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Upper Bound on the Density of Ruelle Resonances for Anosov Flows

Frédéric Faure[1], **Johannes Sjöstrand**[2]

[1] Institut Fourier, UMR 5582, 100 rue des Maths, BP 74, 38402 St Martin d'Hères, France.
E-mail: frederic.faure@ujf-grenoble.fr
[2] IMB, UMR 5584, Université de Bourgogne, 9, Avenue Alain Savary, BP 47870, 21078 Dijon Cedex, France.
E-mail: Johannes.Sjostrand@u-bourgogne.fr

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Abstract: Using a semiclassical approach we show that the spectrum of a smooth Anosov vector field V on a compact manifold is discrete (in suitable anisotropic Sobolev spaces) and then we provide an upper bound for the density of eigenvalues of the operator $(-i)V$, called Ruelle resonances, close to the real axis and for large real parts.

Résumé: Par une approche semiclassique on montre que le spectre d'un champ de vecteur d'Anosov V sur une variété compacte est discret (dans des espaces de Sobolev anisotropes adaptés). On montre ensuite une majoration de la densité de valeurs propres de l'opérateur $(-i)V$, appelées résonances de Ruelle, près de l'axe réel et pour les grandes parties réelles.

Contents

1. Introduction

Chaotic behavior of certain dynamical systems is due to hyperbolicity of the trajectories. This means that the individual trajectories are unstable under small perturbations of the initial point [9,29] and have a complicated and unpredictable behavior. However, the evolution of a cloud of points seems to be simpler: it will spread and equidistribute according to an invariant measure, called an equilibrium measure (or S.R.B. measure). Also from the physical point of view, such a cloud reflects the unavoidable lack of knowledge about the initial point. Following this idea, D. Ruelle [40,41], has shown in the 70's that instead of considering individual trajectories, it is much more natural to study the evolution of densities under a linear operator called the Ruelle transfer operator or the Perron-Frobenius operator.

For dynamical systems with strong chaotic properties, such as uniformly expanding maps or uniformly hyperbolic maps, Ruelle, Bowen, Fried, Rugh and others, using symbolic dynamics techniques (Markov partitions), have shown that the transfer operator has a discrete spectrum of eigenvalues. This spectral description has an important meaning for the dynamics since each eigenvector corresponds to an invariant distribution (up to a time factor). From this spectral characterization of the transfer operator, one can derive other specific properties of the dynamics such as decay of time correlation functions, central limit theorem, mixing, etc. In particular a spectral gap implies exponential decay of correlations.

This spectral approach has recently (2002–2005) been improved by M. Blank, S. Gouëzel, G. Keller, C. Liverani [6,10,22,32], V. Baladi and M. Tsujii [3,4] (see [4] for some historical remarks) and in [17], through the construction of functional spaces adapted to the dynamics, independent of every symbolic dynamics.

The case of flows i.e. dynamical systems with continuous time is more delicate (see [18] for historical remarks). This is due to the direction of time flow which is neutral (i.e. two nearby points on the same trajectory will not diverge from one another). In 1998 Dolgopyat [13,14] showed the exponential decay of correlation functions for certain Anosov flows, using techniques of oscillatory integrals and symbolic dynamics. In 2004 Liverani [31] adapted Dolgopyat's ideas to his functional analytic approach, to treat the case of contact Anosov flows. In 2005 M. Tsujii [50] obtained an explicit estimate for the spectral gap for the suspension of an expanding map and in 2008 [51,52] he obtained such an estimate in the case of contact Anosov flows.

Microlocal approach to transfer operators. It also appeared recently [15–17] that for hyperbolic dynamics on a manifold X, the transfer operators are Fourier integral operators and using standard tools of microlocal analysis, some of their spectral properties can be obtained from the study of "the associated classical symplectic dynamics", namely the initial hyperbolic dynamics on X lifted to the cotangent space T^*X (the phase space).

The simple idea behind this, crudely speaking, is that a transfer operator transports a "wave packet" associated to a point in phase space into another wave packet corresponding to the image point under the symplectic dynamics.

Following this approach, we studied hyperbolic diffeomorphisms in [16,17]. The aim of the present paper is to show that microlocal analysis in the semi-classical limit is also well adapted to hyperbolic systems with a neutral direction and with the inverse of the Fourier component in the neutral direction being a natural semi-classical parameter. In the paper [15] one of us has considered a partially expanding map and showed that a spectral gap develops in the limit of large oscillations in the neutral direction (which is a semiclassical limit). In this paper we consider a hyperbolic flow on a manifold X, generated by a vector field V.

In this paper as well as in [17], one aim is to make more precise the connection between the spectral study of Ruelle resonances and the spectral study in quantum chaos [35,55], in particular to emphasize the importance of the symplectic properties of the dynamics in the cotangent space T^*X on the spectral properties of the transfer operator, and long time behavior of the dynamics.

1.1. The main results. The results of this paper will concern the following situation. Let X be an n-dimensional smooth compact connected Riemannian manifold, with $n \geq 3$. Let ϕ_t be the flow on X generated by a **smooth** vector field $V \in C^\infty(X; TX)$:

$$V(x) = \frac{d}{dt}(\phi_t(x))_{/t=0} \in T_x X, \qquad x \in X. \tag{1.1}$$

We assume that the flow ϕ_t is **Anosov**.

Let us first review some facts about Anosov flows and the dynamics induced on the cotangent space T^*X.

1.1.1. Anosov flows. We recall the definition (see [29] p. 545, or [37] p. 8). See Fig. 1.

Definition 1.1. *On a smooth Riemannian manifold (X, g), a vector field V generates an* **Anosov flow** $(\phi_t)_{t\in\mathbb{R}}$ *(or* **uniformly hyperbolic flow***) if:*

- *For each $x \in X$, there exists a decomposition*

$$T_x X = E_u(x) \oplus E_s(x) \oplus E_0(x), \tag{1.2}$$

where $E_0(x)$ is the one dimensional subspace generated by $V(x)$.
- *The decomposition (1.1) is invariant by ϕ_t for every t:*

$$\forall x \in X, \quad (D_x\phi_t)(E_u(x)) = E_u(\phi_t(x)) \quad \text{and} \quad (D_x\phi_t)(E_s(x)) = E_s(\phi_t(x)).$$

- *There exist constants $c > 0, \theta > 0$ such that for every $x \in X$,*

$$\begin{aligned} |D_x\phi_t(v_s)|_g &\leq c e^{-\theta t}|v_s|_g, \qquad \forall v_s \in E_s(x), \quad t \geq 0 \\ |D_x\phi_t(v_u)|_g &\leq c e^{-\theta |t|}|v_u|_g, \qquad \forall v_u \in E_u(x), \quad t \leq 0, \end{aligned} \tag{1.3}$$

meaning that E_s is the stable distribution and E_u the unstable distribution for positive time.

Let us recall some facts:

- Standard examples of Anosov flows are **suspensions of Anosov diffeomorphisms** (see [37] p. 8), or **geodesic flows on manifolds M with sectional negative curvature** (see [37] p. 9, or [29] p. 549, p. 551). Notice that in this case, the geodesic flow is Anosov on the unit cotangent bundle $X = T_1^*M$.

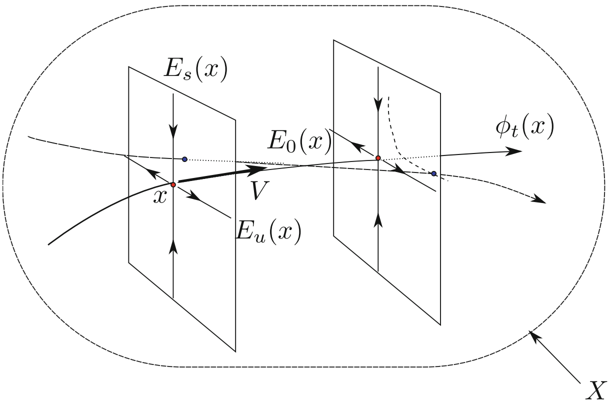

Fig. 1. Picture of an Anosov flow in X and instability of trajectories

- The global hyperbolic structure of Anosov flows or Anosov diffeomorphisms is a very strong geometric property, so that manifolds carrying such dynamics satisfy strong topological conditions and the list of known examples is not so long. See [7] for a detailed discussion and references on that question.
- Let

$$d_u = \dim E_u(x), \qquad d_s = \dim E_s(x),$$

  (they are independent of $x \in X$). Equation (1.2) implies $d_u + d_s + 1 = \dim X = n$. For every $d_u, d_s \geq 1$ one may construct an example of an Anosov flow: one considers a suspension of a hyperbolic diffeomorphism of $SL_{n-1}(\mathbb{Z})$ on $\mathbb{T}^{n-1}$, with $n = d_u + d_s + 1$, such that there are d_s eigenvalues with modulus $|\lambda| < 1$, and d_u eigenvalues with modulus $|\lambda| > 1$.

Anosov one form α and regularity of the distributions $E_u(x)$, $E_s(x)$. The distribution $E_0(x)$ is smooth since $V(x)$ is assumed to be smooth. The distributions $E_u(x)$, $E_s(x)$ and $E_u(x) \oplus E_s(x)$ are only Hölder continuous in general (see [37] p. 15, [21] p. 211).

The above hypothesis on the flow implies that there is a particular continuous 1-form on X, denoted α called the **Anosov 1-form** and defined by

$$\ker(\alpha(x)) = E_u(x) \oplus E_s(x), \qquad (\alpha(x))(V(x)) = 1, \qquad \forall x \in X. \tag{1.4}$$

Since E_u and E_s are invariant by the flow, then α is invariant as well and (in the sense of distributions)

$$\mathcal{L}_V(\alpha) = 0, \tag{1.5}$$

where $\mathcal{L}_V$ denotes the Lie derivative.

We discuss now some known results about the smoothness of the distributions $E_u(x)$, $E_s(x)$ in some special cases.

- In the case of a **geodesic flow** on a smooth negatively curved Riemannian manifold M (with $X = T_1^*M$), the Anosov one form α is the smooth canonical Liouville one form $\alpha = \sum_{j=1}^{n} \xi_j dx^j$ on T^*X and restricted to T_1^*M. We have used here local coordinates x^j on X and related coordinates ξ_j on T_x^*X. Therefore $E_u(x) \oplus E_s(x)$ is C^∞. The distributions $E_u(x)$, $E_s(x)$ are C^1 individually (see [20] p. 252).

- The previous example is a special case of a contact flow: the flow ϕ_t is **a contact flow** (or **Reeb vector field**, see [34] p. 106, [42, p. 55]) if the associated one form α defined in Eq. (1.4) is C^∞ and if

$$d\alpha_{|(E_u \oplus E_s)} \text{ is non degenerate (i.e. symplectic)} \tag{1.6}$$

meaning that α **is a contact one form**. Equivalently, $dx := \alpha \wedge (d\alpha)^d$ is a volume form on X with $d := \dim(E_u) = \dim(E_s)$. Notice that (1.5) implies that the volume form is invariant by the flow:

$$\mathcal{L}_V(dx) = 0. \tag{1.7}$$

In that case, $E_u(x) \oplus E_s(x) = \ker(\alpha)$ is C^∞ and α determines V by $d\alpha(V) = 0$ and $\alpha(V) = 1$.

Anosov flow lifted on the cotangent space T^*X. We first review how one can lift the vector field V and its flow to the cotangent bundle. We may view V as a 1^{st} order differential operator, in local coordinates, $V = \sum_{j=1}^n V_j(x)\frac{\partial}{\partial x_j}$, where V_j are smooth and real-valued. The operator $\widehat{H} := -iV$ takes the form $\widehat{H} = \sum V_j(x) D_{x_j}$, $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$ and it has the real-valued principal symbol $H_0(x,\xi) = V(\xi) \in C^\infty(T^*X)$. In local coordinates $H_0(x,\xi) = \sum V_j(x)\xi_j$.

Let

$$\mathbf{X} = \sum \frac{\partial H_0}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial H_0}{\partial x_j}\frac{\partial}{\partial \xi_j} \tag{1.8}$$

be the Hamilton field of H_0 and let $\phi_t = \exp tV : X \to X$ and $M_t = \exp t\mathbf{X} : T^*X \to T^*X$ be the flows generated by the vector fields V and $\mathbf{X}$, well defined for $t \in \mathbb{R}$. Then M_t is a lift of ϕ_t in the sense that $\pi \circ M_t = \phi_t \circ \pi$, where $\pi : T^*X \to X$ denotes the natural projection. M_t is a symplectic map in the sense that $M_t^*\omega = \omega$, where $\omega = \sum_j d\xi_j \wedge dx_j$ is the symplectic 2-form. In fact M_t is the map naturally induced on T^*X by ϕ_t: we have

$$M_t(x,\xi) = (\phi_t(x), ((D\phi_t(x))^{\mathrm{t}})^{-1}(\xi)). \tag{1.9}$$

Let

$$T_x^*X = E_u^*(x) \oplus E_s^*(x) \oplus E_0^*(x) \tag{1.10}$$

be the decomposition dual to (1.2) in the sense that

$$\begin{gathered} (E_0^*(x))(E_u(x) \oplus E_s(x)) = 0, \\ (E_u^*(x))(E_u(x) \oplus E_0(x)) = 0, \quad (E_s^*(x))(E_s(x) \oplus E_0(x)) = 0. \end{gathered} \tag{1.11}$$

Here $E_0^*(x)$ is dual to $E_0(x)$ spanned by $\alpha(x)$, while $E_u^*(x)$ and $E_s^*(x)$ are dual to $E_s(x)$ and $E_u(x)$ respectively, so that

$$\begin{aligned} \dim E_0^*(x) &= \dim E_0(x) = 1, \\ \dim E_u^*(x) &= \dim E_s(x) = d_s, \\ \dim E_s^*(x) &= \dim E_u(x) = d_u. \end{aligned} \tag{1.12}$$

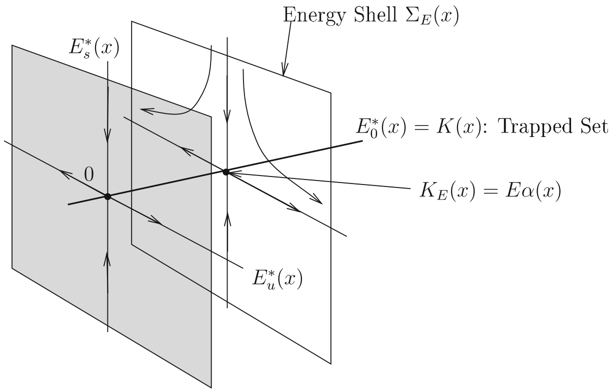

Fig. 2. Picture of T_x^*X for a given $x \in X$, and a given energy $E \in \mathbb{R}$

From (1.9) it follows that $M_t(x, \xi)$ is linear in ξ and combining (1.9) with (1.3) we see that with the notation $\rho = (\rho_x, \rho_\xi)$ to denote a point ρ in the cotangent bundle,

$$\begin{aligned} |M_t(\rho)_\xi| &\leq Ce^{-\theta t}|\rho_\xi|, \quad \rho \in E_s^*, \ t \geq 0, \\ |M_t(\rho)_\xi| &\leq Ce^{-\theta|t|}|\rho_\xi|, \quad \rho \in E_u^*, \ t \leq 0. \end{aligned} \tag{1.13}$$

Here $|\cdot|$ denotes the natural dual norm on the fibers of the cotangent bundle.

We remark that for every $E \in \mathbb{R}$, the energy shell

$$\Sigma_E := H_0^{-1}(E) \tag{1.14}$$

is invariant under the flow M_t (since H_0 is invariant under its own Hamilton flow). In each fiber T^*X, Σ_E is of the form[1]

$$\Sigma_E\,(x) := \Sigma_E \cap T_x^*X = E\alpha(x) + E_s^*(x) \oplus E_u^*(x), \tag{1.15}$$

where α is the Anosov 1 form in (1.4). The set

$$K_E := \Sigma_E \cap E_0^* = \{E\alpha(x);\ x \in X\} \tag{1.16}$$

is the trapped set for the flow M_t restricted to Σ_E in the sense that if $\rho \in \Sigma_E$ then $\{M_t(\rho);\ t \in \mathbb{R}\}$ is bounded if and only if $\rho \in K_E$. See Fig. 2.

The operator $\widehat{H} = -iV$, defined in the sense of distributions as an unbounded operator on $L^2(X)$, is the generator of the group of Ruelle transfer operators $\widehat{M}_t$, given by $\widehat{M}_t u = u \circ \phi_{-t}, t \in \mathbb{R}$. We remark that $\widehat{M}_t$ is also a Fourier integral operator that quantizes the canonical transformation M_t [28].

[1] Since $E_0\left(E_u^* \oplus E_s^*\right) = 0$, $V \in E_0$ and $H_0\,(x, \xi) = V_0\,(\xi)$, then $H_0\left(E_u^* \oplus E_s^*\right) = 0$. Also $H_0\,(\alpha\,(x)) = 1$, then $H_0\,(E\alpha\,(x)) = E$.

1.1.2. Anisotropic Sobolev spaces. The escape function in our approach is provided by the following lemma that will be proved in Sect. 2 using very much the contraction and expansion properties (1.13). Roughly speaking, an escape function on phase space T^*X should decrease along the flow M_t outside the trapped set (1.16). Escape functions are used in order to establish existence of resonances in specific Sobolev spaces and have been introduced by B. Helffer and J. Sjöstrand [25] and used in many situations [43,45–47]. In [24], the authors consider the geodesic flow associated to Schottky groups and provide an upper bound for the density of Ruelle resonances, see also [8].

Lemma 1.2. *Let $u, n_0, s \in \mathbb{R}$ with $u < n_0 < s$ and $u < 0 < s$. Let N_0 be an arbitrarily small conic neighborhood in $T^*X\backslash 0$ of the neutral direction E_0^*. There exists a smooth function $m(x,\xi) \in C^\infty(T^*X)$ called an **order function,** taking values in the interval $[u, s]$, and an **escape function** on T^*X defined by:*

$$G_m(x,\xi) := m(x,\xi) \log \sqrt{1 + (f(x,\xi))^2}, \tag{1.17}$$

where $f \in C^\infty(T^*X)$ and for $|\xi| \geq 1$, $f > 0$ is positively homogeneous of degree 1 in ξ (i.e. $f(x, \lambda\xi) = \lambda f(x,\xi)$ for $\lambda \geq 1$). $f(x,\xi) = |\xi|$ in a conical neighborhood of E_u^* and E_s^*. $f(x,\xi) = |H_0(x,\xi)|$ in a conical neighborhood of E_0^*, such that:

1. *For $|\xi| \geq 1$, $m(x,\xi)$ depends only on the direction $\frac{\xi}{|\xi|} \in S_x^*X$ and takes the value u (respect. n_0, s) in a small neighborhood of E_u^* (respect. E_0^*, E_s^*). See Fig. 3(a).*
2. *G_m decreases strictly and uniformly along the trajectories of the flow M_t in the cotangent space, except in a conical vicinity N_0 of the neutral direction E_0^* and for small $|\xi|$: there exists $R > 0$ such that*

$$\forall (x,\xi) \in T^*X,\ \text{if } |\xi| \geq R,\ \xi \notin N_0 \quad \text{then } \mathbf{X}(G_m)(x,\xi) < -C_m < 0, \tag{1.18}$$

with

$$C_m := c \min(|u|, s) \tag{1.19}$$

and $c > 0$ independent of u, n_0, s.

3. *More generally*

$$\forall (x,\xi) \in T^*X \text{ with } |\xi| \geq R, \qquad \mathbf{X}(G_m)(x,\xi) \leq 0. \tag{1.20}$$

See Fig. 3(b).

The order function $m(x,\xi)$ belongs to the classical symbol space $S^0 = S_1^0(T^*X)$ in the sense that for every choice of coordinates $x_1, \ldots, x_n$ that identifies an open set $U \subset X$ with an open set $\tilde{U} \subset \mathbb{R}^n$ and for the corresponding dual coordinates $\xi_1, \ldots, \xi_n$, we have for all multi-indices $\alpha, \beta \in \mathbb{N}^n$,

$$\partial_\xi^\alpha \partial_x^\beta m(x,\xi) = \mathcal{O}(1)\langle\xi\rangle^{-|\alpha|}, \qquad \text{with } \langle\xi\rangle := \sqrt{1 + |\xi|^2}$$

uniformly on $K \times \mathbb{R}^n$ for every $K \Subset \widetilde{U}$.

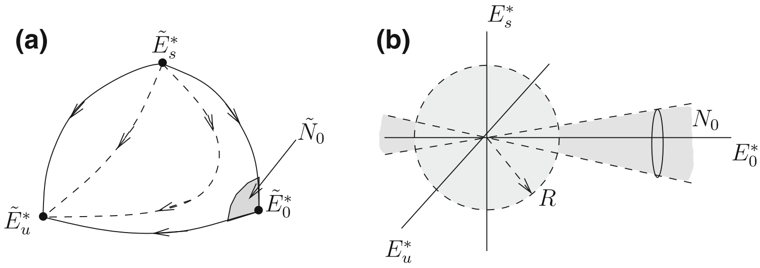

Fig. 3. (**a**) The induced flow $\widetilde{M}_t$ on the cosphere bundle $S^*X := \left(T^*X\backslash\{0\}\right)/\mathbb{R}^+$ which is the bundle of directions of cotangent vectors $\xi/|\xi|$. (Here the picture is restricted to a fiber S_x^*X). (**b**) Picture in the cotangent space T_x^*X which shows in grey the sets outside of which the escape estimate (1.18) holds

Definition 1.3. *Let $m(x,\xi) \in S^0$ be a real-valued and let and $\frac{1}{2} < \rho \leq 1$. A function $p \in C^\infty(T^*X)$ belongs to the* **class $S_\rho^{m(x,\xi)}$ of variable order** *if for every choice of coordinates $x_1,\dots,x_n$ that identifies an open set $U \subset X$ with an open set $\widetilde{U} \subset \mathbb{R}^n$ and for the corresponding dual coordinates $\xi_1,\dots,\xi_n$, we have for all multi-indices $\alpha,\beta \in \mathbb{N}^n$,*

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \leq \mathcal{O}(1)\langle\xi\rangle^{m(x,\xi)-\rho|\alpha|+(1-\rho)|\beta|} \tag{1.21}$$

uniformly on $K \times \mathbb{R}^n$ for every $K \Subset \widetilde{U}$.

We refer to [17, Sect. A.2.2] for a precise description of theorems related to symbols with variable orders.

With m as in Lemma 1.2, we see that the symbol $A_m = \exp G_m$ belongs to the symbol space $S_{1-0}^m := \bigcap_{1>\rho>1/2} S_\rho^m$. As explained in the Appendix in the paper [17, Lem. 6], we can associate to this symbol a pseudodifferential operator $\widehat{A}_m : \mathcal{D}'(X) \to \mathcal{D}'(X), C^\infty(X) \to C^\infty(X)$ whose distribution kernel is smooth outside the diagonal and such that if $\kappa : X \supset U \to \widetilde{U} \subset \mathbb{R}^n$ is a local coordinate chart, then the operator $\hat{B} : C_0^\infty(\widetilde{U}) \ni u \mapsto \left(\widehat{A}_m(u\circ\kappa)\circ\kappa^{-1}\right) \in C^\infty(\widetilde{U})$ is a pseudodifferential operator which up to a smoothing operator is given by **Weyl quantization**[2] [49, (14.5) p. 60]:

[2] In this paper we choose Weyl quantization because it has some well known interesting properties. First a real symbol $p_W \in S^\mu, \mu \in \mathbb{R}$, is quantized in a formally self-adjoint operator $\hat{P} = \mathrm{Op}\,(p_W)$. Secondly, a change of coordinate systems preserving the volume form changes the symbol at a subleading order $S^{\mu-2}$ only. In other words, on a manifold with a fixed smooth density dx, the Weyl symbol p_W of a given pseudo-differential operator $\hat{P}$ is well defined modulo terms in $S^{\mu-2}$.

The Weyl symbol of the operator $\hat{H} = -iV$ considered in this paper is

$$H_W(x,\xi) = V(\xi) + \frac{i}{2}\mathrm{div}\,(V). \tag{1.22}$$

Indeed from [49, (14.7) p. 60], in a given chart where $V = \sum V_j(x)\frac{\partial}{\partial x^j} \equiv V(x)\partial_x$,

$$H_W(x,\xi) = \exp\left(\frac{i}{2}\partial_x\partial_\xi\right)(V(x).\xi) = V(x).\xi + \frac{i}{2}\partial_x V = V(\xi) + \frac{i}{2}\mathrm{div}\,(V)$$

and div (V) depends only on the choice of the volume form, see [48, p. 125]. Notice that this symbol does not depend on the choice of coordinates systems provided the volume form is expressed by $dx = dx_1\dots dx_n$. The first term $H_0(x,\xi) = V(\xi)$ in (1.22) belongs to S^1 and is called the **principal symbol** of $\hat{H}$. The second term $\frac{i}{2}\mathrm{div}\,(V)$ in (1.22) belongs to S^0 and is called the **subprincipal symbol** of $\hat{H}$.

$$\hat{B}u(x) = \mathrm{Op}\,(b)\,u\,(x) := \frac{1}{(2\pi)^n} \iint e^{i(x-y)\cdot\xi} b(\frac{x+y}{2}, \xi) u(y) dy d\xi, \quad (1.23)$$

where the **Weyl symbol** $b \in S^m_{1-0}(T^*X_{|U})$ (if we identify $T^*X_{|U}$ with $\widetilde{U} \times \mathbb{R}^n$ in the natural way) and coincides with A_m modulo a symbol of class $S^{m-1+0}_{1-0} := \bigcap_{\epsilon>0} S^{m-1+\epsilon}_{1-\varepsilon}$. In addition, as we have seen in [17], the quantization $\widehat{A}_m$ can be chosen to be essentially self-adjoint and bijective with an inverse $\widehat{A}_m^{-1}$ which is the quantization of the symbol $1/A_m$ in a similar way. We define the **anisotropic Sobolev space** $H^m(X)$ of variable order m by

$$H^m(X) := \widehat{A}_m^{-1}(L^2(X)), \quad (1.24)$$

and equip it with the natural norm

$$\|u\|_{H^m} = \|\widehat{A}_m u\|,$$

where we follow the convention that all norms are in L^2 if nothing else is specified. Some basic properties of the space H^m, such as embedding properties, are given in [17, Sect. 3.2].

1.1.3. The theorems. Let V be a smooth Anosov vector field on a smooth compact manifold X (we do not assume that the Anosov one form α is contact nor that it is smooth). The first two results are very close to the results already obtained by [10, Thm. 1] (with the slight difference that the authors use Banach spaces). The novelty here is to show that this resonance spectrum fits with the general theory of semiclassical resonances developed by B. Helffer and J. Sjöstrand [25] and initiated by Aguilar, Baslev, Combes [1,5].

Theorem 1.4. ***"Discrete spectrum".*** *Let m be a function which satisfies the hypothesis of Lemma 1.2 p. 9. The generator $\widehat{H} = -iV$ defines a maximal closed unbounded operator on the anisotropic Sobolev space H^m,*

$$\widehat{H} : H^m \to H^m$$

in the sense of distributions with domain given by

$$\mathcal{D}(\widehat{H}) := \{\varphi \in H^m, \quad \widehat{H}\varphi \in H^m\}.$$

It coincides with the closure of $(-iV) : C^\infty \to C^\infty$ in the graph norm for operators. For $z \in \mathbb{C}$ such that $\Im(z) > -(C_m - C)$ with C_m defined in (1.19) and some C independent of m, the operator $(\widehat{H} - z) : \mathcal{D}(\widehat{H}) \cap H^m \to H^m$ is a Fredholm operator with index 0 depending analytically on z. Consequently the operator $\widehat{H}$ ***has a discrete spectrum*** *in the domain $\Im(z) > -(C_m - C)$, consisting of eigenvalues λ_i of finite algebraic multiplicity. Recall that if $|u|$, s are chosen large then C_m is large. See Fig. 4.*

Concerning Fredholm operators we refer to [12, p.122] or [25, App. A p. 220]. The proof of Theorem 1.4 is given in Sect. 3.

Remarks.

- In Proposition 3.1 we shall see that the space H^m is invariant under the operator $\widehat{C}$ of complex conjugation. Since $\widehat{H}$ and $\widehat{C}$ anti-commute it follows that the **set of Ruelle resonances is invariant under reflection in the imaginary axis**.

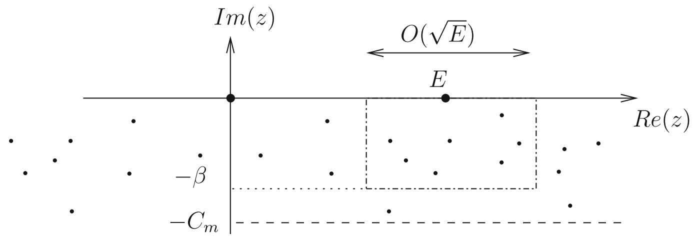

Fig. 4. Spectrum of Ruelle resonances of $\widehat{H} = -iV$. From Theorem 1.8 the number of eigenvalues in the rectangle is $o\left(E^{n-1/2}\right)$ for $E \to \infty$

- We recall a simple and well known result (which follows from the property that $\left\|\widehat{M}_t\right\|_\infty = 1$), that there is **no eigenvalue in the upper half plane** $\Im(z) > 0$ and **no Jordan block on the real axis**. See [10, Thm. 1][17].

The next theorem shows that the spectrum is intrinsic.

Theorem 1.5. ***"The discrete spectrum is intrinsic to the Anosov vector field".*** *Let* $\widetilde{m}, \widetilde{f}, \widetilde{G}_m = \widetilde{m}\log\sqrt{1+\widetilde{f}^2}$ *be another set of functions as in Lemma 1.2 so that Theorem 1.4 applies and* $\widehat{H} : H^{\widetilde{m}} \to H^{\widetilde{m}}$ *has discrete spectrum in the set* $\Im(z) > -(\widetilde{C}_{\widetilde{m}} - \widetilde{C})$*. Then in the set* $\Im(z) > -\min((C_m - C), (\widetilde{C}_{\widetilde{m}} - \widetilde{C}))$ *the eigenvalues of* $\widehat{H} : H^m \to H^m$ *counted with their multiplicity and their respective eigenspaces coincide with those of* $\widehat{H} : H^{\widetilde{m}} \to H^{\widetilde{m}}$.

The eigenvalues λ_i *are called the* ***Ruelle Resonances*** *and we denote the set by* $\mathrm{Res}(\widehat{H})$*. The resolvent* $(z - \widehat{H})^{-1}$ *viewed as an operator* $C^\infty(X) \to \mathcal{D}'(X)$ *has a meromorphic extension from* $\Im(z) \gg 1$ *to* $\mathbb{C}$*. The poles of this extension are the Ruelle resonances.*

The proof of Theorem 1.5 is given in Sect. 4. The next theorem describes the wavefront set of the eigenfunctions associated to λ_i. The wavefront set of a distribution has been introduced by Hörmander. The wavefront set corresponds to the directions in T^*X where the distribution is not C^∞ (i.e. the local Fourier transform is not rapidly decreasing). The wavefront set of a PDO is the directions in T^*X where the symbol is not rapidly decreasing:

Definition 1.6 ([23, p. 77, 49, p. 27]). *If* $(x_0, \xi_0) \in T^*X\backslash 0$*, we say that* $A \in S^m$ *is* **non characteristic (or elliptic)** *at* (x_0, ξ_0) *if* $\left|A(x,\xi)^{-1}\right| \leq C|\xi|^{-m}$ *for* (x, ξ) *in a small conic neighborhood of* (x_0, ξ_0) *and* $|\xi|$ *large. If* $u \in \mathcal{D}'(X)$ *is a distribution, we say that* u **is** C^∞ *at* $(x_0, \xi_0) \in T^*X\backslash 0$ *if there exists* $A \in S^m$ *non characteristic (or elliptic) at* (x_0, ξ_0) *such that* $Op(A)u \in C^\infty(X)$*. The* **wavefront set of the distribution** u *is* $WF(u) := \{(x_0, \xi_0) \in T^*X\backslash 0, \quad u \text{ is not } C^\infty \text{ at } (x_0, \xi_0)\}$*. The* **wavefront set of the operator** $\mathrm{Op}(A)$ *is the smallest closed cone* $\Gamma \subset T^*X\backslash 0$ *such that* $A_{/\complement\Gamma} \in S^{-\infty}\left(\complement\Gamma\right)$.

Theorem 1.7. *The* ***wavefront set*** *of the associated generalized eigenfunctions of* $\widehat{H}$ *is contained in the unstable direction* E_u^*.

Proof. Let us consider an eigen-distribution $\varphi \in H^m$ of $\widehat{H}$. In Lemma 1.2, if we let $n_0, s \to +\infty$, the order function $m(x, \xi)$ can be made arbitrarily large for $|\xi| \geq 1$, in every direction outside a small vicinity of E_u^*. As a result φ (which remains unchanged) is smooth in every direction except E_u^*. From the definition of the wave-front set of a distribution this means that the wave-front set of φ is contained in the unstable direction E_u^*. Theorem 1.7 follows. □

The following theorem is the main result of this paper:

Theorem 1.8. ***"Upper bound for the density of resonances".*** *For every $\beta > 0$, in the limit $E \to +\infty$ we have*

$$\sharp\{\lambda \in \mathrm{Res}(\widehat{H}),\ |\Re(\lambda) - E| \le \sqrt{E},\ \Im(\lambda) > -\beta\} \le o(E^{n-1/2}), \tag{1.25}$$

with $n = \dim X$.

Remarks.

- Since the spectrum is symmetric with respect to the imaginary axis, the estimates (1.25) also hold for $E \to -\infty$.
- The upper bound given in (1.25) results from our method and choice of escape function $A(x, \xi)$. In the proof, $o\left(E^{n-1/2}\right)$ comes from an estimate of a $\mathcal{V} \subset T^*X$ which contains the trapped set Σ_E and whose symplectic volume is of order $\mathrm{Vol}(\mathcal{V}) \simeq \delta E^{-1/2}$, with δ arbitrarily small. Using Weyl inequalities we obtain an upper bound of order $E^n \mathrm{Vol}(\mathcal{V}) \simeq \delta E^{n-1/2}$ in (1.25). It is expected that a better choice of the escape function G_m could improve this upper bound. This is a work in preparation by the authors. For specific models, e.g. geodesic flows on a surface with constant negative curvature, it is observed from zeta functions that the upper bound is $O\left(E^{\frac{n}{2}}\right)$ (see [30]). We reasonably expect this in general.
- From the upper bound (1.25), one can deduce upper bounds in larger spectral domains. For example: for every $\beta > 0$, in the semiclassical limit $E \to +\infty$ we have

$$\sharp\left\{\lambda \in \mathrm{Res}\left(\widehat{H}\right),\ \Re(\lambda) \in [-E, E],\ \Im(\lambda) > -\beta\right\} \le o\left(E^n\right),$$

  with $n = \dim X$.

2. Proof of Lemma 1.2 on the Escape Function

In this section we construct a smooth function G_m on the cotangent space T^*X called the escape function and prove Lemma 1.2. We will denote, $\frac{\xi}{|\xi|}$ the direction of a cotangent vector ξ and $S^*X := (T^*X \backslash \{0\})/\mathbb{R}^+$ the **cosphere bundle** which is the bundle of directions of cotangent vectors $\xi/|\xi|$. S^*X is a compact space. The images of $E_u^*, E_s^*, E_0^*, N_0 \subset T^*X$ by the projection $T^*X\backslash\{0\} \to S^*X$ are denoted respectively $\widetilde{E}_u^*, \widetilde{E}_s^*, \widetilde{E}_0^*, \widetilde{N}_0 \subset S^*X$, see Fig. 3(a) in Subsect. 1.1.2.

Remarks on Lemma 1.2.

- It is important to notice that we can choose m such that the value of C_m **is arbitrarily large** (by making $s, |u| \to \infty$) and that the neighborhood $\widetilde{N}_0$ **is arbitrarily small**.
- The value of n_0 could be chosen to be $n_0 = 0$ for simplicity but it is interesting to observe that letting $n_0, s \to +\infty$, the order function $m(x, \xi)$ can be made arbitrarily large for $|\xi| \ge 1$, outside a small vicinity of E_u^*. We will use this in the proof of Theorem 1.7 in order to show that the wavefront of the eigen-distributions are included in E_u^*.

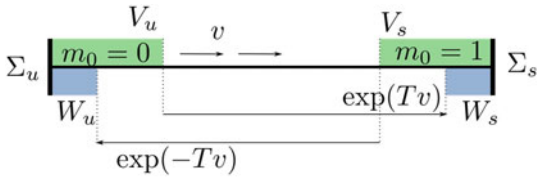

Fig. 5. Illustration for the proof of Lemma 2.1. The horizontal axis is a schematic picture of M and this shows the construction and properties of the sets V_u, V_s and W_u, W_s

- Inspection of the proof shows that with an adapted norm $|\xi|$ obtained by averaging, c can be chosen arbitrarily close to θ, defined in (1.13).
- The constancy of m in the vicinity of the stable/unstable/neutral directions allows us to have a smooth escape function G_m despite the fact that the distributions $E_s^*(x)$, $E_u^*(x)$, $E_0^*(x)$ have only Hölder regularity in general.

We first define a function $m(x,\xi)$ called the **order function** following closely [17] Sect. 3.1 (and [19] p. 196).

The function m. The following lemma is useful for the construction of escape functions. Let M be a compact manifold and let v be a smooth vector field on M. We denote $\exp(tv) : M \to M$ the flow at time t generated by v. Let Σ_u, Σ_s be compact disjoint subsets of M such that

$$\begin{aligned} &\mathrm{dist}\,(\exp(tv)(\rho), \Sigma_s) \to 0,\ t \to +\infty \text{ when } \rho \in M \setminus \Sigma_u, \\ &\mathrm{dist}\,(\exp(tv)(\rho), \Sigma_u) \to 0,\ t \to -\infty \text{ when } \rho \in M \setminus \Sigma_s. \end{aligned}$$

Lemma 2.1. *Let $V_u, V_s \subset M$ be open neighborhoods of Σ_u and Σ_s respectively and let $\varepsilon > 0$. Then there exist $W_u \subset V_u$, $W_s \subset V_s$, $m \in C^\infty(M;[0,1])$, $\eta > 0$ such that $v(m) \geq 0$ on M, $v(m) > \eta > 0$ on $M \setminus (W_u \cup W_s)$, $m(\rho) > 1-\varepsilon$ for $\rho \in W_s$ and $m(\rho) < \varepsilon$ for $\rho \in W_u$.*

Proof. After shrinking V_u, V_s we may assume that $V_u \cap V_s = \emptyset$ and

$$t \geq 0 \Rightarrow \exp(tv)(V_s) \subset V_s, \text{ and } t \leq 0 \Rightarrow \exp(tv)(V_u) \subset V_u. \tag{2.1}$$

Let $T > 0$ and let $W_s := M \backslash \exp(Tv)(V_u) = \exp(Tv)(M \backslash V_u)$ and $W_u := M \backslash \exp(-Tv)(V_s) = \exp(-Tv)(M \backslash V_s)$. See Fig. 5.

If T is large enough one has $W_u \subset V_u$, $W_s \subset V_s$ and $W_s \cap W_u = \emptyset$. Let $m_0 \in C^\infty(M;[0,1])$ be equal to 1 on V_s and equal to 0 on V_u. Put

$$m = \frac{1}{2T}\int_{-T}^{T} m_0 \circ \exp(tv)\,dt. \tag{2.2}$$

Then

$$v(m)(\rho) = \frac{1}{2T}\left(m_0(\exp(Tv)(\rho)) - m_0(\exp(-Tv)(\rho))\right). \tag{2.3}$$

- Let $\rho \in M \setminus (W_u \cup W_s)$. From (2.3) we see that $v(m)(\rho) = \frac{1}{2T}(1-0)) = \frac{1}{2T} > 0$.

For $\rho \in M$ let

$$\mathcal{I}(\rho) := \{t \in \mathbb{R}, \exp(tv)(\rho) \in M \setminus (V_u \cup V_s)\}.$$

This is a closed connected interval by (2.1) and moreover its length is uniformly bounded:

$$\exists \tau > 0, \quad \forall \rho \in M, \quad |\max(\mathcal{I}(\rho)) - \min(\mathcal{I}(\rho))| \leq \tau.$$

In other words, τ is an upper bound for the travel Time in the domain $M \setminus (V_u \cup V_s)$. To prove the lemma, we have to consider two more cases:

- Let $\rho \in W_u$. If $t \leq T - \tau$ then $m_0(\exp(tv)(\rho)) = 0$ and

$$m(\rho) = \frac{1}{2T}\left(\int_{-T}^{T-\tau} \underbrace{m_0(\exp(tv)\rho)}_{=0} dt + \int_{T-\tau}^{T} \underbrace{m_0(\exp(tv)\rho)}_{\leq 1} dt\right) \leq \frac{\tau}{2T} < \varepsilon,$$

  where the last inequality holds if one chooses T large enough. One has $m_0(\exp(-Tv)(\rho)) = 0$ therefore (2.3) implies that $v(m)(\rho) \geq 0$.
- Let $\rho \in W_s$. One shows similarly that

$$m(\rho) = \frac{1}{2T}\left(\int_{-T}^{-T+\tau} \underbrace{m_0(\exp(tv)\rho)}_{\geq 0} dt + \int_{-T+\tau}^{T} \underbrace{m_0(\exp(tv)\rho)}_{=1} dt\right) \geq \frac{2T-\tau}{2T} > 1 - \varepsilon,$$

  for T large enough, and $v(m)(\rho) \geq 0$. □

We now apply Lemma 2.1 to the case when $M = S^*X$ and v is the image $\widetilde{\mathbf{X}}$ on S^*X of our Hamilton field $\mathbf{X}$. See Fig. 6.

- We first take $\Sigma_u = \Sigma_u^1 = \widetilde{E}_s^*$ and let $\Sigma_s = \Sigma_s^1 \subset M$ be the set of limit points $\lim_{j\to+\infty} \exp t_j v(\rho)$, where $\rho \in M \setminus \Sigma_u^1$ and $t_j \to +\infty$. Σ_s^1 is the union of $\widetilde{E}_0^*$, $\widetilde{E}_u^*$ and all trajectories $\exp(\mathbb{R}v)(\rho)$, where ρ has the property that $\exp tv(\rho)$ converges to $\widetilde{E}_0^*$ when $t \to -\infty$ and to $\widetilde{E}_u^*$ when $t \to +\infty$. Equivalently, Σ_s^1 is the image $\widetilde{E_u^* \oplus E_0^*}$ in S^*X of $E_u^* \oplus E_0^*$. Applying the lemma, we get $m_1 = m \in C^\infty(M; [0, 1])$ such that $m_1 < \varepsilon$ outside an arbitrarily small neighborhood W_u^1 of $\Sigma_u^1 = \widetilde{E}_s^*$, $m_1 > 1 - \varepsilon$ outside an arbitrarily small neighborhood W_s^1 of $\Sigma_s^1 = \widetilde{E_u^* \oplus E_0^*}$ and $\widetilde{\mathbf{X}}(m_1) \geq 0$ everywhere with strict inequality $\widetilde{\mathbf{X}}(m_1) > \eta > 0$ outside $W_s^1 \cup W_u^1$.
- Similarly, we can find $m_2 = m \in C^\infty(M; [0, 1])$, such that $m_2 < \varepsilon$ outside an arbitrarily small neighborhood W_u^2 of $\Sigma_u^2 = \widetilde{E_s \oplus E_0^*}$, $m_2 > 1 - \varepsilon$ outside an arbitrarily small neighborhood W_s^2 of $\Sigma_s^2 = \widetilde{E}_u^*$ and $\widetilde{\mathbf{X}}(m_2) \geq 0$ everywhere with strict inequality $\widetilde{\mathbf{X}}(m_2) > \eta > 0$ outside $W_s^2 \cup W_u^2$.

  Let $u < n_0 < s$ and put

$$\widetilde{m} := s + (n_0 - s)\, m_1 + (u - n_0)\, m_2,$$
$$\widetilde{N}_s := W_u^1 \cap W_u^2, \quad \widetilde{N}_0 := W_s^1 \cap W_u^2, \quad \widetilde{N}_u := W_s^1 \cap W_s^2.$$

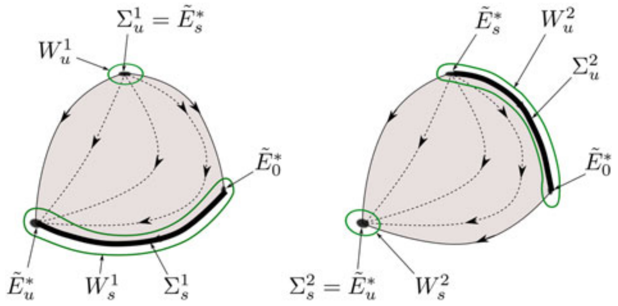

Fig. 6. Representation of different sets on S^*X used in the proof

Then

- on $S^*X\backslash\left(\widetilde{N}_s \cup \widetilde{N}_0 \cup \widetilde{N}_u\right) = \left(S^*X\backslash\left(W_u^1 \cup W_s^1\right)\right) \cup \left(S^*X\backslash\left(W_u^2 \cup W_s^2\right)\right)$ we have $\widetilde{\mathbf{X}}(m_1) > \eta$ or $\widetilde{\mathbf{X}}(m_2) > \eta$, therefore

$$\begin{aligned} \widetilde{\mathbf{X}}(\widetilde{m}) &= (n_0 - s)\,\widetilde{\mathbf{X}}(m_1) + (u - n_0)\,\widetilde{\mathbf{X}}(m_2) \\ &< -\eta \min\left(|n_0 - s|, |u - n_0|\right). \end{aligned} \tag{2.4}$$

- on $\widetilde{N}_s = W_u^1 \cap W_u^2$ we have $m_1 < \varepsilon$ and $m_2 < \varepsilon$ therefore

$$\begin{aligned} \widetilde{m} &> s + (n_0 - s)\,\varepsilon + (u - n_0)\,\varepsilon \\ &= s(1-\varepsilon) + u\varepsilon > \frac{s}{2}, \end{aligned} \tag{2.5}$$

  where the last inequality holds if ε is chosen small enough.

- on $\widetilde{N}_u = W_s^1 \cap W_s^2$ we have $m_1 > 1 - \varepsilon$ and $m_2 > 1 - \varepsilon$ therefore

$$\begin{aligned} \widetilde{m} &< s + (n_0 - s)(1-\varepsilon) + (u - n_0)(1-\varepsilon) \\ &= \varepsilon s + u(1-\varepsilon) < \frac{u}{2}, \end{aligned} \tag{2.6}$$

  where the last inequality holds if ε is chosen small enough.

- on S^*X we have

$$\widetilde{\mathbf{X}}(\widetilde{m}) = (n_0 - s)\,\widetilde{\mathbf{X}}(m_1) + (u - n_0)\,\widetilde{\mathbf{X}}(m_2) \le 0. \tag{2.7}$$

We construct a smooth function m on T^*M satisfying

$$\begin{aligned} m(x,\xi) &= \widetilde{m}\left(\frac{\xi}{|\xi|}\right), \qquad \text{if } |\xi| \ge 1, \\ &= 0 \qquad \text{if } |\xi| \le 1/2. \end{aligned}$$

The symbol G_m. Let

$$G_m(x,\xi) := m(x,\xi)\log\sqrt{1 + (f(x,\xi))^2}$$

with $f \in C^\infty(T^*X)$ such that for $|\xi| \ge 1$, $f > 0$ is positively homogeneous of degree 1 in ξ, and

$$\begin{aligned} \frac{\xi}{|\xi|} \in \widetilde{N}_u \cup \widetilde{N}_s &\Rightarrow f(x,\xi) := |\xi|, \\ \frac{\xi}{|\xi|} \in \widetilde{N}_0 &\Rightarrow f(x,\xi) := |H_0(x,\xi)|. \end{aligned}$$

The consequences of these choices are:

- Since $\mathbf{X}(H_0) = 0$ then $\mathbf{X}\left(\log\sqrt{1+(f(x,\xi))^2}\right) = 0$ for $\frac{\xi}{|\xi|} \in \widetilde{N}_0$.
- Since E_s^* is the stable direction and E_u^* the unstable one,

$$\exists C > 0, \qquad \frac{\xi}{|\xi|} \in \widetilde{N}_s \Rightarrow \mathbf{X}(\log\langle\xi\rangle) < -C, \qquad \frac{\xi}{|\xi|} \in \widetilde{N}_u \Rightarrow \mathbf{X}(\log\langle\xi\rangle) > C. \tag{2.8}$$

- In general $\left|\mathbf{X}\left(\log\sqrt{1+f^2}\right)\right|$ is bounded:

$$\exists C_2 > 0, \qquad \forall \xi \in T^*X, \qquad \left|\mathbf{X}\left(\log\sqrt{1+(f(x,\xi))^2}\right)\right| < C_2.$$

We will show now the uniform escape estimate Eq.(1.18), in Lemma 1.2. One has

$$\mathbf{X}(G_m) = \mathbf{X}(m)\log\sqrt{1+f^2} + m\mathbf{X}\left(\log\sqrt{1+f^2}\right). \tag{2.9}$$

We will first consider each term separately assuming $|\xi| \geq 1$.

- If $\widetilde{\xi} \in S^*X \setminus \left(\widetilde{N}_s \cup \widetilde{N}_u \cup \widetilde{N}_0\right)$ then using (2.4) and the fact that $\left|\mathbf{X}\left(\log\sqrt{1+f^2}\right)\right|$ and m are bounded, there exists $R > 0$ large enough such that for $|\xi| \geq R$,

$$\mathbf{X}(G_m)(x,\xi) < -c\min(s,|u|)$$

with $c > 0$ independent of u, n_0, s.
- If $\widetilde{\xi} \in \widetilde{N}_u$ then from (2.8) and (2.6) there exists $c > 0$ such that

$$\mathbf{X}(G_m) = \underbrace{\mathbf{X}(m)}_{\leq 0}\underbrace{\log\langle\xi\rangle}_{\geq 0} + \underbrace{m}_{<\frac{u}{2}}\underbrace{\mathbf{X}(\log\langle\xi\rangle)}_{>C} < -c|u| < 0.$$

- If $\widetilde{\xi} \in \widetilde{N}_s$ then from (2.8) and (2.5) there exists $c > 0$ such that

$$\mathbf{X}(G_m) = \underbrace{\mathbf{X}(m)}_{\leq 0}\underbrace{\log\langle\xi\rangle}_{\geq 0} + \underbrace{m}_{>\frac{s}{2}}\underbrace{\mathbf{X}(\log\langle\xi\rangle)}_{<-C} < -cs < 0.$$

We have obtained the uniform escape estimate Eq. (1.18), Lemma 1.2. Finally for $\widetilde{\xi} \in \widetilde{N}_0$, we have

$$\mathbf{X}(G_m) = \underbrace{\mathbf{X}(m)}_{\leq 0}\underbrace{\log\sqrt{1+f^2}}_{\geq 0} + m\underbrace{\mathbf{X}\left(\log\sqrt{1+f^2}\right)}_{=0} \leq 0,$$

and we deduce (1.20), Lemma 1.2. We have finished the proof of Lemma 1.2.

3. Proof of Theorem 1.4 about the Discrete Spectrum of Resonances

Here are the different steps that we will follow in the proof.

1. The operator $\widehat{H}$ on the Sobolev space $H^m = \widehat{A}_m^{-1}\left(L^2(X)\right)$ is unitarily equivalent to the operator $\widehat{P} := \widehat{A}_m \widehat{H} \widehat{A}_m^{-1}$ on $L^2(X)$. We will show that $\widehat{P}$ is a pseudo-differential operator. We will compute the symbol $P(x,\xi)$ of $\widehat{P}$ in Lemma 3.2. The important fact is that the derivative of the escape function appears in the imaginary part of the symbol $P(x,\xi)$.
2. For $\Im(z) \gg 1$, using the Gårding inequality, we will show that $\left(\widehat{P} - z\right)$ is invertible and therefore that $\widehat{P}$ has no spectrum in the domain $\Im(z) \gg 1$.
3. Using the Gårding inequality again for a modified operator and analytic Fredholm theory we will show that $\left(\widehat{P} - z\right)$ is invertible for $\Im(z) > -(C_m - C)$ for some constant C independent of m, except for a discrete set of points $z = \lambda_i$ that are eigenvalues of finite multiplicity.

We begin by the following proposition which is a very simple observation.

Proposition 3.1. ***"Symmetry".*** *The conjugation operator $\widehat{C}$ defined by $\hat{C}\varphi = \overline{\varphi}$ on $C^\infty(X)$ and extended to $\mathcal{D}'(X)$ by duality, leaves the space H^m invariant. If $\widehat{H}\psi = \lambda\psi$, $\psi \in H^m$ then $\widetilde{\psi} := \widehat{C}\psi \in H^m$ is also an eigenfunction with eigenvalue $\widetilde{\lambda} = -\overline{\lambda}$. The spectrum of Ruelle resonances is therefore symmetric under reflexion in the imaginary axis.*

Proof. Observe that if $\hat{B}$ is a pseudo-differential operator with symbol $b(x,\xi)$ then from (1.23) $\hat{C}\hat{B}\hat{C}$ is a pseudo-differential operator with symbol $\overline{b(x,-\xi)}$. From its construction the symbol $A_m(x,\xi)$ is real and $A_m(x,-\xi) = A_m(x,\xi)$. Therefore $\widehat{C}\widehat{A}\widehat{C} = \widehat{A}$. Since $\hat{C} = \widehat{A}\widehat{C}\widehat{A}^{-1}$ is an isometry on $L^2(X)$ we conclude that $\hat{C}$ is an isometry on the space $H^m(X) = \widehat{A}^{-1}\left(L^2(X)\right)$. We have the following relation:

$$\widehat{H}\widehat{C} + \widehat{C}\widehat{H} = 0. \tag{3.1}$$

Indeed since the vector field V is real, for every $\varphi \in C^\infty(X)$ one has $\widehat{H}\widehat{C}\varphi = -iV(\overline{\varphi}) = \overline{iV(\varphi)} = -\overline{(-iV(\varphi))} = -\widehat{C}\widehat{H}\varphi$. Then using (3.1), one has $\widehat{H}\widetilde{\psi} = \widehat{H}\widehat{C}\psi = -\widehat{C}\widehat{H}\psi = -\overline{\lambda}\widetilde{\psi}$. $\square$

3.1. Conjugation by the escape function and unique closed extension of $\widehat{P}$ on $L^2(X)$. Let us define

$$\widehat{P} := \widehat{A}_m \widehat{H} \widehat{A}_m^{-1}. \tag{3.2}$$

Then the operator $\widehat{P}$ on $L^2(X)$ is unitarily equivalent to $\widehat{H}$ on H^m. Recall that a smooth function $p(x,\xi)$ on T^*X belongs to the **classical symbol space** $S^\mu = S_1^\mu(T^*X)$ if for every choice of coordinates $x_1,\ldots,x_n$ that identifies an open set $U \subset X$ with an open set $\widetilde{U} \subset \mathbb{R}^n$ and for the corresponding dual coordinates $\xi_1,\ldots,\xi_n$, we have

$$\partial_\xi^\alpha \partial_x^\beta p(x,\xi) = \mathcal{O}(1)\langle\xi\rangle^{\mu-|\alpha|}, \qquad \text{with } \langle\xi\rangle := \sqrt{1+|\xi|^2}$$

uniformly on $K \times \mathbb{R}^n$ for every $K \Subset \widetilde{U}$.

The symbol classes with variable order $S_\rho^{m(x,\xi)}$ have been given in Definition 1.3. In the following lemma, the notation $\mathcal{O}_m\left(S^{-1+0}\right)$ means that the term is a symbol in S^{-1+0}. We add the index m to emphasize that it depends on the escape function m whereas $\mathcal{O}\left(S^0\right)$ means that the term is a symbol in S^0 which *does not* depend on m.

Lemma 3.2. *The operator $\widehat{P}$ defined in (3.2) is a PDO in* $\mathrm{Op}\left(S^1\right)$. *With respect to every given system of coordinates its symbol is equal to*

$$P(x,\xi) = H(x,\xi) + i\left(\mathbf{X}(G_m)\right)(x,\xi) + \mathcal{O}_m\left(S^{-1+0}\right), \tag{3.3}$$

where $H(x,\xi)$ is the symbol of $\widehat{H}$:

$$H(x,\xi) = V(\xi) + \mathcal{O}\left(S^0\right),$$

with principal symbol $V(\xi) \in S^1$ and $\mathbf{X}(G_m) \in S^{+0}$. $\mathbf{X}$ is the Hamiltonian vector field of H defined in (1.8).

Proof. The proof consists in making the following two lines precise and rigorous:

$$\begin{aligned}\widehat{P} &= \widehat{A}\widehat{H}\widehat{A}^{-1} = \mathrm{Op}\left(e^{G_m}\right)\widehat{H}\left(\mathrm{Op}\left(e^{G_m}\right)\right)^{-1}\\ &\simeq \left(1+\mathrm{Op}(G_m)+\ldots\right)\widehat{H}\left(1-\mathrm{Op}(G_m)+\ldots\right)\\ &= \widehat{H} + \left[\mathrm{Op}(G_m), \widehat{H}\right] + \ldots = \mathrm{Op}\left(H - i\{G_m, H\} + \ldots\right) = \mathrm{Op}\left(H + i\mathbf{X}(G_m) + \ldots\right).\end{aligned}$$

In order to avoid to work with exponentials of operators, let us define

$$\widehat{A}_{m,t} := \mathrm{Op}\left(e^{tG_m}\right) = \mathrm{Op}\left(e^{G_{tm}}\right) = \widehat{A}_{tm}, \qquad 0 \le t \le 1,$$

and

$$\widehat{H}_{m,t} := \widehat{A}_{m,t}\widehat{H}\widehat{A}_{m,t}^{-1}$$

which interpolates between $\widehat{H} = \widehat{H}_{m,0}$ and $\widehat{P} = \widehat{H}_{m,1}$. We have $\mathrm{Op}(G_m) \in \mathrm{Op}\left(S^{+0}\right)$, $\widehat{A}_{m,t} \in \mathrm{Op}\left(S^{tm(x,\xi)+0}\right)$, $\widehat{A}_{m,t}^{-1} \in \mathrm{Op}\left(S^{-tm(x,\xi)+0}\right)$, $\widehat{H} \in \mathrm{Op}\left(S^1\right)$. We deduce that[3] $\widehat{H}_{m,t} \in \mathrm{Op}\left(S^{1+0}\right)$. Then

$$\begin{aligned}\frac{d\widehat{A}_{m,t}}{dt} &= \mathrm{Op}\left(G_m e^{tG_m}\right) = \mathrm{Op}(G_m)\,\mathrm{Op}\left(e^{tG_m}\right) + \mathrm{Op}\left(\mathcal{O}_m\left(S^{tm-1+0}\right)\right)\\ \left(\frac{d\widehat{A}_{m,t}}{dt}\right)\widehat{A}_{m,t}^{-1} &= -\widehat{A}_{m,t}\left(\frac{d\widehat{A}_{m,t}^{-1}}{dt}\right) = \mathrm{Op}\left(G_m + r_{m,t}\right)\end{aligned}$$

[3] From the theorem of composition of pseudodifferential operators (PDO), see [49, Prop. (3.3) p. 11], if $A \in S_\rho^{m_1}$ and $B \in S_\rho^{m_2}$ then

$$\mathrm{Op}(A)\,\mathrm{Op}(B) = \mathrm{Op}(AB) + \mathcal{O}\left(\mathrm{Op}(S_\rho^{m_1+m_2-(2\rho-1)})\right),$$

i.e. the symbol of $\mathrm{Op}(A)\,\mathrm{Op}(B)$ is the product AB and belongs to $S_\rho^{m_1+m_2}$ modulo terms in $S_\rho^{m_1+m_2-(2\rho-1)}$.

with $r_{m,t} \in S^{-1+0}$ and[4]

$$\frac{d}{dt}\widehat{H}_{m,t} = \left(\frac{d\widehat{A}_{m,t}}{dt}\widehat{A}_{m,t}^{-1}\right)\widehat{A}_{m,t}\widehat{H}\widehat{A}_{m,t}^{-1} + \widehat{A}_{m,t}\widehat{H}\widehat{A}_{m,t}^{-1}\left(\widehat{A}_{m,t}\frac{d\widehat{A}_{m,t}^{-1}}{dt}\right)$$
$$= \left[\mathrm{Op}\left(G_m + r_{m,t}\right), \widehat{H}_{m,t}\right] \in \mathrm{Op}\left(S^{+0}\right).$$

Therefore $\widehat{H}_{m,t} - \widehat{H} = \left(\int_0^t \frac{d}{ds}\widehat{H}_{m,s}ds\right) \in \mathrm{Op}\left(S^{+0}\right)$ and

$$\frac{d}{dt}\widehat{H}_{m,t} = \left[\mathrm{Op}\left(G_m\right), \widehat{H}\right] + \left[\mathrm{Op}\left(r_{m,t}\right), \widehat{H}\right] + \left[\mathrm{Op}\left(G_m + r_{m,t}\right), \widehat{H}_{m,t} - \widehat{H}\right]$$
$$= \left[\mathrm{Op}\left(G_m\right), \widehat{H}\right] + \mathcal{O}_m\left(\mathrm{Op}\left(S^{-1+0}\right)\right).$$

We deduce that

$$\widehat{P} = \widehat{H} + \left(\int_0^1 \frac{d}{dt}\widehat{H}_{m,t}dt\right) = \widehat{H} + \left[\mathrm{Op}\left(G_m\right), \widehat{H}\right] + \mathcal{O}_m\left(\mathrm{Op}\left(S^{-1+0}\right)\right).$$

Since

$$\left[\mathrm{Op}\left(G_m\right), \widehat{H}\right] = \mathrm{Op}\left(i\left(\mathbf{X}\left(G_m\right)\right)(x,\xi) + \mathcal{O}_m\left(S^{-1+0}\right)\right),$$

we get

$$\widehat{P} = \widehat{H} + \mathrm{Op}\left(i\left(\mathbf{X}\left(G_m\right)\right)(x,\xi) + \mathcal{O}_m\left(S^{-1+0}\right)\right).$$

Finally since $\widehat{H} = \mathrm{Op}\left(V(\xi) + \mathcal{O}\left(S^0\right)\right)$ with a remainder in S^0 which is independent of the escape function m, we get (3.3). □

We have shown that $\widehat{P}$ is a unbounded PDO of order 1 that we may first equip with the domain $C^\infty(X)$ which is dense in $L^2(X)$. Lemma A.1, Sect. A.1 shows that $\widehat{P}$ has a unique closed extension as an unbounded operator $\widehat{P}$ on $L^2(X)$. The adjoint $\widehat{P}^*$ is also a PDO of order 1 and it is the unique closed extension from $C^\infty(X)$.

3.2. $\widehat{P}$ has empty spectrum for $\Im(z)$ large enough. Let us write

$$\widehat{P} = \widehat{P}_1 + i\widehat{P}_2$$

with $\widehat{P}_1 := \frac{1}{2}\left(\widehat{P} + \widehat{P}^*\right)$, $\widehat{P}_2 := \frac{i}{2}\left(\widehat{P}^* - \widehat{P}\right)$ formally self-adjoint. From (3.3) and (1.20), the symbol of the operator $\widehat{P}_2$ is

$$P_2(x,\xi) = \mathbf{X}\left(G_m\right)(x,\xi) + \mathcal{O}\left(S^0\right) + \mathcal{O}_m\left(S^{-1+0}\right) \tag{3.4}$$

which belongs to S^{+0} and satisfies

$$\exists C_0, \forall (x,\xi), \quad \Re\left(P_2(x,\xi)\right) \le C_0.$$

[4] From [49, Eqs. (3.24) (3.25) p. 13], if $A \in S_\rho^{m_1}$ and $B \in S_\rho^{m_2}$ then the symbol of $\left[\mathrm{Op}(A), \mathrm{Op}(B)\right]$ is the Poisson bracket $-i\{A,B\}$ modulo $S_\rho^{m_1+m_2-2(2\rho-1)}$, which belongs to $S_\rho^{m_1+m_2-(2\rho-1)}$. We also recall [48, (10.8) p. 43] that $\{A,B\} = -\mathbf{X}_B(A)$ where $\mathbf{X}_B$ is the Hamiltonian vector field generated by B.

From the sharp Gårding inequality (A.2), Sect. A.2 applied here with order $\mu = 1$ (since $P_2 \in S^{+0} \subset S^1$) we deduce that there exists $C > 0$ such that $\left(\widehat{P_2}u|u\right) \leq (C_0 + C)\|u\|^2$ which is written:

$$\left(\widehat{P_2} - (C_0 + C)\, u|u\right) \leq 0. \tag{3.5}$$

Notice that C and C_0 depend on m a priori.

Lemma 3.3. *From the inequality (3.5) we deduce that for every $z \in \mathbb{C}$, $\Im(z) > C + C_0$, the resolvent $\left(\widehat{P} - z\right)^{-1}$ exists (as a bounded operator on $L^2(X)$). Therefore $\widehat{P}$ has* ***empty spectrum*** *for $\Im(z) > C + C_0$.*

Proof. Let $\varepsilon = \Im(z) - (C_0 + C) > 0$. Then for $u \in C^\infty(X)$,

$$\Im\left(\left(\widehat{P} - z\right) u|u\right) = \left(\left(\widehat{P_2} - (C_0 + C)\right) u|u\right) - (\Im(z) - (C_0 + C))\|u\|^2 \leq -\varepsilon \|u\|^2.$$

By Cauchy-Schwarz inequality,

$$\left\|\left(\widehat{P} - z\right) u\right\| \|u\| \geq \left|\left(\left(\widehat{P} - z\right) u|u\right)\right| \geq \left|\Im\left(\left(\widehat{P} - z\right) u|u\right)\right| \geq \varepsilon \|u\|^2.$$

Hence for $u \in C^\infty(X)$,

$$\left\|\left(\widehat{P} - z\right) u\right\| \geq \varepsilon \|u\|. \tag{3.6}$$

By density this extends to all $u \in \mathcal{D}\left(\widehat{P}\right)$ and it follows that $\widehat{P} - z$ is injective with closed range $\mathcal{R}\left(\widehat{P} - z\right)$.

The same argument for the adjoint $\widehat{P}^* = \widehat{P_1} - i\widehat{P_2}$ gives

$$\left\|\left(\widehat{P}^* - \overline{z}\right) u\right\| \geq \varepsilon \|u\|, \qquad \forall u \in \mathcal{D}\left(\widehat{P}^*\right), \tag{3.7}$$

so $\widehat{P}^* - \overline{z}$ is also injective. If $u \in L^2(X)$ is orthogonal to $\mathcal{R}\left(\widehat{P} - z\right)$ then u belongs to the kernel of $\widehat{P}^* - \overline{z}$ which is 0. Hence $\mathcal{R}\left(\widehat{P} - z\right) = L^2(X)$ and $\widehat{P} - z : \mathcal{D}\left(\widehat{P}\right) \to L^2(X)$ is bijective with bounded inverse. □

3.3. The spectrum of $\widehat{P}$ is discrete on $\Im(z) \geq -(C_m - C)$ with some $C \geq 0$ independent of m. As usual [39, p. 113], in order to obtain a discrete spectrum for the operator $\widehat{P}$, it suffices to construct a relatively compact perturbation $\widehat{\chi}$ of the operator such that $\left(\widehat{P} - i\widehat{\chi}\right)$ has no spectrum on $\Im(z) \geq -(C_m - C)$.

Let $\chi_0 : T^*X \to \mathbb{R}^+$ be a smooth non negative function with $\chi_0(x, \xi) = C_m > 0$ for $(x, \xi) \in N_0$ and $\chi_0(x, \xi) = 0$ outside a neighborhood of N_0 where N_0 is defined in Eq. (1.18), Lemma 1.2. See also Fig. 3(b). We can assume that $\chi_0 \in S^0$.

Let $\widehat{\chi_0} := \mathrm{Op}(\chi_0)$. We can assume that $\widehat{\chi_0}$ is self-adjoint. From Eq. (1.18), for every $(x, \xi) \in T^*X$, $|\xi| \geq R$,

$$(\mathbf{X}(G_m)(x, \xi) - \chi_0(x, \xi)) \leq -C_m,$$

hence (3.4) gives for every $(x, \xi) \in T^*X$:

$$P_2(x, \xi) - \chi_0(x, \xi) \leq -C_m + C + \mathcal{O}_m\left(S^{-1+0}\right), \tag{3.8}$$

with some $C \in \mathbb{R}$ independent of m, coming from the $\mathcal{O}\left(S^0\right)$ term in (3.4). Notice that the remainder term $\mathcal{O}_m\left(S^{-1+0}\right)$ can be bounded by a constant but which depends on m.

Since $P_2 \in S^{\mu}, \forall \mu, 0 < \mu < 1$, the sharp Gårding inequality (A.2) implies that for every $u \in C^{\infty}(X)$ there exists $C_\mu > 0$ (C_μ depends on m a priori) such that

$$\left(\left(\widehat{P_2} - \widehat{\chi}_0 + (C_m - C)\right) u|u\right) \le C_\mu \|u\|^2_{H^{\frac{\mu-1}{2}}}.$$

We have also applied the Calderon Vaillancourt theorem[5] to the remainder term $A \in \mathcal{O}_m\left(S^{-1+0}\right)$ in (3.8) to see that $|(\mathrm{Op}(A)\, u|u)| \le C_A \|u\|^2_{H^{\frac{-1+0}{2}}} \le C'_A \|u\|^2_{H^{\frac{\mu-1}{2}}}$ (since $\frac{-1+0}{2} < \frac{\mu-1}{2}$). The right hand side can be written $C_\mu \|u\|^2_{H^{\frac{\mu-1}{2}}} = C_\mu \left(\left\langle \widehat{\xi}\right\rangle^{\mu-1} u|u\right) = (\widehat{\chi}_1 u|u)$ with $\widehat{\chi}_1 = \mathrm{Op}(\chi_1)$, $\chi_1 = C_\mu \langle \xi \rangle^{\mu-1} \in S^{\mu-1}$ and can be absorbed on the left by defining

$$\chi := \chi_0 + \chi_1, \qquad \widehat{\chi} := \mathrm{Op}(\chi).$$

We can assume that $\widehat{\chi}$ is self-adjoint. We obtain:

$$\left(\left(\widehat{P_2} - \widehat{\chi} + (C_m - C)\right) u|u\right) \le 0.$$

As in the proof of Lemma 3.3, Sect. 3.2, we obtain that the resolvent

$$\left(\widehat{P} - i\widehat{\chi} - z\right)^{-1} : L^2(X) \to L^2(X) \text{ is bounded for } \Im(z) > -(C_m - C). \quad (3.9)$$

The following lemma is the central observation in the proof of Theorem 1.4.

Lemma 3.4. *For every $z \in \mathbb{C}$ such that $\Im(z) > -(C_m - C)$, the operator $\widehat{\chi}\left(\widehat{P} - i\widehat{\chi} - z\right)^{-1} : L^2(X) \to L^2(X)$ is compact.*

Proof. Let $z \in \mathbb{C}$ such that $\Im(z) > -(C_m - C)$. On the cone N_0, the operator $\left(\widehat{P} - i\widehat{\chi} - z\right)$ is elliptic of order 1. We can therefore invert it micro-locally on N_0 (that is its principal symbol $V(\xi)$ is non vanishing away from zero, see for instance [23, Chaps. 6,7]), namely construct $E \in S^{-1}$ and $R_1, R_2 \in S^0$ such that

$$\left(\widehat{P} - i\widehat{\chi} - z\right)\widehat{E} = 1 + \widehat{R}_1, \qquad \widehat{E}\left(\widehat{P} - i\widehat{\chi} - z\right) = 1 + \widehat{R}_2,$$
$$\forall j = 1, 2, \qquad \mathrm{WF}\left(\widehat{R}_j\right) \cap N_0 = \emptyset. \quad (3.10)$$

(The wavefront set of a PDO has been defined in Subsect. 1.1.3). In particular $\mathrm{WF}\left(\widehat{\chi}\widehat{R}_j\right) = \emptyset$, therefore $\widehat{\chi}\widehat{R}_2$ is a compact operator. Also $\widehat{E}$ is a compact operator (since $E \in S^{-1}$). Then from (3.10), we write:

$$\left(\widehat{P} - i\chi - z\right)^{-1} = \widehat{E} - \widehat{R}_2\left(\widehat{P} - i\widehat{\chi} - z\right)^{-1},$$
$$\widehat{\chi}\left(\widehat{P} - i\widehat{\chi} - z\right)^{-1} = \underbrace{\widehat{\chi}}_{bounded}\underbrace{\widehat{E}}_{compact} - \underbrace{\widehat{\chi}\widehat{R}_2}_{compact}\underbrace{\left(\widehat{P} - i\chi\widehat{\chi} - z\right)^{-1}}_{bounded},$$

and deduce that $\widehat{\chi}\left(\widehat{P} - i\widehat{\chi} - z\right)^{-1}$ is a compact operator. □

The following lemma finishes the proof of Theorem 1.4.

[5] If $A \in S^{\mu}$ then there exists $C_A > 0$ such that $\forall u \in C^{\infty}(X)$, $|(\mathrm{Op}(A)\, u|u)| \le C_A \left(\mathrm{Op}\left(\langle \xi \rangle^{\mu}\right) u|u\right) = C_A \|u\|_{H^{\frac{\mu}{2}}}$.

Lemma 3.5. $\widehat{P}$ *has* ***discrete spectrum with locally finite multiplicity*** *on* $\Im(z) > -(C_m - C)$.

Proof. For every $z \in \mathbb{C}$, with $\Im(z) > -(C_m - C)$, we have obtained in (3.9) that $\left(\widehat{P} - i\widehat{\chi} - z\right)$ is invertible. We write:

$$\widehat{P} - z = \left(1 + i\widehat{\chi}\left(\widehat{P} - i\widehat{\chi} - z\right)^{-1}\right)\left(\widehat{P} - i\widehat{\chi} - z\right).$$

Here $\left(\widehat{P} - i\widehat{\chi} - z\right) : \mathcal{D}\left(\widehat{P}\right) \to L^2(X)$ is bijective with bounded inverse and hence Fredholm of index 0. Similarly $\left(1 + i\widehat{\chi}\left(\widehat{P} - i\widehat{\chi} - z\right)^{-1}\right) : L^2(X) \to L^2(X)$ is Fredholm of index 0 by Lemma 3.4. Thus

$$\widehat{P} - z : \mathcal{D}\left(\widehat{P}\right) \to L^2(X), \qquad \Im(z) > C_m - C,$$

is a holomorphic family of Fredholm operators (of index 0) invertible for $\Im(z) \gg 0$. It then suffices to apply the analytic Fredholm theorem ([38, p. 201, case (b)], see also [25, p. 220 App. A]). □

4. Proof of Theorem 1.5 that the Eigenvalues are Intrinsic to the Anosov Vector Field V

Let m and G_m be as in Lemma 1.2. Let $\widehat{m} = f(m)$, where $f \in C^\infty(\mathbb{R})$, $f(t) \geq \max(0,t)$, $f' \geq 0$, $f(t) = 0$ for $t \leq u/2$ and $f(t) = t$ for $t \geq s/2$. $\widehat{H}$ viewed as a closed unbounded operator in $H^{\widehat{m}}$ has no spectrum in the half plane $\Im(z) \geq C_1$ for $C_1 \gg 0$. The same holds for $\widehat{H} : L^2 \to L^2$. Since $\widehat{m} \geq 0$ we have $H^{\widehat{m}} \subset L^2$ so if $v \in H^{\widehat{m}}$ then $R_{L^2}(z)v = R_{H^{\widehat{m}}}(z)v$ for $\Im(z) \geq C_1$, where R_{L^2} denotes the resolvent of $\widehat{H} : L^2 \to L^2$ and similarly for $R_{H^{\widehat{m}}}$.

Since $\widehat{m} \geq m$ we also have $H^{\widehat{m}} \subset H^m$, and hence $R_{H^{\widehat{m}}}(z)v = R_{H^m}(z)v$ for $\Im(z) \geq C_1$, $v \in H^{\widehat{m}}$. Especially when $v \in C^\infty$, we get $R_{L^2}(z)v = R_{H^m}(z)v$, $\Im(z) \geq C_1$. Applying Theorem 1.4, we conclude that $R_{L^2}(z)$, viewed as an operator $C^\infty \to \mathcal{D}'$ has a meromorphic extension $R(z)$ from the half plane $\Im(z) \geq C_1$ to the half plane $\Im(z) > -(C_m - C)$ and this extension coincides with R_{H^m} restricted to C^∞.

If γ is a simple positively oriented closed curve in the half plane $\Im(z) > -(C_m - C)$ which avoids the eigenvalues of $\widehat{H} : H^m \to H^m$, then the spectral projection, associated to the spectrum of $\widehat{H}$ inside γ, is given by

$$\pi_\gamma^{H^m} = \frac{1}{2\pi i}\int_\gamma R_{H^m}(z)\,dz.$$

For $v \in C^\infty$, we have

$$\pi_\gamma^{H^m} v = \pi_\gamma v := \frac{1}{2\pi i}\left(\int_\gamma R(z)\,dz\right)v.$$

Now C^∞ is dense in H^m and $\pi_\gamma^{H^m}$ is of finite rank, hence its range is equal to the image $\pi_\gamma(C^\infty)$ of C^∞. The latter space is independent of the choice of H^m. More precisely if $\widetilde{m}$, $\widetilde{f}$ are as in Theorem 1.4 and we choose γ as above, now in the half plane $\Im(z) > -\min((C_m - C), (C_{\widetilde{m}} - C))$ and avoiding the spectra of $\widehat{H} : H^m \to H^m$ and $\widehat{H} : H^{\widetilde{m}} \to H^{\widetilde{m}}$, then the spectral projections $\pi_\gamma^{H^m}$ and $\pi_\gamma^{H^{\widetilde{m}}}$ have the same range. Theorem 1.5 follows.

5. Proof of Theorem 1.8 for the Upper Bound on the Density of Resonances

The asymptotic regime $\Re(z) \gg 1$ which is considered in Theorem 1.8 is a semiclassical regime in the sense that it involves large values of $H(\xi) = V(\xi) \gg 1$, hence large values of $|\xi|$ in the cotangent space T^*X.

For convenience, we will switch to h-semiclassical analysis. Let $0 < h \ll 1$ be a small parameter (we will set $E = 1/h$ in Theorem 1.8). In h-semiclassical analysis a symbol $a(x,\xi)$ is quantized according to

$$Op_h(a(x,\xi)) = \mathrm{Op}(a(x,h\xi)), \tag{5.1}$$

where Op has been defined in (1.23).

In this section we first recall the definition of symbols in h-semiclassical analysis. In Lemma 5.3 we derive the symbol of the operator $\widehat{P}$ with respect to this new calculus. In Sect. 5.3 we give the main idea of the proof and the next sections give the details.

5.1. Semiclassical symbols. We first define the symbol classes we will need in h-semiclassical analysis. We will use calligraphic symbols $\mathcal{S}$ to distinguish these from the symbols of the homogeneous theory S defined in (1.21).

Definition 5.1. *The* **symbol class** $\left(h^{-k}\mathcal{S}_\rho^\mu\right)$ *with* $1/2 < \rho \le 1$*, order* $\mu \in \mathbb{R}$ *and* $k \in \mathbb{R}$ *consists of* C^∞ *functions* $p(x,\xi;h)$ *on* T^*X*, indexed by* $0 < h \ll 1$ *such that in every trivialization* $(x,\xi): T^*X|_U \to \mathbb{R}^{2n}$*, for every compact* $K \subset U$*,*

$$\forall \alpha,\beta,\quad \left|\partial_\xi^\alpha \partial_x^\beta p\right| \le C_{K,\alpha,\beta} h^{-k+(\rho-1)(|\alpha|+|\beta|)} \langle\xi\rangle^{\mu-\rho|\alpha|+(1-\rho)|\beta|}. \tag{5.2}$$

For short we will write $\mathcal{S}_\rho^\mu$ instead of $\left(h^{-k}\mathcal{S}_\rho^\mu\right)$ when $k=0$, and write $\mathcal{S}^\mu$ instead of $\mathcal{S}_\rho^\mu$ when $\rho = 1$.

For symbols of variable orders we have:

Definition 5.2. *Let* $m(x,\xi) \in S^0$, $\frac{1}{2} < \rho \le 1$ *and* $k \in \mathbb{R}$*. A family of functions* $p(x,\xi;h) \in C^\infty(T^*X)$ *indexed by* $0 < h \ll 1$*, belongs to the* **class** $\left(h^{-k}\mathcal{S}_\rho^{m(x,\xi)}\right)$ **of variable order** *if in every trivialization* $(x,\xi): T^*X|_U \to \mathbb{R}^{2n}$*, for every compact* $K \subset U$ *and all multi-indices* $\alpha,\beta \in \mathbb{N}^n$*, there is a constant* $C_{K,\alpha,\beta}$ *such that*

$$\left|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)\right| \le C_{K,\alpha,\beta} h^{-k-(1-\rho)(|\alpha|+|\beta|)} \langle\xi\rangle^{m(x,\xi)-\rho|\alpha|+(1-\rho)|\beta|}, \tag{5.3}$$

for every $(x,\xi) \in T^*X|_U$.

5.2. The symbol of the conjugated operator. In the 1-quantization we have seen in (1.22) that $\hat{H} = \mathrm{Op}\left(V(\xi) + \frac{i}{2}\mathrm{div}(V)\right)$. We rescale the spectral domain $z \in \mathbb{C}$ by defining:

$$z_h := hz, \qquad \widehat{H}_h := h\widehat{H}. \tag{5.4}$$

From (5.1) we get

$$\begin{aligned}\widehat{H}_h &= \mathrm{Op}\left(V(h\xi) + h\frac{i}{2}\mathrm{div}(V)\right)\\ &= \mathrm{Op}_h\left(V(\xi) + \mathcal{O}\left(hS^0\right)\right) \in \mathrm{Op}_h\left(\mathcal{S}^1\right).\end{aligned}$$

From now on we will work with these new variables and **we will often drop the indices** h for short.

We will take again the escape function to be $G_m(x,\xi) := m(x,\xi)\log\sqrt{1+(f(x,\xi))^2}$ as in (1.17) but quantized by the h-quantization giving the h-PDO $\mathrm{Op}_h(G_m)$ with h-semiclassical symbol $G_m \in \mathcal{S}^{+0}$. Also

$$\widehat{A}_m := \mathrm{Op}_h(\exp(G_m))$$

is a h-PDO with symbol $A_m \in \mathcal{S}^{m(x,\xi)}_{1-0}$ (the invertibility of $\widehat{A}_m$ is automatic if h is small enough). Notice that the Sobolev space defined now by

$$H^m_h := \widehat{A}_m^{-1}\left(L^2(X)\right) = \left(\mathrm{Op}_h(A_m)\right)^{-1}\left(L^2(X)\right)$$

is identical[6] to (1.24) as a linear space. However the norm in H^m_h depends on h.

In the following lemma, we will use again the notation $\mathcal{O}_m\left(h^2\mathcal{S}^{-1+0}\right)$ which means that the term is a symbol in $h^2\mathcal{S}^{-1+0}$. We add the subscript m to emphasize that it depends on the escape function m whereas $\mathcal{O}\left(h\mathcal{S}^0\right)$ means that the term is a symbol in $h\mathcal{S}^0$ which *does not* depend on m.

Lemma 5.3. *We define*

$$\widehat{P} := \widehat{A}_m\widehat{H}\widehat{A}_m^{-1},$$

as in (3.2). Its symbol $P \in \mathcal{S}^1$ is

$$P(x,\xi) = V(\xi) + ih\mathbf{X}(G_m)(x,\xi) + \mathcal{O}\left(h\mathcal{S}^0\right) + \mathcal{O}_m\left(h^2\mathcal{S}^{-1+0}\right). \tag{5.5}$$

Proof. The proof is very similar to the proof of Lemma 3.2. Let us define

$$\widehat{A}_{m,t} := \mathrm{Op}_h\left(e^{tG_m}\right) = \mathrm{Op}_h\left(e^{G_{tm}}\right) = \widehat{A}_{tm}, \qquad 0 \le t \le 1$$

and

$$\widehat{H}_{m,t} := \widehat{A}_{m,t}\widehat{H}\widehat{A}_{m,t}^{-1},$$

which interpolates between $\widehat{H} = \widehat{H}_{m,0}$ and $\widehat{P} = \widehat{H}_{m,1}$. We have[7] $\widehat{A}_{m,t} \in \mathrm{Op}_h\left(\mathcal{S}^{tm+0}\right)$, $\widehat{A}_{m,t}^{-1} \in \mathrm{Op}_h\left(\mathcal{S}^{-tm+0}\right)$, $\widehat{H} \in \mathrm{Op}_h\left(\mathcal{S}^1\right)$, therefore $\widehat{H}_{m,t} \in \mathrm{Op}_h\left(\mathcal{S}^{1+0}\right)$. Then

$$\left(\frac{d}{dt}\widehat{A}_{m,t}\right)\widehat{A}_{m,t}^{-1} = -\widehat{A}_{m,t}\left(\frac{d}{dt}\widehat{A}_{m,t}^{-1}\right) = \mathrm{Op}_h\left(G_m + r_{m,t}\right)$$

---

[6] Let us show that the space $H^m_h := \left(\mathrm{Op}_h(A_m)\right)^{-1}\left(L^2(X)\right)$ using h-quantization does not depend on h and is therefore identical to (1.24) obtained with $h = 1$. We have $\mathrm{Op}_h(A_m) \in \mathrm{Op}\left(S^m_\rho\right)$ (symbol class without h). Therefore $\hat{B} := \mathrm{Op}_h(A_m)\left(\mathrm{Op}_{h=1}(A_m)\right)^{-1} \in \mathrm{Op}\left(S^0\right)$ is invertible on L^2 with continuous inverse: $\hat{B}\left(L^2\right) = L^2$. Hence $H^m_h := \left(\mathrm{Op}_h(A_m)\right)^{-1}\left(\hat{B}L^2(X)\right) = \left(\mathrm{Op}_{h=1}(A_m)\right)^{-1}\left(L^2(X)\right) = H^m_{h=1}$.

[7] The theorem of composition of h-semiclassical PDO [17] says that if $A \in \mathcal{S}^{m_1}_\rho$ and $B \in \mathcal{S}^{m_2}_\rho$, then the symbol of $\mathrm{Op}_h(A)\,\mathrm{Op}_h(B)$ is the product AB and belongs to $\mathcal{S}^{m_1+m_2}_\rho$ modulo $h\mathcal{S}^{m_1+m_2-(2\rho-1)}_\rho$.

with $r_{m,t} \in h\mathcal{S}^{-1+0}$ and

$$\frac{d}{dt}\widehat{H}_{m,t} = \left[\mathrm{Op}_h\left(G_m + r_{m,t}\right), \widehat{H}_{m,t}\right].$$

We deduce that $\frac{d}{dt}\widehat{H}_{m,t} \in \mathrm{Op}_h\left(h\mathcal{S}^{+0}\right)$, therefore $\widehat{H}_{m,t} - \widehat{H} = \left(\int_0^t \frac{d}{ds}\widehat{H}_{m,s}ds\right) \in \mathrm{Op}_h\left(h\mathcal{S}^{+0}\right)$ also and

$$\begin{aligned}\frac{d}{dt}\widehat{H}_{m,t} &= \left[\mathrm{Op}_h\left(G_m\right), \widehat{H}\right] + \left[\mathrm{Op}_h\left(r_{m,t}\right), \widehat{H}\right] + \left[\mathrm{Op}_h\left(G_m + r_{m,t}\right), \widehat{H}_{m,t} - \widehat{H}\right] \\ &= \left[\mathrm{Op}_h\left(G_m\right), \widehat{H}\right] + \mathcal{O}_m\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-1+0}\right)\right).\end{aligned}$$

We deduce that

$$\widehat{P} = \widehat{H} + \left(\int_0^1 \frac{d}{dt}\widehat{H}_{m,t}dt\right) = \widehat{H} + \left[\mathrm{Op}_h\left(G_m\right), \widehat{H}\right] + \mathcal{O}_m\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-1+0}\right)\right).$$

Since[8]

$$\left[\mathrm{Op}\left(G_m\right), \widehat{H}\right] = \mathrm{Op}_h\left(ih\left(\mathbf{X}\left(G_m\right)\right)(x,\xi) + \mathcal{O}_m\left(h^2\mathcal{S}^{-1+0}\right)\right),$$

we get

$$\widehat{P} = \widehat{H} + \mathrm{Op}_h\left(ih\left(\mathbf{X}\left(G_m\right)\right)(x,\xi) + \mathcal{O}_m\left(h^2\mathcal{S}^{-1+0}\right)\right).$$

Finally, since $\widehat{H} = \mathrm{Op}_h\left(V(\xi) + \mathcal{O}\left(h\mathcal{S}^0\right)\right)$ with a remainder in $h\mathcal{S}^0$ which is independent of the escape function m, we get (5.5). □

We recall the main properties of the different terms in (5.5). First $V(\xi) \in \mathcal{S}^1$ is real. In each fiber T_x^*X, $V(\xi)$ is linear in ξ and for every $E \in \mathbb{R}$ the characteristic set $\Sigma_E := \{(x,\xi), V(\xi) - E = 0\}$ is the energy shell defined in (1.14) and transverse to E_0^*.

The second term $ih\mathbf{X}(G_m) \in h\mathcal{S}^{+0}$ is purely imaginary and we recall some properties of $\mathbf{X}(G_m)$ obtained in Lemma 1.2:

$$\mathbf{X}\left(G_m\right)(x,\xi) \text{ is } \begin{cases} \leq 0 & \text{for } |\xi| \geq R \\ \leq \mathcal{O}_m(1) & \text{for } |\xi| < R \\ \leq -C_m, & C_m > 0, \text{ for } (x,\xi) \notin \left(D_R \cup N_0\right), \end{cases} \tag{5.6}$$

where $D_R = \{\xi, |\xi| \leq R\}$ and N_0 is defined in Lemma 1.2. With a convenient choice of the order function $m(x,\xi)$ we have:

$$\begin{cases} N_0 & \text{arbitrarily small conical vicinity of } E_0^*, \\ C_m > 0 & \text{arbitrarily large.} \end{cases} \tag{5.7}$$

[8] If $A \in \mathcal{S}_\rho^{m_1}$ and $B \in \mathcal{S}_\rho^{m_2}$, then the symbol of $\left[\mathrm{Op}_h(A), \mathrm{Op}_h(B)\right]$ is the Poisson bracket $-ih\{A, B\} = ih\mathbf{X}_B(A)$ and belongs to $h\mathcal{S}_\rho^{m_1+m_2-(2\rho-1)}$ modulo $h^2\mathcal{S}_\rho^{m_1+m_2-2(2\rho-1)}$. Here $\mathbf{X}_B$ is the Hamiltonian vector field generated by B.

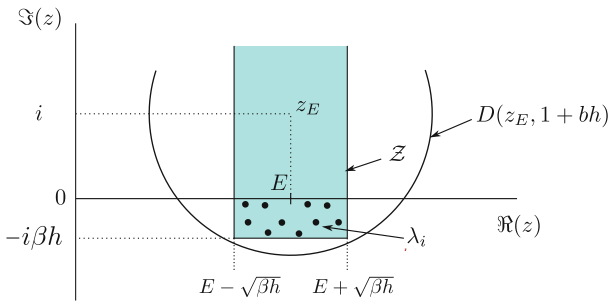

Fig. 7. The objective is to bound from above the number of eigenvalues λ_i in the domain $\mathcal{Z}_\beta$. For that purpose, we will bound the number of resonances in the disk of radius $1 + bh$ and center $z_E = E + i$

5.3. Main idea of the proof. Before giving the details of the proof we give here the main arguments that we will use in order to prove (1.25).

Let us consider the following complex valued function $\widetilde{p}(x, \xi) \in \mathcal{S}^1$ made from the first two leading terms of the symbol (5.5):

$$\widetilde{p}(x, \xi) := V(\xi) + ih\mathbf{X}(G_m). \tag{5.8}$$

Let $E > 0$, $h \ll 1$ and $\beta > 0$. We define the spectral domain $\mathcal{Z} \subset \mathbb{C}$ by:

$$\mathcal{Z} := \left\{ \lambda \in \mathbb{C}, \quad |\Re(\lambda) - E| \leq \sqrt{\beta h}, \quad \Im(\lambda) \geq -\beta h \right\}. \tag{5.9}$$

See Fig. 7. Let

$$\mathcal{V}_{\mathcal{Z}} := \left\{ (x, \xi) \in T^* X, \quad \widetilde{p}(x, \xi) \in \mathcal{Z} \right\}. \tag{5.10}$$

We have from (5.8), (5.6) and assuming $C_m > \beta$,

$$(x, \xi) \in \mathcal{V}_{\mathcal{Z}} \Leftrightarrow \begin{cases} |V(\xi) - E|^2 \leq \beta h \\ h\mathbf{X}(G_m)(x, \xi) \geq -\beta h \end{cases} \Rightarrow \begin{cases} (x, \xi) \in \Sigma_{[E-\sqrt{\beta h}, E+\sqrt{\beta h}]} \\ (x, \xi) \in (D_R \cup N_0) \end{cases}, \tag{5.11}$$

where $\Sigma_{[E-\sqrt{\beta h}, E+\sqrt{\beta h}]} := \left(\bigcup_{|E'-E| \leq \sqrt{\beta h}} \Sigma_{E'} \right)$ is a union of energy shells (1.14). We deduce that the symplectic volume of $\mathcal{V}_{\mathcal{Z}}$ is

$$\mathrm{Vol}(\mathcal{V}_{\mathcal{Z}}) \leq C\, \mathrm{Vol}(N_0 \cap \Sigma_E) \sqrt{h}, \tag{5.12}$$

with some constant $C > 0$ (independent of E and m). See Fig. 8. (Notice that $\mathrm{Vol}(N_0 \cap \Sigma_E)$ is a "Liouville volume" inherited from the symplectic volume on T^*X and the energy function H). We suppose that $E > R$ and that h is small enough so that $\Sigma_{[E-\sqrt{\beta h}, E+\sqrt{\beta h}]} \bigcap D_R = \emptyset$.

Using a max-min formula and Weyl inequalities we will obtain an upper bound for the number of eigenvalues in terms of this upper bound (and choosing $C_m > 4\beta$):

$$\sharp \{\lambda_i \in \mathcal{Z}\} \leq \frac{C_m \mathrm{Vol}(\mathcal{V}_{\mathcal{Z}})}{h^n} = C_m C\, \mathrm{Vol}(X)\, \mathrm{Vol}(N_0 \cap \Sigma_E)\, h^{1/2-n}.$$

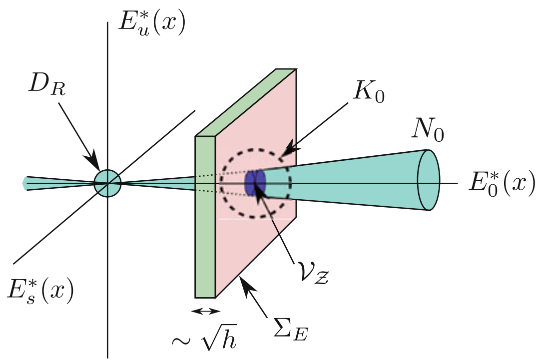

Fig. 8. Picture in T_x^*X with $x \in X$, of the volume $\mathcal{V}_Z$ which supports micro-locally the eigenvalues $\lambda_i \in \mathcal{Z}$ of Fig. 7

Using that Vol $(N_0 \cap \Sigma_E)$ can be chosen arbitrarily small, from Eq. (5.7), we deduce that

$$\sharp \{\lambda_i \in \mathcal{Z}\} \leq o\left(h^{1/2-n}\right),$$

which is precisely (1.25) with $\alpha = 1/h$.

The proof below follows these ideas but is not so simple because $P(x, \xi)$ in Eq. (5.5) is a symbol and not simply a function (symbols belong to a non commutative algebra of star product) and because the term $h\mathbf{X}(G_m)$ is subprincipal. We will have to decompose the phase space T^*X in different parts in order to separate the different contributions as in (5.11). Another technical difficulty is that the width of the volume $\mathcal{V}_{\mathcal{Z}}$ is of order $\sqrt{h}$. We will use FBI quantization which is convenient for a control on phase space at the scale $\sqrt{h}$.

5.4. Proof of Theorem 1.8. We present in reverse order the main steps we will follow in the proof.

Steps of the proof:

- Our purpose is to bound the cardinal of the spectrum $\sigma\left(\widehat{P}\right)$ of the operator $\widehat{P}$ in the rectangular domain $\mathcal{Z}_\beta$ given by (5.9). But as suggested by Fig. 7 and confirmed by Lemma 5.4 below, it suffices to bound the number of eigenvalues of $\widehat{P}$ in the disk

$$D(z_E, 1+bh) := \{z \in \mathbb{C}, \quad |z - z_E| \leq (1+bh)\}, \qquad b > 0,$$

with radius $(1+bh)$ and center:

$$z_E := E + i \in \mathbb{C}.$$

Lemma 5.4. *If $b \geq 2\beta$ and h small enough then*

$$\left(\sigma\left(\widehat{P}\right) \bigcap \mathcal{Z}_\beta\right) \subset D(z_E, 1+bh).$$

Proof. We know that $z \in \sigma\left(\widehat{P}\right) \Rightarrow \Im(z) \leq 0$. Also, the Pythagoras Theorem in the corner of $\mathcal{Z}$ gives the condition $(1+bh)^2 \geq (1+\beta h)^2 + \left(\sqrt{\beta h}\right)^2$ which is fulfilled if $b \geq 2\beta$ and h small enough. □

- In order to bound the number of eigenvalues of $\widehat{P}$ in the disk $D(z_E, 1+bh)$, we will use Weyl inequalities in Corollary 5.9 and a bound for the number of small singular values of the operator $\left(\widehat{P} - z_E\right)$ (i.e. eigenvalues of $\left(\widehat{P} - z_E\right)^* \left(\widehat{P} - z_E\right)$) obtained in Lemma 5.8.
- In order to get this bound on singular values, we will bound from below the expressions $\left\| \left(\widehat{P} - z_E\right) u \right\|^2 = \left(\left(\widehat{P} - z_E\right)^* \left(\widehat{P} - z_E\right) u | u \right)$. From symbolic calculus (see footnote 7) we can compute the symbol of this operator and get:

$$\begin{aligned}
\left(\widehat{P} - z_E\right)^* \left(\widehat{P} - z_E\right) &= \mathrm{Op}\left(|V(\xi) - E|^2 + |1 - h\mathbf{X}(G_m)|^2 \right) && (5.13)\\
&\quad + \mathrm{Op}\left(\mathcal{O}\left(h\mathcal{S}^1\right) + \mathcal{O}_m\left(h^2 \mathcal{S}^{+0}\right) \right) && (5.14)\\
&= \mathrm{Op}\left(|V(\xi) - E|^2 + 1 - 2h\mathbf{X}(G_m) + \mathcal{O}\left(h\mathcal{S}^1\right) \right.\\
&\quad \left. + \mathcal{O}_m\left(h^2 \mathcal{S}^{+0}\right) \right).
\end{aligned}$$

However it is not possible to deduce directly estimates from this symbol because for large $|\xi|$ the remainders $\mathcal{O}\left(h\mathcal{S}^1\right)$ and $\mathcal{O}_m\left(h^2\mathcal{S}^{+0}\right)$ may dominate the important term $2h\mathbf{X}(G_m) \in h\mathcal{S}^{+0}$. Therefore we first have to perform a partition of unity on phase space.

Partition of unity on phase space. Let $K_0 \subset T^*X$ be a compact subset (independent of h) such that $\mathcal{V}_{\mathcal{Z}} \subset K_0$ with $\mathcal{V}_{\mathcal{Z}}$ defined in (5.10). See Fig. 8. Lemma A.3 associates a "quadratic partition of unity of PDO" to the compact set K_0: there exist symbols $\chi_0 \in \mathcal{S}^{-\infty}$ and $\chi_1 \in \mathcal{S}^0$ of self-adjoint operators $\widehat{\chi}_0, \widehat{\chi}_1$ such that

$$\boxed{\widehat{\chi}_0^2 + \widehat{\chi}_1^2 = 1 + \mathrm{Op}\left(h^\infty \mathcal{S}^{-\infty}\right)} \tag{5.15}$$

supp (χ_0) is compact and the compact set K_0, $\chi_0 = 1 + \mathcal{O}(h^\infty)$, $\chi_1 = \mathcal{O}(h^\infty)$. Then from Lemma A.4 called the "IMS localization formula" we have: for every $u \in L^2(X)$,

$$\boxed{\left\| \left(\widehat{P} - z_E\right) u \right\|^2 = \left\| \left(\widehat{P} - z_E\right) \widehat{\chi}_0 u \right\|^2 + \left\| \left(\widehat{P} - z_E\right) \widehat{\chi}_1 u \right\|^2 + \mathcal{O}\left(h^2\right) \|u\|^2.} \tag{5.16}$$

Notice that the remainder is of order h^2, that is of order one higher than one would expect at first sight. We will now study the different terms of (5.16) separately.

Informal remark.. In order to show that Lemma 5.5 below is expected, let us give an informal remark (not necessary for the proof). Using the function $\widetilde{p}(x,\xi) := V(\xi) + ih\mathbf{X}(G_m)$, as in (5.8), which is the dominant term of the symbol $P(x,\xi)$, we write:

$$\begin{aligned}
|\widetilde{p}(x,\xi) - z_E|^2 &= |V(\xi) - E|^2 + |1 - h\mathbf{X}(G_m)|^2 && (5.17)\\
&= |V(\xi) - E|^2 + 1 - 2h\mathbf{X}(G_m) + \mathcal{O}\left(h^2\mathcal{S}^{+0}\right). && (5.18)
\end{aligned}$$

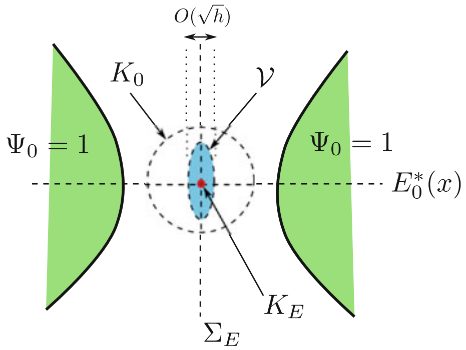

Fig. 9. Picture in $T_x^* X$ with $x \in X$, which shows the partition of unity of phase space used in the proof of Lemma 5.5. The h-essential support of χ_1 is outside the set K_0

If $(x, \xi) \notin K_0$ there are two cases, according to (5.6):

1. $\mathbf{X}(G_m)(x, \xi) \le -C_m$. Then
$$|\widetilde{p}(x, \xi) - z_E|^2 \ge 1 + 2hC_m.$$
2. $|V(\xi) - E|^2 \ge C_0 > 0$ and $\mathbf{X}(G_m) \le \mathcal{O}(1)$ from (5.6). Then
$$|\widetilde{p}(x, \xi) - z_E|^2 \ge 1 + C_0 + \mathcal{O}(h).$$

In both cases we have
$$|\widetilde{p}(x, \xi) - z_E|^2 \ge 1 + 2hC_m. \tag{5.19}$$

Since χ_1 is negligible on K_0, the following Lemma 5.5 is not surprising in the light of property (5.19). It gives a lower bound for the second term in the right side of (5.16).

Lemma 5.5. *For every $u \in L^2(X)$,*
$$\left\| \left(\widehat{P} - z_E\right) \widehat{\chi}_1 u \right\|^2 \ge (1 + 2h(C_m - C)) \|\widehat{\chi}_1 u\|^2 - \mathcal{O}\left(h^\infty\right) \|u\|^2. \tag{5.20}$$

Proof. In order to prove (5.20) we have to consider a partition of unity in order to take into account two contributions as in the discussion after (5.17). Let $\Psi_0 \in \mathcal{S}^0$ which has its support inside the region where $\chi_1 = 1$ and we set $\Psi_0 = 1$ away from a conical neighborhood of the energy shell Σ_E, Eq. (1.14), which is the characteristic set $V(\xi) - E = 0$. See Fig. 9.

Since $(V(\xi) - E)$ is the principal symbol of $(P(x, \xi) - E) \in \mathcal{S}^1$ and is non vanishing on the support of Ψ_0, there exists $\widehat{Q} \in \mathrm{Op}\left(\mathcal{S}^{-1}\right)$ such that
$$\widehat{Q}\left(\widehat{P} - E\right) = \widehat{\Psi}_0 + \widehat{R}, \qquad \widehat{R} \in \mathrm{Op}\left(h^\infty \mathcal{S}^{-\infty}\right).$$

Since $\widehat{Q}$ is continuous in $L^2(X)$, there exists $C_0 > 0$ such that for every $v \in L^2(X)$, $\|v\|^2 \ge \frac{1}{C_0} \left\|\widehat{Q} v\right\|^2$, hence for every $u \in L^2(X)$,
$$\begin{aligned}\left\| \left(\widehat{P} - E\right) \widehat{\chi}_1 u \right\|^2 &\ge \frac{1}{C_0} \left\| \widehat{Q}\left(\widehat{P} - E\right) \widehat{\chi}_1 u \right\|^2 = \frac{1}{C_0} \left\| \left(\widehat{\Psi}_0 + \widehat{R}\right) \widehat{\chi}_1 u \right\|^2 \\ &\ge \frac{1}{2C_0} \left\| \widehat{\Psi}_0 \widehat{\chi}_1 u \right\|^2 - \mathcal{O}\left(h^\infty\right) \|u\|^2. \end{aligned} \tag{5.21}$$

Writing $\widehat{P} = \widehat{P}_1 + i\widehat{P}_2$ with $\widehat{P}_i$ self-adjoint, we have

$$\left\|\left(\widehat{P} - z_E\right)\widehat{\chi}_1 u\right\|^2 = \left(\left(\left(\widehat{P} - E\right)^* + i\right)\left(\left(\widehat{P} - E\right) - i\right)\widehat{\chi}_1 u|\widehat{\chi}_1 u\right)$$
$$= \left\|\left(\widehat{P} - E\right)\widehat{\chi}_1 u\right\|^2 + \|\widehat{\chi}_1 u\|^2 - \left(2\widehat{P}_2\widehat{\chi}_1 u|\widehat{\chi}_1 u\right). \quad (5.22)$$

Using (5.21) in (5.22) we get for every $a > 0$:

$$\begin{aligned}\left\|\left(\widehat{P} - z_E\right)\widehat{\chi}_1 u\right\|^2 &= (1-ah)\left\|\left(\widehat{P} - E\right)\widehat{\chi}_1 u\right\|^2 + ah\left\|\left(\widehat{P} - E\right)\widehat{\chi}_1 u\right\|^2 + \|\widehat{\chi}_1 u\|^2 \\ &\quad - \left(2\widehat{P}_2\widehat{\chi}_1 u|\widehat{\chi}_1 u\right) \\ &\geq (1-ah)\left\|\left(\widehat{P} - E\right)\widehat{\chi}_1 u\right\|^2 + \|\widehat{\chi}_1 u\|^2 - \left(2\widehat{P}_2\widehat{\chi}_1 u|\widehat{\chi}_1 u\right) \\ &\quad + \frac{ah}{2C_0}\left\|\widehat{\Psi}_0\widehat{\chi}_1 u\right\|^2 - \mathcal{O}\left(h^\infty\right)\|u\|^2 \\ &= (1-ah)\left\|\left(\widehat{P} - E\right)\widehat{\chi}_1 u\right\|^2 + \|\widehat{\chi}_1 u\|^2 \\ &\quad + \left(\left(-2\widehat{P}_2 + \frac{ah}{2C_0}\widehat{\Psi}_0^*\widehat{\Psi}_0\right)\widehat{\chi}_1 u|\widehat{\chi}_1 u\right) - \mathcal{O}\left(h^\infty\right)\|u\|^2.\end{aligned} \quad (5.23)$$

Recall from (5.5) that

$$\widehat{P}_2 = \mathrm{Op}\left(h\mathbf{X}\left(G_m\right) + \mathcal{O}\left(h\mathcal{S}^0\right)\right) \in \mathrm{Op}\left(h\mathcal{S}^{+0}\right).$$

Therefore

$$\left(-2\widehat{P}_2 + \frac{ah}{2C_0}\widehat{\Psi}_0^*\widehat{\Psi}_0\right) \in \mathrm{Op}\left(h\mathcal{S}^{+0}\right).$$

Assume $a \geq 4C_0\left(C_m - C\right)$. Then from (5.6) and the hypothesis on Ψ_0, for every $(x,\xi) \in \mathrm{supp}\left(\chi_1\right)$ we have

$$\left(-2P_2 + \frac{ah}{2C_0}\Psi_0^*\Psi_0\right)(x,\xi) \geq \min\left(2h\left(C_m - C\right), \frac{ah}{2C_0}\right) \geq 2h\left(C_m - C\right).$$

We can add a symbol $\Psi_1 \in \mathcal{S}^0$ positive, which vanishes on $\mathrm{supp}\left(\chi_1\right)$ so that $\widehat{\Psi}_1\widehat{\chi}_1 \in \mathrm{Op}\left(h^\infty\mathcal{S}^{-\infty}\right)$ and such that for *every* $(x,\xi) \in T^*X$ we have

$$\left(-2P_2 + \frac{ah}{2C_0}\Psi_0^*\Psi_0 + \Psi_1\right)(x,\xi) \geq \min\left(2h\left(C_m - C\right), \frac{ah}{2C_0}\right) \geq 2h\left(C_m - C\right).$$

The semiclassical sharp Gårding inequality implies that:

$$\forall u \in L^2(X), \quad \left(\left(-2\widehat{P}_2 + \frac{ah}{2C_0}\widehat{\Psi}_0^*\widehat{\Psi}_0\right)\widehat{\chi}_1 u|\widehat{\chi}_1 u\right) \geq \left(2h\left(C_m - C\right) - \mathcal{O}\left(h^2\right)\right)\|\widehat{\chi}_1 u\|^2$$
$$-\mathcal{O}\left(h^\infty\right)\|u\|^2,$$

where the remainder term $\mathcal{O}(h^\infty)\|u\|^2$ comes from $\left(\widehat{\Psi}_1\widehat{\chi}_1 u|\widehat{\chi}_1 u\right)$. With (5.23) we get:

$$\begin{aligned}\left\|\left(\widehat{P}-z_E\right)\widehat{\chi}_1 u\right\|^2 &\geq (1-ah)\left\|\left(\widehat{P}-E\right)\widehat{\chi}_1 u\right\|^2+\|\widehat{\chi}_1 u\|^2\\ &\quad+\left(2h\left(C_m-C\right)-\mathcal{O}\left(h^2\right)\right)\|\widehat{\chi}_1 u\|^2-\mathcal{O}\left(h^\infty\right)\|u\|^2\\ &\geq\left(1+2h\left(C_m-C\right)-\mathcal{O}\left(h^2\right)\right)\|\widehat{\chi}_1 u\|^2-\mathcal{O}\left(h^\infty\right)\|u\|^2.\end{aligned}$$

The term $\mathcal{O}\left(h^2\right)\|\widehat{\chi}_1 u\|^2$ can be absorbed in the constant C. □

Lemma 5.6. *We have*

$$\left\|\left(\widehat{P}-z_E\right)\widehat{\chi}_0 u\right\|^2\geq(1-\mathcal{O}(h))\|\widehat{\chi}_0 u\|^2-\mathcal{O}\left(h^\infty\right)\|u\|^2, \tag{5.24}$$

where the term $\mathcal{O}(h)$ does not depend on m. There exists a family of trace class operators $\widehat{B}_h$ (depending on h) such that

$$\left\|\widehat{B}_h\right\|_{\mathrm{Tr}}\leq\mathcal{O}(1)\,C_m\mathrm{Vol}\left(N_0\cap\Sigma_E\right)h^{1/2-n},\qquad \widehat{B}_h\geq 0, \tag{5.25}$$

(where the constant $\mathcal{O}(1)$ does not depend on the escape function m) and for every $u\in L^2(X)$,

$$\left\|\left(\widehat{P}-z_E\right)\widehat{\chi}_0 u\right\|^2+\left(h\widehat{B}_h u|u\right)\geq\left(1+2h\left(C_m-\mathcal{O}(1)\right)\right)\|\widehat{\chi}_0 u\|^2-\mathcal{O}\left(h^\infty\right)\|u\|^2. \tag{5.26}$$

Remarks. Lemma 5.6 concerns the first term of the right hand side of (5.16). In order to obtain (5.26), which is similar to (5.20), it has been necessary to add a new term which involves a trace class operator $\widehat{B}_h$ to gain positivity in the domain $\mathcal{V}_{\mathcal{Z}}$ (5.11). Equation (5.24) shows that without this term the lower bound is smaller.

Proof. The construction is based on ideas around Anti-Wick quantization, Berezin quantization, FBI transforms, Bargmann-Segal transforms, Gabor frames and Toeplitz operators, see e.g. [26]. We review some definitions in Appendix A.4. We will use the following properties for an operator obtained by Toeplitz quantization of a function $A(x,\xi;h)$ (such that the following expression makes sense). Let

$$\mathrm{Op}_T(A):=\int A(x,\xi;h)\,\widehat{\pi}_{x,\xi}\,dx d\xi.$$

Then for $u\in C^\infty$

$$A(x,\xi)\geq 0\Rightarrow\left(\mathrm{Op}_T(A)\,u|u\right)\geq 0, \tag{5.27}$$

also if $A\in L^1$,

$$\mathrm{Tr}\left(\mathrm{Op}_T(A)\right)=\frac{\mathcal{O}(1)}{h^n}\int A(x,\xi)\,dx d\xi, \tag{5.28}$$

and

$$\left(\forall(x,\xi),\quad A(x,\xi)\geq 0\right)\quad\Rightarrow\quad\left\|\mathrm{Op}_T(A)\right\|_{\mathrm{Tr}}=\mathrm{Tr}\left(\mathrm{Op}_T(A)\right). \tag{5.29}$$

If $\widehat{A} \in \mathrm{Op}_h(\mathcal{S}^m)$ is a PDO with principal symbol a_0 (modulo $h\mathcal{S}^{m-1}$), then there exists $a \in \mathcal{S}^m$ such that

$$\widehat{A} = \mathrm{Op}_T(a) + \widehat{R},$$

where $\widehat{R} \in \mathrm{Op}\left(h^\infty \mathcal{S}^{-\infty}\right)$ is negligible and $a = a_0 \bmod \left(h\mathcal{S}^{m-1}\right)$.

From (5.13) we have

$$\widehat{\chi}_0\left(\widehat{P} - z_E\right)^*\left(\widehat{P} - z_E\right)\widehat{\chi}_0 = \widehat{\chi}_0\widehat{S}\widehat{\chi}_0 + \widehat{R}, \tag{5.30}$$

with $\widehat{R} \in \mathrm{Op}\left(h^\infty \mathcal{S}^{-\infty}\right)$ and

$$\widehat{S} = \mathrm{Op}_T(S),$$

with the Toeplitz symbol

$$S(x, \xi; h) = |V(\xi) - E|^2 + 1 - 2h\mathbf{X}(G_m)(x, \xi) + \mathcal{O}\left(hS^{-\infty}\right) + \mathcal{O}_m\left(h^2\mathcal{S}^{-\infty}\right)$$

(the remainders are in $\mathcal{S}^{-\infty}$ since χ_0 has compact support in (5.30). Since $\mathbf{X}(G_m) \leq 0$ from (5.6), we deduce (5.24) using Gårding's inequality (5.27).

In order to get (5.26), let $0 \leq B_h \in C_0^\infty(T^*X)$ be such that

$$(x, \xi) \in \mathcal{V}_{\mathcal{Z}} \Rightarrow B_h(x, \xi) \geq 2C_m. \tag{5.31}$$

In view of (5.12) B_h can be chosen such that

$$\int_{T^*X} B_h(x, \xi)\, dx d\xi \leq \mathcal{O}(1)\, C_m \mathrm{Vol}(\mathcal{V}_{\mathcal{Z}}) = \mathcal{O}(1)\, C_m \mathrm{Vol}(X)\, \mathrm{Vol}(N_0 \cap \Sigma_E)\sqrt{h}. \tag{5.32}$$

Let $\widehat{B}_h := \mathrm{Op}_T(B_h)$. From (5.29), (5.28) and (5.32) we deduce (5.25). Recall that from (5.11) we have

$$(x, \xi) \notin \mathcal{V}_{\mathcal{Z}} \Rightarrow |V(\xi) - E|^2 \geq hC_m \text{ or } -h\mathbf{X}(G_m) \geq hC_m.$$

Therefore in view of (5.31) for every $(x, \xi) \in T^*X$ we have

$$\begin{aligned} S(x, \xi; h) + hB_h(x, \xi) &= |V(\xi) - E|^2 + 1 - 2h\mathbf{X}(G_m) + hB_h(x, \xi) \\ &\quad + \mathcal{O}\left(hS^{-\infty}\right) + \mathcal{O}_m\left(h^2\mathcal{S}^{-\infty}\right) \\ &\geq 1 + 2hC_m + \mathcal{O}\left(h\mathcal{S}^{-\infty}\right) + \mathcal{O}_m\left(h^2\mathcal{S}^{-\infty}\right). \end{aligned}$$

After multiplying both sides by $\widehat{\chi}_0$, using $\widehat{\chi}_0\widehat{B}_h\widehat{\chi}_0 = \widehat{B}_h + \mathrm{Op}\left(h^\infty\mathcal{S}^{-\infty}\right)$ and Gårding's inequality we deduce that

$$\begin{aligned} \forall u \in L^2(X), \quad \left(\widehat{\chi}_0\widehat{S}\widehat{\chi}_0 u|u\right) + \left(h\widehat{B}_h u|u\right) &\geq \left(\left(\widehat{\chi}_0\left(1 + 2h\left(C_m - \mathcal{O}(1)\right)\right)\widehat{\chi}_0\right)u|u\right) \\ &\quad + \mathcal{O}\left(h^\infty\right)\|u\|^2. \end{aligned}$$

Replacing the first term by (5.30) this gives (5.26). □

Equation (5.16) with (5.20), (5.26), (5.15) gives:

Corollary 5.7. *We have*

$$\forall u \in L^2(X), \qquad \left\| \left(\widehat{P} - z_E\right) u \right\|^2 + \left(h \widehat{B}_h u | u\right) \geq (1 + 2 (C_m - \mathcal{O}(1)) h) \|u\|^2, \tag{5.33}$$

where $\mathcal{O}(1)$ *does not depend on* m. *Moreover*

$$\forall u \in L^2(X), \qquad \left\| \left(\widehat{P} - z_E\right) u \right\|^2 \geq (1 - \mathcal{O}(h)) \|u\|^2. \tag{5.34}$$

Here we have used (5.24) to get (5.34).

Let us show that these last relations imply an upper bound for the number of eigenvalues of the operator $\left(\widehat{P} - z_E\right)^* \left(\widehat{P} - z_E\right)$ smaller than $(1 + 2 (C_m - \mathcal{O}(1)) h)$.

Lemma 5.8. *Let* $s_1 \leq s_2 \leq \cdots$ *be the singular values* $\left(\widehat{P} - z_E\right)$ *sorted from below. More precisely,* $s_1^2 \leq s_2^2 \leq \cdots$ *are the eigenvalues of the positive self-adjoint operator* $\widehat{A} := \left(\widehat{P} - z_E\right)^* \left(\widehat{P} - z_E\right)$ *below the infimum of the essential spectrum of* $\widehat{A}$*, possibly completed with an infinite repetition of that infimum if there are only finitely many such eigenvalues. Then the first eigenvalue is*

$$s_1 \geq 1 - \mathcal{O}(h) \tag{5.35}$$

and

$$\text{if } j > \mathcal{O}(1) C_m \text{Vol}(N_0 \cap \Sigma_E) h^{\frac{1}{2} - n} \text{ then } s_j \geq 1 + (C_m - \mathcal{O}(1)) h, \tag{5.36}$$

where $\mathcal{O}(1)$ *means some constant independent of* m. *In other words the number of singular values of* $\left(\widehat{P} - z_E\right)$ *below* $1 + (C_m - \mathcal{O}(1)) h$ *is* $\mathcal{O}(1) C_m \text{Vol}(N_0 \cap \Sigma_E) h^{\frac{1}{2} - n}$.

Proof. Equation (5.35) is a direct consequence of (5.34). We use the "max-min formula" for self-adjoint operators [39, p. 78] and Eq. (5.33). Put $\lambda_m := C_m - \mathcal{O}(1)$. We have for every j,

$$\begin{aligned} s_j^2 &= \max_{U \subseteq L^2(X),\, \text{codim}(U) \leq j-1} \; \min_{u \in U, \|u\|=1} \left(u, \widehat{A} u\right) \\ &\geq 1 + 2\lambda_m h + \max_{U \subseteq L^2(X),\, \text{codim}(U) \leq j-1} \; \min_{u \in U, \|u\|=1} \left(- \left(h \widehat{B}_h u | u\right)\right) \\ &= 1 + 2\lambda_m h - h \min_{U \subseteq L^2(X),\, \text{codim}(U) \leq j-1} \; \max_{u \in U, \|u\|=1} \left(\left(\widehat{B}_h u | u\right)\right) \\ &= 1 + 2\lambda_m h - h b_j, \end{aligned}$$

where U varies in the set of closed subspaces of $L^2(X)$ and $b_1 \geq b_2 \geq \ldots$ denote the eigenvalues of $\widehat{B}_h$ (possibly completed with an infinite repetition of 0 if there are only finitely many such eigenvalues). We have

$$\left\| \widehat{B}_h \right\|_{\text{Tr}} = \text{Tr}\left(\widehat{B}_h\right) = b_1 + b_2 + \ldots.$$

Equation (5.25) implies that for every $\varepsilon_0 > 0$, if $b_j \geq \varepsilon_0$ then

$$j \varepsilon_0 \leq \text{Tr}\left(\widehat{B}_h\right) \leq \mathcal{O}(1) C_m \text{Vol}(N_0 \cap \Sigma_E) h^{\frac{1}{2} - n}.$$

Equivalently if $j > \frac{1}{\varepsilon_0} \mathcal{O}(1) C_m \text{Vol}(N_0 \cap \Sigma_E) h^{\frac{1}{2} - n}$, then $b_j < \varepsilon_0$ and $s_j^2 \geq 1 + 2\lambda_m h - h b_j \geq 1 + 2(\lambda_m - \varepsilon_0) h$. Taking the square root we get (5.36). □

We deduce now an upper bound for the number of eigenvalues of $\widehat{P}$.

Corollary 5.9. *We have*

$$\sharp\left\{\sigma\left(\widehat{P}\right)\cap D\left(z_E,1+\frac{C_m}{2}h\right)\right\}\leq\mathcal{O}(1)\,C_m\mathrm{Vol}\left(N_0\cap\Sigma_E\right)h^{\frac{1}{2}-n}. \tag{5.37}$$

Proof. Let $\lambda_1,\lambda_2,\lambda_3\ldots$ denote the eigenvalues of $\widehat{P}$ sorted such that $j\to\left|\lambda_j-z_E\right|$ is increasing. The Weyl inequalities (see [43, (a.8), p. 38] for a proof) give

$$\prod_{j=1}^{N}s_j\leq\prod_{j=1}^{N}\left|\lambda_j-z_E\right|,\qquad\forall N, \tag{5.38}$$

where $\left(s_j\right)_j$ are the singular values defined in Lemma 5.8 above. Let

$$\widetilde{N}:=\sharp\left\{\lambda_j:\quad\left|\lambda_j-z_E\right|\leq1+\frac{C_m}{2}h\right\}=\sharp\left\{\sigma\left(P\right)\cap D\left(z_E,1+\frac{C_m}{2}h\right)\right\}$$

and let

$$\widetilde{M}:=\mathcal{O}(1)\,C_m\mathrm{Vol}\left(N_0\cap\Sigma_E\right)h^{\frac{1}{2}-n}$$

be the term which appears in (5.36). We want to show the bound:

$$\widetilde{N}\leq\left(2+\mathcal{O}\left(\frac{1}{C_m}\right)\right)\widetilde{M} \tag{5.39}$$

for $C_m\gg1$. If $\widetilde{N}\leq\widetilde{M}$ then (5.39) is true. Conversely let us suppose that $\widetilde{N}\geq\widetilde{M}$. Using (5.38) we have

$$\left(\prod_{j=1}^{\widetilde{M}}s_j\right)\left(\prod_{j=\widetilde{M}+1}^{\widetilde{N}}s_j\right)\leq\left(1+\frac{C_m}{2}h\right)^{\widetilde{N}}.$$

Then using (5.35) and (5.36) we have

$$\left(1-\mathcal{O}(h)\right)^{\widetilde{M}}\left(1+\left(C_m-\mathcal{O}(1)\right)h\right)^{\widetilde{N}-\widetilde{M}}\leq\left(1+\frac{C_m}{2}h\right)^{\widetilde{N}}.$$

We take the logarithm and since $h\ll1$ we get:

$$-\widetilde{M}\mathcal{O}(h)+\left(\widetilde{N}-\widetilde{M}\right)\left(C_m-\mathcal{O}(1)\right)h\leq\widetilde{N}\frac{C_m}{2}h$$
$$\Leftrightarrow\widetilde{N}\left(\frac{C_m}{2}-\mathcal{O}(1)\right)\leq\widetilde{M}\left(C_m+\mathcal{O}(1)\right).$$

Now since $C_m\gg1$,

$$\Leftrightarrow\widetilde{N}\leq\widetilde{M}\frac{\left(1+\mathcal{O}\left(\frac{1}{C_m}\right)\right)}{\left(\frac{1}{2}+\mathcal{O}\left(\frac{1}{C_m}\right)\right)}=\widetilde{M}\left(2+\mathcal{O}\left(\frac{1}{C_m}\right)\right),$$

so we have obtained (5.39). This implies (5.37). □

From Lemma 5.4 with $b = \frac{C_m}{2}$ and $\beta = \frac{C_m}{4}$ we deduce that the upper bound (5.37) implies an upper bound:

$$\sharp\left\{\lambda_i \in \sigma\left(\widehat{P}\right), \quad |\Re(\lambda_i - E)| \leq \sqrt{\frac{C_m}{4}h}, \quad \Im(\lambda_i) \geq -\frac{C_m}{4}h\right\}$$
$$= \mathcal{O}(1)\, C_m \mathrm{Vol}(N_0 \cap \Sigma_E)\, h^{\frac{1}{2}-n}.$$

Here we take $E = 1$ and $\frac{1}{h} \gg 1$ and return to the original spectral variable $z = \frac{z_h}{h}$ after the scaling (5.4). From (5.7) we can choose the escape function m such that $C_m \gg 1$ is arbitrarily large and $\mathrm{Vol}(N_0 \cap \Sigma_E) < o\left(\frac{1}{C_m}\right)$ is arbitrarily small so that $\mathcal{O}(1)\, C_m \mathrm{Vol}(N_0 \cap \Sigma_E)\, h^{\frac{1}{2}-n} = o\left(h^{\frac{1}{2}-n}\right)$. Since the spectrum does not depend on the escape function m, we get (1.25) with $E = \frac{1}{h}$. We have finished the proof of Theorem 1.8.

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A. Some Results in Operator Theory

A.1. On minimal and maximal extensions. We show here that the pseudodifferential operator $\widehat{P}$ defined in Eq.(3.2), has a unique closed extension on $L^2(X)$. This is well known for elliptic PDO [53, Chap.13, p.125]. The fact that P has order 1 (since it is defined from a vector field on X) is important in the present non elliptic case.

The domain of the minimal closed extension $\widehat{P}_{min}$ of the operator $\widehat{P}$ with domain $C^\infty(X)$ is

$$\mathcal{D}_{min} := \left\{u \in L^2(X), \quad u_j \in C^\infty(X) \to u \text{ in } L^2(X) \text{ and } \widehat{P}u_j \to v \in L^2(X)\right\}. \tag{A.1}$$

The maximal closed extension $\widehat{P}_{max}$ has domain

$$\mathcal{D}_{max} := \left\{u \in L^2(X), \quad \widehat{P}u \in L^2(X)\right\}.$$

(Recall that $\widehat{P}$ is defined a priori on $C^\infty(X)$ and $\mathcal{D}'(X)$).

Lemma A.1. *For a PDO $\widehat{P}$ of order 1 (i.e $\widehat{P} \in \mathrm{Op}\left(S^1\right)$), the minimal and maximal extensions coincide: $\mathcal{D}\left(\widehat{P}\right) := \mathcal{D}_{min} = \mathcal{D}_{max}$, i.e. there is a unique closed extension of the operator $\widehat{P}$ in $L^2(X)$.*

Proof. $\mathcal{D}_{min} \subset \mathcal{D}_{max}$ is clear. Let us check that $\mathcal{D}_{max} \subset \mathcal{D}_{min}$. Let $u \in \mathcal{D}_{max}$, i.e. $u \in L^2(X), v := \widehat{P}u \in L^2(X)$. We will construct a sequence $u_h \in C^\infty(X)$ with $h \to 0$, such that $u_h \to u$ in $L^2(X)$ and show that $\widehat{P}u_h \to v$ in L^2.

Let $\chi : T^*X \to \mathbb{R}^+$ be a C^∞ function such that $\chi(x,\xi) = 1$ for $|\xi| \leq 1$, and $\chi(x,\xi) = 0$ for $|\xi| \geq 2$. For $h > 0$, let the function χ_h on T^*X be defined by $\chi_h(x,\xi) = \chi(x, h\xi)$. Let the **truncation operator** be:

$$\widehat{\chi}_h := \mathrm{Op}(\chi_h).$$

Notice that $\widehat{\chi}_h$ is a smoothing operator which truncates large components in ξ (larger than $1/h$), $\widehat{\chi}_h$ is similar to a convolution in x coordinates.

Let

$$u_h := \widehat{\chi}_h u.$$

It is clear that $u_h \to u$ in $L^2(X)$ as $h \to 0$. We have

$$\widehat{P} u_h = \widehat{P} \widehat{\chi}_h u = \widehat{\chi}_h \widehat{P} u + \left[\widehat{P}, \widehat{\chi}_h\right] u.$$

The first term converges $\widehat{\chi}_h \widehat{P} u \to v = \widehat{P} u$ as $h \to 0$. The principal symbol of the PDO $\left[\widehat{P}, \widehat{\chi}_h\right]$ is

$$\frac{1}{i}\{P, \chi_h\} = \frac{1}{i}\left(\partial_\xi P \partial_x \chi_h - \partial_x P \partial_\xi \chi_h\right).$$

Now we use the fact that $P \in S^1$ has order 1. In the first term, $\partial_\xi P \in S^0$ is bounded and $\partial_x \chi_h$ is non-zero only on a large ring $\frac{1}{h} \leq |\xi| \leq \frac{2}{h}$. In the second term $\partial_x P \in S^1$ has order 1 but $\partial_\xi \chi_h = h \partial_\xi \chi(x, h\xi)$ is non-zero on the same large ring and therefore of order (-1) (since $h \simeq |\xi|^{-1}$ on the ring). Therefore the PDO $\left[\widehat{P}, \widehat{\chi}_h\right]$ converges strongly to zero in $L^2(X)$ as $h \to 0$. Hence $\left[\widehat{P}, \widehat{\chi}_h\right] u \to 0$ as $h \to 0$. We deduce that $\widehat{P} u_h \to v = \widehat{P} u$, and that $u \in \mathcal{D}_{min}$. □

A.2. The sharp Gårding inequality. References: [23, p. 52] or (A.8), [33, p. 99], [54, p. 1157] for a short proof using Toeplitz quantization.

Proposition A.2. *If $\widehat{P}$ is a PDO with symbol $P \in S^\mu$, $\mu \in \mathbb{R}$, $\Re(P) \geq 0$, then there exists $C > 0$ such that*

$$\forall u \in C^\infty(X), \quad \Re\left(\widehat{P} u | u\right) \geq -C \|u\|^2_{H^{\frac{\mu-1}{2}}}, \tag{A.2}$$

where $\|u\|^2_{H^\mu} := \left(\left\langle\widehat{\xi}\right\rangle^\mu u | \left\langle\widehat{\xi}\right\rangle^\mu u\right)_{L^2(X)}$ denotes the norm in the Sobolev space H^μ.

A.3. Quadratic partition of unity on phase space. We denote $\widehat{A} := \mathrm{Op}_h(A)$ for a symbol A.

Lemma A.3. *Let $K_0 \subset T^*X$ compact. There exist symbols $\chi_0 \in \mathcal{S}^{-\infty}$ and $\chi_1 \in \mathcal{S}^0$ of self-adjoint operators $\widehat{\chi}_0, \widehat{\chi}_1$ such that*

$$\widehat{\chi}_0^2 + \widehat{\chi}_1^2 = 1 + \widehat{R}.$$

The symbol $R \in \left(h^\infty \mathcal{S}^{-\infty}\right)$ is negligible, $\mathrm{supp}(\chi_0)$ *is compact and on K_0, $\chi_1(x, \xi) = \mathcal{O}(h^\infty)$, $\chi_0(x, \xi) = 1 + \mathcal{O}(h^\infty)$.*

Proof. Let $K_0 \subset T^*X$ be compact. We can find symbols $0 \leq \chi_0 \in C_0^\infty(T^*X)$ (with compact support) and $0 \leq \chi_1 \in C^\infty(T^*X)$ such that

$$\chi_1 = \begin{cases} 0 & \text{on } K_0 \\ 1 & \text{for } |\xi| \gg 1 \end{cases}$$

and

$$A := \chi_0^2 + \chi_1^2 \text{ is } \begin{cases} > 0 & \text{everywhere} \\ = 1 & \text{for } |\xi| \gg 1. \end{cases}$$

We replace χ_0, χ_1 respectively by $\chi_0 A^{-1/2}, \chi_1 A^{-1/2}$. We obtain $1 = \chi_0^2 + \chi_1^2$.
Let $\widehat{R} := \widehat{\chi}_0^2 + \widehat{\chi}_1^2 - 1$. Then $\widehat{R} \in \mathrm{Op}_h\left(h\mathcal{S}^{-\infty}\right)$. We write $R = hr_0(x, \xi) + h^2 \ldots$.
We replace $\widehat{\chi}_j$, $j = 0, 1$ by

$$\widehat{\chi}_j' := \left(1 + h\widehat{r}_0\right)^{-1/4} \widehat{\chi}_j \left(1 + h\widehat{r}_0\right)^{-1/4},$$

which is also self-adjoint. We obtain

$$\begin{aligned} \widehat{\chi}_0'^2 + \widehat{\chi}_1'^2 &= \left(1 - h\widehat{r}_0\right)\widehat{\chi}_0^2 + \left(1 - h\widehat{r}_1\right)\widehat{\chi}_1^2 + \mathcal{O}\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right)\right) \\ &= 1 + \mathcal{O}\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right)\right). \end{aligned}$$

If we iterate this algorithm, we obtain the lemma. □

A.3.1. I.M.S. localization formula. The following lemma is similar to the "I.M.S localization formula" given in [11, p. 27]. It uses the quadratic partition of phase space obtained in Lemma A.3 above.

Lemma A.4. *Suppose that $\widehat{P} \in \mathrm{Op}_h(\mathcal{S}^{\mu})$ for some $\mu \in \mathbb{R}$ and that $(P - P^*) \in \mathrm{Op}_h(h\mathcal{S}^{\mu})$. Then for every $u \in L^2(X)$, $z \in \mathbb{C}$,*

$$\left\|\left(\widehat{P} - z\right)u\right\|^2 = \left\|\left(\widehat{P} - z\right)\widehat{\chi}_0 u\right\|^2 + \left\|\left(\widehat{P} - z\right)\widehat{\chi}_1 u\right\|^2 + \mathcal{O}\left(h^2\right)\|u\|^2. \quad \text{(A.3)}$$

Proof. For simplicity, we suppose $z = i\beta$ with $\beta \in \mathbb{R}$, i.e. $\Re(z) = 0$ (this is equivalent to replacing $\widehat{P} - \Re(z)$ by some operator $\widehat{P'}$). We use (5.15) and write

$$\left\|\left(\widehat{P} - i\beta\right)u\right\|^2 = \left(\left(\widehat{P} - i\beta\right)^*\left(\widehat{P} - i\beta\right)u|u\right) = \sum_{k=0,1}\left(\left(\widehat{P} - i\beta\right)^*\widehat{\chi}_k^2\left(\widehat{P} - i\beta\right)u|u\right) \quad \text{(A.4)}$$

$$+\mathcal{O}\left(h^{\infty}\right)\|u\|^2. \quad \text{(A.5)}$$

The aim is to move the operators $\widehat{\chi}_k$ outside. One has for $k = 0, 1$:

$$\begin{aligned} &\left(\widehat{P} - i\beta\right)^*\widehat{\chi}_k^2\left(\widehat{P} - i\beta\right) - \widehat{\chi}_k\left(\widehat{P} - i\beta\right)^*\left(\widehat{P} - i\beta\right)\widehat{\chi}_k \\ &\quad = \widehat{\chi}_k\left(\widehat{P} - i\beta\right)^*\widehat{\chi}_k\left(\widehat{P} - i\beta\right) + \left[\widehat{P}^*, \widehat{\chi}_k\right]\widehat{\chi}_k\left(\widehat{P} - i\beta\right) - \widehat{\chi}_k\left(\widehat{P} - i\beta\right)^*\left(\widehat{P} - i\beta\right)\widehat{\chi}_k \\ &\quad = \left[\widehat{P}^*, \widehat{\chi}_k\right]\widehat{\chi}_k\left(\widehat{P} - i\beta\right) - \widehat{\chi}_k\left(\widehat{P} - i\beta\right)^*\left[\widehat{P}, \widehat{\chi}_k\right] \\ &\quad = \left(\underbrace{\left[\widehat{P}^*, \widehat{\chi}_k\right]\widehat{\chi}_k\widehat{P} - \widehat{\chi}_k\widehat{P}^*\left[\widehat{P}, \widehat{\chi}_k\right]}_{\mathrm{I}_k}\right) - i\beta\left(\underbrace{\left[\widehat{P}^*, \widehat{\chi}_k\right]\widehat{\chi}_k + \widehat{\chi}_k\left[\widehat{P}, \widehat{\chi}_k\right]}_{\mathrm{II}_k}\right). \end{aligned} \quad \text{(A.6)}$$

First remark that for every PDO $\widehat{A} \in \mathrm{Op}_h(\mathcal{S}^{\mu})$ with some $\mu \in \mathbb{R}$, then

$$\left[\widehat{A}, \widehat{\chi}_k\right] \in \mathrm{Op}_h\left(h\mathcal{S}^{-\infty}\right).$$

This is obvious for $k = 0$ since $\widehat{\chi}_0 \in \mathrm{Op}_h\left(\mathcal{S}^{-\infty}\right)$ and for $k = 1$ this is because $(\widehat{\chi}_1 - 1) \in \mathrm{Op}_h\left(\mathcal{S}^{-\infty}\right)$ and $\left[\widehat{A}, 1\right] = 0$. We have assumed that

$$\left(\widehat{P}^* - \widehat{P}\right) \in \mathrm{Op}_h\left(h\mathcal{S}^{\mu}\right),$$

therefore

$$\left[\widehat{P}^* - \widehat{P}, \widehat{\chi}_k\right] \in \mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right).$$

Also

$$\left[\widehat{P}, \widehat{\chi}_k\right] \in \mathrm{Op}_h\left(h\mathcal{S}^{-\infty}\right).$$

The first term of (A.6) is

$$\begin{aligned}
\mathrm{I_k} &= \left[\widehat{P}^*, \widehat{\chi}_k\right]\widehat{\chi}_k\widehat{P} - \widehat{\chi}_k\widehat{P}^*\left[\widehat{P}, \widehat{\chi}_k\right] \\
&= \left[\widehat{P}, \widehat{\chi}_k\right]\widehat{\chi}_k\widehat{P} - \widehat{\chi}_k\widehat{P}\left[\widehat{P}, \widehat{\chi}_k\right] + \mathcal{O}\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right)\right) \\
&= \left[\left[\widehat{P}, \widehat{\chi}_k\right], \widehat{\chi}_k\widehat{P}\right] + \mathcal{O}\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right)\right) \\
&= \mathcal{O}\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right)\right).
\end{aligned}$$

The second term of (A.6) is

$$\begin{aligned}
\mathrm{II_k} &= \left[\widehat{P}^*, \widehat{\chi}_k\right]\widehat{\chi}_k + \widehat{\chi}_k\left[\widehat{P}, \widehat{\chi}_k\right] \\
&= \left[\widehat{P}, \widehat{\chi}_k\right]\widehat{\chi}_k + \widehat{\chi}_k\left[\widehat{P}, \widehat{\chi}_k\right] + \mathcal{O}\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right)\right) \\
&= \left[\widehat{P}, \widehat{\chi}_k^2\right] + \mathcal{O}\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right)\right).
\end{aligned}$$

Therefore using (5.15),

$$\begin{aligned}
\mathrm{II}_0 + \mathrm{II}_1 &= \left[\widehat{P}, \widehat{\chi}_0^2 + \widehat{\chi}_1^2\right] + \mathcal{O}\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right)\right) \\
&= \mathcal{O}\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right)\right).
\end{aligned}$$

We have shown that

$$\sum_{k=0,1}\left(\widehat{P} - i\beta\right)^*\widehat{\chi}_k^2\left(\widehat{P} - i\beta\right) = \sum_{k=0,1}\widehat{\chi}_k\left(\widehat{P} - i\beta\right)^*\left(\widehat{P} - i\beta\right)\widehat{\chi}_k + \mathcal{O}\left(\mathrm{Op}_h\left(h^2\mathcal{S}^{-\infty}\right)\right).$$

Coming back to (A.4) we get (A.3). □

A.4. FBI transform and Toeplitz operators. References: [33,54].

The manifold X is equipped with a smooth Riemannian metric so that we have a well-defined exponential map $\exp_x : T_xX \to X$ which is a diffeomorphism from a neighborhood of $0 \in T_xX$ onto a neighborhood of $x \in X$. Define the **coherent state** at point $(x,\xi) \in T^*X$ to be the function of $y \in X$:

$$e_{x,\xi}(y) := \chi(x,y) \exp\left(\frac{i}{h}\xi\left(\exp_x^{-1}(y)\right) - \frac{1}{2h}\langle\xi\rangle \operatorname{dist}(x,y)^2\right), \qquad \langle\xi\rangle := (1+|\xi|)^{1/2},$$

where $\chi \in C^\infty(X\times X)$ is a standard cutoff to a small neighborhood of the diagonal. In the Euclidean case $X = \mathbb{R}^n$, the cutoff is often superfluous and we get the complex Gaussian "wave packet"

$$e_{x,\xi}(y) = \exp\left(\frac{i}{h}\xi.(y-\alpha_x) - \frac{1}{2h}\langle\xi\rangle |y-x|^2\right).$$

We have the following known facts [26,44]:

- There exists $a_0(x,\xi;h) \in h^{-\frac{3n}{2}}\mathcal{S}^{n/2}$ elliptic and $a_0 > 0$ such that

$$u = \int_{T^*X} \left(\widehat{\pi}_{x,\xi} u\right) dx d\xi + \widehat{R}u, \quad \forall u \in L^2(X) \tag{A.7}$$

  with

$$\widehat{\pi}_{x,\xi} := a_0(x,\xi;h)\, e_{x,\xi}\left(e_{x,\xi}|.\right)$$

  and $\widehat{R} \in \mathrm{Op}_h\left(h^\infty \mathcal{S}^{-\infty}\right)$ negligible. dx is the Riemannian volume on X. Equation (A.7) is called **resolution of identity.**
- We can define the **FBI-transform** of $u \in C^\infty(X)$ by

$$(Tu)(x,\xi;h) := \sqrt{a_0(x,\xi;h)}\left(e_{x,\xi}|u\right) = \sqrt{a_0(x,\xi;h)} \int_X \overline{e_{x,\xi}(y)}u(y)\,dy,$$

  which is asymptotically isometric from (A.7):

$$\|Tu\|_{L^2(T^*X)} = \|u\|_{L^2(X)} + \left(u|\hat{R}u\right) = \|u\|_{L^2(X)} + \mathcal{O}\left(h^\infty\right).$$

- $\widehat{\pi}_{x,\xi} \geq 0$ and

$$\left\|\widehat{\pi}_{x,\xi}\right\|_{tr} = \mathrm{Tr}\left(\widehat{\pi}_{x,\xi}\right) = a_0(x,\xi;h)\left\|e_{x,\xi}\right\|^2 = \mathcal{O}(1)\,h^{-n}.$$

- If $\widehat{B} \in \mathrm{Op}_h(\mathcal{S}^m)$ is a PDO with principal symbol b_0 (modulo $h\mathcal{S}^{m-1}$), then

$$\widehat{B} = \int_{T^*X} b(x,\xi;h)\,\widehat{\pi}_{x,\xi}\, dx d\xi + \widehat{R},$$

  where $\widehat{R}$ is negligible as above, $b \in \mathcal{S}^m$ and $b = b_0 \bmod \left(h\mathcal{S}^{m-1}\right)$.

- For a function $A(x,\xi;h)$ (such that the following expression makes sense), we define the **Toeplitz quantization of** A by

$$\mathrm{Op}_T(A) := \int A(x,\xi;h)\,\widehat{\pi}_{x,\xi}\,dx d\xi,$$

then the previous results imply a "Gårding's inequality":

$$A(x,\xi) \geq 0 \Rightarrow \left(\mathrm{Op}_T(A)\,u|u\right) \geq 0 \tag{A.8}$$

and

$$\mathrm{Tr}\left(\mathrm{Op}_T(A)\right) = \frac{\mathcal{O}(1)}{h^n}\int A(x,\xi)\,dx d\xi.$$

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