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Semiclassical origin of the spectral gap for transfer operators of a partially expanding map

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Abstract
We consider a specific family of skew product of partially expanding map on the torus. We study the spectrum of the Ruelle transfer operator and show that in the limit of high frequencies in the neutral direction (this is a semiclassical limit), the spectrum develops a spectral gap, for a generic map. This result has already been obtained by Tsujii (2008 *Ergodic Theory Dyn. Syst.* **28** 291–317). The novelty here is that we use semiclassical analysis which provides a different and quite natural description. We show that the transfer operator is a semiclassical operator with a well-defined 'classical dynamics' on the cotangent space. This classical dynamics has a 'trapped set' which is responsible for the Ruelle resonances spectrum. In particular, we show that the spectral gap is closely related to a specific dynamical property of this trapped set.

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1. Introduction

Chaotic behaviour of certain dynamical systems is due to hyperbolicity of the trajectories. This means that the trajectories of two closed initial points will diverge from each other either in the future or in the past (or both). As a result the behaviour of an individual trajectory seems complicated and unpredictable. However, evolution of a cloud of points seems more simple: it will spread and equidistributes according to an invariant measure, called an equilibrium measure (or S.R.B. measure). Following this idea, Ruelle in the 1970 [Rue78, Rue86] has shown that instead of considering individual trajectories, it is much more natural to consider evolution of densities under a linear operator called the Ruelle transfer operator or the Perron Frobenius operator.

For dynamical systems with strong chaotic properties, such as uniformly expanding maps or uniformly hyperbolic maps, Ruelle, Bowen, Fried, Rugh and others, using symbolic dynamics techniques, have shown that the spectrum of the transfer operator has a discrete spectrum of eigenvalues. This spectral description has an important meaning for the dynamics since each eigenvector corresponds to an invariant distribution (up to a time factor). From this spectral characterization of the transfer operator, one can derive other specific properties of the dynamics such as decay of time correlation functions, central limit theorem, mixing, etc. In particular, a spectral gap implies exponential decay of correlations.

This spectral approach has recently (2002–2005) been improved by Blank, Gouzel, Keller, Liverani [BKL02, GL05, Liv05] and Baladi and Tsujii [Bal05, BT07] (see [BT07] for some historical remarks), through the construction of functional spaces adapted to the dynamics, independent of any symbolic dynamics. The case of dynamical systems with continuous time is more delicate (see [FMT07] for historical remarks). This is due to the direction of time flow which is neutral (i.e. two nearby points on the same trajectory will not diverge from each other). In 1998 Dolgopyat [Dol98, Dol02] showed the exponential decay of correlation functions for certain Anosov flows, using techniques of oscillatory integrals and symbolic dynamics. In 2004 Liverani [Liv04] adapted Dolgopyat's ideas to his functional analytic approach, to treat the case of contact Anosov flows. In 2005 Tsujii [Tsu08] obtained an explicit estimate for the spectral gap for the suspension of an expanding map. Then in 2008 Tsujii [Tsu10b, Tsu10a] obtained an explicit estimate for the spectral gap, in the case of contact Anosov flows.

This work is also closely related to the series of work [GLZ04, SZ04] where the authors study the Zeta function and the density of Ruelle resonances associated with the hyperbolic flow on Schottky manifold and for the quadratic map. In [Chr] the author studies an hyperbolic closed trajectory.

Semiclassical approach for transfer operators. It has appeared recently [FR06, FRS08] that for hyperbolic dynamics, the study of transfer operator is naturally a semiclassical problem in the sense that a transfer operator can be considered as a 'Fourier integral operator' and using standard tools of semiclassical analysis[1], some of its spectral properties can be obtained from the study of 'the associated classical symplectic dynamics', namely the initial hyperbolic dynamics lifted on the cotangent space (the phase space).

The simple idea behind this, crudely speaking, is that a transfer operator transports a 'wave packet' (i.e. localized both in space and in Fourier space) into another wave packet, and this is exactly the characterization of a Fourier integral operator. A wave packet is characterized by a point in phase space (its position and its momentum), hence one is naturally led to study the dynamics in phase space. Moreover, since any function or distribution can be decomposed as a linear superposition of wave packets, the dynamics of wave packets characterizes completely the transfer operators.

Following this approach, in the papers [FR06, FRS08] we studied hyperbolic diffeomorphisms. In [FS10] we studied uniformly hyperbolic flows on a compact manifold. The aim of this paper is to show that semiclassical analysis is well adapted for hyperbolic systems with a neutral direction. We consider here the simplest model: a partially expanding map $f : (x, s) \to f(x, s)$, i.e. a map on a torus $(x, s) \in S^1 \times S^1$ with an expanding direction ($x \in S^1_x$) and a neutral direction ($s \in S^1_s$) (the inverse map f^{-1} is k-valued, with $k \geqslant 2$). The results are presented in section 2. We summarize them in a few lines. First in order to reduce the problem and drop out the neutral direction, we use a Fourier analysis

[1] We recommend reference [Tay96b, chapter 7] for the homogeneous semiclassical theory and [Mar02, EZ03] for the semiclassical theory with a small h parameter.

in $s \in S^1_s$ and decompose the transfer operator $\hat{F}$ on $S^1_x \times S^1_s$ (defined by $\hat{F}\varphi := \varphi \circ f$) as a collection of transfer operators $\hat{F}_\nu$ on the expanding space S^1_x only, with $\nu \in \mathbb{Z}$ being the Fourier parameter and playing the role of the semiclassical parameter. The semiclassical limit is $|\nu| \to \infty$.

Then we introduce a (multivalued) map F_ν on the cotangent space $(x, \xi) \in T^*S^1_x$ which is the canonical map associated with the transfer operator $\hat{F}_\nu$. The fact that the initial map f is expanding along the space S^1_x implies that on the cylinder $T^*S^1_x$ trajectories starting from a large enough value of $|\xi|$ escape towards infinity ($|\xi| \to \infty$). We define the trapped set as the compact set $K = \lim_{n\to\infty} F_\nu^{-n}(K_0)$ where $K_0 \subset T^*S^1$ is an initial large compact set. K contains trajectories which do not escape towards infinity.

Using a standard semiclassical approach (with escape functions on phase space [HS86]) we first show that the operator $\hat{F}_\nu$ has a discrete spectrum called Ruelle resonances (we have to consider $\hat{F}_\nu$ in Sobolev space of distributions). This is theorem 2. This result is well known, but the semiclassical approach we use here is new.

Then we show that a specific hypothesis on the trapped set implies that the operator $\hat{F}_\nu$ develops a 'spectral gap' in the semiclassical limit $\nu \to \infty$ (i.e. its spectral radius reduces). This is theorem 3 illustrated in figure 2. This theorem is very similar to theorem 1.1 in [Tsu08]. With the semiclassical approach, this result is very intuitive: the basic idea (followed in the proof) is that an initial wave packet φ_0 represented as a point on the trapped set K evolves in several wave packets $(\varphi_j)_{j=1\to k}$ under the transfer operator $\hat{F}_\nu$, but in general only one wave packet remains on the trapped set K and the $(k-1)$ other ones escape towards infinity. As a result the probability on the trapped set K decays by a factor $1/k$. This is the origin of the spectral gap at $1/\sqrt{k}$ in figure 2.

2. Model and results

2.1. A partially expanding map

Let $g : S^1 \to S^1$ be a C^∞ diffeomorphism (on $S^1 := \mathbb{R}/\mathbb{Z}$). g can be written as $g : \mathbb{R} \to \mathbb{R}$ with $g(x+1) = g(x) + 1$, $\forall x \in \mathbb{R}$. Let $k \in \mathbb{N}$, $k \geqslant 2$, and let the map $E : S^1 \to S^1$ be defined by

$$E : x \in S^1 \to E(x) = kg(x) \bmod 1. \tag{1}$$

Let

$$E_{\min} := \min_x \left(\frac{dE}{dx}\right)(x) = k \min_x \left(\frac{dg}{dx}(x)\right).$$

We will suppose that the function g is such that $\frac{dg}{dx}(x) > \frac{1}{k}$, i.e.

$$E_{\min} > 1 \tag{2}$$

so that E is a *uniform expanding map* on S^1. The map E is then a $k:1$ map (i.e. every point y has k previous images $x \in E^{-1}(y)$). Let $\tau : S^1 \to \mathbb{R}$ be a C^∞ function, and define a map f on $\mathbb{T}^2 = S^1 \times S^1$ by[2]:

$$f : \begin{pmatrix} x \\ s \end{pmatrix} \longmapsto \begin{pmatrix} x' = E(x) = kg(x) & \bmod 1 \\ s' = s + \dfrac{1}{2\pi}\tau(x) & \bmod 1 \end{pmatrix}. \tag{3}$$

[2] The factor $\frac{1}{2\pi}$ in front of $\tau(x)$ is just for convenience. It will simplify other expressions below.

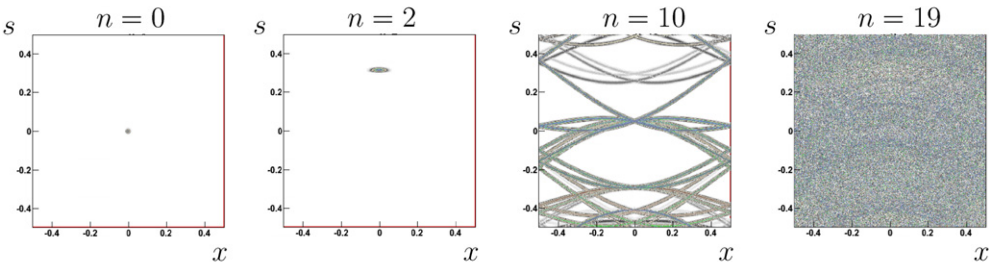

Figure 1. Numerical evolution of an initial small cloud of points on the torus $(x, s) \in \mathbb{T}^2$ under the map f, equation (3), at different time $n = 0, 2, 10, 19$. We have chosen here $E(x) = 2x$ and $\tau(x) = \cos(2\pi x)$. The initial cloud of points is centred around the point $(0, 0)$. For small time n, the cloud of point is transported in the vertical direction s and spreads in the expanding horizontal direction x. Due to instability in x and periodicity, the cloud fills the torus $S^1 \times S^1$ for large time n. On the last image $n = 19$, one observes an invariant absolutely continuous probability measure (called SRB measure, equal to the Lebesgue measure in our example). It reveals the mixing property of the map f in this example (see supplementary data available at stacks.iop.org/Non/24/1/mmedia or www-fourier.ujf-grenoble.fr/~faure/articles/09_html_partially_expanding_maps).

The map f is also a $k : 1$ map. The map f is a very simple example of a *compact group extension* of the expanding map E (see [Dol02, Pes04, p 17]). It is also a special example of a partially hyperbolic map[3]. See figure 1.

2.2. Transfer operator

Instead of studying individual trajectories which have chaotic behaviour, one prefer to study the evolutions of densities induced by the map f. This is the role of the *Perron–Frobenius transfer operator* $\hat{F}^*$ on $C^\infty(\mathbb{T}^2)$ given by:

$$(\hat{F}^*\psi)(\boldsymbol{y}) = \sum_{\boldsymbol{x} \in f^{-1}(\boldsymbol{y})} \frac{1}{|D_{\boldsymbol{x}} f|} \psi(\boldsymbol{x}), \qquad \psi \in C^\infty(\mathbb{T}^2). \tag{4}$$

Indeed if the the function ψ has its support in the vicinity of x then the support of $\hat{F}^*\psi$ is in the vicinity of $y = f(x)$. To explain the Jacobian in the pre-factor, one checks[4] that $\int_{\mathbb{T}^2} (\hat{F}^*\psi)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{\mathbb{T}^2} \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$, i.e. the total measure is preserved.

The operator $\hat{F}^*$ extends to a bounded operator on $L^2(\mathbb{T}^2, \mathrm{d}\boldsymbol{x})$. Its L^2-adjoint written $\hat{F}$ is defined by $(\hat{F}^*\psi, \varphi)_{L^2} = (\psi, \hat{F}\varphi)_{L^2}$, with the scalar product $(\psi, \varphi)_{L^2} := \int_{\mathbb{T}^2} \overline{\psi}(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$. One checks easily that $\hat{F}$ has a simpler expression than $\hat{F}^*$: it is the *pull back operator*, also

[3] A even more general setting would be a C^∞ map $f : M \to M$ on a compact Riemannian manifold M, which is supposed to be partially expanding, i.e. for any $m \in M$, the tangent space $T_m M$ decomposes continuously as

$$T_m M = E_u(m) \oplus E_0(m),$$

where $E_u(m)$ is a (non-invariant) expanding direction (with respect to a Riemannian metric g):

$$|D_m f(v_u)|_g > |v_u|_g, \qquad \forall v_u \in E_u(m)$$

and $E_0(m)$ the neutral direction: there exist a non-zero global section $v_0 \in C^\infty(TM)$ such that $v_0(m) \in E_0(m)$ and $Df(v_0) = v_0$. In our example (3), $M = S^1 \times S^1$, the neutral section is $v_0 = (0, 1)$, and the expanding direction $E_u(m)$ is spanned by the vector $(1, 0)$.

[4] Since $\boldsymbol{y} = f(\boldsymbol{x})$, then $\mathrm{d}\boldsymbol{y} = |D_{\boldsymbol{x}} f|\,\mathrm{d}\boldsymbol{x}$, and

$$\int_{\mathbb{T}^2} (\hat{F}^*\psi)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = \int_{\mathbb{T}^2} \sum_{\boldsymbol{x} \in f^{-1}(\boldsymbol{y})} \frac{1}{|D_{\boldsymbol{x}} f|} \psi(\boldsymbol{x}) |D_{\boldsymbol{x}} f|\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{T}^2} \psi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

called the *Koopman operator,* or *Ruelle transfer operator* and given by

$$(\hat{F}\psi)(\boldsymbol{x}) = \psi(f(\boldsymbol{x})). \tag{5}$$

2.3. The reduced transfer operator

The particular form of map (3) allows some simplifications. Observe that for a function of the form

$$\psi(x,s) = \varphi(x)\mathrm{e}^{\mathrm{i}2\pi\nu s}$$

with $\nu \in \mathbb{Z}$ (i.e. a Fourier mode in s), then

$$(\hat{F}\psi)(x,s) = \varphi(E(x))\mathrm{e}^{\mathrm{i}\nu\tau(x)}\mathrm{e}^{\mathrm{i}2\pi\nu s}.$$

Therefore, the operator $\hat{F}$ preserves the following decomposition in Fourier modes:

$$L^2(\mathbb{T}^2) = \bigoplus_{\nu\in\mathbb{Z}} \mathcal{H}_\nu, \qquad \mathcal{H}_\nu := \{\varphi(x)\mathrm{e}^{\mathrm{i}2\pi\nu s},\ \varphi \in L^2(S^1)\}. \tag{6}$$

The space $\mathcal{H}_\nu$ and $L^2(S^1)$ are unitary equivalent. For $\nu \in \mathbb{Z}$ given, the operator $\hat{F}$ restricted to the space $\mathcal{H}_\nu \equiv L^2(S^1)$, written $\hat{F}_\nu$ is[5]

$$(\hat{F}_\nu\varphi)(x) := \varphi(E(x))\mathrm{e}^{\mathrm{i}\nu\tau(x)}, \qquad \varphi \in L^2(S^1) \equiv \mathcal{H}_\nu, \tag{7}$$

and with respect to the orthogonal decomposition (6), we can write

$$\hat{F} = \bigoplus_{\nu\in\mathbb{Z}} \hat{F}_\nu.$$

We will study the spectrum of this family of operators $\hat{F}_\nu$, with parameter $\nu \in \mathbb{Z}$, and consider more generally a real parameter $\nu \in \mathbb{R}$. We will see that the parameter ν is a *semiclassical parameter*, and $\nu \to \infty$ is the *semiclassical limit*. (if $\nu \neq 0$, $\nu = 1/\hbar$ in usual notations [Mar02]).

Remarks.

- For $\nu = 0$, $\hat{F}_0$ has an obvious eigenfunction $\varphi(x) = 1$, with eigenvalue 1. Except in special cases (e.g. $\tau = 0$ or τ cohomologous to 0, see appendix A), there is no other obvious eigenvalues for $\hat{F}_\nu$ in $L^2(S^1)$ even in the semiclassical limit $\nu \to \infty$. By this, we mean that the author is not aware of the existence of analytical expressions for the eigenvalues.

2.4. Main results on the spectrum of the transfer operator $\hat{F}_\nu$

We first observe that by duality[6], the operator $\hat{F}_\nu$ defined in (7) extends to the distribution space $\mathcal{D}'(S^1)$:

$$\hat{F}_\nu(\alpha)(\overline{\varphi}) = \alpha(\overline{\hat{F}_\nu^*(\varphi)}), \qquad \alpha \in \mathcal{D}'(S^1), \quad \varphi \in C^\infty(S^1), \tag{8}$$

where the L^2-adjoint $\hat{F}_\nu^*$ is given by

$$(\hat{F}_\nu^*\varphi)(y) = \sum_{x\in E^{-1}(y)} \frac{\mathrm{e}^{-\mathrm{i}\nu\tau(x)}}{E'(x)}\varphi(x), \qquad \varphi \in C^\infty(S^1). \tag{9}$$

[5] Note that the operator $\hat{F}_\nu$ appears to be a transfer operator for the expanding map E with an additional weight function $\mathrm{e}^{\mathrm{i}\nu\tau(x)}$.

[6] The complex conjugation in (8) is because duality is related to scalar product on $L^2(S^1)$ by $\alpha(\overline{\varphi}) := \int_{S^1} \overline{\varphi}\alpha = (\varphi,\alpha)_{L^2}$ for any $\alpha, \varphi \in L^2(S^1)$.

Before giving the main results, recall that for $m \in \mathbb{R}$, the Sobolev space $H^m(S^1) \subset \mathcal{D}'(S^1)$ consists in distributions (or continuous functions if $m > 1/2$) ψ such that their Fourier series $\hat{\psi}(\xi)$ satisfy $\|\psi\|^2_{H^m} := \sum_{\xi\in 2\pi\mathbb{Z}} |\langle\xi\rangle^m \hat{\psi}(\xi)|^2 < \infty$, with $\langle\xi\rangle := (1+\xi^2)^{1/2}$. It can equivalently be written ([Tay96a, p 271]).

$$H^m(S^1) := \langle\hat{\xi}\rangle^{-m}(L^2(S^1))$$

with the differential operator $\hat{\xi} := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$.

The following theorem is well known [Rue86]. We will, however, provide a new proof based on semiclassical analysis.

Theorem 2 (Discrete spectrum of resonances). *Let $m < 0$. For any $\nu \in \mathbb{Z}$, the operator $\hat{F}_\nu$ leaves the Sobolev space $H^m(S^1)$ invariant, and*

$$\hat{F}_\nu : H^m(S^1) \to H^m(S^1)$$

is a bounded operator and can be written

$$\hat{F}_\nu = \hat{R}_\nu + \hat{K}_\nu, \tag{10}$$

where $\hat{K}_\nu$ is a compact operator, and $\hat{R}_\nu$ has a small norm:

$$\|\hat{R}_\nu\|_{H^m} \leqslant r_m := \frac{1}{E_{\min}^{|m|}}\sqrt{\frac{k}{E_{\min}}} \tag{11}$$

(the interesting situation is $m \ll 0$, since the norm $\|\hat{R}_\nu\|_{H^m}$ shrinks to zero for $m \to -\infty$).

Therefore, $\hat{F}_\nu$ has an essential spectral radius less than r_m, which means that $\hat{F}_\nu$ has discrete (possibly empty) spectrum of generalized eigenvalues λ_i outside the circle of radius r_m (see [Tay96a, prop 6.9, p 499]). The eigenvalues λ_i are called Ruelle resonances. *Together with their associated eigenspace, they do not depend on m and are intrinsic to the transfer operator $\hat{F}_\nu$.*

The following theorem is analogous to theorem 1.1 in [Tsu08]. However, the approach and the proof we propose are different and rely on semiclassical analysis.

Theorem 3 (Spectral gap in the semiclassical limit). *If the map f is* partially captive *(definition given in p 1489) (and m sufficiently negative), then the spectral radius of the operator $\hat{F}_\nu : H^m(S^1) \to H^m(S^1)$ does not depend on m and satisfies in the semiclassical limit $\nu \to \infty$:*

$$r_s(\hat{F}_\nu) \leqslant \frac{1}{\sqrt{E_{\min}}} + o(1), \tag{12}$$

which is strictly smaller than 1 from (2). The following result gives a more precise information about the norm of the operator that is controlled uniformly in ν and will be used in theorem 5. For any $\rho > \frac{1}{\sqrt{E_{\min}}}$, there exists $c > 0, n_0 \in \mathbb{N}$, $\nu_0 \in \mathbb{N}$, $m_0 < 0$ such that for any $|\nu| \geqslant \nu_0$, $m < m_0$ we have $\|\hat{F}_\nu^{n_0}\|_{H^m_\nu} \leqslant \rho^{n_0}$. Then for any $n \in \mathbb{N}$,

$$\|\hat{F}_\nu^n\|_{H^m_\nu} \leqslant c\rho^n, \tag{13}$$

where $\|\cdot\|_{H^m_\nu}$ is the Sobolev norm defined by $\|\psi\|^2_{H^m_\nu} := \sum_{\xi\in 2\pi\mathbb{Z}} |\langle\frac{1}{\nu}\xi\rangle^m \hat{\psi}(\xi)|^2$.

Remarks.

- This remark concerns the regularity of the eigenfunctions of $\hat{F}_\nu$. Let λ_i be a generalized eigenvalue of $\hat{F}_\nu$. Let φ_i denotes a generalized eigenfunction of $\hat{F}_\nu$ associated with λ_i (i.e. $\hat{F}_\nu\varphi_i = \lambda_i\varphi_i$ if λ_i is an eigenvalue). A corollary of theorem 2 is that for any m such that $m < m_0 < 0$ where m_0 is given by $r_{m_0} = |\lambda_i|$ (as defined in (11)) then λ_i is an eigenvalue of $\hat{F}_\nu$ and therefore φ_i belongs to the Sobolev space $H^m(S^1)$.

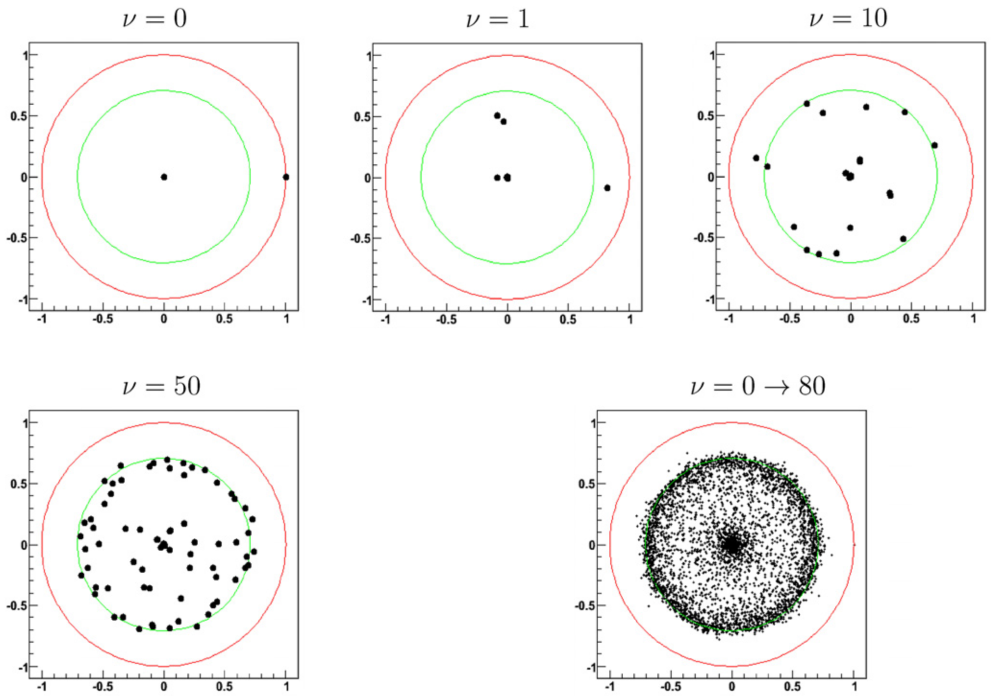

Figure 2. Black dots are numerical computation of the eigenvalues $\lambda_i \in \mathbb{C}$ of $\hat{F}_\nu$ for different values of $\nu \in \mathbb{N}$, and union of these in the last image. We have chosen here $E(x) = 2x$ i.e. $k = 2$, and $\tau(x) = \cos(2\pi x)$. The external circle has radius 1. The internal circle has radius $1/\sqrt{E_{\min}} = 1/\sqrt{2}$ and represents the upper bound given in equation (12). As $\nu \in \mathbb{R}$ moves continuously, the resonances move in a spectacular way (see supplementary data available at stacks.iop.org/Non/24/1/mmedia or www-fourier.ujf-grenoble.fr/∼faure/articles/09_html_partially_expanding_maps).

- By duality we have similar spectral results for the Perron Frobenius operator $\hat{F}_\nu^*$ in the dual spaces $(H^m(S^1))^* = H^{-m}(S^1)$, with $m < 0$. The eigenvalues of $\hat{F}_\nu^*$ are $\overline{\lambda_i}$. An associated generalized eigenfunctions $\tilde{\varphi}_i$ of $\hat{F}_\nu^*$ belongs to $H^{-m}(S^1)$ if $m < m_0 < 0$ where m_0 is given by $r_{m_0} = |\lambda_i|$ as above, therefore $\tilde{\varphi}_i$ belongs to $\bigcap_{m<m_0} H^{-m} = H^\infty = C^\infty(S^1)$.
- In the proof of theorem 3, we will obtain that a general bound for $r_s(\hat{F}_\nu)$ (with no hypothesis on f) is given by

$$r_s(\hat{F}_\nu) \leqslant \frac{1}{\sqrt{E_{\min}}} \exp\left(\frac{1}{2} \lim_{n\to\infty} \left(\frac{\log \mathcal{N}(n)}{n}\right)\right) + o(1), \tag{14}$$

  where the function $\mathcal{N}(n)$ will be defined in equation (44). This bound is similar to the bound given in [Tsu08, theorem 1.1] by Tsujii.
- In [Tsu08, theorem 1.2] Tsujii shows that *the partially captive property*, i.e. $\lim_{n\to\infty}(\frac{\log \mathcal{N}(n)}{n}) = 0$, *is true for almost all functions* τ. He does not use the terminology 'partially captive' but it is equivalent. In [Tsu08, theorem 1.2] Tsujii proves this property for linear map $E(x) = kx$ but in [Tsu08, remark 1.5] he argues that this should be true for nonlinear map as well. See also [Wei10]. However, the author does not know any smooth function τ for which this property holds. From numerical computations, it seems that it holds for $\tau(x) = \cos(2\pi x)$, used in figure 2.

- From the definition of $\mathcal{N}(n)$ it is clear that $\mathcal{N}(n) \leqslant k^n$ hence $\exp(\frac{1}{2}\lim_{n\to\infty}(\frac{\log\mathcal{N}(n)}{n})) \leqslant \sqrt{k}$. Also from the definition of $E_{\min}$, it is clear that $E_{\min} \leqslant k$ and therefore the upper bound in (14) is not sharp since it does not give the obvious bound $r_s(\hat{F}_\nu) \leqslant 1$ (see [FRS08, corollary 2]). In [Tsu08, theorem 1.4] Tsujii proves that $\exp(\frac{1}{2}\lim_{n\to\infty}(\frac{\log\mathcal{N}(n)}{n})) < \sqrt{k}$ if and only if τ is not a co-boundary. This implies a spectral gap for the linear model $E(x) = kx$.
- Note that the above results say nothing about the existence of Ruelle resonances λ_i. In the literature there are many results concerning the existence of resonances [Nau08].
- One observes numerically that for large $\nu \in \mathbb{R}$, the eigenvalues $\lambda_i(\nu)$ repulse each other like eigenvalues of random complex matrices. This suggests that many important questions of quantum chaos (e.g. the conjecture of Random Matrices [Boh91]) also concerns the Ruelle resonances of partially hyperbolic dynamics in the semiclassical limit. The semiclassical limit is precisely the limit of small wavelength in the neutral direction.
- Remarks on numerical computation of the Ruelle resonances: one diagonalizes the matrix which expresses the operator $\hat{F}_\nu$ in Fourier basis $\varphi_n(x) := \exp(\mathrm{i}2\pi nx)$, $n \in \mathbb{Z}$. For the example of figure 2 one obtains $\langle\varphi_{n'}|\hat{F}_\nu\varphi_n\rangle = \mathrm{e}^{-\mathrm{i}2\pi\frac{3}{4}(2n-n')}J_{(2n-n')}(\nu)$ where $J_n(x)$ is the Bessel function of first kind [AS54, 9.1.21 p 360]. Corollary 2 in [FR06] guaranties that the eigenvalues of the truncated matrix $|n|, |n'| \leqslant N$ converges towards the Ruelle resonances as $N \to \infty$.
- One can prove [Arn10] that in the semiclassical limit $\nu \to \infty$, the number of Ruelle resonances λ_i (counting multiplicities) outside a fixed radius λ is bounded by a '*Weyl upper bound*':

$$\forall\lambda > 0, \qquad \sharp\{i \in \mathbb{N},\ s.t.\ |\lambda_i| \geqslant \lambda\} \leqslant \left(\frac{\nu}{2\pi}\right)\mu(K) + o(\nu),$$

  where $\mu(K)$ is the Lebesgue measure of the trapped set K defined later in equation (43). As usual in the semiclassical theory of non-self-adjoint operators, see [Sjö90, SZ07], the Weyl law gives an upper bound for the density of resonances but no lower bound. See discussions in [Non08, section 3.1].

2.5. Spectrum of $\hat{F}_\nu$ and dynamical correlation functions

In this section, in order to give some 'physical meaning' to the spectrum of $\hat{F}_\nu$, we recall relations between the spectral results of theorems 2 and 3 and the evolution of correlation functions [Bal00]. This will allow us to interpret the evolution and convergence of clouds of points observed in figure 1.

We first give an immediate corollary of theorem 2. Let $\nu \in \mathbb{Z}$. For $\varepsilon > 0$, let m such that $r_m < \varepsilon$, (r_m is defined in (11)). We denote by π the spectral projector associated with $\hat{F}_\nu : H^m(S^1) \to H^m(S^1)$ outside the disc of radius ε, and $\hat{k}_\nu := \hat{\pi}\hat{F}_\nu$, $\hat{r}_\nu := (1-\hat{\pi})\hat{F}_\nu$. Then we have a spectral decomposition[7]

$$\hat{F}_\nu = \hat{k}_\nu + \hat{r}_\nu, \qquad \hat{k}_\nu\hat{r}_\nu = \hat{r}_\nu\hat{k}_\nu = 0 \tag{15}$$

[7] Note that in (10) the operators $\hat{R}_\nu$,$\hat{K}_\nu$ are different from $\hat{r}_\nu$, $\hat{k}_\nu$. They are not a spectral decomposition of $\hat{F}_\nu$.

and

1. The spectral radius of $\hat{r}_\nu$ is smaller than ε.
2. $\hat{k}_\nu$ has finite rank. Its spectrum has generalized eigenvalues $\lambda_{i,\nu}$ (counting multiplicity), the *Ruelle resonances*, with $\varepsilon < |\lambda_{i,\nu}|$. The general Jordan decomposition of $\hat{k}_\nu$ can be written as

$$\hat{k}_\nu = \sum_{i\geqslant 0,|\lambda_{i,\nu}|>\varepsilon} \left(\lambda_{i,\nu} \sum_{j=1}^{d_i} v_{i,j,\nu} \otimes w_{i,j,\nu} + \sum_{j=1}^{d_i-1} v_{i,j,\nu} \otimes w_{i,j+1,\nu} \right) \tag{16}$$

with d_i the dimension of the Jordan block associated with the eigenvalue $\lambda_{i,\nu}$, with $v_{i,j,\nu} \in H^m$, $w_{i,j,\nu} \in H^{-m}$ ($w_{i,j,\nu}$ is viewed as a linear form on H^m by duality). They satisfy $w_{i,j,\nu}(v_{k,l,\nu}) = \delta_{ik}\delta_{jl}$.

If $\psi_1, \psi_2 \in C^\infty(S^1)$, the *correlation function at time* $n \in \mathbb{N}$ is defined by

$$C_{\psi_2,\psi_1}(n) := (\hat{F}_\nu^{*n}\psi_2, \psi_1)_{L^2(S^1)} = (\psi_2, \hat{F}_\nu^n \psi_1)_{L^2(S^1)},$$

which represents the function ψ_2 evolved n times by the Perron–Frobenius operator $\hat{F}_\nu^*$ and tested against the test function ψ_1.

Lemma 4. *For any $\psi_1, \psi_2 \in C^\infty(S^1)$, $\varepsilon > 0$ such that $\varepsilon \neq |\lambda_i|, \forall i$, one has for $n \to \infty$,*

$$C_{\psi_2,\psi_1}(n) = \sum_{i\geqslant 0,|\lambda_{i,\nu}|>\varepsilon} \sum_{k=0}^{\min(n,d_i-1)} C_n^k \lambda_{i,\nu}^{n-k} \sum_{j=1}^{d_i-k} v_{i,j,\nu}(\overline{\psi_2}) w_{i,j+k,\nu}(\psi_1) + \|\psi_1\|_{H^m} \|\psi_2\|_{H^{-m}} O(\varepsilon^n) \tag{17}$$

with any m such that $r_m < \varepsilon$, and $C_n^k := \frac{n!}{(n-k)!k!}$.

Remarks.

- More generally equation (17) still holds for $\psi_1 \in H^m$ and $\psi_2 \in H^{-m}$, with m such that $r_m < \varepsilon$.
- The right-hand side of equation (17) is complicated by the possible presence of 'Jordan blocks'. In the case where there is no Jordan block ($d_i = 1, \forall i$) it reads more simply: for $n \to \infty$,

$$C_{\psi_2,\psi_1}(n) = \sum_{i\geqslant 0,|\lambda_{i,\nu}|>\varepsilon} \lambda_{i,\nu}^n v_{i,\nu}(\overline{\psi_2}) w_{i,\nu}(\psi_1) + O(\varepsilon^n).$$

Proof. For any $\varepsilon > 0$, let m such that $r_m < \varepsilon$. For any $n \geqslant 1$ we have $\hat{F}_\nu^n = \hat{k}_\nu^n + \hat{r}_\nu^n$ and $\|\hat{r}_\nu^n\|_{H^m} = O(\varepsilon^n)$. If $\psi_1 \in H^m$, $\psi_2 \in H^{-m}$ then we use the Cauchy–Schwartz inequality $|(\psi, \varphi)_{H^{-m}\times H^m}| \leqslant \|\psi\|_{H^{-m}} \|\varphi\|_{H^m}$ to write

$$\begin{aligned} C_{\psi_2,\psi_1}(n) &= (\psi_2, \hat{F}_\nu^n \psi_1)_{L^2} \\ &= (\psi_2, \hat{k}_\nu^n \psi_1)_{L^2} + (\psi_2, \hat{r}_\nu^n \psi_1)_{L^2} \\ &= (\psi_2, \hat{k}_\nu^n \psi_1)_{L^2} + \|\psi_2\|_{H^{-m}} \|\psi_1\|_{H^m} O(\varepsilon^n). \end{aligned} \tag{18}$$

Using the Jordan Block decomposition of $\hat{k}_\nu$, equation (16), we have

$$(\psi_2, \hat{k}_\nu^n \psi_1) = \sum_{i\geqslant 0,|\lambda_{i,\nu}|>\varepsilon} \sum_{k=0}^{\min(n,d_i-1)} C_n^k \lambda_{i,\nu}^{n-k} \sum_{j=1}^{d_i-k} (\psi_2, v_{i,j,\nu}) w_{i,j+k,\nu}(\psi_1), \tag{19}$$

$$= \sum_{i\geqslant 0,|\lambda_{i,\nu}|>\varepsilon} \sum_{k=0}^{\min(n,d_i-1)} C_n^k \lambda_{i,\nu}^{n-k} \sum_{j=1}^{d_i-k} v_{i,j,\nu}(\overline{\psi_2}) w_{i,j+k,\nu}(\psi_1). \tag{20}$$

We have obtained equation (17). □

We can then draw some conclusions for the full map f and its transfer operator $\hat{F}$, defined in equation (5) instead of merely the restricted transfer operator $\hat{F}_\nu$, $\nu \in \mathbb{Z}$. We recall first that $\lambda_{0,0} = 1$ is an obvious eigenvalue for $\hat{F}_{\nu=0}$ with eigenfunction $v_{0,0}(x) = 1$. If $\lambda_{0,0}$ is non-degenerate, we denote $w_{0,0}(x) \in C^\infty(S^1)$ the dual eigenfunction, i.e. $\hat{F}^*_{\nu=0} w_{0,0} = w_{0,0}$.

Theorem 5 (Mixing on $\mathbb{T}^2$). *If the conclusion of theorem 3 for a spectral gap holds then for any $\rho > \frac{1}{\sqrt{E_{\min}}}$ there exists $\nu_0 \in \mathbb{Z}$ such that for any $\Psi_1, \Psi_2 \in C^\infty(\mathbb{T}^2)$, for $n \to \infty$,*

$$C_{\Psi_2,\Psi_1}(n) := (\hat{F}^{*n}\Psi_2, \Psi_1)_{L^2(\mathbb{T}^2)} = (\Psi_2, \hat{F}^n \Psi_1)_{L^2(\mathbb{T}^2)} \tag{21}$$

$$= \sum_{\nu,|\nu|<\nu_0} \sum_{i\geqslant 0,|\lambda_{i,\nu}|>\rho} \sum_{k=0}^{\min(n,d_i-1)} C_n^k \lambda_{i,\nu}^{n-k} \sum_{j=1}^{d_i-k} v_{i,j,\nu}(\overline{\Psi_{2,\nu}}) w_{i,j+k,\nu}(\Psi_{1,\nu}) \tag{22}$$

$$+ O(\rho^n), \tag{23}$$

where $\Psi_{1,\nu} \in C^\infty(S^1)$ (respectively, $\Psi_{2,\nu}$) are the components of Ψ_1 (respectively Ψ_2) with respect to the decomposition (6). If moreover $\lambda_{0,0} = 1$ is the only eigenvalue of $\hat{F}$ on the unit circle (with multiplicity 1) with associated eigen-states $v_{0,0} = 1$, $w_{0,0}(x) \in C^\infty(S^1)$, then the map f on $\mathbb{T}^2$ is 'mixing' i.e. for $n \to \infty$:

$$C_{\Psi_2,\Psi_1}(n) \to \left(\int_{\mathbb{T}^2} \overline{\Psi_2(x,s)}\, \mathrm{d}x\, \mathrm{d}s \right) \left(\int_{\mathbb{T}^2} \Psi_1 \mathrm{d}\mu_{\mathrm{SRB}} \right) \tag{24}$$

with the 'equilibrium Sina–Ruelle–Bowen measure' $\mathrm{d}\mu_{\mathrm{SRB}} = w_{0,0}(x)\, \mathrm{d}x\, \mathrm{d}s$.

Proof. For smooth functions $\Psi_1, \Psi_2 \in C^\infty(\mathbb{T}^2)$ we have for any m, and any $N \geqslant 0$, $\|\Psi_{1,\nu}\|_{H^m_\nu} = O(\frac{1}{\nu^N})$, $\|\Psi_{2,\nu}\|_{H^{-m}_\nu} = O(\frac{1}{\nu^N})$. On the other hand, if the conclusion of theorem 3 for a spectral gap holds then for any $\rho > \frac{1}{\sqrt{E_{\min}}}$ there exists $\nu_0 \in \mathbb{Z}$ such that for any ν, $|\nu| \geqslant \nu_0$, $\hat{F}_\nu$ has no spectrum outside the disc of radius ρ. Then (17) and (13) give: there exists $c > 0$, $n_0 \in \mathbb{N}$, $\nu_0 \in \mathbb{N}$, $m_0 < 0$ such that for any $|\nu| \geqslant \nu_0$, $m < m_0$, any $n \in \mathbb{N}$,

$$(\Psi_{2,\nu}, \hat{F}_\nu^n \Psi_{1,\nu})_{L^2(S^1)} \leqslant \|\Psi_{1,\nu}\|_{H^m_\nu} \|\Psi_{2,\nu}\|_{H^{-m}_\nu} \|\hat{F}_\nu^n\|_{H^m_\nu}, \qquad \text{for } |\nu| \geqslant \nu_0$$

$$\leqslant \frac{C_N}{\nu^N} c\rho^n.$$

Hence for N large enough, $\sum_{|\nu|\geqslant\nu_0} |(\Psi_{2,\nu}, \hat{F}_\nu^n \Psi_{1,\nu})_{L^2(S^1)}| \leqslant O(\rho^n)$. We use lemma 4 with $\varepsilon = \rho$ and obtain

$$C_{\Psi_2,\Psi_1}(n) = (\Psi_2, \hat{F}^n \Psi_1)_{L^2(\mathbb{T}^2)} = \sum_{\nu\in\mathbb{Z}} (\Psi_{2,\nu}, \hat{F}_\nu^n \Psi_{1,\nu})_{L^2(S^1)}$$

$$= \sum_{|\nu|<\nu_0} (\Psi_{2,\nu}, \hat{F}_\nu^n \Psi_{1,\nu})_{L^2(S^1)} + \sum_{|\nu|\geqslant\nu_0} (\Psi_{2,\nu}, \hat{F}_\nu^n \Psi_{1,\nu})_{L^2(S^1)}$$

$$= \sum_{\nu,|\nu|<\nu_0} \sum_{i\geqslant 0,|\lambda_{i,\nu}|>\rho} \sum_{k=0}^{\min(n,d_i-1)} C_n^k \lambda_{i,\nu}^{n-k} \sum_{j=1}^{d_i-k} v_{i,j,\nu}(\overline{\Psi_{2,\nu}}) w_{i,j+k,\nu}(\Psi_{1,\nu}) + O(\rho^n).$$

If $\lambda_{0,0} = 1$ is the only eigenvalue of $\hat{F}$ on the unit circle (with multiplicity 1) and the other ones are $|\lambda_{i,\nu}| < \rho < 1$ this gives

$$C_{\Psi_2,\Psi_1}(n) = v_{0,0}(\overline{\Psi_{2,0}}) w_{0,0}(\Psi_{1,0}) + O(\rho^n)$$

$$= \left(\int_{\mathbb{T}^2} \overline{\Psi_2(x,s)}\, \mathrm{d}x\, \mathrm{d}s \right) \left(\int_{\mathbb{T}^2} \Psi_1\, \mathrm{d}\mu_{\mathrm{SRB}} \right) + O(\rho^n). \qquad \square$$

Remarks

- In figure 1 one observes the mixing property of the dynamics and the equilibrium measure on the last image. In this case ($E(x) = 2x$ is linear) we have $w_{0,0}(x) = 1$ and the equilibrium measure is the Lebesgue measure $\mathrm{d}\mu_{\mathrm{SRB}} = \mathrm{d}x\,\mathrm{d}s$.
- In formula (21), note that this is a finite sum other the eigenvalues outside the disc of radius ρ. This expression shows that the time behaviour of correlation functions is governed by an effective linear operator of finite rank up to $\mathcal{O}(\rho^n)$, $n \to \infty$. This effective linear operator is the spectral projection of $\hat{F}$ on the outer eigenspaces such that $|\lambda_{i,\nu}| > \rho$. The semiclassical theory used in this paper does not give direct information on this effective operator, but on its existence and the fact that it is supported by the low Fourier modes.

3. Proof of theorem 2 on resonances spectrum

In this proof, we follow closely the proof of theorem 4 in [FRS08] although we deal here with expanding map instead of hyperbolic map, and this simplifies a lot, since we can work with ordinary Sobolev spaces and not anisotropic Sobolev spaces. Here $\nu \in \mathbb{Z}$ is fixed.

3.1. Dynamics on the cotangent space T^*S^1

The first step is to realize that in order to study the spectral properties of the transfer operator, we have to study the dynamics lifted on the cotangent space. This basic idea has already been exploited in [FRS08].

In equation (1), the map $E : S^1 \to S^1$ is a $k:1$ map, which means that every point $y \in S^1$ has k inverses denoted by $x_\varepsilon \in E^{-1}(y)$ and given explicitly by

$$x_\varepsilon = E_\varepsilon^{-1}(y) = g^{-1}\left(\frac{y}{k} + \varepsilon\frac{1}{k}\right), \qquad \text{with } \varepsilon = 0, \ldots, k-1, \quad y \in [0, 1[.$$

We will denote the derivative by $E'(x) := \mathrm{d}E/\mathrm{d}x$.

Proposition 6. *In equation (7) $\hat{F}_\nu$ is a* Fourier integral operator *(FIO) acting on $C^\infty(S^1)$.*

The associated canonical map *on the cotangent space $(x, \xi) \in T^*S^1 \equiv S^1 \times \mathbb{R}$ is k-valued and given by*

$$F(x, \xi) = \{F_0(x, \xi), \ldots, F_{k-1}(x, \xi)\}, \qquad (x, \xi) \in S^1 \times \mathbb{R}, \tag{25}$$

where for any $\varepsilon = 0, \ldots, k-1$,

$$F_\varepsilon : \begin{cases} x \to x'_\varepsilon = E_\varepsilon^{-1}(x) = g^{-1}\left(\frac{1}{k}x + \varepsilon\frac{1}{k}\right), \\ \xi \to \xi'_\varepsilon = E'(x'_\varepsilon)\xi = kg'(x'_\varepsilon)\xi. \end{cases} \tag{26}$$

Similarly the adjoint $\hat{F}^*$ is a FIO whose canonical map *is F^{-1}. See figure 3.*

The proof is just that the operator $\varphi \to \varphi \circ E$ on $C^\infty(S^1)$ is one of the simplest example of Fourier integral operator, see [Mar02, example 2, p 150].

The term $\mathrm{e}^{\mathrm{i}\nu\tau(x)}$ in equation (7) does not contribute to the expression of F, since here ν is considered as a fixed parameter, and therefore $\mathrm{e}^{\mathrm{i}\nu\tau(x)}$ acts as a pseudodifferential operator (equivalently as a FIO whose canonical map is the identity).

The map F is the map E^{-1} lifted on the cotangent space T^*S^1 in the canonical way. Indeed, if we denote a point $(x, \xi) \in T^*S^1 \equiv S^1 \times \mathbb{R}$ then using the usual formula for differentials $y = E(x) = kg(x) \Rightarrow \mathrm{d}y = E'(x)\,\mathrm{d}x \Leftrightarrow \xi' = E'(x)\xi$, we deduce the above expression for F.

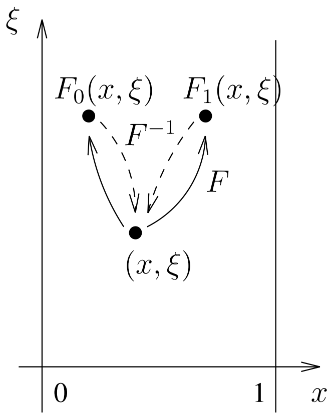

Figure 3. This figure is for $k = 2$. The map $F = \{F_0, \ldots, F_{k-1}\}$ is $1 : k$, and its inverse F^{-1} is $k : 1$ on $T^*S^1 \equiv S^1 \times \mathbb{R}$. The dynamics of F is also the dynamics of wave packets under the transfer operator $\hat{F}_\nu$.

Remarks

- The physical meaning for $\hat{F}_\varepsilon$ being a *Fourier Integral Operator* (F.I.O.) and its relation with the canonical map F can be well understood in term of dynamics of wave packets. We provide a detailed presentation of this in the next section.
- Observe that the dynamics of the map F on $S^1 \times \mathbb{R}$ has a quite simple property: the zero section $\{(x, \xi) \in S^1 \times \mathbb{R},\ \xi = 0\}$ is globally invariant and any other point with $\xi \neq 0$ escapes towards infinity ($\xi \to \pm\infty$) in a controlled manner:

$$|\xi'_\varepsilon| \geqslant E_{\min}|\xi|, \qquad \forall \varepsilon = 0, \ldots, k-1, \tag{27}$$

  where $E_{\min} > 1$ is given in (2).

3.2. Dynamics of wave packets

The reading of this section is not necessary for the rest of the paper. It is intended to the reader non-familiar with Fourier integral operators and associated canonical maps. We show that if $\varphi_{(x,\xi)}$ is a wave packet 'micro-localized' at position $(x, \xi) \in T^*S^1$ of phase space, then $\varphi' := \hat{F}_\nu \varphi_{(x,\xi)}$ is a superposition of k wave packets at positions $(x'_\varepsilon, \xi'_\varepsilon) = F_\varepsilon(x, \xi)$, $\varepsilon = 0, \ldots, k-1$, given by the canonical map F. This is illustrated in figure 3.

Let $\hbar > 0$ be a parameter (called the *semiclassical parameter*), and for $(x, \xi) \in \mathbb{R}^2 \equiv T^*\mathbb{R}$ we define $\varphi_{(x,\xi)} \in \mathcal{S}(\mathbb{R})$ called a *Gaussian wavepacket* by

$$\tilde{\varphi}_{(x,\xi)}(y) := C e^{i\frac{1}{\hbar}y\xi} e^{-\frac{1}{2\hbar}(y-x)^2}, \qquad C > 0.$$

Note that in this last expression the factor $e^{iy(\xi/\hbar)}$ is similar to a Fourier mode with Fourier component $\xi/\hbar$. In other words, the role of $\hbar$ in this factor is just a scaling in Fourier space. This 'artificial' parameter $\hbar$ is independent of ν in this section, whereas we will choose it such that $\hbar = 1/\nu$ in section 4.

We then construct the periodic function $\varphi_{(x,\xi)}(y) := \sum_{k\in\mathbb{Z}} \tilde{\varphi}_{(x,\xi)}(y-k)$ in $C^\infty(S^1)$, and choose C such that $\|\varphi_{(x,\xi)}\|_{L^2(S^1)} = 1$. We easily observe that for $\hbar \ll 1$, $\varphi_{(x,\xi)}(y) = O(\hbar^\infty)$ is negligible if $y \neq x \bmod 1$, where $O(\hbar^\infty)$ means $O(\hbar^N)$ for any $N > 0$. Similarly the $\hbar$-Fourier transform of $\tilde{\varphi}_{(x,\xi)}$ (defined by $(\mathcal{F}\tilde{\varphi}_{(x,\xi)})(\eta) := \frac{1}{\sqrt{2\pi\hbar}} \int e^{-i\frac{\eta y}{\hbar}} \tilde{\varphi}_{(x,\xi)}(y)\, dy$) is $|(\mathcal{F}\tilde{\varphi}_{(x,\xi)})(\eta)| = C \exp(-\frac{(\eta-\xi)^2}{2\hbar})$ and is negligible if $\eta \neq \xi$.

Alternatively this can be seen on the scalar product between two Gaussian wavepackets which is also easy to compute:

$$\begin{aligned}|(\varphi_{(y,\eta)},\varphi_{(x,\xi)})_{L^2(S^1)}| &= |(\tilde\varphi_{(y,\eta)},\tilde\varphi_{(x,\xi)})_{L^2(\mathbb{R})}| + O(\mathrm{e}^{-c/\hbar}), \qquad c>0\\ &= \exp\left(-\frac{1}{4\hbar}((x-y)^2+(\xi-\eta)^2)\right) + O(\mathrm{e}^{-c/\hbar})\\ &= O(\hbar^\infty) \qquad \text{if } (y,\eta)\neq(x,\xi) \end{aligned} \tag{28}$$

Definition 7 ([Mar02], chapter 3). *If $\psi_\hbar \in L^2(\mathbb{R})$, $\hbar\to 0$, is a sequence of functions, we say that ψ is* micro-locally small *near $(x_0,\xi_0)\in T^*\mathbb{R}$ if*

$$|(\tilde\varphi_{(y,\eta)},\psi)_{L^2(\mathbb{R})}| = O(\hbar^\infty)$$

uniformly for (y,η) in a neighbourhood of (x_0,ξ_0). The complementary of such points (x_0,ξ_0) is called the micro-support *of ψ.*

For example (28) shows that the micro-support of a Gaussian wavepacket $\tilde\varphi_{(x,\xi)}$ is the point $(x,\xi)\in T^*\mathbb{R}$.

Proposition 8. *The micro-support of $\hat F_\nu \varphi_{(x,\xi)}$ is the set of points*

$$F(x,\xi) = \{F_0(x,\xi),\ldots,F_{k-1}(x,\xi)\}\in T^*S^1.$$

We have indeed uniformly in a neighbourhood of $(y,\eta)\neq F_\varepsilon(x,\xi)$, $\quad\forall\varepsilon=0,\ldots,k-1$:

$$|(\varphi_{(y,\eta)},\hat F_\nu\varphi_{(x,\xi)})_{L^2(S^1)}| = O(\hbar^\infty).$$

Proof. We have

$$(\tilde\varphi_{(y,\eta)},\hat F_\nu\tilde\varphi_{(x,\xi)})_{L^2(\mathbb{R})} = C^2\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\frac{1}{\hbar}\eta z}\mathrm{e}^{-\frac{1}{2\hbar}(y-z)^2}\mathrm{e}^{+\mathrm{i}\frac{1}{\hbar}\xi E(z)}\mathrm{e}^{-\frac{1}{2\hbar}(x-E(z))^2}\mathrm{e}^{\mathrm{i}\nu\tau(z)}\,\mathrm{d}z. \tag{29}$$

In the semiclassical limit $\hbar\ll 1$, because of the quadratic terms in the exponential, this is clearly negligible if $z\neq y$ or $x\neq E(z)$ hence if $y\neq E_\varepsilon^{-1}(x)$ mod 1, $\forall\varepsilon$. But due to the high oscillatory terms with the phase function $\varphi(z) := -\eta z+\xi E(z)$ this is also negligible from the *non-stationary phase approximation*[8], if $\varphi'(z) = -\eta+\xi E'(z)\neq 0$ i.e. if $\eta\neq E'(y)\xi$. Together this gives $(y,\eta)\neq F_\varepsilon(x,\xi)$, $\varepsilon=0,\ldots,k-1$. □

3.3. The escape function

The *class of symbols*[9] S^m, with *order* $m\in\mathbb{R}$, consists of functions on the cotangent space $A\in C^\infty(S^1\times\mathbb{R})$ such that

$$|\partial_\xi^\alpha\partial_x^\beta A|_\infty \leqslant C_{\alpha,\beta}\langle\xi\rangle^{m-|\alpha|}, \qquad \langle\xi\rangle = (1+\xi^2)^{1/2}.$$

Lemma 9. *Let $m<0$ and define the C^∞ function on T^*S^1:*

$$A_m(x,\xi) := \langle\xi\rangle^m \ \in S^m$$

with $\langle\xi\rangle = (1+\xi^2)^{1/2}$. A_m decreases with $|\xi|$ and belongs to the symbol class S^m. Equation (27) implies that the function A_m decreases strictly *along the trajectories of F outside the zero section:*

$$\forall R>0,\quad \forall|\xi|>R,\quad \forall\varepsilon=0,\ldots,k-1 \quad \frac{A_m(F_\varepsilon(x,\xi))}{A_m(x,\xi)}\leqslant C^{|m|}<1,$$

$$\text{with } C=\sqrt{\frac{R^2+1}{R^2E_{\min}^2+1}}<1. \tag{30}$$

One has $C\to 1/E_{\min}$ for $R\to\infty$.

[8] Let $\varphi\in C^\infty(\mathbb{R})$ real valued, such that $\varphi'\neq 0$ every where. It $u\in C_0^\infty(\mathbb{R})$ then the integral $\int \mathrm{e}^{\mathrm{i}\frac{1}{\hbar}\varphi(x)}u(x)\mathrm{d}x$ is rapidly decreasing when $\hbar\to 0$ [EZ03, chaptre 3].

[9] See [Tay96b, p 2].

Proof.

$$\frac{A_m(F_\varepsilon(x,\xi))}{A_m(x,\xi)} = \frac{(1+\xi^2)^{|m|/2}}{(1+(\xi'_\varepsilon)^2)^{|m|/2}} \leqslant \frac{(1+\xi^2)^{|m|/2}}{(1+E_{\min}^2\xi^2)^{|m|/2}} \leqslant \left(\frac{1+R^2}{1+E_{\min}^2R^2}\right)^{|m|/2} = C^{|m|}. \qquad \square$$

The symbol A_m can be quantized into a pseudodifferential operator $\hat{A}_m$ (PDO for short) which is self-adjoint and invertible on $C^\infty(S^1)$ using the quantization rule ([Tay96b, p 2])

$$(\hat{A}\varphi)(x) = \int A(x,\xi)\mathrm{e}^{\mathrm{i}(x-y)\xi}\varphi(y)\,\mathrm{d}y\,\mathrm{d}\mu(\xi), \tag{31}$$

with measure $\mathrm{d}\mu(\xi) = \sum_{n\in\mathbb{Z}}\delta(\xi-2\pi n)$. In our simple case, this is very explicit: in Fourier space, $\hat{A}_m$ is simply the multiplication by $\langle\xi\rangle^m$.

Recall that the *Sobolev space* $H^m(S^1)$ is defined by ([Tay96a, p 271]):

$$H^m(S^1) := \hat{A}_m^{-1}(L^2(S^1)).$$

The following commutative diagram:

$$\begin{array}{ccc} L^2(S^1) & \overset{\hat{Q}_m}{\rightarrow} & L^2(S^1) \\ \downarrow \hat{A}_m^{-1} & \circlearrowleft & \downarrow \hat{A}_m^{-1} \\ H^m(S^1) & \overset{\hat{F}_\nu}{\rightarrow} & H^m(S^1) \end{array}$$

shows that $\hat{F}_\nu : H^m(S^1) \to H^m(S^1)$ is unitary equivalent to

$$\hat{Q}_m := \hat{A}_m\hat{F}_\nu\hat{A}_m^{-1} : L^2(S^1) \to L^2(S^1).$$

We will therefore study the operator $\hat{Q}_m$. Note that $\hat{Q}_m$ is defined *a priori* on a dense domain $(C^\infty(S^1))$. Define

$$\hat{P} := \hat{Q}_m^*\hat{Q}_m = \hat{A}_m^{-1}(\hat{F}_\nu^*\hat{A}_m^2\hat{F}_\nu)\hat{A}_m^{-1} = \hat{A}_m^{-1}\hat{B}\hat{A}_m^{-1}, \tag{32}$$

where appears the operator

$$\hat{B} := \hat{F}_\nu^*\hat{A}_m^2\hat{F}_\nu. \tag{33}$$

The Egorov theorem will help us to treat this operator (see [Tay96b, p 24]). This is a simple but crucial step in the proof: as explained in [FRS08], the Egorov theorem is the main theorem used in order to establish both the existence of a discrete spectrum of resonances and properties of them. However, there is a difference with [FRS08]: for the expanding map we consider here, the operator $\hat{F}_\nu$ is not invertible and the canonical map F is k-valued. Therefore we have to state the Egorov theorem in an appropriate way (we restrict, however, the statement to our simple context).

Lemma 10 (Egorov theorem). $\hat{B} := \hat{F}_\nu^*\hat{A}_m^2\hat{F}_\nu$ *is a pseudodifferential operator with symbol in* S^{2m} *given by*

$$B(x,\xi) = \left(\sum_{\varepsilon=0,\ldots,k-1}\frac{1}{E'(x'_\varepsilon)}A_m^2(F_\varepsilon(x,\xi))\right) + R \tag{34}$$

with $R \in S^{2m-1}$ *has a subleading order.*

Proof. As we explained in proposition 6, $\hat{F}_\nu$ and $\hat{F}_\nu^*$ are Fourier integral operators (FIO) whose canonical map are respectively F and F^{-1}. The pseudodifferential operator (PDO) $\hat{A}_m$ can also be considered as a FIO whose canonical map is the identity. By composition we deduce that $\hat{B} = \hat{F}_\nu^*\hat{A}_m^2\hat{F}_\nu$ is a FIO whose canonical map is the identity since $F^{-1}\circ F = I$.

See figure 3. Therefore $\hat{B}$ is a PDO. Using (7), (9) and (26) we obtain that the principal symbol of $\hat{B}$ is

$$\sum_{\varepsilon=0,\dots,k-1} \frac{1}{E'(x'_\varepsilon)} A_m^2(F_\varepsilon(x,\xi)). \tag{35}$$

□

Remark. Contrary to (33), $\hat{F}_\nu \hat{A}_m \hat{F}_\nu^*$ is not a PDO, but a FIO whose canonical map $F \circ F^{-1}$ is $k-$valued (see figure 3).

Now by *theorem of composition of PDO* ([Tay96b, p 11], (32) and (34) imply that $\hat{P}$ is a PDO of order 0 with principal symbol:

$$P(x,\xi) = \frac{B(x,\xi)}{A_m^2(x,\xi)} = \left(\sum_{\varepsilon=0,\dots,k-1} \frac{1}{E'(x'_\varepsilon)} \frac{A_m^2(F_\varepsilon(x,\xi))}{A_m^2(x,\xi)} \right).$$

Estimate (30) together with (2) give the following upper bound:

$$\forall |\xi| > R, \qquad |P(x,\xi)| \leqslant C^{2|m|} \sum_{\varepsilon=0,\dots,k-1} \frac{1}{E'(x'_\varepsilon)} \leqslant C^{2|m|} \frac{k}{E_{\min}}$$

(This upper bound goes to zero as $m \to -\infty$). From L^2*-continuity theorem for PDO* we deduce that for any $\alpha > 0$ (see [FRS08, lemma 38])

$$\hat{P} = \hat{k}_\alpha + \hat{p}_\alpha$$

with $\hat{k}_\alpha$ a smoothing operator (hence compact) and $\|\hat{p}_\alpha\| \leqslant C^{2|m|} \frac{k}{E_{\min}} + \alpha$. If $\hat{Q}_m = \hat{U}|\hat{Q}|$ is the polar decomposition of $\hat{Q}_m$, with $\hat{U}$ unitary, then from (32) $\hat{P} = |\hat{Q}|^2 \Leftrightarrow |\hat{Q}| = \sqrt{\hat{P}}$ and the spectral theorem ([Tay96b, p 75]) gives that $|\hat{Q}|$ has a similar decomposition

$$|\hat{Q}| = \hat{k}'_\alpha + \hat{q}_\alpha$$

with $\hat{k}'_\alpha$ smoothing and $\|\hat{q}_\alpha\| \leqslant C^{|m|} \sqrt{\frac{k}{E_{\min}}} + \alpha$, with any $\alpha > 0$. Since $\|\hat{U}\| = 1$ we deduce a similar decomposition for $\hat{Q}_m = \hat{U}|\hat{Q}| : L^2(S^1) \to L^2(S^1)$ and we deduce (10) and (11) for $\hat{F}_\nu : H^m \to H^m$. We also use the fact that $C \to 1/E_{\min}$ for $R \to \infty$ in (30).

The fact that the eigenvalues λ_i and their generalized eigenspaces do not depend on the choice of space H^m is due to density of Sobolev spaces. We refer to the argument given in the proof of corollary 1 in [FRS08]. This completes the proof of theorem 2.

4. Proof of theorem 3 on spectral gap

We will follow step by step the same analysis as in the previous section. The main difference now is that in theorem 3, $\nu \gg 1$ is a semiclassical parameter. In other words, we just perform a linear rescaling in cotangent space: $\xi_h := \hbar\xi$ with

$$\hbar := \frac{1}{\nu} \ll 1.$$

Therefore, our quantization rule for a symbol $A(x,\xi_h) \in S^m$, equation (31) writes now (see [Mar02, p 22])

$$(\hat{A}\varphi)(x) = \int A(x,\xi_h) \mathrm{e}^{\mathrm{i}(x-y)\xi_h/\hbar} \varphi(y)\, \mathrm{d}y\, \mathrm{d}\mu(\xi_h) \tag{36}$$

with measure $\mathrm{d}\mu(\xi_h) = \sum_{n\in\mathbb{Z}} \delta(\xi_h - 2\pi n\hbar)$. For simplicity we will write ξ for ξ_h below.

4.1. Dynamics on the cotangent space T^*S^1

If we consider again equation (29) then the multiplicative term $e^{i\nu\tau(x)} = e^{i\tau(x)/\hbar}$ contributes now to the stationary phase approximation (the phase function is $\varphi(z) = -\eta z + \xi E(z) + \tau(z)$ so $0 = \varphi'(z) = -\eta + \xi E'(z) + \tau'(z)$ implies $\eta = E'(z)\xi + \tau'(z)$) and modifies the expression of the canonical map F. We obtain:

Proposition 11. *In equation (7) $\hat{F}_\nu$ is a* semiclassical Fourier integral operator *acting on $C^\infty(S^1)$ (with semiclassical parameter $\hbar := 1/\nu \ll 1$). The associated* canonical map *on the cotangent space $(x,\xi) \in T^*S^1 \equiv S^1 \times \mathbb{R}$ is k-valued and given by*

$$F(x,\xi) = \{F_0(x,\xi), \ldots, F_{k-1}(x,\xi)\}, \qquad (x,\xi) \in S^1 \times \mathbb{R}, \tag{37}$$

$$F_\varepsilon : \begin{cases} x \to x'_\varepsilon = E_\varepsilon^{-1}(x), \\ \xi \to \xi'_\varepsilon = E'(x'_\varepsilon)\xi + \dfrac{d\tau}{dx}(x'_\varepsilon), \end{cases} \qquad \varepsilon = 0, \ldots, k-1. \tag{38}$$

Similarly $\hat{F}_\nu^*$ is a FIO whose canonical map is F^{-1}.

Note that for simplicity we have kept the same notation for the canonical map F although it differs from (26).

Since the map F is k-valued, a trajectory is a tree. Let us precise the notation:

Definition 12. *For $\varepsilon = (\ldots \varepsilon_3, \varepsilon_2, \varepsilon_1) \in \{0, \ldots, k-1\}^{\mathbb{N}^*}$, a point $(x,\xi) \in S^1 \times \mathbb{R}$ and time $n \in \mathbb{N}^*$ let us denote*

$$F_\varepsilon^n(x,\xi) := F_{\varepsilon_n} F_{\varepsilon_{n-1}} \ldots F_{\varepsilon_1}(x,\xi). \tag{39}$$

For a given sequence $\varepsilon \in \{0, \ldots, k-1\}^{\mathbb{N}^*}$, a trajectory *issued from the point (x,ξ) is $\{F_\varepsilon^n(x,\xi),\ n \in \mathbb{N}\}$.*

Note that at time $n \in \mathbb{N}$, there are k^n points issued from a given point (x,ξ):

$$F^n(x,\xi) := \{F_\varepsilon^n(x,\xi),\ \varepsilon \in \{0, \ldots, k-1\}^n\}. \tag{40}$$

The new term $\frac{d\tau}{dx}(x'_\varepsilon)$ in the expression of ξ'_ε, equation (38), complicates significantly the dynamics near the zero section $\xi = 0$. However, a trajectory from an initial point with $|\xi|$ large enough still escape towards infinity:

Lemma 13. *For any $1 < \kappa < E_{\min}$, there exists $R \geqslant 0$ such that for any $|\xi| > R$, any $\varepsilon = 0, \ldots k-1$,*

$$|\xi'_\varepsilon| > \kappa|\xi|. \tag{41}$$

Proof. From (38), one has $\xi'_\varepsilon = E'(x'_\varepsilon)\xi + \tau'(x'_\varepsilon)$, so $\xi'_\varepsilon - \kappa\xi = (E'(x'_\varepsilon) - \kappa)\xi + \tau'(x'_\varepsilon) \geqslant (E_{\min} - \kappa)\xi + \min\tau' > 0$ if $\xi > -\frac{\min\tau'}{(E_{\min}-\kappa)} \geqslant 0$, and similarly $\xi'_\varepsilon - \kappa\xi \leqslant (E_{\min} - \kappa)\xi + \max\tau' < 0$ if $\xi < -\frac{\max\tau'}{(E_{\min}-\kappa)}$. □

We will denote the set

$$\mathcal{Z} := S^1 \times [-R, R] \tag{42}$$

outside of which trajectories escape in a controlled manner (41). See figure 4.

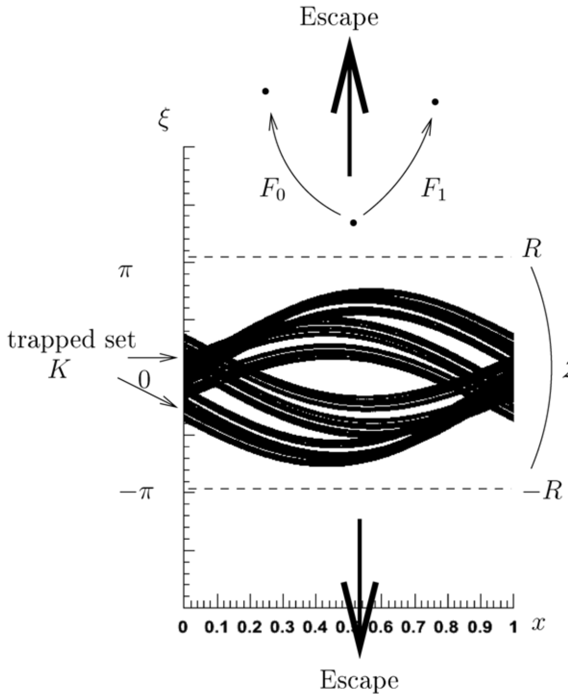

Figure 4. The trapped set K in the cotangent space $S^1 \times \mathbb{R}$. We have chosen here $E(x) = 2x$ and $\tau(x) = \cos(2\pi x)$.

4.2. The trapped set K

We will be interested now in the trajectories of F which do not escape towards infinity.

Definition 14. *We define* the trapped set

$$K := \bigcap_{n\in\mathbb{N}} (F^{-1})^n(\mathcal{Z}), \tag{43}$$

which contains points for which a trajectory at least does not escape towards infinity. See figure 4. The definition of K does not depend on the compact set $\mathcal{Z}$ (if $\mathcal{Z}$ is chosen large enough).

Since the map F is multivalued, some trajectories may escape from the trapped set. We will need a characterization of how many such trajectories succeed to escape: For $n \in \mathbb{N}$, let

$$\mathcal{N}(n) := \max_{(x,\xi)} \sharp\{F^n_\varepsilon(x,\xi) \in \mathcal{Z}, \varepsilon \in \{0,\ldots,k-1\}^n\}. \tag{44}$$

See figure 5 for an illustration of $\mathcal{N}(n)$. Of course $\mathcal{N}(n) \leqslant k^n$.

Definition 15. *The map F (or f) is* partially captive *if*[10]

$$\frac{\log \mathcal{N}(n)}{n} \underset{n\to\infty}{\longrightarrow} 0. \tag{45}$$

This property is the hypothesis of theorem 3.

[10] It can be shown that $\log \mathcal{N}(n)$ is sub-additive and therefore $\lim_{n\to\infty} \inf(\frac{\log \mathcal{N}(n)}{n}) = \lim_{n\to\infty}(\frac{\log \mathcal{N}(n)}{n})$ [RS72, p 217, ex 11].

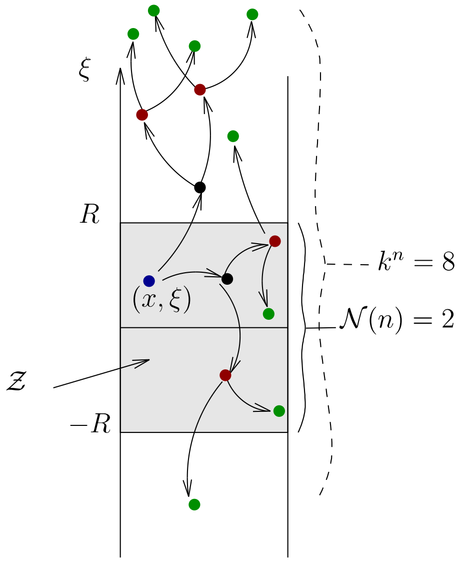

Figure 5. This Figure illustrates the *trajectories* $F^n_\varepsilon(x, \xi)$ issued from an initial point (x, ξ) after time n. Here $k = 2$ and $n = 3$. The property for the map F of being 'partially captive' according to definition 15 is related to the number of points $\mathcal{N}(n)$ which do not escape from the compact zone $\mathcal{Z}$ after time n.

Remarks.

- $\mathcal{N}(n)$ defined in (44) is the number of trajectories which do not escape outside the vicinity $\mathcal{Z}$ of the trapped set before time n. Since there are k^n trajectories which start from a given point $(x, \xi) \in S^1 \times \mathbb{R}$, the property for the map F to be 'partially captive', i.e. $\frac{\log \mathcal{N}(n)}{n} \underset{n\to\infty}{\longrightarrow} 0$, means that 'most' of the trajectories escape from the trapped set K. See figure 5. Another description of the trapped set K and of the partially captive property will be given in appendix B. Note that the function $\mathcal{N}(n)$, equation (44) depends on the set $\mathcal{Z}$ but property (45) does not.
- If the function τ is trivial in (38), i.e. $\tau = 0$, then obviously all the trajectories issued from a point (x, ξ) on the line $\xi = 0$ remains on this line (the trapped set). Therefore
$$\sharp\{F^n_\varepsilon(x, \xi) \in \mathcal{Z}, \varepsilon \in \{0, 1, \ldots, k-1\}^n\} = k^n$$
and the map F is not partially captive (but could be called 'totally captive'). This is also true if the function τ is a 'co-boundary', i.e. if $\tau(x) = \eta(E(x)) - \eta(x)$ with $\eta \in C^\infty(S^1)$ as discussed in appendix A.
- Tsujii has studied a dynamical system very similar to (38) in [Tsu01], but this model is not volume preserving. He establishes there that the SRB measure on the trapped set is absolutely continuous for almost every τ.

4.3. The escape function

Let $m < 0$ and consider the C^∞ function on T^*S^1:

$$\begin{aligned} A_m(x, \xi) &:= \langle\xi\rangle^m \qquad \text{for } |\xi| > R + \eta, \\ &:= 1 \qquad \text{for } \xi \leqslant R, \end{aligned}$$

where $\eta > 0$ is small and with $\langle \xi \rangle := (1+\xi^2)^{1/2}$. A_m decreases with $|\xi|$ and belongs to the symbol class S^m.

Equation (41) implies that the function A_m *decreases strictly* along the trajectories of F outside the trapped set (similarly to equation (30)):

$$\forall |\xi| > R, \qquad \forall \varepsilon = 0,\dots,k-1 \qquad \frac{A_m(F_\varepsilon(x,\xi))}{A_m(x,\xi)} \leqslant C^{|m|} < 1,$$
$$\text{with} \qquad C = \sqrt{\frac{R^2+1}{\kappa^2 R^2+1}} < 1. \tag{46}$$

And for any point we have the general bound:

$$\forall \varepsilon = 0,\dots,k-1, \quad \forall (x,\xi) \in T^*S^1, \quad \frac{A_m(F_\varepsilon(x,\xi))}{A_m(x,\xi)} \leqslant 1. \tag{47}$$

Using the quantization rule (36), the symbol A_m can be quantized giving a pseudodifferential operator $\hat{A}_m$ which is self-adjoint and invertible on $C^\infty(S^1)$. In our case $\hat{A}_m$ is simply a multiplication operator by $A_m(\xi)$ in Fourier space.

Let us consider the (usual) Sobolev space

$$H_\nu^m(S^1) := \hat{A}_m^{-1}(L^2(S^1)).$$

Then $\hat{F}_\nu : H_\nu^m(S^1) \to H_\nu^m(S^1)$ is unitary equivalent to

$$\hat{Q} := \hat{A}_m \hat{F}_\nu \hat{A}_m^{-1} : L^2(S^1) \to L^2(S^1).$$

Let $n \in \mathbb{N}^*$ (a fixed time which will be made large at the end of the proof) and define

$$\hat{P}^{(n)} := \hat{Q}^{*n}\hat{Q}^n = \hat{A}_m^{-1}\hat{F}_\nu^{*n}\hat{A}_m^2\hat{F}_\nu^n\hat{A}_m^{-1}. \tag{48}$$

Using *Egorov theorem* (lemma 10) and theorem of composition of PDO [EZ03, chaptre 4], we obtain that $\hat{P}^{(n)}$ is a PDO of order 0 with principal symbol

$$P^{(n)}(x,\xi) = \left(\sum_{\varepsilon\in\{0,\dots,k-1\}^n} \frac{1}{E_n'(x)} \frac{A_m^2(F_\varepsilon^n(x,\xi))}{A_m^2(x,\xi)} \right), \tag{49}$$

where $E_n'(x) := \prod_{j=1}^n E'(E_{\varepsilon_j}^{-j}(x))$ is the expanding rate of the trajectory at time n. Equation (2) implies that $E_n'(x) \geqslant E_{\min}^n$. Now we will bound this (positive) symbol from above, considering different cases for the trajectory $F_\varepsilon^n(x,\xi)$, as illustrated in figure 5.

1. If $(x,\xi) \notin \mathcal{Z}$ then (46) gives
$$\frac{A^2(F_\varepsilon^n(x,\xi))}{A^2(x,\xi)} = \frac{A^2(F_\varepsilon^n(x,\xi))}{A^2(F_\varepsilon^{n-1}(x,\xi))} \cdots \frac{A^2(F_\varepsilon(x,\xi))}{A^2(x,\xi)} \leqslant (C^{2|m|})^n \tag{50}$$
therefore
$$P^{(n)}(x,\xi) \leqslant \frac{k^n}{E_{\min}^n}(C^{2|m|})^n.$$
2. If $(x,\xi) \in \mathcal{Z}$ but $F_\varepsilon^{n-1}(x,\xi) \notin \mathcal{Z}$ then $\frac{(A^2\circ F_\varepsilon^n)(x,\xi)}{(A^2\circ F_\varepsilon^{n-1})(x,\xi)} \leqslant C^{2|m|}$ from (46). Using also (47) we have
$$\frac{A^2(F_\varepsilon^n(x,\xi))}{A^2(x,\xi)} = \frac{A^2(F_\varepsilon^n(x,\xi))}{A^2(F_\varepsilon^{n-1}(x,\xi))} \cdots \frac{A^2(F_\varepsilon(x,\xi))}{A^2(x,\xi)} \leqslant C^{2|m|}. \tag{51}$$
3. In the other cases ($(x,\xi) \in \mathcal{Z}$ and $F_\varepsilon^{n-1}(x,\xi) \in \mathcal{Z}$) we can only use (47) to bound:
$$\frac{A^2(F_\varepsilon^n(x,\xi))}{A^2(x,\xi)} \leqslant 1 \tag{52}$$

From definition (44) we have

$$\sharp\{F_\varepsilon^{n-1}(x,\xi) \in \mathcal{Z}, \qquad \varepsilon \in \{0,1\}^n\} \leqslant \mathcal{N}(n-1).$$

For $(x,\xi) \in \mathcal{Z}$, we split the sum equation (49) accordingly to cases 1,2 or 3 above. Note that $(C^{2|m|})^n \leqslant C^{2|m|}$. This gives

$$P^{(n)}(x,\xi) \leqslant \frac{1}{E_{\min}^n}((k^n - \mathcal{N}(n-1))C^{2|m|} + \mathcal{N}(n-1)) \leqslant \mathcal{B} \tag{53}$$

with the bound

$$\mathcal{B} := \left(\frac{k}{E_{\min}}\right)^n C^{2|m|} + \frac{\mathcal{N}(n-1)}{E_{\min}^n}. \tag{54}$$

Then

$$\sup_{(x,\xi)} |P^{(n)}(x,\xi)| \leqslant \mathcal{B}$$

With L^2 *continuity theorem* for pseudodifferential operators [Mar02] this implies that in the limit $\hbar \to 0$

$$\|\hat{P}^{(n)}\|_{L^2\to L^2} \leqslant \mathcal{B} + \mathcal{O}_n(\hbar). \tag{55}$$

Polar decomposition of $\hat{Q}^n$ gives

$$\|\hat{Q}^n\|_{L^2\to L^2} \leqslant \||\hat{Q}^n|\| = \sqrt{\|\hat{P}^{(n)}\|} \leqslant (\mathcal{B} + \mathcal{O}_n(\hbar))^{1/2}. \tag{56}$$

We make now the assumption that F be *partially captive*, equation (45). For any $c > 0$, and some n we have $\frac{\log \mathcal{N}(n)}{n} < c$ hence $\mathcal{N}(n) < \mathrm{e}^{cn}$. In other words for any $\rho > \frac{1}{\sqrt{E_{\min}}}$, there exists $n \in \mathbb{N}$ such that $\frac{\mathcal{N}(n-1)}{E_{\min}^n} < \rho^{2n}$.

We let $\hbar = 1/\nu \to 0$ first, and after $m \to -\infty$ giving $C^{|m|} \to 0$, and also we let n be large enough. Then from (56) and (54) we have obtained that for any $\rho > \frac{1}{\sqrt{E_{\min}}}$, there exists $n_0 \in \mathbb{N}$, $\nu_0 \in \mathbb{N}$, $m_0 < 0$ such that for any $|\nu| \geqslant \nu_0$, $m < m_0$,

$$\|\hat{F}_\nu^{n_0}\|_{H_\nu^m} = \|\hat{Q}^{n_0}\|_{L^2} \leqslant \rho^{n_0}. \tag{57}$$

Also, there exists $c > 0$ independent of $|\nu| \geqslant \nu_0$, such that for any r such that $0 \leqslant r < n_0$ we have $\|\hat{Q}^r\|_{L^2} < c$. As a consequence for any $n \in \mathbb{N}$ we write $n = kn_0 + r$ with $0 \leqslant r < n_0$ and

$$\|\hat{F}_\nu^n\|_{H_\nu^m} = \|\hat{Q}^n\|_{L^2} \leqslant \|\hat{Q}^{n_0}\|_{L^2}^k \|\hat{Q}^r\|_{L^2} \leqslant \rho^n \frac{\|\hat{Q}^r\|_{L^2}}{\rho^r} \leqslant c\rho^n$$

with $c > 0$ independent of $|\nu| \geqslant \nu_0$. We have obtained (13).

For any n the spectral radius of $\hat{Q}$ satisfies [RS72, p 192]

$$r_s(\hat{Q}) \leqslant \|\hat{Q}^n\|^{1/n} \leqslant c^{1/n}\rho.$$

So we get that for $\hbar = 1/\nu \to 0$,

$$r_s(\hat{F}_\nu) = r_s(\hat{Q}) \leqslant \frac{1}{\sqrt{E_{\min}}} + o(1).$$

Without the assumption that F be partially captive, equation (45), we have more generally

$$\left(\frac{\mathcal{N}(n-1)}{E_{\min}^n}\right)^{1/2n} = \frac{1}{\sqrt{E_{\min}}} \exp\left(\frac{1}{2n}\log \mathcal{N}(n-1)\right)$$

and

$$r_s(\hat{Q}) \leqslant \sqrt{\frac{1}{E_{\min}} \exp\left(\lim_{n\to\infty} \inf \left(\frac{\log \mathcal{N}(n)}{n}\right)\right)} + o(1). \tag{58}$$

We have completed the proof of theorem 3.

Acknowledgments

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Appendix A. Equivalence classes of dynamics

Let us make a simple and well-known observation about equivalent classes of dynamics. The map f we consider in equation (3) depends on $k \in \mathbb{N}$ and on the functions $E : S^1 \to S^1$, $\tau : S^1 \to \mathbb{R}$. To emphasize this dependence, we denote $f_{(E,\tau)}$. The transfer operator (7) is also denoted by $\hat{F}_{(E,\tau)}$.

In this appendix we characterize an equivalence class of functions (E, τ) such that in a given equivalence class the maps $f_{(E,\tau)}$ are C^∞ conjugated together, the transfer operators $\hat{F}_{(E,\tau)}$ are also conjugated and the resonances spectrum are therefore the same.

Let $\eta : S^1 \to \mathbb{R}$ be a smooth function. Let us consider the map $T : S^1 \times S^1 \to S^1 \times S^1$ defined by

$$T(x, s) = \left(x, s + \frac{1}{2\pi}\eta(x)\right).$$

Then using (3) one obtains that

$$(T^{-1} \circ f_{(E,\tau)} \circ T)(x, s) = \left(E(x), s + \frac{1}{2\pi}(\tau(x) + \eta(x) - \eta(E(x)))\right).$$

Therefore

$$(T^{-1} \circ f_{(E,\tau)} \circ T) = f_{(E,\zeta)},$$

i.e. $f_{(E,\zeta)} \sim f_{(E,\tau)}$, with

$$\zeta = \tau + (\eta - \eta \circ E).$$

The function τ has been modified by a '*co-boundary term*' [KH95, p 100]. The function ζ is said to be *cohomologous* to τ.

Lemma 16. *With (7) we also obtain that the transfer operator $\hat{F}_{(E,\zeta)}$ of $f_{(E,\zeta)}$ on $C^\infty(S^1)$ is given by*

$$\hat{F}_{(E,\zeta)} = \hat{\chi}\hat{F}_{(E,\tau)}\hat{\chi}^{-1} \tag{59}$$

with the operator $\hat{\chi} : C^\infty(S^1) \to C^\infty(S^1)$ defined by

$$(\hat{\chi}\varphi)(x) = \mathrm{e}^{\mathrm{i}\nu\eta(x)}\varphi(x).$$

Proof. $(\hat{\chi}\hat{F}_{(E,\tau)}\hat{\chi}^{-1}\varphi)(x) = (\varphi(E(x))\mathrm{e}^{-\mathrm{i}\nu\eta(E(x))})\mathrm{e}^{\mathrm{i}\nu\tau(x)}\mathrm{e}^{\mathrm{i}\nu\eta(x)} = (\hat{F}_{(E,\zeta)}\varphi)(x).$ □

The conjugation (59) immediately implies that $\hat{F}_{(E,\zeta)}$ and $\hat{F}_{(E,\tau)}$ have the same spectrum of Ruelle resonances.

Observe that $\hat{\chi}$ is a FIO whose associated canonical map on $T^*S^1 \equiv S^1 \times \mathbb{R}$ is given by ($\nu \gg 1$ is considered as a semiclassical parameter):

$$\chi : (x, \xi) \in (S^1 \times \mathbb{R}) \to \left(x, \xi + \frac{\mathrm{d}\eta}{\mathrm{d}x}\right) \in (S^1 \times \mathbb{R}).$$

Therefore at the level of canonical maps on T^*S^1:

$$F_{(E,\zeta)} = \chi \circ F_{(E,\tau)} \circ \chi^{-1}. \tag{60}$$

The conjugation (60) implies in particular that the corresponding trapped sets (43) are related by

$$K_{(E,\zeta)} = \chi(K_{(E,\tau)}).$$

Appendix B. Description of the trapped set

In this section we provide further description of the trapped set K defined in equation (43) as well as the dynamics of the canonical map F restricted on it.

Appendix B.1. Dynamics on the cover $\mathbb{R}^2$

The dynamics of F on the cylinder $T^*S^1 = S^1 \times \mathbb{R}$ has been defined in equation (38). This is a multivalued map. It is convenient to consider the *lifted dynamics on the cover* $\mathbb{R}^2$ which is a diffeomorphism given by

$$\tilde{F} : \begin{cases} x \to x' = E^{-1}(x), \\ \xi \to \xi' = E'(x')\xi + \dfrac{d\tau}{dx}(x'), \end{cases} \tag{61}$$

where $E = kg : \mathbb{R} \to \mathbb{R}$ is map (1) lifted on $\mathbb{R}$. It is invertible from (2). Let us suppose for simplicity that $E(0) = 0$.

One easily establishes the following properties of the map $\tilde{F}$, illustrated in figure 6:

- The point

$$I := (0, \xi_I) := \left(0, -\frac{\tau'(0)}{(E'(0) - 1)}\right)$$

is the unique *fixed point* of $\tilde{F}$. It is hyperbolic with *unstable manifold*

$$W_u = \{(0, \xi), \xi \in \mathbb{R}\}$$

and *stable manifold*

$$W_s = \{(x, S(x)), x \in \mathbb{R}\}, \tag{62}$$

where the C^∞ function $S(x)$ is defined by the following *co-homological equation*, deduced directly from (61)

$$S(E^{-1}(x)) = E'(E^{-1}(x))S(x) + \tau'(E^{-1}(x)), \qquad S(0) = \xi_I.$$

This gives

$$S(x) = \frac{1}{E'(E^{-1}(x))}(S(E^{-1}(x)) - \tau'(E^{-1}(x)))$$

and recursively we deduce that

$$S(x) = -\sum_{p=1}^{\infty} \frac{1}{E'^{(-p)}(x)} \tau'(E^{(-p)}x), \tag{63}$$

where

$$x_{(-p)} := E^{(-p)}(x) := (\underbrace{E^{-1} \circ \ldots \circ E^{-1}}_{p})(x) \tag{64}$$

and

$$E'^{(-p)}(x) := E'(x_{-p}) \ldots E'(x_{-2})E'(x_{-1}) \tag{65}$$

is the product of derivatives. In the case of $E(x) = 2x$, one obtains simply

$$S(x) = -\sum_{p=1}^{\infty} \frac{1}{2^p} \tau' \left(\frac{x}{2^p}\right). \tag{66}$$

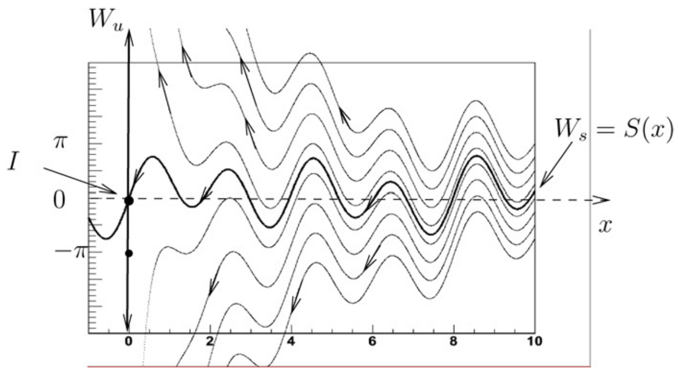

Figure 6. The fixed point $I = (0, \xi_0)$, the stable manifold W_s and unstable manifold W_u of the lifted map $\tilde{F}$, equation (61), in the example $E(x) = 2x$, $\tau(x) = \cos(2\pi x)$.

- If $\mathcal{P} : (x, \xi) \in \mathbb{R}^2 \to (x \bmod 1, \xi) \in S^1 \times \mathbb{R} \equiv T^*S^1$ denotes the projection, then *trapped set* K, defined in equation (43) is obtained by wrapping the stable manifold around the cylinder and taking the closure:

$$K = \overline{\mathcal{P}(W_s)} \tag{67}$$

Compare figures 6 and 4.

- If $X_0 = (x_0, \xi_0) \in \mathbb{R}^2$ is an initial point on the plane, and $\mathcal{P}(X_0)$ denotes its image on the cylinder, then at time $n \in \mathbb{N}$, the k^n evolutions of the point $\mathcal{P}(X_0)$ under the map F^n are the images of the evolutions $\tilde{F}^n(X_k)$ of the translated points $X_p = X_0 + (p, 0)$, with $p = 0 \to k^n - 1$:

$$F^n(\mathcal{P}(X_0)) = \{\mathcal{P}(\tilde{F}^n(X_p)),\ X_p = X_0 + (p, 0),\ p \in \{0, \ldots, k^n - 1\}\}$$

and more precisely, using notation of equation (39) for these points, one has the relation:

$$F^n_\varepsilon(\mathcal{P}(X_0)) = \mathcal{P}(\tilde{F}^n(X_p)), \tag{68}$$

where ε is the number p written in base k:

$$\varepsilon = \varepsilon_{n-1} \ldots \varepsilon_1 \varepsilon_0 = p_{\mathrm{base}\,k} \in \{0, \ldots, k-1\}^n$$

Figure 7 illustrates this correspondence.

- For an initial point $X_0 = (x_0, \xi_0) \in \mathbb{R}^2$, then $X_n = (x_n, \xi_n) = \tilde{F}^n(X_0)$ satisfies

$$x_n = E^{(-n)}(x_0), \qquad \xi_n - S(x_n) = (E'^{(-n)}(x_0))(\xi_0 - S(x_0)) \tag{69}$$

with $E^{(-n)}(x)$, $E'^{(-n)}(x)$ given by (64) and (65). Hence

$$|\xi_n - S(x_n)| \geqslant E^n_{\min} |\xi_0 - S(x_0)|.$$

This last inequality describes how fast the trajectories above or below the separatrix W_s escape towards infinity in figure 6.

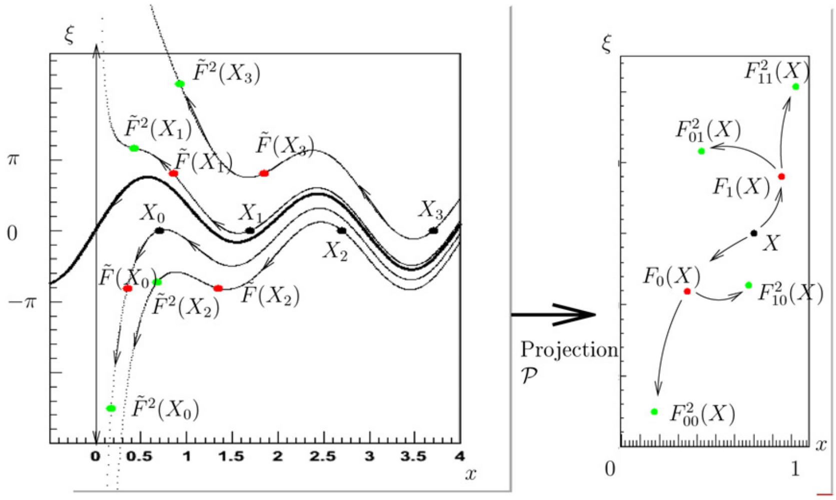

Figure 7. This picture shows how the dynamics of a point $X = \mathcal{P}(X_0) \in S^1 \times \mathbb{R}$ under the map F is related by equation (68) to the dynamics of its lifted images $X_k = X_0 + (k, 0)$ under $\tilde{F}$ on the cover $\mathbb{R}^2$.

Appendix B.2. Partially captive property

Here we rephrase the property of partial captivity, definition 15, in terms of a property on the separatrix function $S(x)$ defined in equation (62) and given in equation (63).

Proposition 17. *For $n \in \mathbb{N}$, and $\tilde{R} > 0$, let*

$$\tilde{\mathcal{N}}_{\tilde{R}}(n) = \max_{(x,\xi)\in\mathbb{R}^2} \sharp \left\{ p \in \{0, \ldots, k^n - 1\},\ |\xi - S(x+p)| \leqslant \frac{\tilde{R}}{E'^{(-n)}(x+p)} \right\}. \quad (70)$$

Then the map F is partially captive *(see definition p 1489) if and only if*

$$\lim_{n\to\infty} \frac{\log \tilde{\mathcal{N}}_{\tilde{R}}(n)}{n} = 0 \quad (71)$$

for every $\tilde{R}$.

Proof. From (44) and using (68) and (69) one obtains

$$\mathcal{N}(n) = \max_{(x,\xi)} \sharp\{F^n_\varepsilon((x,\xi)) \in \mathcal{Z}, \qquad \varepsilon \in \{0, \ldots, k-1\}^n\}$$
$$= \max_{(x,\xi)} \sharp\{|\xi_{n,p}| \leqslant R,\ p \in \{0, \ldots, k^n - 1\}\}$$

with $(x_{n,p}, \xi_{n,p}) := \tilde{F}^n((x+p, \xi))$ given by

$$x_{n,p} = E^{(-n)}(x+p)$$

and

$$\xi_{n,p} = S(x_{n,p}) + (E'^{(-n)}(x+p))(\xi - S(x+p)).$$

Therefore

$$|\xi_{n,p}| \leqslant R \Leftrightarrow \left|\frac{S(x_{n,p})}{E'^{(-n)}(x+p)} + \xi - S(x+p)\right| \leqslant \frac{R}{E'^{(-n)}(x+p)}.$$

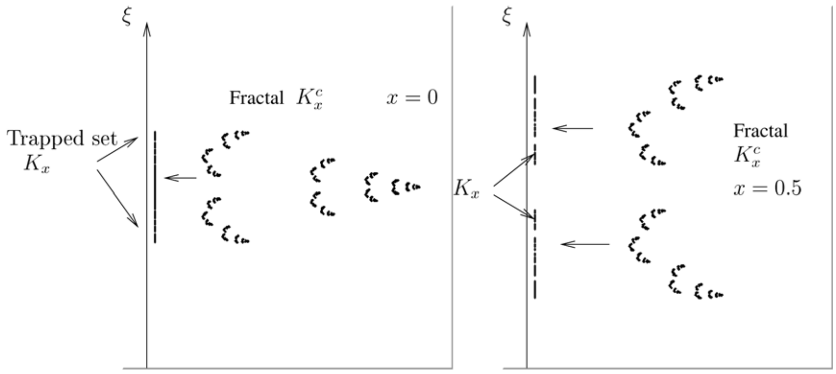

Figure 8. This picture represents the set $K_x^c \subset \mathbb{C}$ defined by equation (72). The trapped set K at position x is obtained by the projection on the imaginary axis $K_x = \Im(K_x^c)$. In the supplementary data (available at stacks.iop.org/Non/24/1473/mmedia or www-fourier.ujf-grenoble.fr/~faure/articles/09_html_partially_expanding_maps) one can observe the motion of the fractal K_x^c as $x \in \mathbb{R}$ increases smoothly.

But S is a bound function on $\mathbb{R}$, $|S(x)| \leqslant S_{\max}$. Therefore the last equation is equivalent to $|\xi - S(x+p)| \leqslant \frac{\tilde{R}}{E'^{(-n)}(x+p)}$ with some $\tilde{R} > 0$, and conversely. This implies that (71) is equivalent to (45). □

Appendix B.3. Fractal aspect of the trapped set

The characterization equation (70) concerns the discrete set of points $S(x+m)$, $m \in \mathbb{Z}$. From equation (67) these points are the slice of the trapped set $K = \cup_{x \in S^1} K_x$:

$$K_x = \overline{\{S(x+m),\ m \in \mathbb{Z}\}}.$$

Since we present here merely an observation and not a proof, we consider, for simplicity, the simple model with $E(x) = 2x$, and $\tau(x) = \cos(2\pi x)$. From equation (66), these points are given by

$$S(x+m) = \sum_{p=1}^{\infty} \frac{2\pi}{2^p} \sin\left(\frac{2\pi}{2^p}(x+m)\right) = \Im\left(\sum_{p=1}^{\infty} \frac{2\pi}{2^p} \exp\left(\frac{\mathrm{i}2\pi}{2^p}(x+m)\right)\right).$$

Therefore, the slice K_x is the projection on the imaginary axis of the following set:

$$K_x^c = \overline{\{S^c(x+m), m \in \mathbb{Z}\}}, \tag{72}$$

$$S^c(x+m) = \sum_{p=1}^{\infty} \frac{2\pi}{2^p} \mathrm{e}^{\mathrm{i}2\pi \frac{1}{2^p}(x+m)}.$$

In figure 8 we observe that K_x^c is *a fractal set.* (a proof of this would require the theory of 'iterated function systems' [Fal97]). Compare figures 8 with 4.

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