

ON DYNAMICS OF $Out(F_n)$ ON $PSL_2(\mathbb{C})$ CHARACTERS

BY

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ABSTRACT

We introduce and study an open set of $PSL_2(\mathbb{C})$ characters of a nonabelian free group, on which the action of the outer automorphism group is properly discontinuous, and which is strictly larger than the set of discrete, faithful convex-cocompact (i.e. Schottky) characters. This implies, in particular, that the outer automorphism group does not act ergodically on the set of characters with dense image. Hence there is a difference between the geometric (discrete vs. dense) decomposition of the characters, and a natural dynamical decomposition.

1. Introduction

Let F_n be the free group on $n \geq 2$ generators. Its automorphism group $Aut(F_n)$ acts naturally, by precomposition, on $Hom(F_n, G) \equiv G^n$ for any group G . The outer automorphism group $Out(F_n) = Aut(F_n)/Inn(F_n)$ acts on the quotient $Hom(F_n, G)/G$, where G is understood to act by inner automorphisms. When G is a Lie group we consider instead the character variety $\mathcal{X}(F_n, G) \equiv Hom(F_n, G)//G$, the quotient in the sense of geometric invariant theory.

When G is a compact Lie group and $n \geq 3$, Gelander [19] showed that this action is ergodic, settling a conjecture of Goldman [23], who had proved it for $G = SU(2)$. When G is noncompact, the situation is different because there is a

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natural decomposition of $\mathcal{X}(F_n, G)$, up to measure 0, into (characters of) **dense** and **discrete** representations, and in the cases of interest to us the action on the discrete set is not ergodic, indeed even has a nontrivial domain of discontinuity.

See Lubotzky [33] for a comprehensive survey on the dynamics of representation spaces, from algebraic, geometric and computational points of view.

We will focus on the case of $G = \mathrm{PSL}_2(\mathbb{C})$, where the interior of the discrete set is the set of Schottky representations. In this case one can ask if the action is ergodic, or even topologically transitive, in the set of dense representations, but this turns out to be the wrong question. In particular:

THEOREM 1.1: *There is an open subset of $\mathcal{X}(F_n, \mathrm{PSL}_2(\mathbb{C}))$, strictly larger than the set of Schottky characters, which is $\mathrm{Out}(F_n)$ invariant, and on which $\mathrm{Out}(F_n)$ acts properly discontinuously.*

In other words, the natural “geometric” decomposition of $\mathcal{X}(F_n, \mathrm{PSL}_2(\mathbb{C}))$, in terms of discreteness vs. density of the image group, is distinct from the “dynamical” decomposition, in terms of proper discontinuity vs. chaotic action of $\mathrm{Out}(F_n)$.

The subset promised in Theorem 1.1 will be the set of **primitive-stable** representations (see definitions below). It is quite easy to see that this set is open and $\mathrm{Out}(F_n)$ invariant, and that the action on it is properly discontinuous (Theorems 3.2, 3.3). Thus the main content of this note is the observation, via a lemma of Whitehead on free groups and a little bit of hyperbolic geometry, that it contains non-Schottky (and in particular nondiscrete) elements. This will be carried out in Section 4.

One should compare this with results of Goldman [22] on the rank 2 case for $\mathrm{SL}_2(\mathbb{R})$ characters, and with work of Bowditch [5] on the complex rank 2 case. Bowditch, and Tan–Wong–Zhang [46] and [47], studied a condition very similar to primitive stability; we will compare the two in Section 5.

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Souto in particular pointed out the connection to Otal’s work. The referee made a number of helpful and incisive suggestions.

2. Background and notation

In the remainder of the paper we fix $n \geq 2$ and let $G = PSL_2(\mathbb{C})$. We also fix a free generating set $X = \{x_1, \dots, x_n\}$ of F_n .

Note that $Hom(F_n, G)$ can be identified with G^n via $\rho \mapsto (\rho(x_1), \dots, \rho(x_n))$ once the generators are fixed. The quotient $\mathcal{X}(F_n) = Hom(F_n, G)/G$, which we denote $\mathcal{X}(F_n, G)$ or just $\mathcal{X}(F_n)$, is obtained by considering characters of representations, which in our case corresponds to trace functions. This quotient naturally has the structure of an algebraic variety, and differs from the purely topological quotient $Hom(F_n, G)/G$ only at reducible points, i.e. representations whose images fix a point on $\hat{\mathbb{C}}$ (see Kapovich [26] or Morgan–Shalen [40]). Characters of reducible representations form a subset of measure 0 outside all of the sets we shall be considering, and so we shall be able to ignore the distinction between these two quotients.

Geometric decomposition. There is a natural $Out(F_n)$ -invariant decomposition of $\mathcal{X}(F_n)$ in terms of the geometry of the action of $\rho(F_n)$ on \mathbb{H}^3 . Namely, let $\mathcal{D}(F_n) = \mathcal{D}(F_n, G)$ denote the (characters of) discrete and faithful representations, and let $\mathcal{E}(F_n) = \mathcal{E}(F_n, G)$ denote those of representations with dense image in G .

It is fairly well-known (see [6, 7]) that

LEMMA 2.1: $\mathcal{E}(F_n)$ is nonempty and open, $\mathcal{D}(F_n)$ is closed, and

$$\mathcal{X}(F_n) \setminus (\mathcal{D}(F_n) \cup \mathcal{E}(F_n))$$

has measure 0.

The idea of the measure 0 statement is this: If ρ is not faithful it satisfies some relation; this is a nontrivial algebraic condition, so defines a subvariety of measure 0. There are a countable number of such relations. If ρ is not discrete, consider the identity component of the closure of $\rho(F_n)$ in G . This is a connected Lie subgroup of G and hence is either all of G (so $\rho(F_n)$ is dense), or solvable (i.e. elementary, fixing a point in the Riemann sphere), or conjugate to $PSL_2(\mathbb{R})$. The latter cases are again detected by algebraic conditions.

Openness of $\mathcal{E}(F_n)$ follows from the Kazhdan–Margulis–Zassenhaus lemma [27], which furnishes a neighborhood U of the identity in G in which any set of n elements generates either an elementary or an indiscrete group. Generating an elementary group is a nontrivial algebraic condition on $U^n \subset \text{Hom}(F_n, G)$ (for $n \geq 2$), and as above for an indiscrete group not to be dense is also a nontrivial algebraic condition. Hence an open dense subset W of U^n consists of dense representations. Now given any ρ with dense image, there exist elements $h_1, \dots, h_n \in F_n$ such that $(\rho(h_1), \dots, \rho(h_n)) \in W$, and since W is open the same is true for ρ' sufficiently close to ρ — hence ρ' also generates a dense subgroup.

The fact that $\mathcal{D}(F_n)$ is closed follows from Jørgensen’s inequality [25], or alternately from the Kazhdan–Margulis–Zassenhaus lemma. Lemma 2.1 in fact holds for much more general target groups G — see [6, 7] for details.

Note, when G is compact $\mathcal{D}(F_n, G)$ is empty, and in this case Gelander proved that $\text{Out}(F_n)$ acts ergodically on $\mathcal{X}(F_n, G)$ and hence on $\mathcal{E}(F_n, G)$. Our main theorem will show that, in general, the action on $\mathcal{E}(F_n, G)$ is not ergodic.

Schottky groups. A **Schottky group** (or representation) ρ is one which is obtained by a “ping-pong” configuration in the sphere at infinity $\partial\mathbb{H}^3$. That is, suppose that $D_1, D'_1, \dots, D_n, D'_n$ are $2n$ disjoint closed (topological) disks in $\partial\mathbb{H}^3$ and $g_1, \dots, g_n \in \text{PSL}_2(\mathbb{C})$ are isometries such that $g_i(D_i)$ is the closure of the complement of D'_i . Then $\{g_1, \dots, g_n\}$ generate a free discrete group of rank n , called a Schottky group. The representation sending $x_i \mapsto g_i$ is discrete and faithful, and moreover, an open neighborhood of it in $\text{Hom}(F_n, G)$ consists of similar representations. We let $\mathcal{S}(F_n)$ denote the open set of all characters of Schottky representations.

Sullivan [45] proved that

THEOREM 2.2: $\mathcal{S}(F_n)$ is the interior of $\mathcal{D}(F_n)$.

(This theorem is not known to hold for the higher-dimensional hyperbolic setting; see §5.)

To obtain a geometric restatement of the Schottky condition, consider the **limit set** for the action of any discrete group of isometries on \mathbb{H}^3 , namely the minimal closed invariant subset of $\partial\mathbb{H}^3$. The convex hull in \mathbb{H}^3 of this limit set is invariant, and its quotient by the group is called the **convex core** of the quotient manifold. The deep usefulness of the convex core in the study of

hyperbolic 3-manifolds was first exploited by Thurston. The Schottky condition on ρ is equivalent to the condition that the convex core of $\mathbb{H}^3/\rho(F_n)$ is a compact handlebody of genus n (see Marden [34, pp. 98–99] for a discussion and references).

It is a well-known consequence of the deformation theory of Kleinian groups (see e.g. Bers [3] for background) that $Out(F_n)$ acts properly discontinuously on $\mathcal{S}(F_n)$: that is, that the set $\{\psi \in Out(F_n) : \psi(K) \cap K \neq \emptyset\}$ is finite for any compact $K \subset \mathcal{S}(F_n)$. We note that this will also follow from Theorem 3.3.

3. Primitive-stable representations

Let Γ be a bouquet of n oriented circles labeled by our fixed generating set. We let $\mathcal{B} = \mathcal{B}(\Gamma)$ denote the set of bi-infinite (oriented) geodesics in Γ , as in Bestvina–Feighn–Handel [4]. Each such geodesic lifts to an F_n -invariant set of bi-infinite geodesics in $\tilde{\Gamma}$, the universal covering tree (and the Cayley graph of F_n with respect to X).

Let ∂F_n be the boundary at infinity of F_n , or the space of ends of the tree $\tilde{\Gamma}$. We have a natural action of F_n on ∂F_n . Each element of \mathcal{B} can be identified with an F_n -invariant subset of $\partial F_n \times \partial F_n \setminus \Delta$ (with Δ the diagonal), i.e. the pairs of endpoints of its lifts. $Out(F_n)$ acts naturally on \mathcal{B} (and in general on F_n -invariant subsets of $\partial F_n \times \partial F_n \setminus \Delta$).

Equivalently we can identify \mathcal{B} with the set of bi-infinite reduced words in the generators, modulo shift. To every conjugacy class $[w]$ in F_n is associated an element of \mathcal{B} named \overline{w} , namely the periodic word determined by concatenating infinitely many copies of a cyclically reduced representative of w .

An element of F_n is called **primitive** if it is a member of a free generating set, or equivalently if it is the image of a standard generator by an element of $Aut(F_n)$. Let $\mathcal{P} = \mathcal{P}(F_n)$ denote the subset of \mathcal{B} consisting of \overline{w} for conjugacy classes $[w]$ of primitive elements. Note that \mathcal{P} is $Out(F_n)$ -invariant.

Given a representation $\rho : F_n \rightarrow PSL_2(\mathbb{C})$ and a basepoint $x \in \mathbb{H}^3$, there is a unique map $\tau_{\rho,x} : \tilde{\Gamma} \rightarrow \mathbb{H}^3$ mapping the origin of $\tilde{\Gamma}$ to x , ρ -equivariant, and mapping each edge to a geodesic segment. Every element of \mathcal{B} is represented by an F_n -invariant family of leaves in $\tilde{\Gamma}$, which map to a family of broken geodesic paths in \mathbb{H}^3 .

Definition 3.1: A representation $\rho : F_n \rightarrow \mathrm{PSL}_2(\mathbb{C})$ is **primitive-stable** if there are constants K, δ and a basepoint $x \in \mathbb{H}^3$ such that $\tau_{\rho, x}$ takes all leaves of \mathcal{P} to (K, δ) -quasi geodesics.

Note that if there is one such basepoint then any basepoint will do, at the expense of increasing δ . This condition is invariant under conjugacy and so makes sense for $[\rho] \in \mathcal{X}(F_n)$. Moreover, the property is $\mathrm{Out}(F_n)$ -invariant since \mathcal{P} is $\mathrm{Out}(F_n)$ -invariant. Primitive-stability is a strengthening of the negation of **redundancy**, whose relevance was explained to me by Alex Lubotzky (see §5).

Let us establish some basic facts.

LEMMA 3.2:

- (1) If ρ is Schottky then it is primitive-stable.
- (2) Primitive-stability is an open condition in $\mathcal{X}(F_n)$.
- (3) If ρ is primitive-stable then, for every proper free factor A of F_n , $\rho|_A$ is Schottky.

Proof. To see (1), note that if ρ is discrete and faithful, then $\tau_{\rho, x}$ is the lift to universal covers of a homotopy-equivalence from Γ to the quotient manifold $N_\rho = \mathbb{H}^3/\rho(F_n)$. If ρ is Schottky, then the convex core of N_ρ is compact and hence its homotopy-equivalence to the image of Γ lifts to a quasi-isometry of the convex hull of the group to $\tau_{\rho, x}(\tilde{\Gamma})$. It follows that *all* leaves in \mathcal{B} map to uniform quasi-geodesics, and in particular the primitive ones.

Next we prove (2). Let ρ be primitive-stable, and fix a basepoint x and quasi-geodesic constants K, δ as in the definition. Let $\tau = \tau_{\rho, x}$.

Let L be a primitive leaf, with vertex sequence $\{v_i \in \tilde{\Gamma}\}$, and let $p_i = \tau(v_i)$. The condition that $\tau|_L$ is quasi-geodesic is equivalent to the following statement: there exist constants $c > 0$ and $k \in \mathbb{N}$ such that, if P_i is the hyperplane perpendicularly bisecting the segment $[p_i, p_{i+k}]$, then for all j P_{jk} separates $P_{(j-1)k}$ from $P_{(j+1)k}$, and $d(P_{jk}, P_{(j+1)k}) > c$. This is an easy exercise, and we note that K, δ determine k, c , and vice versa.

Now consider a representation ρ' close to ρ , and let $\tau' = \tau_{\rho', x}$. Up to the action of F_n there are only finitely many sequences of tree edges of length $2k$, and hence the relative position (i.e. up to isometry) of P_i and P_{i+k} , over all primitive leaves and all i , is determined by the ρ image of a finite number of words of F_n . These images each vary continuously with ρ , and hence for ρ'

sufficiently close to ρ , we have that the separation and distance properties for the P'_i still hold, with modified constants. Hence the primitive leaves are still (uniformly) quasi-geodesically mapped by τ' .

Finally we prove (3). Let A be a proper free factor, so that $F_n = A * B$ with A and B nontrivial. Suppose ρ is primitive-stable. If A is cyclic, then $\rho|_A$ being Schottky is equivalent to A 's generator having loxodromic image, and this is an immediate consequence of having a quasi-geodesic orbit. Hence we may now assume A has rank at least 2. By (2), there is a neighborhood U of ρ consisting of primitive-stable elements. Suppose $\rho|_A$ were not Schottky. Since $\mathcal{S}(A)$ is the interior of $\mathcal{D}(A)$ by Sullivan's theorem 2.2, and since $\mathcal{D}(A) \cup \mathcal{E}(A)$ is dense in $\mathcal{X}(A)$ by Lemma 2.1, we can perturb $\rho|_A$ arbitrarily slightly to get a dense representation. Leaving $\rho|_B$ unchanged we obtain $\rho' \in U$ with $\rho'|_A$ dense. Now let $g_m \in A$ be an infinite sequence of reduced words with $\rho'(g_m) \rightarrow id$. For any generator b of B , a sequence of elementary automorphisms multiplying b by generators of A (Nielsen moves) takes b to $g_m b$, which therefore is primitive. Note that each $g_m b$ is cyclically reduced, so primitive-stability of ρ' implies that the axes of $g_m b$ are uniformly quasi-geodesically mapped by $\tau_{\rho',x}$. But this contradicts the fact that $\rho'(g_m) \rightarrow id$ while the length of g_m goes to infinity. ■

Let

$$\mathcal{PS} = \mathcal{PS}(F_n) \subset \mathcal{X}(F_n)$$

be the set of conjugacy classes of primitive-stable representations. We have shown that \mathcal{PS} is an open $Out(F_n)$ invariant set containing the Schottky set. In fact,

THEOREM 3.3: *The action of $Out(F_n)$ on $\mathcal{PS}(F_n)$ is properly discontinuous.*

Proof. Let $\ell_\rho(g)$ denote the translation length of the geodesic representative of $\rho(g)$ for $g \in F_n$, and let $\|g\|$ denote the minimal combinatorial length, with respect to the fixed generators of F_n , of any element in the conjugacy class of g (equivalently it is the word length of g after being cyclically reduced).

Let C be a compact set in $\mathcal{PS}(F_n)$. For each $[\rho] \in C$ we have a positive lower bound for $\ell_\rho(w)/\|w\|$ over primitive elements of F_n , and a continuity argument as in part (2) of Lemma 3.2 implies that a uniform lower bound

$$\frac{\ell_\rho(w)}{\|w\|} > r > 0$$

holds over all $[\rho]$ in C . Now, on the other hand, an upper bound on this ratio holds trivially for any ρ by the triangle inequality applied to any $\tau_{\rho,x}$. Continuity again gives us a uniform upper bound

$$\frac{\ell_\rho(w)}{\|w\|} < R$$

for $[\rho] \in C$ (here, one should choose a compact preimage of C in $Hom(F_n, G)$, which is easy to do).

Now if $[\Phi] \in Out(F_n)$ satisfies $[\Phi](C) \cap C \neq \emptyset$, we apply the inequalities to conclude, for $[\rho]$ in this intersection, that

$$\|\Phi(w)\| \leq (1/r)\ell_\rho(\Phi(w)) = (1/r)\ell_{\rho \circ \Phi}(w) \leq (R/r)\|w\|.$$

The proof is then completed by the lemma below.

LEMMA 3.4: *For any A , the set*

$$\{f \in Aut(F_n) : \|f(w)\| \leq A\|w\| \quad \forall \text{ primitive } w\}$$

has finite image in $Out(F_n)$.

Proof. In fact much less is needed; it suffices to have the inequality only for w with $\|w\| \leq 2$. Let x_1, \dots, x_n be generators of F_n , and consider the action of F_n on the tree $\tilde{\Gamma}$ (its Cayley graph). For any $i, j \leq n$, let D denote the distance between the axis of $f(x_i)$ and the axis of $f(x_j)$. A look at the action on the tree indicates, if $D > 0$, that $\|f(x_i x_j)\| = 2D + \|f(x_i)\| + \|f(x_j)\|$. Hence, since $\|f(x_i x_j)\| \leq 2A$, we get an upper bound on D .

An upper bound on the pairwise distances between the axes of the $f(x_i)$ implies that there is a point on $\tilde{\Gamma}$ which is a bounded distance from all the axes simultaneously (minimize the sum of distances to the axes, which is proper and convex unless all the axes coincide). After conjugating f we may assume that this point is the origin. Now the bound $\|f(x_i)\| \leq A$ implies a finite number of choices for f . ■

4. Whitehead’s lemma and indiscrete primitive-stable representations

In this section we will prove that \mathcal{PS} is strictly bigger than the set of characters of Schottky representations. In particular, we will define the notion of a **blocking** curve on the boundary of a handlebody, and show

THEOREM 4.1: *Let $\rho : F_n \rightarrow G$ be discrete, faithful and geometrically finite with one cusp c , which is a blocking curve. Then ρ is primitive-stable.*

We will see below (Lemmas 4.4 and 4.5) that blocking curves are a non-empty class. Using the deformation theory of hyperbolic 3-manifolds one can then obtain primitive-stable points in the boundary of Schottky space — see the end of the section for details.

Note that, since $\mathcal{PS}(F_n)$ is open, this implies the existence of a rich class of primitive-stable representations, including dense ones (by Lemma 2.1), as well as discrete faithful ones with degenerate ends (since these are topologically generic in the boundary of Schottky space — see [11] and [12, Cor. A]).

First let us recall Whitehead’s criterion and define the notion of **blocking**. Whitehead studied the question of which elements in the free group are primitive. He found a necessary combinatorial condition, as part of an algorithm that decides the question of primitivity.

THE WHITEHEAD GRAPH. As before, we fix a generating set $X = \{x_1, \dots, x_n\}$ of F_n . For a word g in the generators and their inverses, the **Whitehead** graph $Wh(g) = Wh(g, X)$ is the graph with $2n$ vertices labeled $x_1, x_1^{-1}, \dots, x_n, x_n^{-1}$, and an edge from x to y^{-1} for each string xy that appears in g or in a cyclic permutation of g . For more information see Whitehead [48, 49], Stallings [44] and Otal [42].

Whitehead proved

LEMMA 4.2 (Whitehead): *Let g be a cyclically reduced word. If $Wh(g)$ is connected and has no cutpoint, then g is not primitive.*

Define the “reduced” Whitehead graph $Wh'(g)$ to be the same as $Wh(g)$ except that we do not count cyclic permutations of g . In other words, we do not consider the word xy where x is the last letter of g and y is the first, so $Wh'(g)$ may have one fewer edge than $Wh(g)$.

Let us say that a reduced word g is **primitive-blocking** if it does not appear as a subword of any cyclically reduced primitive word. An immediate corollary of Lemma 4.2 is:

COROLLARY 4.3: *If g is a reduced word with $Wh'(g)$ connected and without cutpoints, then g is primitive-blocking.*

Let us also say that g is **blocking** if some power g^k is primitive-blocking. A curve on the surface of the handlebody of genus n is blocking if a reduced representative of its conjugacy class in the fundamental group is blocking (with respect to our given generators).

An instructive example of a blocking curve occurs for even rank, when the handlebody is homeomorphic to the product of an interval with a surface with one boundary component:

LEMMA 4.4: *Let Σ be a surface with one boundary component. The curve $\partial\Sigma \times \{1/2\}$ in the handlebody $\Sigma \times [0, 1]$ is blocking with respect to standard generators of $\pi_1(\Sigma)$; in fact its square is primitive-blocking.*

Proof. Using standard generators $a_1, b_1, \dots, a_k, b_k$ for $\pi_1(\Sigma)$, the boundary is represented by $c = [a_1, b_1] \cdots [a_k, b_k]$. $\text{Wh}'(c)$ is a cycle minus one edge (corresponding to $b_k^{-1}a_1$), and $\text{Wh}'(c^2)$ contains the missing edge, and so by Whitehead's lemma is blocking. ■

One can construct other blocking curves on the boundary of any handlebody by explicit games with train tracks. We omit this approach, and instead study the relationship of the blocking condition to the Masur Domain in the measured lamination space.

LAMINATIONS AND THE WHITEHEAD CONDITION. Let $\mathcal{PML}(\partial H)$ denote Thurston's space of projectivized measured laminations on the boundary of the handlebody [15, 18]. Within this we have the Masur domain \mathcal{O} consisting of those laminations that have positive intersection number with every lamination that is a limit of meridians of H [36]. This is an open set of full measure in $\mathcal{PML}(\partial H)$ [36, 28].

Using Otal's work [42], we can extend Whitehead's condition to laminations on the boundary as follows. Any free generating set of F_n is dual to a system of disks on H , which cut it into a 3-ball (Nielsen). Given such a system $\Delta = \{\delta_1, \dots, \delta_n\}$ of disks and a lamination μ , realize both μ and the disk boundaries in minimal position — e.g. fix a hyperbolic metric on ∂H and make them geodesics. Otal calls μ **tight** with respect to Δ if there are no **waves** on Δ which are disjoint from μ . A wave on Δ is an arc properly embedded in $\partial H \setminus \Delta$, which is homotopic, rel endpoints, through H but not through ∂H into Δ . In particular, if μ is tight then no arcs of $\mu \setminus \Delta$ are waves. Hence, if a closed

curve is tight with respect to Δ , then its itinerary through the disks describes a **cyclically reduced** word in F_n with respect to the dual generators.

We define $Wh(\mu, \Delta)$ as follows: Cutting along Δ , ∂H becomes a planar surface with $2n$ boundary components, each labeled by δ_i^+ or δ_i^- . Let $w(\mu, \Delta)$ denote the union of the boundary components and the arcs of $\mu \setminus \Delta$. Then $Wh(\mu, \Delta)$ is obtained from $w(\mu, \Delta)$ by making the boundary components into vertices and identifying arcs of μ that connect the same boundary components. In particular, $Wh(\mu, \Delta)$ can be given a planar embedding (by choosing one arc from each equivalence class). If μ is a single closed curve, Δ is dual to the original generators and μ is tight with respect to Δ , then this is equivalent to the original definition.

Otal proved the following in [42]. We give a proof, since Otal’s thesis is hard to obtain.

LEMMA 4.5 (Otal): *If $\mu \in \mathcal{O}(H)$, then there is a generating set with dual disks Δ such that μ is tight with respect to Δ , and $Wh(\mu, \Delta)$ is connected and has no cutpoints.*

Proof. First note that $\inf_{\delta} \{i(\mu, \delta)\}$, where δ runs over meridians of H , is positive and realized. For if $\{\delta_i\}$ is a minimizing sequence such that infinitely many of the δ_i are distinct, then an accumulation point in $\mathcal{PML}(S)$ will have intersection number 0 with μ , contradicting $\mu \in \mathcal{O}(H)$. The same holds for disk systems, so we may choose a disk system Δ that minimizes $i(\mu, \Delta)$.

Now μ cannot have a wave with respect to Δ . If it did, then a surgery along such a wave would produce a new Δ' whose intersection number with μ is strictly smaller, contradicting the choice of Δ .

If $Wh(\mu, \Delta)$ is disconnected, then there is a loop β in the planar surface $P = \partial H \setminus \mathcal{N}(\Delta)$ which separates the boundary components, and does not intersect μ (here \mathcal{N} denotes a regular neighborhood). This gives a meridian that misses μ in ∂H , again contradicting $\mu \in \mathcal{O}(H)$.

If $Wh(\mu, \Delta)$ is connected but has a cutpoint, this is represented by a boundary component $\gamma \subset \partial P$, equal to one copy δ_i^{\pm} of a component of Δ (we are blurring the distinction between the disks in Δ and their boundaries in ∂H). Let $\bar{\gamma}$ denote the other copy. Then γ separates the planar complex $w(\mu, \Delta)$, and we may let X_1 be a component of $w(\mu, \Delta) \setminus \gamma$ which does not contain $\bar{\gamma}$. Now $X_1 \cup \gamma$ cuts P into a union of disks-with-holes, one of which, E , must contain $\bar{\gamma}$. Let β be the boundary component of E that separates $\bar{\gamma}$ from X_1 . Any arcs of

$w(\mu, \Delta)$ passing through β must pass through γ (since γ separates the interior of E from the rest of the graph), but arcs of X_1 incident to γ do not meet β ; hence $i(\beta, \mu)$ is strictly less than $i(\gamma, \mu)$.

Because β separates γ from $\bar{\gamma}$, cutting along β and regluing γ to $\bar{\gamma}$ yields again a connected planar surface — hence $\Delta \cup \{\beta\} \setminus \{\gamma\}$ is a new disk system, with strictly smaller intersection number with μ . Again this is a contradiction, so we conclude that $Wh(\mu, \Delta)$ is connected and without cutpoints. ■

Call a lamination λ **blocking**, with respect to Δ (or the dual generators), if λ has no waves with respect to Δ , and there is some k such that every length k subword of the infinite word determined by a leaf of λ passing across Δ does not appear in a cyclically reduced primitive word. Note that, for simple closed curves, this coincides with the previous definition of blocking. An immediate corollary of the above lemma is:

LEMMA 4.6: *A connected Masur-domain lamination (e.g. a simple closed curve or a filling lamination) on the boundary of a handlebody is blocking with respect to some generating set.*

Proof. Given $\mu \in \mathcal{O}(H)$ let Δ be as in Lemma 4.5. In a connected measured lamination every leaf is dense. Thus a sufficiently long leaf of μ would traverse every edge of $Wh(\mu, \Delta)$, and so the corresponding word is blocking by Corollary 4.3. Note for a simple closed curve this argument shows that its square is primitive-blocking. ■

BLOCKING CUSPS ARE PRIMITIVE-STABLE. We can now provide the proof of Theorem 4.1, namely that a geometrically finite representation with a single blocking cusp is primitive-stable.

Proof of Theorem 4.1. Let $N_\rho = \mathbb{H}^3/\rho(F_n)$ be the quotient manifold, and C_ρ its convex core. The geometrically finite hypothesis means that C_ρ is a union of a compact handlebody H and a subset of a parabolic cusp P (namely a vertical slab in a horoball modulo \mathbb{Z}) along an annulus A with core curve c in ∂H , which we are further assuming is a blocking curve.

We will prove that all primitive elements of F_n are represented by geodesics in a fixed compact set $K \subset C_\rho$. The idea is that in order to leave a compact set, a primitive element must wind around the cusp, and this will be prohibited by the blocking property.

Let γ be a closed geodesic in N_ρ . Then $\gamma \subset C_\rho$. The orthogonal projection $P \rightarrow \partial P$ gives a retraction $\pi : C_\rho \rightarrow H$. Let $\hat{\gamma} = \pi(\gamma)$.

CLAIM: $\hat{\gamma}$ is uniformly quasi-geodesic in H , with constants independent of γ . More precisely, the lift $\tilde{\gamma}$ of $\hat{\gamma}$ to the universal cover \tilde{H} is uniformly quasi-geodesic with respect to the path metric. This follows from a basic fact about any family Q of disjoint horoballs in \mathbb{H}^3 :

LEMMA 4.7: Let Q be a family of disjoint open horoballs in \mathbb{H}^m , and let $\pi_Q : \mathbb{H}^m \rightarrow \mathbb{H}^m \setminus Q$ be given by orthogonal projection from Q to ∂Q and identity in $\mathbb{H}^m \setminus Q$. If L is a geodesic in \mathbb{H}^m , then $\pi_Q(L)$ is a quasi-geodesic in $\mathbb{H}^m \setminus Q$ with its path metric, with constants independent of Q or L .

Proof. This is closely related to statements in Farb [17] and Klarreich [29] and can also be proved in greater generality, e.g. for uniformly separated quasi-convex subsets of a δ -hyperbolic space. We will sketch a proof for completeness.

Note first, there is a constant r_0 such that, if P_1 and P_2 are horoballs in \mathbb{H}^m with $d(P_1, P_2) \geq 1$, then any two geodesic segments connecting P_1 to P_2 in their common exterior lie within r_0 -neighborhoods of each other.

If $Q' \subset Q$ is obtained by retracting horoballs to concentric horoballs at depth bounded by r_0 , then it suffices to prove the theorem for Q' . This is because any arc on $\pi_Q(L)$ can be approximated in a controlled way by an arc on $\pi_{Q'}(L)$. Moreover, given L it suffices to prove the theorem for the union $Q_L \subset Q$ of horoballs that intersect L , since $\pi_Q(L) = \pi_{Q_L}(L)$ and $\mathbb{H}^m \setminus Q \subset \mathbb{H}^m \setminus Q_L$. We can therefore reduce to the case that any two components of Q are separated by distance at least 1, and L penetrates each component of Q to depth at least r_0 .

Let β be a geodesic in $\mathbb{H}^m \setminus Q$ with endpoints on $\pi_Q(L)$. It is therefore a concatenation of hyperbolic geodesics in $\mathbb{H}^m \setminus Q$ with endpoints on ∂Q , alternating with geodesics on ∂Q in the path (Euclidean) metric. Let α be one of the hyperbolic geodesic segments, with endpoints on horoballs $P_1, P_2 \in Q$. Let α' be the component of $L \setminus (P_1 \cup P_2)$ with endpoints on P_1 and P_2 , respectively — then α and α' are within r_0 of each other. If α' meets another component P_3 of Q , then it penetrates it by at least r_0 ; but this means that α meets P_3 as well, a contradiction. Hence α' is in fact a component of $L \setminus Q$. This means that α can be replaced by an arc traveling along ∂P_1 , L and ∂P_2 of uniformly comparable length. Replacing all hyperbolic segments in this way, and then straightening

the arcs of intersection of the resulting path with ∂Q , we obtain a segment of $\pi_Q(L)$ whose length is comparable with that of β . Since the endpoints of β were arbitrary points of $\pi_Q(L)$, this gives uniform quasi-geodesic constants for $\pi_Q(L)$. ■

Now, H retracts to a spine of the manifold which can be identified with the bouquet Γ , so \tilde{H} is naturally quasi-isometric to the Cayley graph $\tilde{\Gamma}$ of F_n . In a tree, uniform quasi-geodesics are uniformly close to their geodesic representatives.

Returning to our geodesic γ in N_ρ , suppose that it travels a large distance from the compact core H . In particular, a component a of $\gamma \cap P$ is very long, so its image $\pi(a)$ must wind many times around A (in fact the number of times is exponential in the length of a). By Lemma 4.7, the lift $\tilde{\gamma}$ of $\hat{\gamma} = \pi(\gamma)$ is quasi-geodesic in \tilde{H} . Since it contains a high power of the core curve of A , its retraction to the tree contains a high power of some representative c' of the curve c . This subword $(c')^m$ must be uniformly close to a segment of the form $(c^*)^m$, where c^* is a cyclically reduced representative of c . It follows that the geodesic representative of $\tilde{\gamma}$ in the tree is uniformly close to a segment of this form as well. Since a tree is 0-hyperbolic, it follows that the geodesic representative of $\tilde{\gamma}$ in the tree actually contains a high power of c^* .

But since c is blocking, this means that γ cannot have been primitive. We conclude that all primitive geodesics are trapped in a fixed compact core of C_ρ . The retraction of this core to the spine of H therefore lifts to a quasi-isometry, and stability of quasi-geodesics again implies that each primitive geodesic in the tree is uniformly quasi-geodesic. ■

Proof of Theorem 1.1. In view of Lemma 3.2 and Theorem 3.3, all that remains to prove is the existence of a non-Schottky primitive stable representation. By Theorem 4.1, any discrete, faithful and geometrically finite representation with a single blocking cusp would do.

Given a curve c on the boundary of a handlebody H , a sufficient condition that it be realizeable as the single cusp of a geometrically finite representation is that it be homotopically nontrivial in the handlebody, and that any homotopically nontrivial proper annulus in H with both boundary components on c must be parallel into c . This is a consequence of Thurston's geometrization theorem (see e.g. [26]) or of Maskit [35].

Otal [42] shows that any curve in the Masur Domain has complement satisfying these topological conditions (even more strongly, that $(H, \partial H \setminus c)$ is acylindrical, so there are no essential annuli at all in $(H, \partial H \setminus c)$), and hence can appear as the single cusp of a geometrically finite representation. By Lemma 4.6 such a curve is also blocking with respect to some generating set, and hence gives us a primitive-stable representation.

When n is even the example of Lemma 4.4, where $c = \partial\Sigma$, also suffices, even though c is not in the Masur domain. Note that we can explicitly construct Fuchsian representations for which c is the unique cusp, and Lemma 4.4 provides the blocking property. ■

5. Further remarks and questions

Having established that $Out(F_n)$ acts properly discontinuously on $\mathcal{PS}(F_n)$, and that $\mathcal{PS}(F_n)$ is strictly larger than $\mathcal{S}(F_n)$, one is naturally led to study the dynamical decomposition of $\mathcal{X}(F_n)$. In particular, we ask if \mathcal{PS} is the maximal domain of discontinuity, and what happens in its complement. We have only partial results in this direction.

OUTSIDE \mathcal{PS} . The polar opposite of the primitive-stable characters are the **redundant characters** $\mathcal{R}(F_n)$, defined (after Lubotzky) as follows: $[\rho]$ is redundant if there is a proper free factor A of F_n such that $\rho(A)$ is dense. Note that $\mathcal{R}(F_n)$ is $Out(F_n)$ -invariant. Clearly $\mathcal{R}(F_n)$ and $\mathcal{PS}(F_n)$ are disjoint, by Lemma 3.2.

The set $\mathcal{E}(F_n)$ of representations with dense image is open (Lemma 2.1), and it follows (applying this to the factors) that $\mathcal{R}(F_n)$ is open. The action of $Out(F_n)$ on $\mathcal{R}(F_n)$ cannot be properly discontinuous, and in fact there is a larger set for which we can show this.

Let $\mathcal{PS}'(F_n)$ be the set of (conjugacy classes of) representations ρ which are Schottky on every proper free factor. Hence $\mathcal{PS}(F_n) \subset \mathcal{PS}'(F_n)$ by Lemma 3.2, and $\mathcal{PS}'(F_n)$ is still in the complement of $\mathcal{R}(F_n)$. Let $\mathcal{R}'(F_n) = \mathcal{X}(F_n) \setminus \mathcal{PS}'(F_n)$.

LEMMA 5.1:

- (1) If $n \geq 3$ then $\mathcal{R}(F_n)$ is dense in $\mathcal{R}'(F_n)$.
- (2) No point of $\mathcal{R}'(F_n)$ can be in a domain of discontinuity for $Out(F_n)$. Equivalently, any open invariant set in $\mathcal{X}(F_n)$ on which $Out(F_n)$ acts properly discontinuously must be contained in $\mathcal{PS}'(F_n)$.

The proof is quite analogous to part (2) of Lemma 3.2. In fact we note that part (2) of Lemma 3.2 is an immediate corollary of Lemma 5.1 and Theorem 3.3.

Proof. For (1), let $[\rho] \in \mathcal{R}'(F_n)$ and let A be a proper free factor such that $\rho|_A$ is not Schottky. We may assume A has rank at least 2 since $n \geq 3$. Hence (as in the proof of Lemma 3.2) $\rho|_A$ is approximated by representations with dense image. It follows that ρ itself is approximated by representations dense on A , so \mathcal{R} is dense in \mathcal{R}' .

For (2), we will show that for every neighborhood U of $[\rho] \in \mathcal{R}'(F_n)$ there is an infinite set of elements $[\phi] \in \text{Out}(F_n)$ such that $[\phi](U) \cap U \neq \emptyset$. Since $\mathcal{X}(F_n)$ is locally compact, this implies $[\rho]$ cannot be in any open set on which $\text{Out}(F_n)$ acts properly discontinuously.

Suppose first $n \geq 3$. Since $\mathcal{R}(F_n)$ is dense in $\mathcal{R}'(F_n)$, it suffices to consider the case that $\rho \in \mathcal{R}(F_n)$. Again let A be a proper free factor on which ρ is dense. We can assume that $F_n = A * B$ where B is generated by one element b . Now let $g_m \in A$ be a sequence such that $\rho(g_m) \rightarrow id$, and let $\phi_m \in \text{Aut}(F_n)$ be the automorphism that is the identity on A and sends b to $g_m b$. Note that $[\phi_m]$ has infinite order in $\text{Out}(F_n)$. The number of powers of ϕ_m that take ρ to any fixed neighborhood of itself goes to ∞ as $m \rightarrow \infty$, because $\rho(g_m) \rightarrow id$. This concludes the proof for $n \geq 3$.

If $n = 2$, $\mathcal{R}(F_n)$ is empty. However, every $[\rho] \in \mathcal{R}'(F_n)$ has primitive element mapping to a nonloxodromic, so ρ may be approximated by a representation ρ' sending a generator to an irrational elliptic. The same argument as above can then be applied to ρ' . ■

We remark that \mathcal{PS}' is indeed strictly larger than \mathcal{PS} , at least for even rank: If $n = 2g$, represent the handlebody as an I -bundle over a genus g surface Σ with one puncture, and let $\rho : \pi_1(\Sigma) \rightarrow \text{PSL}_2(\mathbb{C})$ be a degenerate surface group with no nonperipheral parabolics (i.e. every parabolic in $\rho(\pi_1(\Sigma))$ is conjugate to the image of an element of $\pi_1(\partial\Sigma)$, ρ is discrete and faithful, and at least one end of the resulting manifold is geometrically infinite). The puncture cannot be in any proper free factor of F_n , because it is represented by a curve on the boundary of the handlebody whose complement is incompressible. Hence the restriction of ρ to any proper free factor A has no parabolics. By the Thurston–Canary Covering Theorem [14] and the Tameness Theorem [1, 10], $\rho(A)$ cannot be geometrically infinite, so it must be Schottky, and hence $[\rho] \in \mathcal{PS}'(F_n)$.

On the other hand, every nonperipheral nonseparating simple curve on Σ is primitive, and the ending lamination of a degenerate end can be approximated by such curves. Hence the uniform quasi-geodesic condition fails on primitive elements, and $[\rho]$ cannot be in $\mathcal{PS}(F_n)$.

Note that this example can be approximated by elements of $\mathcal{PS}(F_n)$, as well as by $\mathcal{R}'(F_n)$ (the latter by using Cusps are Dense [37] or the Density Theorem for surface groups [9]). This leads us to ask:

QUESTION 5.2: Is $\mathcal{PS}(F_n)$ the interior of $\mathcal{PS}'(F_n)$?

In particular, in view of Lemma 5.1, a positive answer would imply

CONJECTURE 5.3: $\mathcal{PS}(F_n)$ is the domain of discontinuity of $Out(F_n)$ acting on $\mathcal{X}(F_n)$.

We remark that, a priori, there may not be any such domain, i.e. there may be no maximal set on which the action is properly discontinuous.

RANK 2. For the free group of rank 2, we have already seen that some of our statements are slightly different. In particular, $\mathcal{R}(F_2)$ is empty since no one-generator subgroup of $PSL_2(\mathbb{C})$ is dense. Moreover, $\mathcal{PS}'(F_2)$ is exactly the set for which every generator is loxodromic, and this is dense in $\mathcal{X}(F_2)$ since it is the complement of countably many proper algebraic sets.

Question 5.2 in particular asks, therefore, whether $\mathcal{PS}(F_2)$ is dense. It is not clear (to the author) whether this is true, but there is some evidence against it (see below).

Another important feature of rank 2 is that the conjugacy class of the commutator of the generators, and its inverse, are permuted by automorphisms. It follows that the trace of the commutator is an $Out(F_2)$ -invariant function on $\mathcal{X}(F_2)$, and one can therefore study level sets of this function.

The domain of discontinuity of $Out(F_2)$ was studied by Bowditch and Tan–Wong–Zhang [46] and [47]. Bowditch defines the following condition on $[\rho] \in \mathcal{X}(F_n)$, which Tan–Wong–Zhang call condition BQ:

- (1) $\rho(x)$ is loxodromic for all primitive $x \in F_2$.
- (2) The number of conjugacy classes of primitive elements x such that $|\text{tr}(\rho(x))| \leq 2$ is finite.

They show, using Bowditch's work, that $Out(F_2)$ acts properly discontinuously on the invariant open set BQ . It is still unclear whether BQ is the largest such set.

Note that condition (1) is equivalent to membership in $\mathcal{PS}'(F_2)$. It is evident that $\mathcal{PS}(F_2) \subset BQ$, and it seems plausible that they are equal.

Note also that computer experiments indicate that the intersection of BQ with a level set of the commutator trace function is *not* dense in the level set (see Dumas [16]). In particular, the slice corresponding to trace -2 consists of **type-preserving representations** of the punctured-torus group, i.e. those with parabolic commutator, and empirically it seems that BQ in this slice coincides with the quasi-Fuchsian representations (which are all primitive-stable too, by Theorem 4.1 combined with Lemma 4.4). Bowditch has conjectured that this is in fact the case, and this seems to be a difficult problem. At any rate this appears to be evidence against the density of $\mathcal{PS}(F_2)$.

ERGODICITY. The question of the ergodic decomposition of $Out(F_n)$ on $\mathcal{X}(F_n)$ is still open. Note, in rank 2, the decomposition must occur along level sets of the commutator trace function. In rank 3 and higher our observations indicate that the simplest possible situation is that, outside $\mathcal{PS}(F_n)$, the action is ergodic, which we pose as a variation of Lubotzky's original question:

QUESTION 5.4: Let $n \geq 3$. Is there a decomposition of $\mathcal{X}(F_n)$ into a domain where the action is properly discontinuous, and a set where it is ergodic? More pointedly, does $Out(F_n)$ act ergodically on the complement of $\mathcal{PS}(F_n)$ in $\mathcal{X}(F_n)$?

In Gelander–Minsky [20] we show that in fact the action on $\mathcal{R}(F_n)$ is ergodic and topologically minimal. So if for example $\mathcal{PS}' \setminus \mathcal{PS}$ and $\mathcal{R}' \setminus \mathcal{R}$ have measure 0, we would have a positive answer for the above question.

UNDERSTANDING $\mathcal{PS}(F_n)$. It would also be nice to have a clearer understanding of the boundary of $\mathcal{PS}(F_n)$, and of which discrete representations $\mathcal{PS}(F_n)$ contains.

From Lemma 3.2 we know that any discrete faithful representation with cusp curves that have compressible complement cannot be in $\mathcal{PS}(F_n)$. We've also mentioned the degenerate surface groups which are in $\mathcal{PS}'(F_n)$ but not $\mathcal{PS}(F_n)$.

If, however, ρ is discrete and faithful without parabolics and is not Schottky, then it has an ending lamination which must lie in the Masur domain, and hence is blocking by Lemma 4.6. Hence it would be plausible to expect:

CONJECTURE 5.5: *Every discrete faithful representation of F_n without parabolics is primitive-stable.*

More generally, a discrete faithful representation has a possibly disconnected ending lamination, whose closed curve components are parabolics. All the examples we have considered suggest this conjecture:

CONJECTURE 5.6: *A discrete faithful representation of F_n is primitive-stable if and only if every component of its ending lamination is blocking.*

NOTE: Since the first version of this article appeared, Jeon–Kim–Lecuire–Oshika [24] gave a positive solution to Conjectures 5.5 and 5.6, using the work of Mj [39] on Cannon–Thurston maps.

It might also be interesting to think about which representations with discrete image (but not necessarily faithful) are primitive-stable. In [38] we construct many primitive-stable representations whose images uniformize knot complements. What properties of a marked 3-manifold correspond to primitive stability?

Another interesting question is:

QUESTION 5.7: How do we produce computer pictures of $\mathcal{PS}(F_n)$?

For rank $n = 2$, the character variety has complex dimension 2, and one can try to draw slices of dimension 1. Komori–Sugawa–Wada–Yamashita developed a program for drawing Bers slices, which are parts of the discrete faithful locus [31, 30], and Dumas refined this using Bowditch’s work [16]. In particular, what Dumas’ program is really doing is drawing slices of Bowditch’s domain BQ. If indeed $BQ = \mathcal{PS}(F_2)$, then this produces images of $\mathcal{PS}(F_2)$ as well.

OTHER TARGET GROUPS. The discussion can be extended to other noncompact Lie groups, with moderate success. Let us consider first the case of $\text{Isom}_+(\mathbb{H}^d) \cong SO(d, 1)$ for all $d \geq 2$, where $d = 3$ is the case we have been considering. The definition of \mathcal{PS} is unchanged, and stability of quasi-geodesics works in all dimensions in the same way. Lemma 2.1 and Theorem 3.3 still

hold. However, Sullivan's theorem (Theorem 2.2) equating Schottky representations with those in the interior of $\mathcal{D}(F_n)$ is no longer available. Schottky representations in higher dimensions can be replaced by **convex-cocompact** representations: discrete and faithful, with convex hull of the limit set having a compact quotient. Now the conclusions of Lemma 3.2 must be changed somewhat: A convex-cocompact representation is certainly still primitive-stable, but it is not clear that $\text{int}(\mathcal{D}(F_n)) \subset \mathcal{PS}(F_n)$. For a primitive-stable ρ , the proof of Lemma 3.2 shows that ρ restricted to each proper free factor A is in $\text{int}(D(A))$, but not that it is convex-cocompact.

For $d \geq 3$, the natural embedding of $\text{Isom}_+(\mathbb{H}^3)$ in $\text{Isom}_+(\mathbb{H}^d)$ clearly preserves primitive-stability and nondiscreteness, so it is still true that \mathcal{PS} contains indiscrete representations in higher dimension (and hence dense ones, by Lemma 2.1).

The case of $d = 2$ is slightly trickier. When n is even, we have given an example of a blocking curve that is the boundary of a one-holed surface, and so Theorem 4.1 shows that a finite-area Fuchsian structure on this surface, which gives an element of $\mathcal{X}(F_n, \text{Isom}_+(\mathbb{H}^2)) = \mathcal{X}(F_n, \text{PSL}_2(\mathbb{R}))$, is primitive-stable but not Schottky. However, when n is odd we have no such example, and it is unclear to me if \mathcal{PS} contains indiscrete elements. For $n = 2$ and $d = 2$, Goldman [22] has given the ergodic decomposition for each level set of the trace function. Our indiscrete primitive-stable examples in this case are punctured-torus groups whose boundary curve is elliptic, associated to hyperbolic structures on the torus with one cone point. Goldman shows that the action on the set of representations corresponding to cone structures is proper, but we do not know if all of them are primitive-stable.

For other noncompact rank-1 semisimple Lie groups, namely isometry groups of the nonhomogeneous negatively curved symmetric spaces, primitive-stability can again be defined in the same way. The hyperbolic plane always embeds geodesically in such a space, with its full isometry group acting. This is not easily extracted from the literature but is well-known; see Mostow [41], Bridson–Haefliger [8, Chap. II.10] and Allcock [2] for the requisite machinery, or note that these spaces are just the real, complex, and quaternionic hyperbolic spaces, and the Cayley plane, in all of which one can restrict to a real subspace. Thus, in these cases our examples for F_{2g} can be used.

Other cases, such as higher rank semisimple groups, presumably require a rethinking of the definitions, but there is again no reason to think that the

geometric decomposition of Lemma 2.1 should be the right dynamical decomposition for the action of $Out(F_n)$.

A completely different picture holds in the setting of nonlocally connected groups. In the case of $G = SL_2(K)$, with K a non-Archimedean local field of characteristic $\neq 2$, as well as $G = Aut(T)$ for a tree T , Glasner [21] has shown that $Aut(F_n)$ acts ergodically on $Hom(F_n, G)$.

OTHER DOMAIN GROUPS. If we replace F_n by $\pi = \pi_1(S)$ for a closed orientable surface S , $Out(F_n)$ is replaced by $Out(\pi) = MCG(S)$, and the primitive elements are replaced by their natural analogue, the simple curves in the surface. The Schottky representations are replaced by the quasi-Fuchsian representations $QF(S)$. We can define $\mathcal{PS}(\pi)$ in a similar way, but now there is no good reason to think that $\mathcal{PS}(\pi)$ is strictly larger than the quasi-Fuchsian locus. Indeed, every boundary point of QF can be shown not to lie in $\mathcal{PS}(\pi)$ (nor in any domain of discontinuity for $MCG(S)$ — see also Souto–Storm [43] for a related result), and this is because all parabolics are simple curves, and all ending laminations are limits of simple curves. It is still open as far as I know whether in fact $\mathcal{PS}(\pi)$ is equal to QF ; this is closely related to (but formally weaker than) Bowditch’s conjecture in this setting.

The situation is different for a nonorientable surface S . Just recently, Lee has shown [32] that in this case there are primitive-stable elements in the boundary of QF , namely those representations corresponding to twisted I -bundles over S for which no simple curve is parabolic.

One can also consider $\mathcal{X}(H, PSL_2(\mathbb{C}))$ where H is any fundamental group of a hyperbolic 3-manifold, and the dynamics are by $Out(H)$; see Canary–Storm [13] (which uses techniques quite different from ours). In some sense, the more complicated the group, the weaker we should expect the correspondence between the geometric and dynamical decompositions. An extreme example is when H is a nonuniform lattice in $PSL_2(\mathbb{C})$. In this case $\mathcal{X}(H)$ has positive dimension, while Mostow rigidity tells us that $\mathcal{D}(H)$ is a single point. On the other hand, Mostow also tells us that $Out(H)$ is finite in this case, so that it acts properly discontinuously on the whole of $\mathcal{X}(H)$.

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