{tC_{X,V}}$, $\|\mathcal{L}^{-t}\|_{\mathcal{H}_W} \leq Ce^{\left(C'_{X,V}+Cr\lambda\right)t}$. The generator $A = -X + V$ has **"future" discrete spectrum** on $\{z \in \mathbb{C},\quad C_{X,V} - r\lambda < \mathrm{Re}(z) < C_{X,V}\}$. and **"past" discrete spect.** on $\left\{z \in \mathbb{C},\quad -C'_{X,V} < \mathrm{Re}(z) < -C'_{X,V} - r\lambda_{max}\right\}$, with*

$$C_{X,V} := \overline{\max}\left(\frac{1}{2}\mathrm{div}X + \mathrm{Re}(V)\right) := \lim_{t\to+\infty}\max_{m\in M}\frac{1}{t}\int_0^t\left(\frac{1}{2}\mathrm{div}X + \mathrm{Re}(V)\right)(\phi^s(m))\,ds,, \tag{5.5}$$

$$C'_{X,V} = \overline{\min}\left(\frac{1}{2}\mathrm{div}X + \mathrm{Re}(V)\right), \tag{5.6}$$



Remark 5.18. If $V' + \overline{V} + \mathrm{div}X = 0$ then the **future** spectrum λ_j^+ of $A = -X + V$ is related to the **past** spectrum $\lambda_j'^-$ of $A' = -X + V'$ by $\lambda_j'^- = -\overline{\lambda_j^+}$. In particular for $\mathrm{Re}(V) = -\frac{1}{2}\mathrm{div}X$ (called "half-density correction"), the operator $A = -X - \frac{1}{2}\mathrm{div}X + i\mathrm{Im}(V)$ has the same past and future spectrum, i.e. $\lambda_j^- = -\overline{\lambda_j^+}, \forall j$.

5.5.8 Sketch of proof of Theorem 5.17

Recall the isometries

$$\mathcal{H}_W(M) \xrightarrow{\mathcal{T}} L^2\left(T^*M; W^2\right) \xrightarrow{\mathcal{M}_W} L^2\left(T^*M\right)$$

We consider the Schwartz kernel of the lifted transfer operator:

$$
\begin{aligned}
\left|\left\langle\delta_{\varrho'},\left(W\mathcal{T}\mathcal{L}^t\mathcal{T}^*W^{-1}\right)\delta_{\varrho}\right\rangle\right| &= \frac{W(\varrho')}{W(\varrho)}\left|\left\langle\varphi_{\varrho'},\mathcal{L}^t\varphi_{\varrho}\right\rangle\right| \\
&\underset{microlocal\,of\,\mathcal{L}^t}{\leq} \frac{W(\varrho')}{W(\varrho)}C_{N,t}\left\langle \mathrm{dist}_g\left(\rho',\tilde{\phi}^t(\rho)\right)\right\rangle^{-N} \\
&= C_{N,t}\left(\frac{W\left(\tilde{\phi}^t(\rho)\right)}{W(\rho)}\right)\left(\frac{W(\rho')}{W\left(\tilde{\phi}^t(\rho)\right)}\right)\left\langle \mathrm{dist}_g\left(\rho',\tilde{\phi}^t(\rho)\right)\right\rangle^{-N} \\
&\underset{decay\&temperate}{\leq} C_{N,t}\left\langle \mathrm{dist}_g\left(\rho',\tilde{\phi}^t(\rho)\right)\right\rangle^{N_0-N}
\end{aligned}
$$

i.e. it decays outside the graph of $\tilde{\phi}^t$. Apply Schur test and deduce that the operator is bounded. More precisely the kernel decays as $e^{(C-\lambda r)t}$ outside the trapped set. From this we deduce the **discrete Ruelle Pollicott spectrum** on $\mathrm{Re}(z) \geq C_{X,V} - \lambda r$. For $t \leq 0$ it is bounded by $e^{(C-\lambda_{max}r)t}$ **.** We deduce **bounded resolvent** on $\mathrm{Re}(z) \leq C - \lambda_{max} r$.

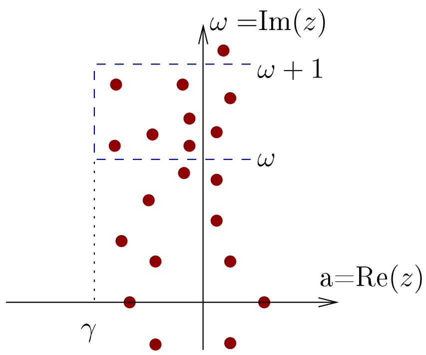
5.5.9 Fractal Weyl law (after J. Sjöstrand 90)

Here is another result that concerns the density of Ruelle eigenvalues.

Theorem 5.19 (F.-Tsujii 17[17]. "**Upper bound for the density of eigenvalues**"). $\forall\gamma \in \mathbb{R}, \exists C > 0, \forall\omega \geq 1,$

$$
\sharp\left\{z\in\sigma(A);\quad \mathrm{Re}(z) > \gamma,\ \mathrm{Im}(z)\in[\omega,\omega+1]\right\} \leq C\,|\omega|^{\frac{n}{1+\beta_0}}.
$$

with $\beta_0 \in]0,1]$ *is Hölder exponent of* $E_u \oplus E_s$.



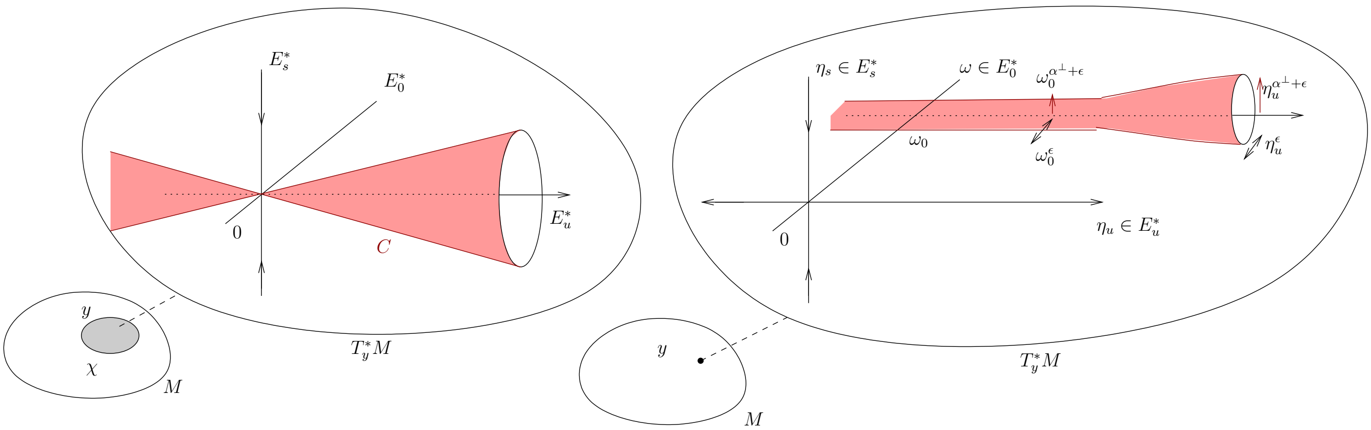
5.5.10 About the wave front set of resonances

The next result concerns the region in T^*M where Ruelle eigenfunctions are not negligible. It is called the "**wave-front set**".

Theorem 5.20 (F.-Tsujii 17[17]. "Parabolic wave front set")**.** *$\forall C, N, \epsilon, \exists C_N$, for any (generalized) Ruelle Pollicott eigenfunction u with* $\mathrm{Re}(z) > -C$ *then* $\forall (y,\eta) \in T^*M$*,*

$$|(\mathcal{T}u)(y,\eta)| \leq \frac{C_N}{\left\langle |\eta|^{-\epsilon} \mathrm{dist}_g \left(\rho, E_u^* + \underbrace{\mathrm{Im}(z)\mathscr{A}}_{\omega_0} \right) \right\rangle^N} \|u\|_{\mathcal{H}_W(M)}$$

Remark 5.21. We choose parameter $\alpha = \frac{1}{1+\min(\beta_u,\beta_s)}$ (but expect $\alpha = \frac{1}{1+\beta_u}$) so that **uncertainty principle absorbs Hölder fluctuations**. Previous results [12] showed that the wave-font set is contained in some conical neighborhood of E_u^*. (Figure (a)). Theorem 5.20 is therefore more precise (Figure (b)).



(a) (b)

5.5.11 Band structure for the Ruelle spectrum

In Section @@ we have seen that the Ruelle spectrum of the geodesic flow on homogeneous space is structured in vertical bands. This has been obtained from representation theory that is not available in non constant curvature. Netherveless there is a similar structure that is due to the important property that the trapped set is a smooth symplectic manifold of T^*M [10, 15, 13, 16].

Theorem 5.22 ("**Band spectrum**" (F.-Tsujii 12,13))**.** *For Anosov* ***geodesic flow (contact)****,* $\forall\epsilon > 0$*,*

$$\left(\sigma\left(A\right)\cap\left\{z,\left|\mathrm{Im}z\right|>C_{\epsilon}\right\}\right)\subset\bigcup_{k\geq0}\underbrace{\left[\gamma_{k}^{-}-\epsilon,\gamma_{k}^{+}+\epsilon\right]\times i\mathbb{R}}_{bande\mathbf{B}_{k}}$$

with

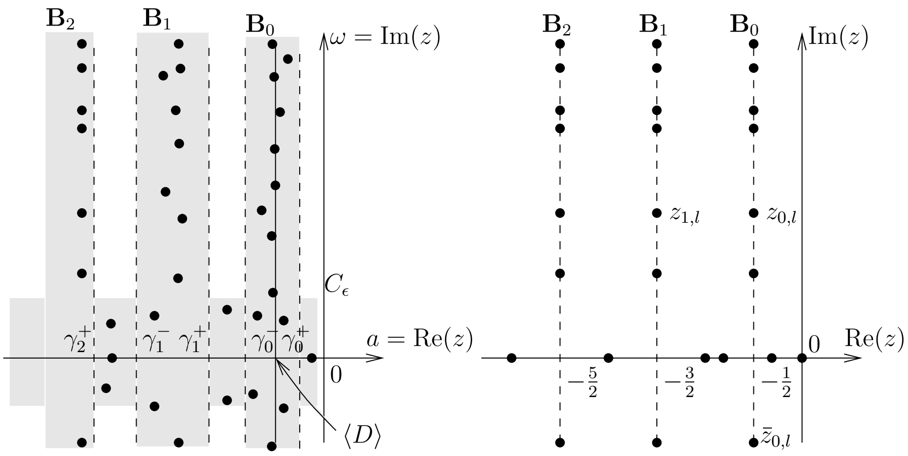
$$\gamma_{k}^{+}=\lim_{t\to\infty}\sup_{x\in M}\frac{1}{t}\left(\int_{0}^{t}D\circ\phi^{-s}ds\right)(x)-k\log\left\Vert D\phi^{t}\left(x\right)_{/E_{u}}\right\Vert_{\min},$$

$$\gamma_{k}^{-}=\lim_{t\to\infty}\inf_{x\in M}\frac{1}{t}\left(\int_{0}^{t}D\circ\phi^{-s}ds\right)(x)-k\log\left\Vert D\phi^{t}\left(x\right)_{/E_{u}}\right\Vert$$

$D\left(x\right):=V\left(x\right)-\frac{1}{2}\mathrm{div}X_{/E_{u}(x)}$, $\qquad \left\Vert L\right\Vert_{\min}:=\left\Vert L^{-1}\right\Vert^{-1}$.

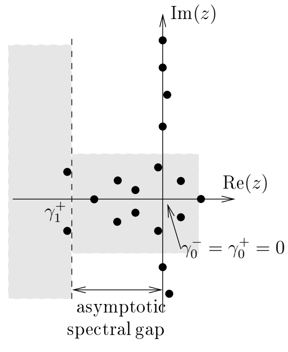
Case of $\Gamma\backslash SL_2(R)$ $z_{k,l}=-\frac{1}{2}-k+i\sqrt{\mu_l-\frac{1}{4}}$

$\Delta u_l=\mu_l u_l$



Remark 5.23. :

- $D\left(x\right)=V\left(x\right)-\frac{1}{2}\mathrm{div}X_{/E_{u}(x)}$ is an **effective damping function** (due to escape to high frequencies).
- The choice $V=\frac{1}{2}\mathrm{div}X_{/E_{u}(x)}$ (>0, Hölder) gives $D\left(x\right)=0$ and $\gamma_{0}^{\pm}=0$: eigenvalues in band $\mathbf{B}_{0}$ accumulate on the axis $\mathrm{Re}\left(z\right)=0$.



5.6 Relation with periodic orbits γ. Trace formula and zeta functions

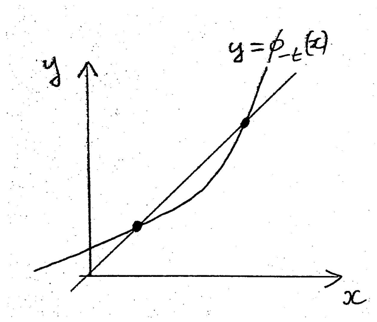
We write the transfer operator as

$$\begin{aligned}\left(\mathcal{L}^t u\right)(m) &\underset{(5.2)}{=} e^{V_{[-t,0]}(m)}.\left(u\left(\phi^{-t}(m)\right)\right)\\ &= \int_M K_t(m,m')\,v(m')\,dm'\end{aligned}$$

with the distributional Schwartz kernel given by $K_t(m,m') = e^{V_{[-t,0]}(m)}\delta\left(m' - \phi^{-t}(m)\right)$ (this is the "graph of the flow").

For $t>0$, the **"flat trace"** defined by:

$$\mathrm{Tr}^\flat\left(\mathcal{L}^t\right) := \int_M K_t(m,m)\,dm$$



A point $m \in M$ such that $\phi^T(m) = m$ is called a **periodic point** with period T. If $T>0$ and $\phi^t(m) \neq m$ for $0<t<T$ then T is called the primitive period and the set of points $\gamma := \{\phi^t(m), t \in [0,T[\}$ is called a **primitive orbit** with period $|\gamma| := T$. Repetitions are also periodic orbits with periods $n|\gamma|$ with $n \geq 1$.

Theorem 5.24. ***"Atiyah-Bott trace formula"*:**

$$\mathrm{Tr}^\flat\left(\mathcal{L}^t\right) = \sum_{\gamma:o.p.}|\gamma|\sum_{n\geq 1}\frac{e^{\int^t V}.\delta(t-n|\gamma|)}{\left|\det\left(1 - D_{(u,s)}\phi^{-t}(\gamma)\right)\right|}$$

This is a distribution in $\mathcal{D}'(\mathbb{R}_t)$.

Proof. We have

$$\mathrm{Tr}^\flat\left(\mathcal{L}^t\right) := \int_M K_t(m,m)\,dm = \int_M e^{V_{[-t,0]}(m)}\delta\left(m - \phi^{-t}(m)\right)dm$$

We have $\delta(m - \phi^{-t}(m)) \neq 0$ iff $m = \phi^t(m)$, i.e. m is a periodic point with period t. Hence the integral reduces to periodic points. We use local coordinates $(x,z) \in \mathbb{R}^{n+1}$ such that the periodic points are at $x=0$ and $z=\text{cste}$ is tangent to $E_u(m)\oplus E_s(m)$. Then $D\phi^{-t}(x,z)\equiv\left(D\phi^{-t}_{(u,s)}(x),z\right)$ and $Df_m=\left(\mathrm{Id}-D\phi^{-t}_{(u,s)},0\right)$. We use that if $f:\mathbb{R}^n\to\mathbb{R}^n$ with fixed point $f(0)=0$, with the change of variable $y=f(x)$, we write $\int\delta(f(x))\,dx=\frac{1}{|\det Df(0)|}\int\delta(y)\,dy=\frac{1}{|\det Df(0)|}$. □

Question: relation between the periodic orbits γ and the Ruelle spectrum of $A=-X+V$, generator of $\mathcal{L}^t=e^{tA}$?

To answer this question we will use zeta functions.

5.6.1 dynamical zeta functions

In linear algebra, the eigenvalues of a matrix $\mathbf{A}$ are zeroes of the holomorphic function

$$\mathbf{d}(z):=\det(z-\mathbf{A})=\mathbf{d}(z_0).\exp\left(\lim_{\varepsilon\to0}\left[-\int_\varepsilon^\infty\frac{1}{t}e^{-zt}\mathrm{Tr}\left(e^{t\mathbf{A}}\right)dt\right]_{z_0}^{z}\right),\quad z_0\notin\mathrm{Spec}(\mathbf{A}).$$

Proof. Write $(z-A)^{-1}=\int_0^\infty e^{-(z-A)t}dt$, and $d(z)=\det(z-A)=\exp(\mathrm{Tr}(\log(z-A)))$ hence $\frac{d}{dz}\log d(z)=\mathrm{Tr}(z-A)^{-1}=\int_0^\infty e^{-zt}\mathrm{Tr}\left(e^{tA}\right)dt$. □

Similarly, for $\mathrm{Re}(z)\gg1$ we define the "**spectral determinant**" or **dynamical zeta function** (we may think $d(z)=$"$\det(z-A)$"):

$$\begin{aligned}d(z):&=\exp\left(-\int_{|\gamma|_{min}}^\infty\frac{1}{t}e^{-zt}\mathrm{Tr}^\flat\left(e^{tA}\right)dt\right)\\ &\underset{(??)}{=}\exp\left(-\sum_\gamma\sum_{n\geq1}\frac{e^{\int^t V}.e^{-zn|\gamma|}}{n\left|\det\left(1-D_{(u,s)}\phi_{n|\gamma|}(\gamma)\right)\right|}\right)\end{aligned}$$

Here is a recent result that we may have expected from the toy models of previous chapters.

Theorem 5.25 (Giullieti, Liverani, Pollicott 12, Dyatlov-Zworski.13)**.** *For an Anosov vector field X, $d(z)$ has an **analytic extension** on $\mathbb{C}$. Its zeroes are **Ruelle resonances** with multiplicities.*

Remark 5.26. in 2008, Baladi-Tsujii [2] have obtained a similar result for Anosov diffeomorphisms.

5.6.2 Application: counting periodic orbits

- **Rem:** $\left|\det\left(1-D_{(u,s)}\phi^{-t}\right)\right|\underset{t\gg1}{\asymp}\left|\det\left(D\phi^{-t}_{|E_u(x)}\right)\right|=e^{\left(\mathrm{div}X_{/E_u}\right)_{[-t,0]}}$ so the choice $V=J=\mathrm{div}X_{/E_u}$ gives the **counting formula** $\mathrm{Tr}^\flat\left(\mathcal{L}^t\right)\asymp\sum_{\gamma:o.p.}|\gamma|\sum_{n\geq1}\delta(t-n|\gamma|)$

The objective is to express in term of Ruelle spectrum the **counting function**:

$$\pi(T):=\sharp\{\gamma:periodic-orbit,\quad|\gamma|\leq T\}=\sum_{\gamma,|\gamma|\leq T}1$$

Observe that $\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1} \underset{t\infty}{\simeq} \det\left(D\phi_{t/E_u}\right)^{-1}$. The choice of potential $V = \mathrm{div}X_{/E_u}$ gives $e^{\int^t V} = \det\left(D\phi_{t/E_u}\right)$ and $e^{\int^t V}\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1} \simeq 1$.

Theorem 5.27. *[18](with pinching hypothesis) there exists $\delta > 0$ s.t.*

$$\pi(T) = \mathrm{Ei}(h_{top}T) + O\left(e^{(h_{top}-\delta)T}\right) \underset{T\to\infty}{\sim} \frac{e^{h_{top}T}}{h_{top}T}$$

*with $\mathrm{Ei}(x) := \int_{x_0}^{x} \frac{e^y}{y}dy$ and h_{top} dominant eigenvalue of $A = -X + \mathrm{div}X_{/E_u(x)}$ called **topological entropy**.*

5.6.3 Semiclassical zeta function

Observe that we have $\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1} \underset{t\infty}{\simeq} \det\left(D\phi_{t/E_u}\right)^{-1/2}\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1/2}$ and $\det\left(D\phi_{t/E_u}\right)^{-1/2} = e^{-\frac{1}{2}\int^t \mathrm{div}X_{/E_u}}e^{-\frac{1}{2}\int^t V_0}$ so in (**??**) we have

$$e^{\int^t V}\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1} \underset{t\infty}{\simeq} e^{\int^t D}\left|\det\left(1-D_{(u,s)}\phi_t(\gamma)\right)\right|^{-1/2}.$$

We define the "**Gutzwiller-Voros zeta function**" or "**semi-classical zeta function**" by

$$d_{G-V}(z) := \exp\left(-\sum_{\gamma}\sum_{n\geq 1}\frac{e^{-zn|\gamma|}e^{\int^t D}}{n\left|\det\left(1-D_{(u,s)}\phi_{n|\gamma|}(\gamma)\right)\right|^{1/2}}\right) \tag{5.7}$$

Theorem 5.28. *[16]The semiclassical zeta function $d_{G-V}(z)$ has an meromorphic extension on $\mathbb{C}$. On $\mathrm{Re}(z) > \gamma_1^+$, $d_{G-V}(z)$ has finite number of poles and its zeroes coincide (up to finite number) with the Ruelle eigenvalues of A.*

See figure **??**. The motivation for studying $d_{G-V}(z)$ comes from the Gutzwiller semiclassical trace formula in quantum chaos. Also in the case of surface with constant curvature, and $V = V_0 = \frac{1}{2}$, we have $D_{(u,s)}\phi_{n|\gamma|}(\gamma) = \begin{pmatrix} e^{|\gamma|n} & 0 \\ 0 & e^{-|\gamma|n}\end{pmatrix}$. This gives

$$\begin{aligned} d_{G-V}(z) &\underset{(5.7)}{=} \exp\left(-\sum_{\gamma}\sum_{n\geq 1}\sum_{m\geq 0}\frac{1}{n}e^{-n|\gamma|\left(z+\frac{1}{2}+m\right)}\right) \\ &= \prod_{\gamma}\prod_{m\geq 0}\left(1-e^{-\left(z+\frac{1}{2}+m\right)|\gamma|}\right) =: \zeta_{Selberg}\left(z+\frac{1}{2}\right) \end{aligned}$$

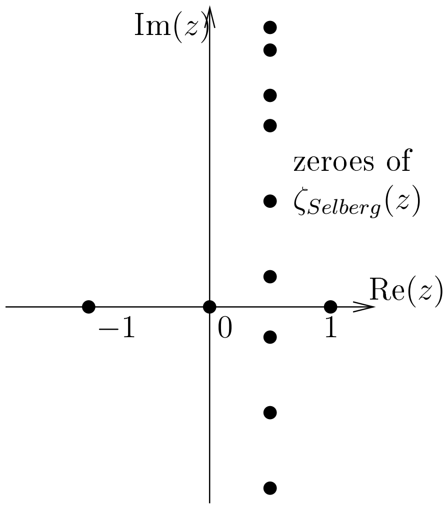


Figure 5.1: Zeroes of the holomorphic function $\zeta_{Selberg}$.

Proof. Put $x = e^{-|\gamma|n}$ and use that $\left|\det\begin{pmatrix} 1-x^{-1} & 0 \\ 0 & 1-x \end{pmatrix}\right|^{-1/2} = x^{1/2}(1-x)^{-1} = x^{1/2}\sum_{m\geq 0} x^m$. □

Therefore $d_{G-V}(z)$ **"generalizes" the Selberg zeta function** $\zeta_{Selberg}$ for case of variable curvature (or contact Anosov flows). Compare figure 5.1 with figure ??.

5.6.4 Correlation functions

For $\forall u, v \in C^\infty(M)$,

$$\left\langle u, \mathcal{L}^t v \right\rangle_{L^2} = \langle u, \underbrace{\left(\Pi_{\text{Band B}_0}\mathcal{L}^t\right)}_{\text{operateur quantique}} v\rangle + O_{u,v}\left(e^{\left(\gamma_1^+ + \epsilon\right)t}\right)$$

Interpretations:

- **Emergence of an effective quantum dynamics** from classical correlation functions
- Operators $(\Pi_{\text{Band B}_0}\mathcal{L}^t)$ and $(\Pi_{\text{Band B}_0}A)$ are a **natural quantization** of the geodesic flow (exact trace formula, Egorov theorem etc..)

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