MICROLOCAL ANALYSIS OF ASYMPTOTICALLY HYPERBOLIC SPACES AND HIGH ENERGY RESOLVENT ESTIMATES

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Abstract. In this paper we describe a new method for analyzing the Laplacian on asymptotically hyperbolic spaces, which was introduced in [18]. This new method in particular constructs the analytic continuation of the resolvent for even metrics (in the sense of Guillarmou), and gives high energy estimates in strips. The key idea is an extension across the boundary for a problem obtained from the Laplacian shifted by the spectral parameter. The extended problem is non-elliptic – indeed, on the other side it is related to the Klein-Gordon equation on an asymptotically de Sitter space – but nonetheless it can be analyzed by methods of Fredholm theory. This method is a special case of a more general approach to the analysis of PDEs which includes, for instance, Kerr-de Sitter and Minkowski type spaces; see [18] for details. The present paper is self-contained, and deals with asymptotically hyperbolic spaces without burdening the reader with material only needed for the analysis of the Lorentzian problems considered in [18].

1. Introduction

In this paper we describe a new method for analyzing the Laplacian on asymptotically hyperbolic, or conformally compact, spaces, which was introduced in [18]. This new method in particular constructs the analytic continuation of the resolvent for even metrics (in the sense of Guillarmou [9]), and gives high energy estimates in strips. The key idea is an extension across the boundary for a problem obtained from the Laplacian shifted by the spectral parameter. The extended problem is non-elliptic – indeed, on the other side it is related to the Klein-Gordon equation on an asymptotically de Sitter space – but nonetheless it can be analyzed by methods of Fredholm theory. In [18] these methods, with some additional ingredients, were used to analyze the wave equation on Kerr-de Sitter space-times; the present setting is described there as the simplest application of the tools introduced. The purpose of the present paper is to give a self-contained treatment of conformally compact spaces, without burdening the reader with the additional machinery required for the Kerr-de Sitter analysis.

We start by recalling the definition of manifolds with *even* conformally compact metrics. These are Riemannian metrics g_0 on the interior of an n-dimensional compact manifold with boundary X_0 such that near the boundary Y, with a product

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decomposition nearby and a defining function x, they are of the form

$$g_0 = \frac{dx^2 + h}{x^2},$$

where h is a family of metrics on $Y = \partial X_0$ depending on x in an even manner, i.e. only even powers of x show up in the Taylor series. (There is a much more natural way to phrase the evenness condition, see [9, Definition 1.2].) We also write $X_{0,\mathrm{even}}$ for the manifold X_0 when the smooth structure has been changed so that x^2 is a boundary defining function; thus, a smooth function on X_0 is even if and only if it is smooth when regarded as a function on $X_{0,\mathrm{even}}$. The analytic continuation of the resolvent in this category (but without the evenness condition) was obtained by Mazzeo and Melrose [11] (Agmon [1] and Perry [16, 17] had similar results in the restricted setting of hyperbolic quotients), with the possibility of some essential singularities at pure imaginary half-integers noticed by Borthwick and Perry [2]. Guillarmou [9] showed that for even metrics the latter do not exist, but generically they do exist for non-even metrics, by a more careful analysis utilizing the work of Graham and Zworski [8]. Further, if the manifold is actually asymptotic to hyperbolic space (note that hyperbolic space is of this form in view of the Poincaré model), Melrose, Sá Barreto and Vasy [13] proved high energy resolvent estimates in strips around the real axis via a parametrix construction; these are exactly the estimates that allow expansions for solutions of the wave equation in terms of resonances. Estimates just on the real axis were obtained by Cardoso and Vodev for more general conformal infinities [3, 22]. One implication of our methods is a generalization of these results: we allow general conformal infinities, and obtain estimates in arbitrary strips.

Below $\dot{\mathcal{C}}^\infty(X_0)$ denotes 'Schwartz functions' on X_0, i.e. $\mathcal{C}^\infty$ functions vanishing with all derivatives at ∂X_0, and $\mathcal{C}^{-\infty}(X_0)$ is the dual space of 'tempered distributions' (these spaces are naturally identified for X_0 and $X_{0,\mathrm{even}}$), while $H^s(X_{0,\mathrm{even}})$ is the standard Sobolev space on $X_{0,\mathrm{even}}$ (corresponding to extension across the boundary, see e.g. [10, Appendix B], where these are denoted by $\bar{H}^s(X^\circ_{0,\mathrm{even}})$). For instance, $\|u\|^2_{H^1(X_{0,\mathrm{even}})} = \|u\|^2_{L^2(X_{0,\mathrm{even}})} + \|du\|^2_{L^2(X_{0,\mathrm{even}})}$, with the norms taken with respect to any *smooth* Riemannian metric on $X_{0,\mathrm{even}}$ (all choices yield equivalent norms by compactness). Here we point out that while $x^2 g_0$ is a smooth non-degenerate section of the pull-back of T^*X_0 to $X_{0,\mathrm{even}}$ (which essentially means that it is a smooth, in $X_{0,\mathrm{even}}$, non-degenerate linear combination of dx and dy_j in local coordinates), as $\mu = x^2$ means $d\mu = 2x\,dx$, it is actually not a smooth section of $T^*X_{0,\mathrm{even}}$. However, $x^{n+1}\,|dg_0|$ is a smooth non-degenerate density, so $L^2(X_{0,\mathrm{even}})$ (up to norm equivalence) is the L^2 space given by the density $x^{n+1}\,|dg_0|$, i.e. is $x^{-(n+1)/2}L^2_{g_0}(X_0)$, i.e.

$$\|x^{-(n+1)/2}u\|_{L^2(X_{0,\mathrm{even}})} \sim \|u\|_{L^2_{g_0}(X)}.$$

Further, in local coordinates (μ, y), using $2\partial_\mu = x^{-1}\partial_x$, the $H^1(X_{0,\mathrm{even}})$ norm of u is equivalent to

$$\|u\|^2_{L^2(X_{0,\mathrm{even}})} + \|x^{-1}\partial_x u\|^2_{L^2(X_{0,\mathrm{even}})} + \sum_{j=1}^{n-1} \|\partial_{y_j} u\|^2_{L^2(X_{0,\mathrm{even}})}.$$

We also let $H^s_\hbar(X_{0,\mathrm{even}})$ be the standard semiclassical Sobolev space, i.e. for h bounded away from 0 this is equipped with a norm equivalent to the standard fixed (h-independent) norm on $H^s(X_{0,\mathrm{even}})$, but the uniform behavior as $h \to 0$ is different; e.g. locally the $H^1_\hbar(X)$ norm is given by $\|u\|^2_{H^1_\hbar} = \sum_j \|hD_j u\|^2_{L^2} + \|u\|^2_{L^2}$, see [6, 7]. Thus, in (1.1), for $s = 1$ (which is possible when $C < 1/2$, i.e. if one only considers the continuation into a small strip beyond the continuous spectrum),

$$s=1 \qquad \Longrightarrow \qquad \|u\|_{H^{s-1}_{|\sigma|^{-1}}(X_{0,\mathrm{even}})} = \|u\|_{L^2(X_{0,\mathrm{even}})}$$
$$\text{and } \|u\|^2_{H^{s}_{|\sigma|^{-1}}(X_{0,\mathrm{even}})} = \|u\|^2_{L^2(X_{0,\mathrm{even}})} + |\sigma|^{-2}\|du\|^2_{L^2(X_{0,\mathrm{even}})},$$

with the norms taken with respect to any smooth Riemannian metric on $X_{0,\mathrm{even}}$.

Theorem. *(See Theorem 5.1 for the full statement.) Suppose that X_0 is an n-dimensional manifold with boundary Y with an even Riemannian conformally compact metric g_0. Then the inverse of*

$$\Delta_{g_0} - \left(\frac{n-1}{2}\right)^2 - \sigma^2,$$

written as $\mathcal{R}(\sigma) : L^2 \to L^2$, has a meromorphic continuation from $\operatorname{Im}\sigma \gg 0$ to $\mathbb{C}$,

$$\mathcal{R}(\sigma) : \dot{\mathcal{C}}^\infty(X_0) \to \mathcal{C}^{-\infty}(X_0),$$

with poles with finite rank residues. If in addition (X_0, g_0) is non-trapping, then non-trapping estimates hold in every strip $|\operatorname{Im}\sigma| < C$, $|\operatorname{Re}\sigma| \gg 0$: for $s > \frac{1}{2} + C$,

$$\|x^{-(n-1)/2+\imath\sigma}\mathcal{R}(\sigma)f\|_{H^{s}_{|\sigma|^{-1}}(X_{0,\mathrm{even}})} \leq \tilde{C}|\sigma|^{-1}\|x^{-(n+3)/2+\imath\sigma}f\|_{H^{s-1}_{|\sigma|^{-1}}(X_{0,\mathrm{even}})}. \tag{1.1}$$

If f has compact support in X_0°, the $s-1$ norm on f can be replaced by the $s-2$ norm. For suitable $\delta_0 > 0$, the estimates are valid in regions $-C < \operatorname{Im}\sigma < \delta_0|\operatorname{Re}\sigma|$ if the multipliers $x^{\imath\sigma}$ are slightly adjusted.

Further, as stated in Theorem 5.1, the resolvent is *semiclassically outgoing* with a loss of h^{-1}, in the sense of recent results of Datchev and Vasy [4] and [5]. This means that for mild trapping (where, in a strip near the spectrum, one has polynomially bounded resolvent for a compactly localized version of the trapped model) one obtains resolvent bounds of the same kind as for the above-mentioned trapped models, and lossless estimates microlocally away from the trapping. In particular, one obtains logarithmic losses compared to non-trapping on the spectrum for hyperbolic trapping in the sense of [23, Section 1.2], and polynomial losses in strips, since for the compactly localized model this was recently shown by Wunsch and Zworski [23].

Our method is to change the smooth structure, replacing x by $\mu = x^2$, conjugate the operator by an appropriate weight as well as remove a vanishing factor of μ, and show that the new operator continues smoothly and non-degenerately (in an appropriate sense) across $\mu = 0$, i.e. Y, to a (non-elliptic) problem which we can analyze utilizing by now almost standard tools of microlocal analysis. These steps are reflected in the form of the estimate (1.1); μ shows up in the use of evenness, conjugation due to the presence of $x^{-(n+1)/2+\imath\sigma}$, and the two halves of the vanishing factor of μ being removed in $x^{\pm 1}$ on the left and right hand sides.

While it might seem somewhat ad hoc, this construction in fact has origins in wave propagation in one higher dimensional (i.e. $n+1$-dimensional) Lorentzian spaces – either Minkowski space, or de Sitter space blown up at a point at future infinity. Namely in both cases the wave equation (and the Klein-Gordon equation on de Sitter space) is a totally characteristic, or b-, PDE, and after a Mellin transform this gives a PDE on the sphere at infinity in the Minkowski case, and on the front face of the blow-up in the de Sitter setting. These are exactly the PDE arising by the process described in the previous paragraph, with the original manifold X_0 lying in the interior of the light cone in Minkowski space (so there are two copies, at future and past infinity) and in the interior of the backward light cone from the blow-up point in the de Sitter case; see [18] for more detail. This relationship, restricted to the X_0-region, was exploited in [20, Section 7], where the work of Mazzeo and Melrose was used to construct the Poisson operator on asymptotically de Sitter spaces. Conceptually the main novelty here is that we work directly with the extended problem, which turns out to simplify the analysis of Mazzeo and Melrose in many ways and give a new explanation for Guillarmou's results as well as yield high energy estimates.

We briefly describe this extended operator, P_σ. It has radial points at the conormal bundle $N^*Y \setminus o$ of Y in the sense of microlocal analysis, i.e. the Hamilton vector field is radial at these points, i.e. is a multiple of the generator of dilations of the fibers of the cotangent bundle there. However, tools exist to deal with these, going back to Melrose's geometric treatment of scattering theory on asymptotically Euclidean spaces [12]. Note that $N^*Y \setminus o$ consists of two components, Λ_+, resp. Λ_-, and in $S^*X = (T^*X \setminus o)/\mathbb{R}^+$ the images, L_+, resp. L_-, of these are sources, resp. sinks, for the Hamilton flow. At $L_\pm$ one has choices regarding the direction one wants to propagate estimates (into or out of the radial points), which directly correspond to working with strong or weak Sobolev spaces. For the present problem, the relevant choice is propagating estimates *away from* the radial points, thus working with the 'good' Sobolev spaces (which can be taken to have as positive order as one wishes; there is a minimum amount of regularity imposed by our choice of propagation direction, cf. the requirement $s > \frac{1}{2} + C$ above (1.1)). All other points are either elliptic, or microhyperbolic. It remains to either deal with the non-compactness of the 'far end' of the n-dimensional de Sitter space — or instead, as is indeed more convenient when one wants to deal with more singular geometries, adding complex absorbing potentials, in the spirit of works of Nonnenmacher and Zworski [15] and Wunsch and Zworski [23]. In fact, the complex absorption could be replaced by adding a space-like boundary, see [18], but for many microlocal purposes complex absorption is more desirable, hence we follow the latter method. However, crucially, these complex absorbing techniques (or the addition of a space-like boundary) already enter in the non-semiclassical problem in our case, as we are in a non-elliptic setting.

One can reverse the direction of the argument and analyze the wave equation on an n-dimensional even asymptotically de Sitter space X_0' by extending it across the boundary, much like the the Riemannian conformally compact space X_0 is extended in this approach. Then, performing microlocal propagation in the opposite direction, which amounts to working with the adjoint operators that we already need in order to prove existence of solutions for the Riemannian spaces, we obtain existence, uniqueness and structure results for asymptotically de Sitter spaces, recovering a large part of the results of [20]. Here we only briefly indicate this method of analysis in Remark 5.3.

In other words, we establish a Riemannian-Lorentzian duality, that will have counterparts both in the pseudo-Riemannian setting of higher signature and in higher rank symmetric spaces, though in the latter the analysis might become more complicated. Note that asymptotically hyperbolic and de Sitter spaces are not connected by a 'complex rotation' (in the sense of an actual deformation); they are smooth continuations of each other in the sense we just discussed.

To emphasize the simplicity of our method, we list all of the microlocal techniques (which are relevant both in the classical and in the semiclassical setting) that we use on a *compact manifold without boundary*; in all cases *only microlocal Sobolev estimates* matter (not parametrices, etc.):

(i) Microlocal elliptic regularity.
(ii) Microhyperbolic propagation of singularities.
(iii) *Rough* analysis at a Lagrangian invariant under the Hamilton flow which roughly behaves like a collection of radial points, though the internal structure does not matter, in the spirit of [12, Section 9].
(iv) Complex absorbing 'potentials' in the spirit of [15] and [23].

These are almost 'off the shelf' in terms of modern microlocal analysis, and thus our approach, from a microlocal perspective, is quite simple. We use these to show that on the continuation across the boundary of the conformally compact space we have a Fredholm problem, on a perhaps slightly exotic function space, which however is (perhaps apart from the complex absorption) the simplest possible coisotropic function space based on a Sobolev space, with order dictated by the radial points. Also, we propagate the estimates along bicharacteristics in different directions depending on the component $\Sigma_\pm$ of the characteristic set under consideration; correspondingly the sign of the complex absorbing 'potential' will vary with $\Sigma_\pm$, which is perhaps slightly unusual. However, this is completely parallel to solving the standard Cauchy, or forward, problem for the wave equation, where one propagates estimates in *opposite* directions relative to the Hamilton vector field in the two components of the characteristic set.

The complex absorption we use modifies the operator P_σ outside $X_{0,\mathrm{even}}$. However, while $(P_\sigma - \imath Q_\sigma)^{-1}$ depends on Q_σ, its behavior on $X_{0,\mathrm{even}}$, and even near $X_{0,\mathrm{even}}$, is independent of this choice; see the proof of Section 5 for a detailed explanation. In particular, although $(P_\sigma - \imath Q_\sigma)^{-1}$ may have resonances other than those of $\mathcal{R}(\sigma)$, the resonant states of these additional resonances are supported outside $X_{0,\mathrm{even}}$, hence do not affect the singular behavior of the resolvent in $X_{0,\mathrm{even}}$.

While the results are stated for the scalar equation, analogous results hold for operators on natural vector bundles, such as the Laplacian on differential forms. This is so because the results work if the principal symbol of the extended problem is scalar with the demanded properties, and the principal symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*)$ is either scalar at the 'radial sets', or instead satisfies appropriate estimates (as an endomorphism of the pull-back of the vector bundle to the cotangent bundle) at this location; see Remark 3.1. The only change in terms of results on asymptotically hyperbolic spaces is that the threshold $(n-1)^2/4$ is shifted; in terms of the explicit conjugation of Section 5 this is so because of the change in the first order term in (3.2).

In Section 3 we describe in detail the setup of conformally compact spaces and the extension across the boundary. Then in Section 4 we describe the in detail the necessary microlocal analysis for the extended operator. Finally, in Section 5 we translate these results back to asymptotically hyperbolic spaces.

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2. Notation

We start by briefly recalling the basic pseudodifferential objects, in part to establish notation. As a general reference for microlocal analysis, we refer to [10], while for semiclassical analysis, we refer to [6, 7].

First, $S^k(\mathbb{R}^p;\mathbb{R}^\ell)$ is the set of $\mathcal{C}^\infty$ functions on $\mathbb{R}^p_z\times\mathbb{R}^\ell_\zeta$ satisfying uniform bounds

$$|D_z^\alpha D_\zeta^\beta a|\leq C_{\alpha\beta}\langle\zeta\rangle^{k-|\beta|},\ \alpha\in\mathbb{N}^p,\ \beta\in\mathbb{N}^\ell.$$

If $O\subset\mathbb{R}^p$ and $\Gamma\subset\mathbb{R}^\ell_\zeta$ are open, we define $S^k(O;\Gamma)$ by requiring these estimates to hold only for $z\in O$ and $\zeta\in\Gamma$. (We could instead require uniform estimates on compact subsets; this makes no difference here.) The class of classical (or one-step polyhomogeneous) symbols is the subset $S^k_{\mathrm{cl}}(\mathbb{R}^p;\mathbb{R}^\ell)$ of $S^k(\mathbb{R}^p;\mathbb{R}^\ell)$ consisting of symbols possessing an asymptotic expansion

$$a(z,r\omega)\sim\sum a_j(z,\omega)r^{k-j}, \tag{2.1}$$

where $a_j\in\mathcal{C}^\infty(\mathbb{R}^p\times\mathbb{S}^{\ell-1})$. Then on $\mathbb{R}^n_z$, pseudodifferential operators $A\in\Psi^k(\mathbb{R}^n)$ are of the form

$$A=\mathrm{Op}(a);\ (\mathrm{Op}(a)u)(z)=(2\pi)^{-n}\int_{\mathbb{R}^n}e^{i(z-z')\cdot\zeta}a(z,\zeta)\,u(z')\,d\zeta\,dz',$$
$$u\in\mathcal{S}(\mathbb{R}^n),\ a\in S^k(\mathbb{R}^n;\mathbb{R}^n);$$

understood as an oscillatory integral. Classical pseudodifferential operators, $A\in\Psi^k_{\mathrm{cl}}(\mathbb{R}^n)$, form the subset where a is a classical symbol. The principal symbol $\sigma_k(A)$ of $A\in\Psi^k(\mathbb{R}^n)$ is the equivalence class $[a]$ of a in $S^k(\mathbb{R}^n;\mathbb{R}^n)/S^{k-1}(\mathbb{R}^n;\mathbb{R}^n)$. For classical a, one can instead regard $a_0(z,\omega)r^k$ as the principal symbol; it is a $\mathcal{C}^\infty$ function on $\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$, which is homogeneous of degree k with respect to the $\mathbb{R}^+$-action given by dilations in the second factor, $\mathbb{R}^n\setminus\{0\}$. The principal symbol is multiplicative, i.e. $\sigma_{k+k'}(AB)=\sigma_k(A)\sigma_{k'}(B)$. Moreover, the principal symbol of a commutator is given by the Poisson bracket (or equivalently by the Hamilton vector field): $\sigma_{k+k'-1}(\imath[A,B])=\mathsf{H}_{\sigma_k(A)}\sigma_{k'}(B)$, with $\mathsf{H}_a=\sum_{j=1}^n((\partial_{\zeta_j}a)\partial_{z_j}-(\partial_{z_j}a)\partial_{\zeta_j})$. Note that for a homogeneous of order k, H_a is homogeneous of order $k-1$.

There are two very important properties: non-degeneracy (called ellipticity) and extreme degeneracy (captured by the operator wave front set) of an operator. One says that A is elliptic at $\alpha\in\mathbb{R}^n\times(\mathbb{R}^n\setminus\{0\})$ if there exists an open cone Γ (conic with respect to the $\mathbb{R}^+$-action on $\mathbb{R}^n\setminus o$) around α and $R>0$, $C>0$ such that $|a(x,\xi)|\geq C|\xi|^k$ for $|\xi|>R$, $(x,\xi)\in\Gamma$, where $[a]=\sigma_k(A)$. If A is classical, and a is taken to be homogeneous, this just amounts to $a(\alpha)\neq 0$.

On the other hand, for $A=\mathrm{Op}(a)$ and $\alpha\in\mathbb{R}^n\times(\mathbb{R}^n\setminus o)$ one says that $\alpha\notin\mathrm{WF}'(A)$ if there exists an open cone Γ around α such that $a|_\Gamma\in S^{-\infty}(\Gamma)$, i.e. $a|_\Gamma$ is rapidly decreasing, with all derivatives, as $|\xi|\to\infty$, $(x,\xi)\in\Gamma$. Note that both the elliptic set $\mathrm{ell}(A)$ of A (i.e. the set of points where A is elliptic) and $\mathrm{WF}'(A)$ are conic.

Differential operators on $\mathbb{R}^n$ form the subset of $\Psi(\mathbb{R}^n)$ in which a is polynomial in the second factor, $\mathbb{R}^n_\zeta$, so locally

$$A = \sum_{|\alpha|\leq k} a_\alpha(z) D_z^\alpha, \qquad \sigma_k(A) = \sum_{|\alpha|=k} a_\alpha(z)\zeta^\alpha.$$

If X is a manifold, one can transfer these definitions to X by localization and requiring that the Schwartz kernels are $\mathcal{C}^\infty$ densities away from the diagonal in $X^2 = X \times X$; then $\sigma_k(A)$ is in $S^k(T^*X)/S^{k-1}(T^*X)$, resp. $S^k_{\mathrm{hom}}(T^*X \setminus o)$ when $A \in \Psi^k(X)$, resp. $A \in \Psi^k_{\mathrm{cl}}(X)$; here o is the zero section, and hom stands for symbols homogeneous with respect to the $\mathbb{R}^+$ action. If A is a differential operator, then the classical (i.e. homogeneous) version of the principal symbol is a homogeneous polynomial in the fibers of the cotangent bundle of degree k. The notions of $\mathrm{ell}(A)$ and $\mathrm{WF}'(A)$ extend to give conic subsets of $T^*X \setminus o$; equivalently they are subsets of the cosphere bundle $S^*X = (T^*X \setminus o)/\mathbb{R}^+$. We can also work with operators depending on a parameter $\lambda \in O$ by replacing $a \in S^k(\mathbb{R}^n; \mathbb{R}^n)$ by $a \in S^k(\mathbb{R}^n \times O; \mathbb{R}^n)$, with $\mathrm{Op}(a_\lambda) \in \Psi^k(\mathbb{R}^n)$ smoothly dependent on $\lambda \in O$. In the case of differential operators, a_α would simply depend smoothly on the parameter λ.

We next consider the semiclassical operator algebra. We adopt the convention that $\hbar$ denotes semiclassical objects, while h is the actual semiclassical parameter. This algebra, $\Psi_\hbar(\mathbb{R}^n)$, is given by

$$A_h = \mathrm{Op}_\hbar(a);\ \mathrm{Op}_\hbar(a)u(z) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta/h} a(z,\zeta,h)\, u(z')\, d\zeta\, dz',$$
$$u \in \mathcal{S}(\mathbb{R}^n),\ a \in \mathcal{C}^\infty([0,1)_h; S^k(\mathbb{R}^n; \mathbb{R}^n_\zeta));$$

its classical subalgebra, $\Psi_{\hbar,\mathrm{cl}}(\mathbb{R}^n)$ corresponds to $a \in \mathcal{C}^\infty([0,1)_h; S^k_{\mathrm{cl}}(\mathbb{R}^n; \mathbb{R}^n_\zeta))$. The semiclassical principal symbol is now $\sigma_{\hbar,k}(A) = a|_{h=0} \in S^k(\mathbb{R}^n \times \mathbb{R}^n)$. In the setting of a general manifold X, $\mathbb{R}^n \times \mathbb{R}^n$ is replaced by T^*X. Correspondingly, $\mathrm{WF}'_\hbar(A)$ and $\mathrm{ell}_\hbar(A)$ are subsets of T^*X. We can again add an extra parameter $\lambda \in O$, so $a \in \mathcal{C}^\infty([0,1)_h; S^k(\mathbb{R}^n \times O; \mathbb{R}^n_\zeta))$; then in the invariant setting the principal symbol is $a|_{h=0} \in S^k(T^*X \times O)$.

Differential operators now take the form

$$A_{h,\lambda} = \sum_{|\alpha|\leq k} a_\alpha(z,\lambda;h)(hD_z)^\alpha. \tag{2.2}$$

Such a family has two principal symbols, the standard one (but taking into account the semiclassical degeneration, i.e. based on $(hD_z)^\alpha$ rather than D_z^α), which depends on h and is homogeneous, and the semiclassical one, which is at $h = 0$, and is not homogeneous:

$$\sigma_k(A_{h,\lambda}) = \sum_{|\alpha|=k} a_\alpha(z,\lambda;h)\zeta^\alpha,$$
$$\sigma_\hbar(A_{h,\lambda}) = \sum_{|\alpha|\leq k} a_\alpha(z,\lambda;0)\zeta^\alpha.$$

However, the restriction of $\sigma_k(A_{h,\lambda})$ to $h = 0$ is the principal symbol of $\sigma_\hbar(A_{h,\lambda})$. In the special case in which $\sigma_k(A_{h,\lambda})$ is independent of h (which is true in the setting considered below), one can simply regard the usual principal symbol as the principal part of the semiclassical symbol.

This is a convenient place to recall from [12] that it is often useful to consider the radial compactification of the fibers of the cotangent bundle to balls (or hemispheres, in the exposition of [12]). Thus, one adds a sphere at infinity to the fiber T^*_qX of T^*X over each $q \in X$. This sphere is naturally identified with S^*_qX, and we obtain compact fibers $\overline{T}^*_qX$ with boundary S^*_qX, with the smooth structure near S^*_qX arising from reciprocal polar coordinates $(\tilde{\rho}, \omega) = (r^{-1}, \omega)$ for $\tilde{\rho} > 0$, but extending to $\tilde{\rho} = 0$, and with S^*_qX given by $\tilde{\rho} = 0$. Thus, with $X = \mathbb{R}^n$ the classical expansion (2.1) becomes

$$a(z, \tilde{\rho}, \omega) \sim \tilde{\rho}^{-k} \sum a_j(z, \omega)\tilde{\rho}^j,$$

where $a_j \in \mathcal{C}^\infty(\mathbb{R}^p \times \mathbb{S}^{\ell-1})$, so in particular for $k = 0$, this is simply the Taylor series expansion at S^*X of a function smooth up to $S^*X = \partial\overline{T}^*X$. In the semiclassical context then one considers $\overline{T}^*X \times [0, 1)$, and notes that 'classical' semiclassical operators of order 0 are given locally by $\mathrm{Op}_\hbar(a)$ with a extending to be smooth up to the boundaries of this space, with semiclassical symbol given by restriction to $\overline{T}^*X \times \{0\}$, and standard symbol given by restriction to $S^*X \times [0, 1)$. Thus, the claim regarding the limit of the semiclassical symbol at infinity is simply a matching statement of the two symbols at the corner $S^*X \times \{0\}$ in this compactified picture.

Finally, we recall that if $P = \sum_{|\alpha|\leq k} a_\alpha(z)D_z^\alpha$ is an order k differential operator, then the behavior of $P - \lambda$ as $\lambda \to \infty$ can be converted to a semiclassical problem by considering

$$P_{\hbar,\sigma} = h^k(P - \lambda) = \sum_{|\alpha|\leq k} h^{k-|\alpha|}a_\alpha(z)(hD_z)^\alpha - \sigma,$$

where $\sigma = h^k\lambda$. Here there is freedom in choosing h, e.g. $h = |\lambda|^{1/k}$, in which case $|\sigma| = 1$, but it is often useful to leave some flexibility in the choice so that $h \sim |\lambda|^{1/k}$ only, and thus σ is in a compact subset of $\mathbb{C}$ disjoint from 0. Note that

$$\sigma_\hbar(P_{\hbar,\sigma}) = \sum_{|\alpha|=k} a_\alpha(z)\zeta^\alpha - \sigma.$$

If we do not want to explicitly multiply by h^k, we write the full high-energy principal symbol of $P - \lambda$ as

$$\sigma_{\mathrm{full}}(P_\lambda) = \sum_{|\alpha|=k} a_\alpha(z)\zeta^\alpha - \lambda.$$

More generally, if $P(\lambda) = \sum_{|\alpha|+|\beta|\leq k} a_\alpha(z)\lambda^\beta D_z^\alpha$ is an order k differential operator depending on a large parameter λ, we let

$$\sigma_{\mathrm{full}}(P(\lambda)) = \sum_{|\alpha|+|\beta|=k} a_\alpha(z)\lambda^\beta\zeta^\alpha$$

be the full large-parameter symbol. With $\lambda = h^{-1}\sigma$,

$$P_{\hbar,\sigma} = h^kP(\lambda) = \sum_{|\alpha|+|\beta|\leq k} h^{k-|\alpha|-|\beta|}a_\alpha(z)\sigma^\beta(hD_z)^\alpha$$

is a semiclassical differential operator with semiclassical symbol

$$\sigma_\hbar(P_{\hbar,\sigma}) = \sum_{|\alpha|+|\beta|=k} a_\alpha(z)\sigma^\beta\zeta^\alpha.$$

Note that the full large-parameter symbol and the semiclassical symbol are 'the same', i.e. they are simply related to each other.

3. Conformally compact spaces

3.1. **From the Laplacian to the extended operator.** Suppose that g_0 is an even asymptotically hyperbolic metric on X_0, with $\dim X_0 = n$. Then we may choose a product decomposition near the boundary such that

$$g_0 = \frac{dx^2 + h}{x^2} \tag{3.1}$$

there, where h is an even family of metrics; it is convenient to take x to be a globally defined boundary defining function. Then the dual metric is

$$G_0 = x^2(\partial_x^2 + H),$$

with H the dual metric family of h (depending on x as a parameter), and

$$|dg_0| = \sqrt{|\det g_0|}\,dx\,dy = x^{-n}\sqrt{|\det h|}\,dx\,dy$$

so

$$\Delta_{g_0} = (xD_x)^2 + \imath(n-1+x^2\gamma)(xD_x) + x^2\Delta_h, \tag{3.2}$$

with γ even, and Δ_h the x-dependent family of Laplacians of h on Y.

We show now that if we change the smooth structure on X_0 by declaring that only even functions of x are smooth, i.e. introducing $\mu = x^2$ as the boundary defining function, then after a suitable conjugation and division by a vanishing factor the resulting operator smoothly and non-degenerately continues across the boundary, i.e. continues to $X_{-\delta_0} = (-\delta_0, 0)_\mu \times Y \sqcup X_{0,\mathrm{even}}$, where $X_{0,\mathrm{even}}$ is the manifold X_0 with the new smooth structure.

First, changing to coordinates (μ, y), $\mu = x^2$, we obtain

$$\Delta_{g_0} = 4(\mu D_\mu)^2 + 2\imath(n-1+\mu\gamma)(\mu D_\mu) + \mu\Delta_h, \tag{3.3}$$

Now we conjugate by $\mu^{-\imath\sigma/2+(n+1)/4}$ to obtain

$$\begin{aligned}
&\mu^{\imath\sigma/2-(n+1)/4}(\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+(n+1)/4}\\
&= 4(\mu D_\mu - \sigma/2 - \imath(n+1)/4)^2 + 2\imath(n-1+\mu\gamma)(\mu D_\mu - \sigma/2 - \imath(n+1)/4)\\
&\qquad + \mu\Delta_h - \frac{(n-1)^2}{4} - \sigma^2\\
&= 4(\mu D_\mu)^2 - 4\sigma(\mu D_\mu) + \mu\Delta_h - 4\imath(\mu D_\mu) + 2\imath\sigma - 1\\
&\qquad + 2\imath\mu\gamma(\mu D_\mu - \sigma/2 - \imath(n+1)/4).
\end{aligned}$$

Next we multiply by $\mu^{-1/2}$ from both sides to obtain

$$\begin{aligned}
&\mu^{-1/2}\mu^{\imath\sigma/2-(n+1)/4}(\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+(n+1)/4}\mu^{-1/2}\\
&= 4\mu D_\mu^2 - \mu^{-1} - 4\sigma D_\mu - 2\imath\sigma\mu^{-1} + \Delta_h - 4\imath D_\mu + 2\mu^{-1} + 2\imath\sigma\mu^{-1} - \mu^{-1}\\
&\qquad + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-1)/4)\\
&= 4\mu D_\mu^2 - 4\sigma D_\mu + \Delta_h - 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \sigma/2 - \imath(n-1)/4).
\end{aligned} \tag{3.4}$$

This operator is in $\mathrm{Diff}^2(X_{0,\mathrm{even}})$, and now it continues smoothly across the boundary, by extending h and γ in an arbitrary smooth manner. This form suffices for analyzing the problem for σ in a compact set, or indeed for σ going to infinity in a strip near the reals. However, it is convenient to modify it as we would like the resulting operator to be semiclassically elliptic when σ is away from the reals. We achieve this via conjugation by a smooth function, with exponent depending on σ. The latter would make no difference even semiclassically in the real regime as it is conjugation by an elliptic semiclassical FIO. However, in the non-real regime (where we would like ellipticity) it does matter; the present operator is not semiclassically elliptic at the zero section. So finally we conjugate by $(1+\mu)^{\imath\sigma/4}$ to obtain

$$\begin{aligned} P_\sigma = 4(1+a_1)\mu D_\mu^2 - 4(1+a_2)\sigma D_\mu - (1+a_3)\sigma^2 + \Delta_h \\ - 4\imath D_\mu + b_1\mu D_\mu + b_2\sigma + c_1 \end{aligned} \tag{3.5}$$

with a_j smooth, real, vanishing at $\mu=0$, b_j and c_1 smooth. In fact, we have $a_1\equiv 0$, but it is sometimes convenient to have more flexibility in the form of the operator since this means that we do not need to start from the relatively rigid form (3.2).

Writing covectors as

$$\xi\,d\mu + \eta\,dy,$$

the principal symbol of $P_\sigma\in\mathrm{Diff}^2(X_{-\delta_0})$, including in the high energy sense ($\sigma\to\infty$), is

$$p_{\mathrm{full}} = 4(1+a_1)\mu\xi^2 - 4(1+a_2)\sigma\xi - (1+a_3)\sigma^2 + |\eta|^2_{\mu,y}, \tag{3.6}$$

and is real for σ real. The Hamilton vector field is

$$\begin{aligned} \mathsf{H}_{p_{\mathrm{full}}} &= 4(2(1+a_1)\mu\xi - (1+a_2)\sigma)\partial_\mu + \tilde{\mathsf{H}}_{|\eta|^2_{\mu,y}} \\ &\quad - \Big(4(1+a_1+\mu\frac{\partial a_1}{\partial\mu})\xi^2 - 4\frac{\partial a_2}{\partial\mu}\sigma\xi + \frac{\partial a_3}{\partial\mu}\sigma^2 + \frac{\partial|\eta|^2_{\mu,y}}{\partial\mu}\Big)\partial_\xi \\ &\quad - \Big(4\frac{\partial a_1}{\partial y}\mu\xi^2 - 4\frac{\partial a_2}{\partial y}\sigma\xi - \frac{\partial a_3}{\partial y}\sigma^2\Big)\partial_\eta, \end{aligned} \tag{3.7}$$

where $\tilde{\mathsf{H}}$ indicates that this is the Hamilton vector field in T^*Y, i.e. with μ considered a parameter. Correspondingly, the standard, 'classical', principal symbol is

$$p = \sigma_2(P_\sigma) = 4(1+a_1)\mu\xi^2 + |\eta|^2_{\mu,y}, \tag{3.8}$$

which is real, independent of σ, while the Hamilton vector field is

$$\begin{aligned} \mathsf{H}_p &= 8(1+a_1)\mu\xi\partial_\mu + \tilde{\mathsf{H}}_{|\eta|^2_{\mu,y}} \\ &\quad - \Big(4(1+a_1+\mu\frac{\partial a_1}{\partial\mu})\xi^2 + \frac{\partial|\eta|^2_{\mu,y}}{\partial\mu}\Big)\partial_\xi - 4\frac{\partial a_1}{\partial y}\mu\xi^2\partial_\eta. \end{aligned} \tag{3.9}$$

It is useful to keep in mind that as $\Delta_{g_0}-\sigma^2-(n-1)^2/4$ is formally self-adjoint relative to the metric density $|dg_0|$ for σ real, so the same holds for $\mu^{-1/2}(\Delta_{g_0}-\sigma^2-(n-1)^2/4)\mu^{-1/2}$ (as μ is real), and indeed for its conjugate by $\mu^{-\imath\sigma/2}(1+\mu)^{\imath\sigma/4}$ for σ real since this is merely unitary conjugation. As for f real, A formally self-adjoint relative to $|dg_0|$, $f^{-1}Af$ is formally self-adjoint relative to $f^2|dg_0|$, we then deduce that for σ real, P_σ is formally self-adjoint relative to

$$\mu^{(n+1)/2}|dg_0| = \frac{1}{2}|dh|\,|d\mu|,$$

as $x^{-n}\,dx = \frac{1}{2}\mu^{-(n+1)/2}\,d\mu$. Note that $\mu^{(n+1)/2}|dg_0|$ thus extends to a $\mathcal{C}^\infty$ density to $X_{-\delta_0}$, and we deduce that with respect to the extended density, $\sigma_1(\frac{1}{2\imath}(P_\sigma-P_\sigma^*))|_{\mu\geq 0}$

vanishes when $\sigma \in \mathbb{R}$. Since in general $P_\sigma - P_{\operatorname{Re}\sigma}$ differs from $-4\imath(1+a_2)\operatorname{Im}\sigma D_\mu$ by a zeroth order operator, we conclude that

$$\sigma_1\Big(\frac{1}{2\imath}(P_\sigma - P_\sigma^*)\Big)\Big|_{\mu=0} = -4(\operatorname{Im}\sigma)\xi. \tag{3.10}$$

We still need to check that μ can be appropriately chosen in the interior away from the region of validity of the product decomposition (3.1) (where we had no requirements so far on μ). This only matters for semiclassical purposes, and (being smooth and non-zero in the interior) the factor $\mu^{-1/2}$ multiplying from both sides does not affect any of the relevant properties (semiclassical ellipticity and possible non-trapping properties), so can be ignored — the same is true for σ-independent powers of μ.

Thus, near $\mu = 0$, but μ bounded away from 0, the only semiclassically non-trivial action we have done was to conjugate the operator by $e^{-\imath\sigma\phi}$ where $e^\phi = \mu^{1/2}(1+\mu)^{-1/4}$; we need to extend ϕ into the interior. But the semiclassical principal symbol of the conjugated operator is, with $\sigma = z/h$,

$$(\zeta - z\,d\phi, \zeta - z\,d\phi)_{G_0} - z^2 = |\zeta|^2_{G_0} - 2z(\zeta, d\phi)_{G_0} - (1 - |d\phi|^2_{G_0})z^2. \tag{3.11}$$

For z non-real this is elliptic if $|d\phi|_{G_0} < 1$. Indeed, if (3.11) vanishes then from the vanishing imaginary part we get

$$2\operatorname{Im} z((\zeta, d\phi)_{G_0} + (1 - |d\phi|^2_{G_0})\operatorname{Re} z) = 0, \tag{3.12}$$

and then the real part is

$$\begin{aligned} &|\zeta|^2_{G_0} - 2\operatorname{Re} z(\zeta, d\phi)_{G_0} - (1 - |d\phi|^2_{G_0})((\operatorname{Re} z)^2 - (\operatorname{Im} z)^2) \\ &= |\zeta|^2_{G_0} + (1 - |d\phi|^2_{G_0})((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2), \end{aligned} \tag{3.13}$$

which cannot vanish if $|d\phi|_{G_0} < 1$. But, reading off the dual metric from the principal symbol of (3.3),

$$\frac{1}{4}\Big|d(\log\mu - \frac{1}{2}\log(1+\mu))\Big|^2_{G_0} = \Big(1 - \frac{\mu}{2(1+\mu)}\Big)^2 < 1$$

for $\mu > 0$, with a strict bound as long as μ is bounded away from 0. Correspondingly, $\mu^{1/2}(1+\mu)^{-1/4}$ can be extended to a function e^ϕ on all of X_0 so that semiclassical ellipticity for z away from the reals is preserved, and we may even require that ϕ is constant on a fixed (but arbitrarily large) compact subset of X_0°. Then, after conjugation by $e^{-\imath\sigma\phi}$,

$$P_{h,z} = e^{\imath z\phi/h}\mu^{-(n+1)/4-1/2}(h^2\Delta_{g_0} - z)\mu^{(n+1)/4-1/2}e^{-\imath z\phi/h} \tag{3.14}$$

is semiclassically elliptic in $\mu > 0$ (as well as in $\mu \le 0$, μ near 0, where this is already guaranteed), as desired.

Remark 3.1. We have not considered vector bundles over X_0. However, for instance for the Laplacian on the differential form bundles it is straightforward to check that slightly changing the power of μ in the conjugation the resulting operator extends smoothly across ∂X_0, has scalar principal symbol of the form (3.6), and the principal symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*)$, which plays a role below, is also as in the scalar setting, so all the results in fact go through.

3.2. **Local dynamics near the radial set.** Let

$$N^*S\setminus o=\Lambda_+\cup\Lambda_-,\qquad \Lambda_\pm=N^*S\cap\{\pm\xi>0\},\qquad S=\{\mu=0\};$$

thus $S\subset X_{-\delta_0}$ can be identified with $Y=\partial X_0(=\partial X_{0,\mathrm{even}})$. Note that $p=0$ at $\Lambda_\pm$ and H_p is radial there since

$$N^*S=\{(\mu,y,\xi,\eta):\ \mu=0,\ \eta=0\},$$

so

$$\mathsf{H}_p|_{N^*S}=-4\xi^2\partial_\xi.$$

This corresponds to $dp=4\xi^2\,d\mu$ at N^*S, so the characteristic set $\Sigma=\{p=0\}$ is smooth at N^*S.

Let $L_\pm$ be the image of $\Lambda_\pm$ in $S^*X_{-\delta_0}$. Next we analyze the Hamilton flow at $\Lambda_\pm$. First,

$$\mathsf{H}_p|\eta|^2_{\mu,y}=8(1+a_1)\mu\xi\partial_\mu|\eta|^2_{\mu,y}-4\frac{\partial a_1}{\partial y}\mu\xi^2\cdot_h\eta \tag{3.15}$$

and

$$\mathsf{H}_p\mu=8(1+a_1)\xi\mu. \tag{3.16}$$

In terms of linearizing the flow at N^*S, p and μ are equivalent as $dp=4\xi^2\,d\mu$ there, so one can simply use $\hat{p}=p/|\xi|^2$ (which is homogeneous of degree 0, like μ), in place of μ. Finally,

$$\mathsf{H}_p|\xi|=-4\,\mathrm{sgn}(\xi)+b, \tag{3.17}$$

with b vanishing at $\Lambda_\pm$.

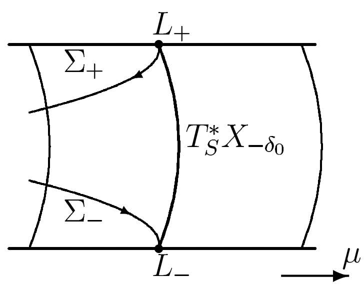

FIGURE 1. The cotangent bundle of $X_{-\delta_0}$ near $S=\{\mu=0\}$. It is drawn in a fiber-radially compactified view. The boundary of the fiber compactificaton is the cosphere bundle $S^*X_{-\delta_0}$; it is the surface of the cylinder shown. $\Sigma_\pm$ are the components of the (classical) characteristic set containing $L_\pm$. They lie in $\mu\leq 0$, only meeting $S^*_SX_{-\delta_0}$ at $L_\pm$. Semiclassically, i.e. in the interior of $\overline{T}^*X_{-\delta_0}$, for $z=h^{-1}\sigma>0$, only the component of the semiclassical characteristic set containing L_+ can enter $\mu>0$. This is reversed for $z<0$.

It is convenient to rehomogenize (3.15) in terms of $\hat{\eta}=\eta/|\xi|$. This can be phrased more invariantly by working with $S^*X_{-\delta_0}=(T^*X_{-\delta_0}\setminus o)/\mathbb{R}^+$, briefly discussed in Section 2. Let $L_\pm$ be the image of $\Lambda_\pm$ in $S^*X_{-\delta_0}$. Homogeneous degree zero functions on $T^*X_{-\delta_0}\setminus o$, such as $\hat{p}$, can be regarded as functions on $S^*X_{-\delta_0}$. For semiclassical purposes, it is best to consider $S^*X_{-\delta_0}$ as the boundary at fiber infinity of the fiber-radial compactification $\overline{T}^*X_{-\delta_0}$ of $T^*X_{-\delta_0}$, also discussed in Section 2.

Then at fiber infinity near N^*S, we can take $(|\xi|^{-1}, \hat{\eta})$ as (projective, rather than polar) coordinates on the fibers of the cotangent bundle, with $\tilde{\rho} = |\xi|^{-1}$ defining $S^*X_{-\delta_0}$ in $\overline{T}^*X_{-\delta_0}$. Then $W = |\xi|^{-1}\mathsf{H}_p$ is a $\mathcal{C}^\infty$ vector field in this region and

$$|\xi|^{-1}\mathsf{H}_p|\hat{\eta}|^2_{\mu,y} = 2|\hat{\eta}|^2_{\mu,y}\mathsf{H}_p|\xi|^{-1} + |\xi|^{-3}\mathsf{H}_p|\eta|^2_{\mu,y} = 8(\operatorname{sgn}\xi)|\hat{\eta}|^2 + \tilde{a}, \tag{3.18}$$

where $\tilde{a}$ vanishes cubically at N^*S. In similar notation we have

$$\mathsf{H}_p\tilde{\rho} = 4\operatorname{sgn}(\xi) + \tilde{a}', \qquad \tilde{\rho} = |\xi|^{-1}, \tag{3.19}$$

and

$$|\xi|^{-1}\mathsf{H}_p\mu = 8(\operatorname{sgn}\xi)\mu + \tilde{a}'', \tag{3.20}$$

with $\tilde{a}'$ smooth (indeed, homogeneous degree zero without the compactification) vanishing at N^*S, and $\tilde{a}''$ is also smooth, vanishing quadratically at N^*S. As the vanishing of $\hat{\eta}, |\xi|^{-1}$ and μ defines ∂N^*S, we conclude that $L_- = \partial\Lambda_-$ is a sink, while $L_+ = \partial\Lambda_+$ is a source, in the sense that all nearby bicharacteristics (in fact, including semiclassical (null)bicharacteristics, since $\mathsf{H}_p|\xi|^{-1}$ contains the additional information needed; see (3.29)) converge to $L_\pm$ as the parameter along the bicharacteristic goes to $\mp\infty$. In particular, the quadratic defining function of $L_\pm$ given by

$$\rho_0 = \widehat{\tilde{p}} + \hat{p}^2, \text{ where } \hat{p} = |\xi|^{-2}p,\ \widehat{\tilde{p}} = |\hat{\eta}|^2,$$

satisfies

$$(\operatorname{sgn}\xi)W\rho_0 \geq 8\rho_0 + \mathcal{O}(\rho_0^{3/2}). \tag{3.21}$$

We also need information on the principal symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*)$ at the radial points. At $L_\pm$ this is given by

$$\sigma_1(\frac{1}{2\imath}(P_\sigma - P_\sigma^*))|_{N^*S} = -(4\operatorname{sgn}(\xi))\operatorname{Im}\sigma|\xi|; \tag{3.22}$$

here $(4\operatorname{sgn}(\xi))$ is pulled out due to (3.19), namely its size relative to $\mathsf{H}_p|\xi|^{-1}$ matters. This corresponds to the fact that $(\mu \pm \imath 0)^{\imath\sigma}$, which are Lagrangian distributions associated to $\Lambda_\pm$, solve the PDE (3.5) modulo an error that is two orders lower than what one might a priori expect, i.e. $P_\sigma(\mu \pm \imath 0)^{\imath\sigma} \in (\mu \pm \imath 0)^{\imath\sigma}\mathcal{C}^\infty(X_{-\delta_0})$. Note that P_σ is second order, so one should lose two orders a priori, i.e. get an element of $(\mu \pm \imath 0)^{\imath\sigma-2}\mathcal{C}^\infty(X_{-\delta_0})$; the characteristic nature of $\Lambda_\pm$ reduces the loss to 1, and the particular choice of exponent eliminates the loss. This has much in common with $e^{\imath\lambda/x}x^{(n-1)/2}$ being an approximate solution in asymptotically Euclidean scattering, see [12].

3.3. **Global behavior of the characteristic set.** By (3.8), points with $\xi = 0$ cannot lie in the characteristic set. Thus, with

$$\Sigma_\pm = \Sigma \cap \{\pm\xi > 0\},$$

$\Sigma = \Sigma_+ \cup \Sigma_-$ and $\Lambda_\pm \subset \Sigma_\pm$. Further, the characteristic set lies in $\mu \leq 0$, and intersects $\mu = 0$ only in $\Lambda_\pm$.

Moreover, as $\mathsf{H}_p\mu = 8(1 + a_1)\xi\mu$ and $\xi \neq 0$ on Σ, and μ only vanishes at $\Lambda_+ \cup \Lambda_-$ there, for $\epsilon_0 > 0$ sufficiently small the $\mathcal{C}^\infty$ function μ provides a negative global escape function on $\mu \geq -\epsilon_0$ which is decreasing on Σ_+, increasing on Σ_-. Correspondingly, bicharacteristics in Σ_- travel from $\mu = -\epsilon_0$ to L_-, while in Σ_+ they travel from L_+ to $\mu = -\epsilon_0$.

3.4. **High energy, or semiclassical, asymptotics.** We are also interested in the high energy behavior, as $|\sigma| \to \infty$. For the associated semiclassical problem one obtains a family of operators

$$P_{h,z} = h^2 P_{h^{-1}z},$$

with $h = |\sigma|^{-1}$, and z corresponding to $\sigma/|\sigma|$ in the unit circle in $\mathbb{C}$. Then the semiclassical principal symbol $p_{\hbar,z}$ of $P_{h,z}$ is a function on $T^*X_{-\delta_0}$, whose asymptotics at fiber infinity of $T^*X_{-\delta_0}$ is given by the classical principal symbol p. We are interested in $\operatorname{Im}\sigma \geq -C$, which in semiclassical notation corresponds to $\operatorname{Im} z \geq -Ch$. It is sometimes convenient to think of $p_{\hbar,z}$, and its rescaled Hamilton vector field, as objects on $\overline{T}^* X_{-\delta_0}$. Thus,

$$p_{\hbar,z} = \sigma_{2,\hbar}(P_{h,z}) = 4(1+a_1)\mu\xi^2 - 4(1+a_2)z\xi - (1+a_3)z^2 + |\eta|^2_{\mu,y}, \tag{3.23}$$

so

$$\operatorname{Im} p_{\hbar,z} = -2\operatorname{Im} z(2(1+a_2)\xi + (1+a_3)\operatorname{Re} z). \tag{3.24}$$

In particular, for z non-real, $\operatorname{Im} p_{\hbar,z} = 0$ implies $2(1+a_2)\xi + (1+a_3)\operatorname{Re} z = 0$, so

$$\begin{aligned}\operatorname{Re} p_{\hbar,z} = ((1+a_1)(1+a_3)^2(1+a_2)^{-2}\mu + (1+2a_2)(1+a_3))(\operatorname{Re} z)^2 \\ + (1+a_3)(\operatorname{Im} z)^2 + |\eta|^2_{\mu,y} > 0\end{aligned} \tag{3.25}$$

near $\mu = 0$, i.e. $p_{\hbar,z}$ is semiclassically elliptic on $T^*X_{-\delta_0}$, but *not* at fiber infinity, i.e. at $S^*X_{-\delta_0}$ (standard ellipticity is lost only in $\mu \leq 0$, of course). In $\mu > 0$ we have semiclassical ellipticity (and automatically classical ellipticity) by our choice of ϕ following (3.11). Explicitly, if we introduce for instance

$$(\mu, y, \nu, \hat{\eta}), \qquad \nu = |\xi|^{-1},\ \hat{\eta} = \eta/|\xi|, \tag{3.26}$$

as valid projective coordinates in a (large!) neighborhood of $L_\pm$ in $\overline{T}^* X_{-\delta_0}$, then

$$\nu^2 p_{\hbar,z} = 4(1+a_1)\mu - 4(1+a_2)(\operatorname{sgn}\xi)z\nu - (1+a_3)z^2\nu^2 + |\hat{\eta}|^2_{y,\mu}$$

so

$$\nu^2 \operatorname{Im} p_{\hbar,z} = -4(1+a_2)(\operatorname{sgn}\xi)\nu\operatorname{Im} z - 2(1+a_3)\nu^2\operatorname{Re} z\operatorname{Im} z$$

which automatically vanishes at $\nu = 0$, i.e. at $S^*X_{-\delta_0}$. Thus, for σ large and pure imaginary, the semiclassical problem adds no complexity to the 'classical' quantum problem, but of course it does not simplify it. In fact, we need somewhat more information at the characteristic set, which is thus at $\nu = 0$ when $\operatorname{Im} z$ is bounded away from 0:

$$\begin{aligned}&\nu \text{ small},\ \operatorname{Im} z \geq 0 \Rightarrow (\operatorname{sgn}\xi)\operatorname{Im} p_{\hbar,z} \leq 0 \Rightarrow \pm\operatorname{Im} p_{\hbar,z} \leq 0 \text{ near } \Sigma_{\hbar,\pm},\\ &\nu \text{ small},\ \operatorname{Im} z \leq 0 \Rightarrow (\operatorname{sgn}\xi)\operatorname{Im} p_{\hbar,z} \geq 0 \Rightarrow \pm\operatorname{Im} p_{\hbar,z} \geq 0 \text{ near } \Sigma_{\hbar,\pm},\end{aligned}$$

which, as we recall in Section 4, means that for $P_{h,z}$ with $\operatorname{Im} z > 0$ one can propagate estimates forwards along the bicharacteristics where $\xi > 0$ (in particular, away from L_+, as the latter is a source) and backwards where $\xi < 0$ (in particular, away from L_-, as the latter is a sink), while for $P^*_{h,z}$ the directions are reversed since its semiclassical symbol is $\overline{p_{\hbar,z}}$. The directions are also reversed if $\operatorname{Im} z$ switches sign. This is important because it gives invertibility for $z = \imath$ (corresponding to $\operatorname{Im}\sigma$ large positive, i.e. the physical halfplane), but does not give invertibility for $z = -\imath$ negative.

We now return to the claim that even semiclassically, for z almost real (i.e. when z is not bounded away from the reals; we are not fixing z as we let h vary!), when the operator is not semiclassically elliptic on $T^*X_{-\delta_0}$ as mentioned above, the characteristic set can be divided into two components $\Sigma_{\hbar,\pm}$, with $L_\pm$ in different components. The vanishing of the factor following $\operatorname{Im} z$ in (3.24) gives a hypersurface that separates $\Sigma_\hbar$ into two parts. Indeed, this is the hypersurface given by

$$2(1+a_2)\xi + (1+a_3)\operatorname{Re} z = 0, \tag{3.27}$$

on which, by (3.25), $\operatorname{Re} p_{\hbar,z}$ cannot vanish, so

$$\Sigma_\hbar = \Sigma_{\hbar,+} \cup \Sigma_{\hbar,-}, \qquad \Sigma_{\hbar,\pm} = \Sigma_\hbar \cap \{\pm(2(1+a_2)\xi + (1+a_3)\operatorname{Re} z) > 0\}.$$

Farther in $\mu > 0$, the hypersurface is given, due to (3.12), by

$$(\zeta, d\phi)_{G_0} + (1 - |d\phi|^2_{G_0})\operatorname{Re} z = 0,$$

and on it, by (3.13), the real part is $|\zeta|^2_{G_0} + (1-|d\phi|^2_{G_0})((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2) > 0$; correspondingly

$$\Sigma_\hbar = \Sigma_{\hbar,+} \cup \Sigma_{\hbar,-}, \qquad \Sigma_{\hbar,\pm} = \Sigma_\hbar \cap \{\pm((\zeta, d\phi)_{G_0} + (1-|d\phi|^2_{G_0})\operatorname{Re} z) > 0\}.$$

In fact, more generally, the real part is

$$\begin{aligned}
&|\zeta|^2_{G_0} - 2\operatorname{Re} z(\zeta, d\phi)_{G_0} - (1-|d\phi|^2_{G_0})((\operatorname{Re} z)^2 - (\operatorname{Im} z)^2)\\
&= |\zeta|^2_{G_0} - 2\operatorname{Re} z((\zeta, d\phi)_{G_0} + (1-|d\phi|^2_{G_0})\operatorname{Re} z) + (1-|d\phi|^2_{G_0})((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2),
\end{aligned}$$

so for $\pm\operatorname{Re} z > 0$, $\mp((\zeta, d\phi)_{G_0} + (1-|d\phi|^2_{G_0})\operatorname{Re} z) > 0$ implies that $p_{\hbar,z}$ does not vanish. Correspondingly, only one of the two components of $\Sigma_{\hbar,\pm}$ enter $\mu > 0$, namely for $\operatorname{Re} z > 0$, only $\Sigma_{\hbar,+}$ enters, while for $\operatorname{Re} z < 0$, only $\Sigma_{\hbar,-}$ enters.

We finally need more information about the global semiclassical dynamics.

Lemma 3.2. *There exists $\epsilon_0 > 0$ such that the following holds. All semiclassical null-bicharacteristics in $(\Sigma_{\hbar,+} \setminus L_+) \cap \{-\epsilon_0 \le \mu \le \epsilon_0\}$ go to either L_+ or to $\mu = \epsilon_0$ in the backward direction and to $\mu = \epsilon_0$ or $\mu = -\epsilon_0$ in the forward direction, while all semiclassical null-bicharacteristics in $(\Sigma_{\hbar,-} \setminus L_-) \cap \{-\epsilon_0 \le \mu \le \epsilon_0\}$ go to L_- or $\mu = \epsilon_0$ in the forward direction and to $\mu = \epsilon_0$ or $\mu = -\epsilon_0$ in the backward direction.*

For $\operatorname{Re} z > 0$, only $\Sigma_{\hbar,+}$ enters $\mu > 0$, so the $\mu = \epsilon_0$ possibility only applies to $\Sigma_{\hbar,+}$ then, while for $\operatorname{Re} z < 0$, the analogous remark applies to $\Sigma_{\hbar,-}$.

Proof. We assume that $\operatorname{Re} z > 0$ for the sake of definiteness. Observe that the semiclassical Hamilton vector field is

$$\begin{aligned}
\mathsf{H}_{p_{\hbar,z}} &= 4(2(1+a_1)\mu\xi - (1+a_2)z)\partial_\mu + \tilde{\mathsf{H}}_{|\eta|^2_{\mu,y}}\\
&\quad - \Big(4(1+a_1+\mu\frac{\partial a_1}{\partial\mu})\xi^2 - 4\frac{\partial a_2}{\partial\mu}z\xi + \frac{\partial a_3}{\partial\mu}z^2 + \frac{\partial|\eta|^2_{\mu,y}}{\partial\mu}\Big)\partial_\xi\\
&\quad - \Big(4\frac{\partial a_1}{\partial y}\mu\xi^2 - 4\frac{\partial a_2}{\partial y}z\xi - \frac{\partial a_3}{\partial y}z^2\Big)\partial_\eta;
\end{aligned} \tag{3.28}$$

here we are concerned about z real. Near $S^*X_{-\delta_0} = \partial\overline{T}^*X_{-\delta_0}$, using the coordinates (3.26) (which are valid near the characteristic set)

$$\begin{aligned} W_\hbar = \nu\mathsf{H}_{p_{\hbar,z}} &= 4(2(1+a_1)\mu(\operatorname{sgn}\xi) - (1+a_2)z\nu)\partial_\mu + \nu\tilde{\mathsf{H}}_{|\eta|^2_{\mu,y}} \\ &\quad + (\operatorname{sgn}\xi)\Big(4(1+a_1+\mu\frac{\partial a_1}{\partial\mu}) - 4\frac{\partial a_2}{\partial\mu}z(\operatorname{sgn}\xi)\nu + \frac{\partial a_3}{\partial\mu}z^2\nu^2 \\ &\qquad + \frac{\partial|\hat{\eta}|^2_{\mu,y}}{\partial\mu}\Big)(\nu\partial_\nu + \hat{\eta}\partial_{\hat{\eta}}) \\ &\quad - \Big(4\frac{\partial a_1}{\partial y}\mu - 4(\operatorname{sgn}\xi)\frac{\partial a_2}{\partial y}z\nu - \frac{\partial a_3}{\partial y}z^2\nu^2\Big)\partial_{\hat{\eta}}, \end{aligned} \tag{3.29}$$

with $\nu\tilde{\mathsf{H}}_{|\eta|^2_{\mu,y}} = \sum_{ij} H_{ij}\hat{\eta}_i\partial_{y_j} - \sum_{ijk}\frac{\partial H_{ij}}{\partial y_k}\hat{\eta}_i\hat{\eta}_j\partial_{\hat{\eta}_k}$ smooth. Thus, $W_\hbar$ is a smooth vector field on the compactified cotangent bundle, $\overline{T}^*X_{-\delta_0}$ which is tangent to its boundary, $S^*X_{-\delta_0}$, and $W_\hbar - W = \nu W^\sharp$ (with W considered as a homogeneous degree zero vector field) with $W^\sharp$ smooth and tangent to $S^*X_{-\delta_0}$. In particular, by (3.19) and (3.21), using that $\tilde{\rho}^2 + \rho_0$ is a quadratic defining function of $L_\pm$,

$$(\operatorname{sgn}\xi)W_\hbar(\tilde{\rho}^2+\rho_0) \geq 8(\tilde{\rho}^2+\rho_0) - \mathcal{O}((\tilde{\rho}^2+\rho_0)^{3/2})$$

shows that there is $\epsilon_1 > 0$ such that in $\tilde{\rho}^2 + \rho_0 \leq \epsilon_1$, $\xi > 0$, $\tilde{\rho}^2 + \rho_0$ is strictly increasing along the Hamilton flow except at L_+, while in $\tilde{\rho}^2 + \rho_0 \leq \epsilon_1$, $\xi < 0$, $\tilde{\rho}^2 + \rho_0$ is strictly decreasing along the Hamilton flow except at L_-. Indeed, all null-bicharacteristics in this neighborhood of $L_\pm$ except the constant ones at $L_\pm$ tend to $L_\pm$ in one direction and to $\tilde{\rho}^2 + \rho_0 = \epsilon_1$ in the other direction.

Choosing $\epsilon_0' > 0$ sufficiently small, the characteristic set in $\overline{T}^*X_{-\delta_0} \cap \{-\epsilon_0' \leq \mu \leq \epsilon_0'\}$ is disjoint from $S^*X_{-\delta_0} \setminus \{\tilde{\rho}^2 + \rho_0 \leq \epsilon_1\}$, and indeed only contains points in $\Sigma_{\hbar,+}$ as $\operatorname{Re} z > 0$. Since $\mathsf{H}_{p_{\hbar,z}}\mu = 4(2(1+a_1)\mu\xi - (1+a_2)z)$, it is negative on $\overline{T}^*_{\{\mu=0\}}X_{-\delta_0} \setminus S^*X_{-\delta_0}$. In particular, there is a neighborhood U of $\mu = 0$ in $\Sigma_{\hbar,+} \setminus S^*X_{-\delta_0}$ on which the same sign is preserved; since the characteristic set in $\overline{T}^*X_{-\delta_0} \setminus \{\tilde{\rho}^2+\rho_0 < \epsilon_1\}$ is compact, and is indeed a subset of $T^*X_{-\delta_0} \setminus \{\tilde{\rho}^2+\rho_0 < \epsilon_1\}$, we deduce that $|\mu|$ is bounded below on $\Sigma \setminus (U \cup \{\tilde{\rho}^2 + \rho_0 < \epsilon_1\})$, say $|\mu| \geq \epsilon_0'' > 0$ there, so with $\epsilon_0 = \min(\epsilon_0', \epsilon_0'')$, $\mathsf{H}_{p_{\hbar,z}}\mu < 0$ on $\Sigma_{\hbar,+} \cap \{-\epsilon_0 \leq \mu \leq \epsilon_0\} \setminus \{\tilde{\rho}^2+\rho_0^2 < \epsilon_1\}$. As $\mathsf{H}_{p_{\hbar,z}}\mu < 0$ at $\mu = 0$, bicharacteristics can only cross $\mu = 0$ in the outward direction.

Thus, if γ is a bicharacteristic in $\Sigma_{\hbar,+}$, there are two possibilities. If γ is disjoint from $\{\tilde{\rho}^2+\rho_0 < \epsilon_1\}$, it has to go to $\mu = \epsilon_0$ in the backward direction and to $\mu = -\epsilon_0$ in the forward direction. If γ has a point in $\{\tilde{\rho}^2 + \rho_0 < \epsilon_1\}$, then it has to go to L_+ in the backward direction and to $\tilde{\rho}^2 + \rho_0 = \epsilon_1$ in the forward direction; if $|\mu| \geq \epsilon_0$ by the time $\tilde{\rho}^2 + \rho_0 = \epsilon_1$ is reached, the result is proved, and otherwise $\mathsf{H}_{p_{\hbar,z}}\mu < 0$ in $\tilde{\rho}^2 + \rho_0 \geq \epsilon_1$, $|\mu| \leq \epsilon_0$, shows that the bicharacteristic goes to $\mu = -\epsilon_0$ in the forward direction.

If γ is a bicharacteristic in $\Sigma_{\hbar,-}$, only the second possibility exists, and the bicharacteristic cannot leave $\{\tilde{\rho}^2 + \rho_0 < \epsilon_1\}$ in $|\mu| \leq \epsilon_0$, so it reaches $\mu = -\epsilon_0$ in the backward direction (as the characteristic set is in $\mu \leq 0$). □

If we assume that g_0 is a non-trapping metric, i.e. bicharacteristics of g_0 in $T^*X_0^\circ \setminus o$ tend to ∂X_0 in both the forward and the backward directions, then $\mu = \epsilon_0$ can be excluded from the statement of the lemma, and the above argument gives the following stronger conclusion: for sufficiently small $\epsilon_0 > 0$, and for $\operatorname{Re} z > 0$, any bicharacteristic in $\Sigma_{\hbar,+}$ in $-\epsilon_0 \leq \mu$ has to go to L_+ in the backward direction, and to $\mu = -\epsilon_0$ in the forward direction (with the exception of the constant bicharacteristics at L_+), while in $\Sigma_{\hbar,-}$, all bicharacteristics in $-\epsilon_0 \leq \mu$ lie in $-\epsilon_0 \leq \mu \leq 0$, and go to L_- in the forward direction and to $\mu = -\epsilon_0$ in the backward direction (with the exception of the constant bicharacteristics at L_-).

In fact, for applications, it is also useful to remark that for sufficiently small $\epsilon_0 > 0$, and for $\alpha \in T^*X_0$,

$$0 < \mu(\alpha) < \epsilon_0,\ p_{\hbar,z}(\alpha) = 0 \text{ and } (\mathsf{H}_{p_{\hbar,z}}\mu)(\alpha) = 0 \Rightarrow (\mathsf{H}^2_{p_{\hbar,z}}\mu)(\alpha) < 0. \tag{3.30}$$

Indeed, as $\mathsf{H}_{p_{\hbar,z}}\mu = 4(2(1+a_1)\mu\xi - (1+a_2)z)$, the hypotheses imply $z = 2(1+a_1)(1+a_2)^{-1}\mu\xi$ and

$$\begin{aligned} 0 &= p_{\hbar,z} \\ &= 4(1+a_1)\mu\xi^2 - 8(1+a_1)\mu\xi^2 - 4(1+a_1)^2(1+a_2)^{-2}(1+a_3)\mu^2\xi^2 + |\eta|^2_{\mu,y} \\ &= -4(1+a_1)\mu\xi^2 - 4(1+a_1)^2(1+a_2)^{-2}(1+a_3)\mu^2\xi^2 + |\eta|^2_{\mu,y}, \end{aligned}$$

so $|\eta|^2_{\mu,y} = 4(1+b)\mu\xi^2$, with b vanishing at $\mu = 0$. Thus, at points where $\mathsf{H}_{p_{\hbar,z}}\mu$ vanishes, writing $a_j = \mu\tilde{a}_j$,

$$\mathsf{H}^2_{p_{\hbar,z}}\mu = 8(1+a_1)\mu\mathsf{H}_{p_{\hbar,z}}\xi + 8\mu^2\xi\mathsf{H}_{p_{\hbar,z}}\tilde{a}_1 - 4z\mu\mathsf{H}_{p_{\hbar,z}}\tilde{a}_2 = 8(1+a_1)\mu\mathsf{H}_{p_{\hbar,z}}\xi + \mathcal{O}(\mu^2\xi^2). \tag{3.31}$$

Now

$$\mathsf{H}_{p_{\hbar,z}}\xi = -(4(1+a_1+\mu\frac{\partial a_1}{\partial\mu})\xi^2 - 4\frac{\partial a_2}{\partial\mu}z\xi + \frac{\partial a_3}{\partial\mu}z^2 + \frac{\partial|\eta|^2_{\mu,y}}{\partial\mu}).$$

Since $z\xi$ is $\mathcal{O}(\mu\xi^2)$ due to $\mathsf{H}_{p_{\hbar,z}}\mu = 0$, z^2 is $\mathcal{O}(\mu^2\xi^2)$ for the same reason, and $|\eta|^2$ and $\partial_\mu|\eta|^2$ are $\mathcal{O}(\mu\xi^2)$ due to $p_{\hbar,z} = 0$, we deduce that $\mathsf{H}_{p_{\hbar,z}}\xi < 0$ for sufficiently small $|\mu|$, so (3.31) implies (3.30). Thus, μ can be used for gluing constructions as in [4].

3.5. **Complex absorption.** The final step of fitting P_σ into our general microlocal framework is moving the problem to a compact manifold, and adding a complex absorbing second order operator. We thus consider a compact manifold without boundary X for which $X_{\mu_0} = \{\mu > \mu_0\}$, $\mu_0 = -\epsilon_0 < 0$, with $\epsilon_0 > 0$ as above, is identified as an open subset with smooth boundary; it is convenient to take X to be the double of X_{μ_0}, so there are two copies of $X_{0,\mathrm{even}}$ in X.

In the case of hyperbolic space, this doubling process can be realized from the perspective of $(n+1)$-dimensional Minkowski space. Then, as mentioned in the introduction, the Poincaré model shows up in two copies, namely in the interior of the future and past light cone inside the sphere at infinity, while de Sitter space as the 'equatorial belt', i.e. the exterior of the light cone at the sphere at infinity. One can take the Minkowski equatorial plane, $t = 0$, as $\mu = \mu_0$, and place the complex absorption there, thereby decoupling the future and past hemispheres. See [18] for more detail.

It is convenient to separate the 'classical' (i.e. quantum!) and 'semiclassical' problems, for in the former setting trapping for g_0 does not matter, while in the latter it does.

We then introduce a 'complex absorption' operator $Q_\sigma \in \Psi^2_{\mathrm{cl}}(X)$ with real principal symbol q supported in, say, $\mu < -\epsilon_1$, with the Schwartz kernel also supported in the corresponding region (i.e. in both factors on the product space this condition

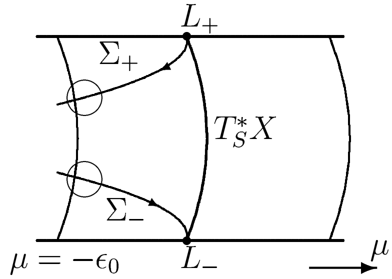

FIGURE 2. The cotangent bundle near $S = \{\mu = 0\}$. It is drawn in a fiber-radially compactified view, as in Figure 1. The circles on the left show the support of q; it has opposite signs on the two disks corresponding to the opposite directions of propagation relative to the Hamilton vector field.

holds on the support) such that $p \pm \imath q$ is elliptic near ∂X_{μ_0}, i.e. near $\mu = \mu_0$, and which satisfies that $\pm q \geq 0$ near $\Sigma_\pm$. This can easily be done since $\Sigma_\pm$ are disjoint, and away from these p is elliptic, hence so is $p \pm \imath q$ regardless of the choice of q; we simply need to make q to have support sufficiently close to $\Sigma_\pm$, elliptic on $\Sigma_\pm$ at $\mu = -\epsilon_0$, with the appropriate sign near $\Sigma_\pm$. Having done this, we extend p and q to X in such a way that $p \pm \imath q$ are elliptic near ∂X_{μ_0}; the region we added is thus irrelevant at the level of bicharacteristic dynamics (of p) in so far as it is decoupled from the dynamics in X_0, and indeed also for analysis as we see shortly (in so far as we have two essentially decoupled copies of the same problem). This is accomplished, for instance, by using the doubling construction to define p on $X \setminus X_{\mu_0}$ (in a smooth fashion at ∂X_{μ_0}, as can be easily arranged; the holomorphic dependence of P_σ on σ is still easily preserved), and then, noting that the characteristic set of p still has two connected components, making q elliptic on the characteristic set of p near ∂X_{μ_0}, with the same sign in each component as near ∂X_{μ_0}. (An alternative would be to make q elliptic on the characteristic set of p near $X \setminus X_{\mu_0}$; it is just slightly more complicated to write down such a q when the high energy behavior is taken into account. With the present choice, due to the doubling, there are essentially two copies of the problem on X_0: the original, and the one from the doubling.) Finally we take Q_σ be any operator with principal symbol q with Schwartz kernel satisfying the desired support conditions and which depends on σ holomorphically. We may choose Q_σ to be independent of σ so Q_σ is indeed holomorphic; in this case we may further replace it by $\frac{1}{2}(Q_\sigma + Q_\sigma^*)$ if self-adjointness is desired.

In view of Subsection 3.3 we have arranged the following. For $\alpha \in S^*X \cap \Sigma$, let $\gamma_+(\alpha)$, resp. $\gamma_-(\alpha)$ denote the image of the forward, resp. backward, half-bicharacteristic of p from α. We write $\gamma_\pm(\alpha) \to L_\pm$ (and say $\gamma_\pm(\alpha)$ tends to $L_\pm$) if given any neighborhood O of $L_\pm$, $\gamma_\pm(\alpha) \cap O \neq \emptyset$; by the source/sink property this implies that the points on the curve are in O for sufficiently large (in absolute value) parameter values. Then, with $\mathrm{ell}(Q_\sigma)$ denoting the elliptic set of Q_σ,

$$\begin{aligned} &\alpha \in \Sigma_- \setminus L_- \Rightarrow \gamma_+(\alpha) \to L_- \text{ and } \gamma_-(\alpha) \cap \mathrm{ell}(Q_\sigma) \neq \emptyset, \\ &\alpha \in \Sigma_+ \setminus L_+ \Rightarrow \gamma_-(\alpha) \to L_+ \text{ and } \gamma_+(\alpha) \cap \mathrm{ell}(Q_\sigma) \neq \emptyset. \end{aligned} \tag{3.32}$$

That is, all forward and backward half-(null)bicharacteristics of P_σ either enter the elliptic set of Q_σ, or go to $\Lambda_\pm$, i.e. $L_\pm$ in S^*X. The point of the arrangements regarding Q_σ and the flow is that we are able to propagate estimates forward near where $q \geq 0$, backward near where $q \leq 0$, so by our hypotheses we can always propagate estimates for $P_\sigma - \imath Q_\sigma$ from $\Lambda_\pm$ towards the elliptic set of Q_σ. On the other hand, for $P_\sigma^* + \imath Q_\sigma^*$, we can propagate estimates from the elliptic set of Q_σ towards $\Lambda_\pm$. This behavior of $P_\sigma - \imath Q_\sigma$ vs. $P_\sigma^* + \imath Q_\sigma^*$ is important for duality reasons.

An alternative to the complex absorption would be simply adding a boundary at $\mu = \mu_0$; this is easy to do since this is a space-like hypersurface, but this is slightly unpleasant from the point of view of microlocal analysis as one has to work on a manifold with boundary (though as mentioned this is easily done, see [18]).

For the semiclassical problem, when z is almost real (namely when $\operatorname{Im} z$ is bounded away from 0 we only need to make sure we do not mess up the semiclassical ellipticity in $T^*X_{-\delta_0}$) we need to increase the requirements on Q_σ, and what we need to do depends on whether g_0 is non-trapping.

If g_0 is non-trapping, we choose Q_σ such that $h^2 Q_{h^{-1}z} \in \Psi^2_{\hbar,\mathrm{cl}}(X)$ with semiclassical principal symbol $q_{\hbar,z}$, and in addition to the above requirement for the classical symbol, we need semiclassical ellipticity near $\mu = \mu_0$, i.e. that $p_{\hbar,z} - \imath q_{\hbar,z}$ and its complex conjugate are elliptic near ∂X_{μ_0}, i.e. near $\mu = \mu_0$, and which satisfies that for z real $\pm q_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\pm}$. Again, we extend P_σ and Q_σ to X in such a way that $p - \imath q$ and $p_{\hbar,z} - \imath q_{\hbar,z}$ (and thus their complex conjugates) are elliptic near ∂X_{μ_0}; the region we added is thus irrelevant. This is straightforward to arrange if one ignores that one wants Q_σ to be holomorphic: one easily constructs a function $q_{\hbar,z}$ on T^*X (taking into account the disjointness of $\Sigma_{\hbar,\pm}$), and defines $Q_{h^{-1}z}$ to be h^{-2} times the semiclassical quantization of $q_{\hbar,z}$ (or any other operator with the same semiclassical and standard principal symbols). Indeed, for our purposes this would suffice since we want high energy estimates for the analytic continuation resolvent on the original space X_0 (which we will know exists by the non-semiclassical argument), and as we shall see, the resolvent is given by the same formula in terms of $(P_\sigma - \imath Q_\sigma)^{-1}$ independently whether Q_σ is holomorphic in σ (as long as it satisfies the other properties), so there is no need to ensure the holomorphy of Q_σ. However, it is instructive to have an example of a holomorphic family Q_σ in a strip at least: in view of (3.24) we can take (with $C > 0$)

$$q_{h,z} = 2(2(1+a_2)\xi + (1+a_3)z)(\xi^2 + |\eta|^2 + z^2 + C^2h^2)^{1/2}\chi(\mu),$$

where $\chi \geq 0$ is supported near μ_0; the corresponding full symbol is

$$\sigma_{\mathrm{full}}(Q_\sigma) = 2(2(1+a_2)\xi + (1+a_3)\sigma)(\xi^2 + |\eta|^2 + \sigma^2)^{1/2}\chi(\mu),$$

and Q_σ is taken as a quantization of this full symbol. Here the square root is defined on $\mathbb{C}\setminus[0,-\infty)$, with real part of the result being positive, and correspondingly $q_{h,z}$ is defined away from $h^{-1}z \in \pm\imath[C,+\infty)$. Note that $\xi^2 + |\eta|^2 + \sigma^2$ is an elliptic symbol in $(\xi, \eta, \operatorname{Re}\sigma, \operatorname{Im}\sigma)$ as long as $|\operatorname{Im}\sigma| < C'|\operatorname{Re}\sigma|$, so the corresponding statement also holds for its square root. While $q_{h,z}$ is only holomorphic away from $h^{-1}z \in \pm\imath[C,+\infty)$, the full (and indeed the semiclassical and standard principal) symbols are actually holomorphic in cones near infinity, and indeed e.g. via convolutions by the Fourier transform of a compactly supported function can be extended to be holomorphic in $\mathbb{C}$, but this is of no importance here.

If g_0 is trapping, we need to add complex absorption inside X_0 as well, at $\mu = \epsilon_0$, so we relax the requirement that Q_σ is supported in $\mu < -\epsilon_0/2$ to support in $|\mu| > \epsilon_0/2$, but we require in addition to the other classical requirements that

$p_{\hbar,z} - \imath q_{\hbar,z}$ and its complex conjugate are elliptic near $\mu = \pm\epsilon_0$, and which satisfies that $\pm q_{\hbar,z} \geq 0$ on $\Sigma_{\hbar,\pm}$. This can be achieved as above for μ near μ_0. Again, we extend P_σ and Q_σ to X in such a way that $p - \imath q$ and $p_{\hbar,z} - \imath q_{\hbar,z}$ (and thus their complex conjugates) are elliptic near ∂X_{μ_0}.

In either of these semiclassical cases we have arranged that for sufficiently small $\delta_0 > 0$, $p_{\hbar,z} - \imath q_{\hbar,z}$ and its complex conjugate are *semiclassically non-trapping* for $|\operatorname{Im} z| < \delta_0$, namely the bicharacteristics from any point in $\Sigma_\hbar \setminus (L_+ \cup L_-)$ flow to $\operatorname{ell}(q_{\hbar,z}) \cup L_-$ (i.e. either enter $\operatorname{ell}(q_{\hbar,z})$ at some finite time, or tend to L_-) in the forward direction, and to $\operatorname{ell}(q_{\hbar,z}) \cup L_+$ in the backward direction. Here $\delta_0 > 0$ arises from the particularly simple choice of $q_{\hbar,z}$ for which semiclassical ellipticity is easy to check for $\operatorname{Im} z > 0$ (bounded away from 0) and small; a more careful analysis would give a specific value of δ_0, and a more careful choice of $q_{\hbar,z}$ would give a better result.

4. Microlocal analysis

4.1. **Elliptic and microhyperbolic points.** First, recall the basic elliptic and microhyperbolic regularity results. Let $\mathrm{WF}^s(u)$ denote the H^s wave front set of a distribution $u \in \mathcal{C}^{-\infty}(X)$, i.e. $\alpha \notin \mathrm{WF}^s(u)$ if there exists $A \in \Psi^0(X)$ elliptic at α such that $Au \in H^s$. Elliptic regularity states that

$$P_\sigma - \imath Q_\sigma \text{ elliptic at } \alpha,\ \alpha \notin \mathrm{WF}^{s-2}((P_\sigma - \imath Q_\sigma)u) \Rightarrow \alpha \notin \mathrm{WF}^s(u).$$

In particular, if $(P_\sigma - \imath Q_\sigma)u \in H^{s-2}$ and $p - \imath q$ is elliptic at α then $\alpha \notin \mathrm{WF}^s(u)$. Analogous conclusions apply to $P_\sigma^* + \imath Q_\sigma^*$; since both p and q are real, $p - \imath q$ is elliptic if and only if $p + \imath q$ is.

We also have real principal type propagation, in the usual form valid outside $\operatorname{supp} q$:

$$\mathrm{WF}^s(u) \setminus (\mathrm{WF}^{s-1}((P_\sigma - \imath Q_\sigma)u) \cup \operatorname{supp} q)$$

is a union of maximally extended bicharacteristics of H_p in the characteristic set $\Sigma = \{p = 0\}$ of P_σ. Putting it differently,

$$\alpha \notin \mathrm{WF}^s(u) \cup \mathrm{WF}^{s-1}((P_\sigma - \imath Q_\sigma)u) \cup \operatorname{supp} q \Rightarrow \tilde\gamma(\alpha) \cap \mathrm{WF}^s(u) = \emptyset,$$

where $\tilde\gamma(\alpha)$ is the component of the bicharacteristic $\gamma(\alpha)$ of p in the complement of $\mathrm{WF}^{s-1}((P_\sigma - \imath Q_\sigma)u) \cup \operatorname{supp} q$. If $(P_\sigma - \imath Q_\sigma)u \in H^{s-1}$, then $\mathrm{WF}^{s-1}((P_\sigma - \imath Q_\sigma)u) = \emptyset$ can be dropped from all statements above; if $q = 0$ one can thus replace $\tilde\gamma$ by γ.

In general, the result does not hold for non-zero q. However, it holds in one direction (backward/forward) of propagation along H_p if q has the correct sign. Thus, let $\tilde\gamma_\pm(\alpha)$ be a forward $(+)$ or backward $(-)$ bicharacteristic from α, defined on an interval I. If $\pm q \geq 0$ on a neighborhood of $\tilde\gamma_\pm(\alpha)$ (i.e. $q \geq 0$ on a neighborhood of $\tilde\gamma_+(\alpha)$, or $q \leq 0$ on a neighborhood of $\tilde\gamma_-(\alpha)$) then (for the corresponding sign)

$$\alpha \notin \mathrm{WF}^s(u) \text{ and } \mathrm{WF}^{s-1}((P_\sigma - \imath Q_\sigma)u) \cap \tilde\gamma_\pm(\alpha) = \emptyset \Rightarrow \tilde\gamma_\pm(\alpha) \cap \mathrm{WF}^s(u) = \emptyset,$$

i.e. one can propagate regularity forward if $q \geq 0$, backward if $q \leq 0$. A proof of this claim that is completely analogous to Hörmander's positive commutator proof in the real principal type setting can easily be given; see [15] and [4] in the semiclassical setting; the changes are minor in the 'classical' setting. Note that at points where $q \neq 0$, just $\alpha \notin \mathrm{WF}^{s-1}((P_\sigma - \imath Q_\sigma)u)$ implies $\alpha \notin \mathrm{WF}^{s+1}(u)$ (stronger than stated above), but at points with $q = 0$ such an elliptic estimate is unavailable (unless P_σ is elliptic).

As $P_\sigma^* + \imath Q_\sigma^*$ has symbol $p + \imath q$, one can propagate regularity in the opposite direction as compared to $P_\sigma - \imath Q_\sigma$. Thus, if $\mp q \geq 0$ on a neighborhood of $\tilde{\gamma}_\pm(\alpha)$ (i.e. $q \leq 0$ on a neighborhood of $\tilde{\gamma}_+(\alpha)$, or $q \geq 0$ on a neighborhood of $\tilde{\gamma}_-(\alpha)$) then (for the corresponding sign)

$$\alpha \notin \mathrm{WF}^s(u) \text{ and } \mathrm{WF}^{s-1}((P_\sigma^* + \imath Q_\sigma^*)u) \cap \tilde{\gamma}_\pm(\alpha) = \emptyset \Rightarrow \tilde{\gamma}_\pm(\alpha) \cap \mathrm{WF}^s(u) = \emptyset.$$

4.2. **Analysis near $\Lambda_\pm$.** The last ingredient in the classical setting is an analogue of Melrose's regularity result at radial sets which have the same features as ours. Although it is not stated in this generality in Melrose's paper [12], the proof is easily adapted. Thus, the results are:

At $\Lambda_\pm$, for $s \geq m > (1 - \operatorname{Im}\sigma)/2$, we can propagate estimates *away* from $\Lambda_\pm$:

Proposition 4.1. *Suppose $s \geq m > (1 - \operatorname{Im}\sigma)/2$, and $\mathrm{WF}^m(u) \cap \Lambda_\pm = \emptyset$. Then*

$$\Lambda_\pm \cap \mathrm{WF}^{s-1}(P_\sigma u) = \emptyset \Rightarrow \Lambda_\pm \cap \mathrm{WF}^s(u) = \emptyset.$$

This is completely analogous to Melrose's estimates in asymptotically Euclidean scattering theory at the radial sets [12, Section 9]. Note that the H^s regularity of u at $\Lambda_\pm$ is 'free' in the sense that we do not need to impose H^s assumptions on u anywhere; merely H^m at $\Lambda_\pm$ does the job; of course, on $P_\sigma u$ one must make the H^{s-1} assumption, i.e. the loss of one derivative compared to the elliptic setting. At the cost of changing regularity, one can propagate estimate *towards* $\Lambda_\pm$. Keeping in mind that taking P_σ^* in place of P_σ, principal symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*)$ switches sign, we have the following:

Proposition 4.2. *For $s < (1 + \operatorname{Im}\sigma)/2$, and O a neighborhood of $\Lambda_\pm$,*

$$\mathrm{WF}^s(u) \cap (O \setminus \Lambda_\pm) = \emptyset,\ \mathrm{WF}^{s-1}(P_\sigma^* u) \cap \Lambda_\pm = \emptyset \Rightarrow \mathrm{WF}^s(u) \cap \Lambda_\pm = \emptyset.$$

Proof of Propositions 4.1-4.2. The proof is a positive commutator estimate. Consider commutants $C_\epsilon^* C_\epsilon$ with $C_\epsilon \in \Psi^{s-1/2-\delta}(X)$ for $\epsilon > 0$, uniformly bounded in $\Psi^{s-1/2}(X)$ as $\epsilon \to 0$; with the ϵ-dependence used to regularize the argument. More precisely, let

$$c = \phi(\rho_0)\tilde{\rho}^{-s+1/2}, \qquad c_\epsilon = c(1 + \epsilon\tilde{\rho}^{-1})^{-\delta},$$

where $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ is identically 1 near 0, $\phi' \leq 0$ and ϕ is supported sufficiently close to 0 so that

$$\rho_0 \in \operatorname{supp} d\phi \Rightarrow \pm\tilde{\rho}\,\mathsf{H}_p\rho_0 > 0; \tag{4.1}$$

such ϕ exists by (3.21). To avoid using the sharp Gårding inequality, we choose ϕ so that $\sqrt{-\phi\phi'}$ is $\mathcal{C}^\infty$. Note that the sign of $\mathsf{H}_p\tilde{\rho}^{-s+1/2}$ depends on the sign of $-s+1/2$ which explains the difference between $s > 1/2$ and $s < 1/2$ in Propositions 4.1-4.2 when there are no other contributions to the threshold value of s. The contribution of the principal symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*)$, however, shifts the critical value $1/2$.

Now let $C \in \Psi^{s-1/2}(X)$ have principal symbol c, and have $\mathrm{WF}'(C) \subset \operatorname{supp}\phi\circ\rho_0$, and let $C_\epsilon = CS_\epsilon$, $S_\epsilon \in \Psi^{-\delta}(X)$ uniformly bounded in $\Psi^0(X)$ for $\epsilon > 0$, converging to Id in $\Psi^{\delta'}(X)$ for $\delta' > 0$ as $\epsilon \to 0$, with principal symbol $(1 + \epsilon\tilde{\rho}^{-1})^{-\delta}$. Thus, the principal symbol of C_ϵ is c_ϵ.

First, consider Proposition 4.1. Then

$$\begin{aligned}&\sigma_{2s}(\imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma)) = \sigma_1(\imath(P_\sigma^* - P_\sigma))c_\epsilon^2 + 2c_\epsilon \mathsf{H}_p c_\epsilon\\ &= \pm 8\left(-\operatorname{Im}\sigma\phi + \left(-s + \frac{1}{2}\right)\phi \pm \frac{1}{4}(\tilde{\rho}\mathsf{H}_p\rho_0)\phi' + \delta\frac{\epsilon}{\tilde{\rho}+\epsilon}\phi\right)\phi\tilde{\rho}^{-2s}(1 + \epsilon\tilde{\rho}^{-1})^{-\delta},\end{aligned}$$

so

$$\begin{aligned}&\pm\sigma_{2s}(\imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma))\\ &\qquad \leq -8\left(s-\frac{1}{2}+\operatorname{Im}\sigma-\delta\right)\tilde\rho^{-2s}(1+\epsilon\tilde\rho^{-1})^{-\delta}\phi^2\\ &\qquad\qquad + 2(\pm\tilde\rho\mathsf{H}_p\rho_0)\tilde\rho^{-2s}(1+\epsilon\tilde\rho^{-1})^{-\delta}\phi'\phi.\end{aligned}\tag{4.2}$$

Here the first term on the right hand side is negative if $s-1/2+\operatorname{Im}\sigma-\delta>0$ and this is the same sign as that of ϕ' term; the presence of δ (needed for the regularization) is the reason for the appearance of m in the estimate. Thus,

$$\pm\imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma) = -S_\epsilon^*(B^*B + B_1^*B_1 + B_{2,\epsilon}^*B_{2,\epsilon})S_\epsilon + F_\epsilon,$$

with $B, B_1, B_{2,\epsilon}\in\Psi^s(X)$, $B_{2,\epsilon}$ uniformly bounded in $\Psi^s(X)$ as $\epsilon\to 0$, F_ϵ uniformly bounded in $\Psi^{2s-1}(X)$, and $\sigma_s(B)$ an elliptic multiple of $\phi(\rho_0)\tilde\rho^{-s}$. Computing the pairing, using an extra regularization (insert a regularizer $\Lambda_r\in\Psi^{-1}(X)$, uniformly bounded in $\Psi^0(X)$, converging to Id in $\Psi^\delta(X)$ to justify integration by parts, and use that $[\Lambda_r, P_\sigma^*]$ is uniformly bounded in $\Psi^1(X)$, converging to 0 strongly, cf. [19, Lemma 17.1] and its use in [19, Lemma 17.2]) yields

$$\langle\imath(P_\sigma^* C_\epsilon^* C_\epsilon - C_\epsilon^* C_\epsilon P_\sigma)u,u\rangle = \langle\imath C_\epsilon^* C_\epsilon u, P_\sigma u\rangle - \langle\imath P_\sigma, C_\epsilon^* C_\epsilon u\rangle.\tag{4.3}$$

Using Cauchy-Schwartz on the right hand side, a standard functional analytic argument (see, for instance, Melrose [12, Proof of Proposition 7 and Section 9]) gives an estimate for Bu, showing u is in H^s on the elliptic set of B, provided u is microlocally in $H^{s-\delta}$. A standard inductive argument, starting with $s-\delta=m$ and improving regularity by $\leq 1/2$ in each step proves Proposition 4.1.

For Proposition 4.2, when applied to P_σ in place of P_σ^* (so the assumption is $s<(1-\operatorname{Im}\sigma)/2$), the argument is similar, but we want to change the sign of the first term on the right hand side of (4.2), i.e. we want it to be positive. This is satisfied if $s-1/2+\operatorname{Im}\sigma-\delta<0$, hence (as $\delta>0$) if $s-1/2+\operatorname{Im}\sigma<0$, so regularization is not an issue. On the other hand, ϕ' now has the wrong sign, so one needs to make an assumption on $\operatorname{supp} d\phi$; one can arrange that this is in $O\setminus\Lambda$ by making ϕ have sufficiently small support, but identically 1 near 0. Since the details are standard, see [12, Section 9], we leave these to the reader. When interchanging P_σ and P_σ^*, we need to take into account the switch of the sign of the principal symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*)$, which causes the sign change in front of $\operatorname{Im}\sigma$ in the statement of the proposition. □

4.3. **Global estimates.** For our Fredholm results, we actually need estimates. However, these can be easily obtained from regularity results as in e.g. [10, Proof of Theorem 26.1.7] by the closed graph theorem. It should be noted that of course one really proved versions of the relevant estimates when proving regularity, but the closed graph theorem provides a particularly simple way of combining these (though it comes at the cost of using a theorem which in principle is unnecessary).

So suppose $s\geq m>(1-\operatorname{Im}\sigma)/2$, $u\in H^m$ and $(P_\sigma-\imath Q_\sigma)u\in H^{s-1}$. The above results give that, first, $\mathrm{WF}^s(u)$ (indeed, $\mathrm{WF}^{s+1}(u)$) is disjoint from the elliptic set of $P_\sigma-\imath Q_\sigma$. Next $\Lambda_\pm$ is disjoint from $\mathrm{WF}^s(u)$, hence so is a neighborhood of $\Lambda_\pm$ as the complement of the wave front set is open. Thus by propagation of singularities and (3.32), taking into account the sign of q along $\Sigma_\pm$, $\mathrm{WF}^s(u)\cap\Sigma_\pm=\emptyset$. Now, by the regularity result, the inclusion map

$$\mathcal{Z}_s=\{u\in H^m:\ (P_\sigma-\imath Q_\sigma)u\in H^{s-1}\}\to H^m,$$

in fact maps to H^s.

Note that $\mathcal{Z}_s$ is complete with the norm $\|u\|^2_{\mathcal{Z}_s} = \|u\|^2_{H^m} + \|(P_\sigma - \imath Q_\sigma)u\|^2_{H^{s-1}}$. Indeed, $\{u_j\}_{j=1}^\infty$ Cauchy in $\mathcal{Z}_s$ means $u_j \to u$ in H^m and $(P_\sigma - \imath Q_\sigma)u_j \to v \in H^{s-1}$. By the first convergence, $(P_\sigma - \imath Q_\sigma)u_j \to (P_\sigma - \imath Q_\sigma)u$ in H^{m-2}, thus, as $s-1 \geq m-2$, $(P_\sigma - \imath Q_\sigma)u_j \to v$ in H^{m-2} shows $(P_\sigma - \imath Q_\sigma)u = v \in H^{s-1}$, and thus, $(P_\sigma - \imath Q_\sigma)u_j \to (P_\sigma - \imath Q_\sigma)u$ in H^{s-1}, so $u_j \to u$ in $\mathcal{Z}_s$.

The graph of the inclusion map, considered as a subset of $\mathcal{Z}_s \times H^s$ is closed, for $(u_j, u_j) \to (u, v) \in \mathcal{Z}_s \times H^s$ implies in particular $u_j \to u$ and $u_j \to v$ in H^m, so $u = v \in \mathcal{Z}_s \cap H^s$. Correspondingly, by the closed graph theorem, the inclusion map is continuous, i.e.

$$\|u\|_{H^s} \leq C(\|(P_\sigma - \imath Q_\sigma)u\|_{H^{s-1}} + \|u\|_{H^m}), \qquad u \in \mathcal{Z}_s. \tag{4.4}$$

This estimate implies that $\mathrm{Ker}(P_\sigma - \imath Q_\sigma)$ in H^s is finite dimensional since elements of this kernel lie in $\mathcal{Z}_s$, and since on the unit ball of this closed subspace of H^s (for $P_\sigma - \imath Q_\sigma : H^s \to H^{s-2}$ is continuous), $\|u\|_{H^s} \leq C\|u\|_{H^m}$, and the inclusion $H^s \to H^m$ is compact. Further, elements of $\mathrm{Ker}(P_\sigma - \imath Q_\sigma)$ are in $\mathcal{C}^\infty(X)$ by our regularity result, and thus this space is independent of the choice of s.

On the other hand, for the adjoint operator, $P_\sigma^* + \imath Q_\sigma^*$ we have that if $s' < (1 + \operatorname{Im}\sigma)/2$ (recall that replacing P_σ by its adjoint switches the sign of the principal symbol of $\frac{1}{2\imath}(P_\sigma - P_\sigma^*)$), $u \in H^{-N}$ and $(P_\sigma^* + \imath Q_\sigma^*)u \in H^{s'-1}$ then first $\mathrm{WF}^{s'}(u)$ (indeed, $\mathrm{WF}^{s'+1}(u)$) is disjoint from the elliptic set of $P_\sigma^* + \imath Q_\sigma^*$. Next, by propagation of singularities and (3.32), taking into account the sign of q along $\Sigma_\pm$, namely the sign of the imaginary part of the principal symbol switched by taking the adjoints, $\mathrm{WF}^{s'}(u) \cap (\Sigma_\pm \setminus \Lambda_\pm) = \emptyset$. Finally, by the result at the radial points $\Lambda_\pm$ is disjoint from $\mathrm{WF}^{s'}(u)$. Thus, the inclusion map

$$\mathcal{W}_{s'} = \{u \in H^{-N} :\ (P_\sigma^* + \imath Q_\sigma^*)u \in H^{s'-1}\} \to H^{-N},$$

in fact maps to $H^{s'}$. We deduce, as above, by the closed graph theorem, that

$$\|u\|_{H^{s'}} \leq C(\|(P_\sigma^* + \imath Q_\sigma^*)u\|_{H^{s'-1}} + \|u\|_{H^{-N}}), \qquad u \in \mathcal{W}_{s'}. \tag{4.5}$$

As above, this estimate implies that $\mathrm{Ker}(P_\sigma^* + \imath Q_\sigma^*)$ in $H^{s'}$ is finite dimensional. Indeed, by our regularity results (elliptic regularity, propagation of singularities, and then regularity at the radial set) elements of $\mathrm{Ker}(P_\sigma^* + \imath Q_\sigma^*)$ have wave front set in $\Lambda_+ \cup \Lambda_-$ and lie in $\cap_{s' < (1+\operatorname{Im}\sigma)/2} H^{s'}$.

The dual of H^s for $s > (1 - \operatorname{Im}\sigma)/2$, is $H^{-s} = H^{s'-1}$, $s' = 1 - s$, so $s' < (1 + \operatorname{Im}\sigma)/2$ in this case, while the dual of H^{s-1}, $s > (1 - \operatorname{Im}\sigma)/2$, is $H^{1-s} = H^{s'}$, with $s' = 1 - s < (1 + \operatorname{Im}\sigma)/2$ again. Thus, the spaces (apart from the residual spaces H^m and H^{-N}, into which the inclusion is compact) in the left, resp. right, side of (4.5), are exactly the duals of those on the right, resp. left, side of (4.4). Thus, by a standard functional analytic argument, see e.g. [10, Proof of Theorem 26.1.7], namely dualization and using the compactness of the inclusion $H^{s'} \to H^{-N}$ for $s' > -N$, (4.5) gives the H^s-solvability, $s = 1 - s'$ (i.e. we demand $u \in H^s$), of

$$(P_\sigma - \imath Q_\sigma)u = f,\ s > (1 - \operatorname{Im}\sigma)/2,$$

for f in the annihilator (in $H^{s-1} = (H^{s'})^*$ with duality induced by the L^2 inner product) of the finite dimensional subspace $\mathrm{Ker}(P_\sigma^* + \imath Q_\sigma^*)$ of $H^{s'}$.

Recall from [10, Proof of Theorem 26.1.7] that this argument has two parts: first for any complementary subspace V of $\mathrm{Ker}(P_\sigma^* + \imath Q_\sigma^*)$ in $H^{s'}$ (i.e. V is closed,

$V \cap \operatorname{Ker}(P_\sigma^* + \imath Q_\sigma^*) = \{0\}$, and $V + \operatorname{Ker}(P_\sigma^* + \imath Q_\sigma^*) = H^{s'}$, e.g. V is the $H^{s'}$ orthocomplement of $\operatorname{Ker}(P_\sigma^* + \imath Q_\sigma^*)$), one can drop $\|u\|_{H^{-N}}$ from the right hand side of (4.5) when $u \in V \cap \mathcal{W}_{s'}$ at the cost of replacing C by a larger constant C'. Indeed, if no C' existed, one would have a sequence $u_j \in V \cap \mathcal{W}_{s'}$ such that $\|u_j\|_{H^{s'}} = 1$ and $\|(P_\sigma^* + \imath Q_\sigma^*)u_j\|_{H^{s'-1}} \to 0$, so $(P_\sigma^* + \imath Q_\sigma^*)u_j \to 0$ in $H^{s'-1}$. By weak compactness of the $H^{s'}$ unit ball, there is a weakly convergent subsequence u_{j_ℓ} converging to some $u \in H^{s'}$, by the closedness (which implies weak closedness) of V, $u \in V$, so $(P_\sigma^* + \imath Q_\sigma^*)u_{j_\ell} \to (P_\sigma^* + \imath Q_\sigma^*)u$ weakly in $H^{s'-2}$, and thus $(P_\sigma^* + \imath Q_\sigma^*)u = 0$ so $u \in V \cap \operatorname{Ker}(P_\sigma^* + \imath Q_\sigma^*) = \{0\}$. On the other hand, by compactness of the inclusion $H^{s'} \to H^{-N}$, $u_{j_\ell} \to u$ strongly in H^{-N}, so $\{u_{j_\ell}\}$ is Cauchy in H^{-N}, hence from (4.5), it is Cauchy in $H^{s'}$, so it converges to u strongly in $H^{s'}$ and hence $\|u\|_{H^{s'}} = 1$. This contradicts $u = 0$, completing the proof of

$$\|u\|_{H^{s'}} \leq C' \|(P_\sigma^* + \imath Q_\sigma^*)u\|_{H^{s'-1}}, \ u \in V \cap \mathcal{W}_{s'}. \tag{4.6}$$

Thus, with $s' = 1 - s$, and for f in the annihilator (in H^{s-1}, via the L^2-pairing) of $\operatorname{Ker}(P_\sigma^* + \imath Q_\sigma^*) \subset H^{s'}$, and for $v \in V \cap \mathcal{W}_{s'}$,

$$|\langle f, v\rangle| \leq \|f\|_{H^{s-1}} \|v\|_{H^{s'}} \leq C' \|f\|_{H^{s-1}} \|(P_\sigma^* + \imath Q_\sigma^*)v\|_{H^{s'-1}}.$$

As adding an element of $\operatorname{Ker}(P_\sigma^* + \imath Q_\sigma^*)$ to v does not change either side, the inequality holds for all $v \in \mathcal{W}_{s'} \subset H^{s'}$. Thus, the conjugate-linear map $(P_\sigma^* + \imath Q_\sigma^*)v \mapsto \langle f, v\rangle$, $v \in \mathcal{W}_{s'}$, which is well-defined, is continuous from $\operatorname{Ran}_{\mathcal{W}_{s'}}(P_\sigma^* + \imath Q_\sigma^*) \subset H^{s'-1}$ to $\mathbb{C}$, and by the Hahn-Banach theorem can be extended to a continuous conjugate linear functional ℓ on $H^{s'-1} = (H^s)^*$, so there exists $u \in H^s$ such that $\langle u, \phi\rangle = \ell(\phi)$ for $\phi \in H^{s'-1}$. In particular, when $\phi = (P_\sigma^* + \imath Q_\sigma^*)\psi$, $\psi \in \mathcal{C}^\infty(X) \subset \mathcal{W}_{s'}$,

$$\langle u, (P_\sigma^* + \imath Q_\sigma^*)\psi\rangle = \ell(\phi) = \langle f, \psi\rangle,$$

so $(P_\sigma - \imath Q_\sigma)u = f$ as claimed.

In order to set up Fredholm theory, let $\tilde{P}$ be any operator with principal symbol $p - \imath q$; e.g. $\tilde{P}$ is $P_{\sigma_0} - \imath Q_{\sigma_0}$ for some σ_0. Then consider

$$\mathcal{X}^s = \{u \in H^s : \ \tilde{P}u \in H^{s-1}\}, \ \mathcal{Y}^s = H^{s-1}, \tag{4.7}$$

with

$$\|u\|_{\mathcal{X}^s}^2 = \|u\|_{H^s}^2 + \|\tilde{P}u\|_{H^{s-1}}^2.$$

Note that the $\mathcal{Z}_s$-norm is equivalent to the $\mathcal{X}^s$-norm, and $\mathcal{Z}_s = \mathcal{X}^s$, by (4.4) and the preceding discussion. Note that $\mathcal{X}^s$ only depends on the principal symbol of $\tilde{P}$. Moreover, $\mathcal{C}^\infty(X)$ is dense in $\mathcal{X}^s$; this follows by considering $R_\epsilon \in \Psi^{-\infty}(X)$, $\epsilon > 0$, such that $R_\epsilon \to \mathrm{Id}$ in $\Psi^\delta(X)$ for $\delta > 0$, R_ϵ uniformly bounded in $\Psi^0(X)$; thus $R_\epsilon \to \mathrm{Id}$ strongly (but not in the operator norm topology) on H^s and H^{s-1}. Then for $u \in \mathcal{X}^s$, $R_\epsilon u \in \mathcal{C}^\infty(X)$ for $\epsilon > 0$, $R_\epsilon u \to u$ in H^s and $\tilde{P}R_\epsilon u = R_\epsilon \tilde{P}u + [\tilde{P}, R_\epsilon]u$, so the first term on the right converges to $\tilde{P}u$ in H^{s-1}, while $[\tilde{P}, R_\epsilon]$ is uniformly bounded in $\Psi^1(X)$, converging to 0 in $\Psi^{1+\delta}(X)$ for $\delta > 0$, so converging to 0 strongly as a map $H^s \to H^{s-1}$. Thus, $[\tilde{P}, R_\epsilon]u \to 0$ in H^{s-1}, and we conclude that $R_\epsilon u \to u$ in $\mathcal{X}^s$. (In fact, $\mathcal{X}^s$ is a first-order coisotropic space, more general function spaces of this nature are discussed by Melrose, Vasy and Wunsch in [14, Appendix A].)

With these preliminaries,

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

is bounded for each σ with $s \geq m > (1 - \operatorname{Im} \sigma)/2$, and is an analytic family of bounded operators in this half-plane of σ's. Further, it is Fredholm for each σ: the kernel in $\mathcal{X}^s$ is finite dimensional, and it surjects onto the annihilator in H^{s-1} of the (finite dimensional) kernel of $P_\sigma^* + \imath Q_\sigma^*$ in H^{1-s}, which thus has finite codimension, and is closed, since for f in this space there exists $u \in H^s$ with $(P_\sigma - \imath Q_\sigma)u = f$, and thus $u \in \mathcal{X}^s$. Restating this as a theorem:

Theorem 4.3. *Let P_σ, Q_σ be as above, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in (4.7). Then*

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

is an analytic family of Fredholm operators on

$$\mathbb{C}_s = \{\sigma \in \mathbb{C} : \ \operatorname{Im} \sigma > 1 - 2s\}. \tag{4.8}$$

Thus, analytic Fredholm theory applies, giving meromorphy of the inverse provided the inverse exists for a particular value of σ.

Remark 4.4. Note that the Fredholm property means that $P_\sigma^* + \imath Q_\sigma^*$ is also Fredholm on the dual spaces; this can also be seen directly from the estimates. The analogue of this remark also applies to the semiclassical discussion below.

4.4. **Semiclassical estimates.** There are semiclassical estimates completely analogous to those in the classical setting; we again phrase these as wave front set statements. Let $H_\hbar^s$ denote the semiclassical Sobolev space of order s, i.e. as a function space this is the space of functions $(u_h)_{h \in I}$, $I \subset (0,1]_h$ with values in the standard Sobolev space H^s, with $A_h u_h$ bounded in L^2 for an elliptic, semiclassically elliptic, operator $A_h \in \Psi_\hbar^s(X)$. (Note that u_h need not be defined for all $h \in (0,1]$; we suppress I from the notation.) Let

$$\mathrm{WF}_\hbar^{s,r}(u) \subset \partial(\overline{T}^* X \times [0,1)_h) = S^*X \times [0,1)_h \cup T^*X \times \{0\}_h$$

denote the semiclassical wave front set of a polynomially bounded family of distributions, i.e. $u = (u_h)_{h \in I}$, $I \subset (0,1]$, satisfying u_h is uniformly bounded in $h^{-N} H_\hbar^{-N}$ for some N. This is defined as follows: we say that $\alpha \notin \mathrm{WF}_\hbar^{s,r}(u)$ if there exists $A \in \Psi_\hbar^0(X)$ elliptic at α such that $Au \in h^r H_\hbar^s$. Note that, in view of the description of the symbols in Section 2, ellipticity at α means the ellipticity of $\sigma_\hbar(A_h)$ if $\alpha \in T^*X \times \{0\}$, that of $\sigma_0(A_h)$ if $\alpha \in S^*X \times (0,1)$, and that of either (and thus both, in view of the compatibility of these symbols) of these when $\alpha \in S^*X \times \{0\}$. The semiclassical wave front set captures global estimates: if u is polynomially bounded and $\mathrm{WF}_\hbar^{s,r}(u) = \emptyset$, then $u \in h^r H_\hbar^s$.

Elliptic regularity states that

$$P_{h,z} - \imath Q_{h,z} \text{ elliptic at } \alpha, \ \alpha \notin \mathrm{WF}_\hbar^{s-2,0}((P_{h,z} - \imath Q_{h,z})u) \Rightarrow \alpha \notin \mathrm{WF}_\hbar^{s,0}(u).$$

Thus, $(P_{h,z} - \imath Q_{h,z}) \in H_\hbar^{s-2}$ and $p_{\hbar,z} - \imath q_{\hbar,z}$ is elliptic at α then $\alpha \notin \mathrm{WF}_\hbar^s(u)$.

We also have real principal type propagation:

$$\mathrm{WF}_\hbar^{s,-1}(u) \setminus (\mathrm{WF}_\hbar^{s-1,0}((P_{h,z} - \imath Q_{h,z})u) \cup \operatorname{supp} q_{\hbar,z})$$

is a union of maximally extended bicharacteristics of H_p in the characteristic set $\Sigma_{\hbar,z} = \{p_{\hbar,z} = 0\}$ of $P_{h,z}$. Put it differently,

$$\alpha \notin \mathrm{WF}^{s,-1}(u) \cup \mathrm{WF}^{s-1,0}((P_{h,z} - \imath Q_{h,z})u) \cup \operatorname{supp} q_{\hbar,z} \Rightarrow \tilde{\gamma}(\alpha) \cap \mathrm{WF}^{s,-1}(u) = \emptyset,$$

where $\tilde{\gamma}(\alpha)$ is the component of the bicharacteristic $\gamma(\alpha)$ of $p_{\hbar,z}$ in the complement of $\mathrm{WF}^{s-1,0}((P_{h,z} - \imath Q_{h,z})u) \cup \operatorname{supp} q_{\hbar,z}$. If $(P_{h,z} - \imath Q_{h,z})u \in H^{s-1}$, then

$\mathrm{WF}^{s-1,0}((P_{h,z}-\imath Q_{h,z})u)=\emptyset$ can be dropped from all statements above; if $q_{\hbar,z}=0$ one can thus replace $\tilde\gamma$ by γ.

In general, the result does not hold for non-zero $q_{\hbar,z}$. However, it holds in one direction (backward/forward) of propagation along $\mathsf{H}_{p_{\hbar,z}}$ if $q_{\hbar_z}$ has the correct sign. Thus, with $\tilde\gamma_\pm(\alpha)$ a forward (+) or backward (−) bicharacteristic from α defined on an interval, if $\pm q_{\hbar,z}\geq 0$ on a neighborhood of $\tilde\gamma_\pm(\alpha)$ then

$$\alpha\notin \mathrm{WF}_\hbar^{s,-1}(u) \text{ and } \tilde\gamma_\pm(\alpha)\cap \mathrm{WF}_\hbar^{s-1,0}((P_{\hbar,z}-\imath Q_{\hbar,z})u)=\emptyset$$
$$\Rightarrow \tilde\gamma_\pm(\alpha)\cap \mathrm{WF}_\hbar^{s,-1}(u)=\emptyset,$$

i.e. one can propagate regularity forward if $q_{\hbar,z}\geq 0$, backward if $q_{\hbar,z}\leq 0$; see [15] and [4]. Again, for $P^*_{\hbar,z}+\imath Q^*_{\hbar,z}$ the directions are reversed, i.e. one can propagate regularity forward if $q_{\hbar,z}\leq 0$, backward if $q_{\hbar,z}\geq 0$.

A semiclassical version of Melrose's regularity result was proved by Zworski and the author in the asymptotically Euclidean setting, [21]. We need a more general result, which is an easy adaptation:

Proposition 4.5. *Suppose $s\geq m>(1-\operatorname{Im}\sigma)/2$, and $\mathrm{WF}_\hbar^{m,-N}(u)\cap L_\pm=\emptyset$ for some N. Then*

$$L_\pm\cap \mathrm{WF}^{s-1,0}(P_{\hbar,z}u)=\emptyset\Rightarrow L_\pm\cap\mathrm{WF}^{s,-1}(u)=\emptyset.$$

Again, at the cost of changing regularity, one can propagate estimate *towards* $L_\pm$.

Proposition 4.6. *For $s<(1+\operatorname{Im}\sigma)/2$, and O a neighborhood of $L_\pm$,*

$$\mathrm{WF}_\hbar^{s,-1}(u)\cap(O\setminus L_\pm)=\emptyset,\ \mathrm{WF}_\hbar^{s-1,0}(P^*_{\hbar,z}u)\cap L_\pm=\emptyset\Rightarrow \mathrm{WF}_\hbar^{s,-1}(u)\cap L_\pm=\emptyset.$$

Proof. We just need to localize in $\tilde\rho$ in addition to ρ_0; such a localization in the classical setting is implied by working on S^*X or with homogeneous symbols. We achieve this by modifying the localizer ϕ in the commutant constructed in the proof of Propositions 4.1-4.2. As already remarked, the proof is much like at radial points in semiclassical scattering on asymptotically Euclidean spaces, studied by Vasy and Zworski [21], but we need to be more careful about localization in ρ_0 and $\tilde\rho$ as we are assuming less about the structure.

First, note that $L_\pm$ is defined by $\tilde\rho=0$, $\rho_0=0$, so $\tilde\rho^2+\rho_0$ is a quadratic defining function of $L_\pm$. Thus, let $\phi\in\mathcal{C}_c^\infty(\mathbb{R})$ be identically 1 near 0, $\phi'\leq 0$ and ϕ supported sufficiently close to 0 so that

$$\tilde\rho^2+\rho_0\in\operatorname{supp}d\phi\Rightarrow \pm\tilde\rho(\mathsf{H}_p\rho_0+2\tilde\rho\mathsf{H}_p\tilde\rho)>0$$

and

$$\tilde\rho^2+\rho_0\in\operatorname{supp}\phi\Rightarrow\pm\tilde\rho\mathsf{H}_p\tilde\rho>0.$$

Such ϕ exists by (3.19) and (3.21) as

$$\pm\tilde\rho(\mathsf{H}_p\rho_0+2\tilde\rho\mathsf{H}_p\tilde\rho)\geq 8\rho_0+8\tilde\rho^2-\mathcal{O}((\tilde\rho^2+\rho_0)^{3/2}).$$

Then let c be given by

$$c=\phi(\rho_0+\tilde\rho^2)\tilde\rho^{-s+1/2},\qquad c_\epsilon=c(1+\epsilon\tilde\rho^{-1})^{-\delta}.$$

The rest of the proof proceeds exactly as for Propositions 4.1-4.2. □

Suppose now that $p_{\hbar,z}$ is semiclassically non-trapping, as discussed at the end of Section 3. Suppose again that $s \geq m > (1-\operatorname{Im}\sigma)/2$, $h^N u_h$ is bounded in $H_\hbar^m$ and $(P_{h,z} - \imath Q_{h,z})u_h \in H_\hbar^{s-1}$. The above results give that, first, $\mathrm{WF}_\hbar^{s,-1}(u)$ (indeed, $\mathrm{WF}_\hbar^{s+1,0}(u)$) is disjoint from the elliptic set of $P_{h,z} - \imath Q_{h,z}$. Next we see that $L_\pm$ is disjoint from $\mathrm{WF}_\hbar^{s,-1}(u)$, hence so is a neighborhood of $L_\pm$. Thus by propagation of singularities and the semiclassically non-trapping property, taking into account the sign of q along $\Sigma_{\hbar,\pm}$, $\mathrm{WF}_\hbar^{s,-1}(u) \cap \Sigma_{\hbar,\pm} = \emptyset$. In summary, $\mathrm{WF}_\hbar^{s,-1}(u) = \emptyset$, i.e. hu_h is bounded in $H_\hbar^s$, i.e.

$$h^N u_h \text{ bounded in } H_\hbar^m,\ (P_{h,z} - \imath Q_{h,z})u_h \in H_\hbar^{s-1} \Rightarrow hu_h \in H_\hbar^s. \tag{4.9}$$

Now suppose that for a decreasing sequence $h_j \to 0$, $w_h \in \operatorname{Ker}(P_{h,z} - \imath Q_{h,z})$ and $\|w_h\|_{H_\hbar^s} = 1$. Then for any N, $u_h = h^{-N} w_h$ satisfies the above hypotheses, and we deduce that hu_h is uniformly bounded in $H_\hbar^s$, i.e. $h^{-N+1} w_h$ is uniformly bounded in $H_\hbar^s$. But for $N > 1$ this contradicts that $\|w_h\|_{H_\hbar^s} = 1$, so such a sequence h_j does not exist. Therefore $\operatorname{Ker}(P_{h,z} - \imath Q_{h,z}) = \{0\}$ for sufficiently small h.

Using semiclassical propagation of singularities in the reverse direction, much as we did in the previous section, we deduce that $\operatorname{Ker}(P_{h,z}^* + \imath Q_{h,z}^*) = \{0\}$ for h sufficiently small. Since $P_{h,z} - \imath Q_{h,z} : \mathcal{X}^s \to \mathcal{Y}^s$ is Fredholm, we deduce immediately that there exists h_0 such that it is invertible for $h < h_0$.

In order to obtain uniform estimates for $(P_{h,z} - \imath Q_{h,z})^{-1}$ as $h \to 0$, it is convenient to 'renormalize' the problem to make the function spaces (and their norms) independent of h so that one can use the uniform boundedness principle. (Again, this could have been avoided if we had just stated the estimates uniformly in u as well, much like the closed graph theorem could have been avoided in the previous section.) So for $r \in \mathbb{R}$ let $\Lambda_h^r \in \Psi_h^r$ be elliptic and invertible, and let

$$P_{h,z}^s - \imath Q_{h,z}^s = \Lambda_h^{s-1}(P_{h,z} - \imath Q_{h,z})\Lambda_h^s.$$

Then, with $\tilde{P} = P_{h_0,z_0}^s \in \Psi^1(X)$, for instance, independent of h,

$$\mathcal{X} = \{u \in L^2 :\ \tilde{P}u \in L^2\},\ \mathcal{Y} = L^2,$$

$P_{h,z}^s - \imath Q_{h,z}^s : \mathcal{X} \to \mathcal{Y}$ is invertible for $h < h_0$ by the above observations. Let $j : \mathcal{X} \to \mathcal{Z} = L^2$ be the inclusion map. Then

$$j \circ h(P_{h,z}^s - \imath Q_{h,z}^s)^{-1} : \mathcal{Y} \to \mathcal{Z}$$

is continuous for each $h < h_0$.

We claim that for each (non-zero) $f \in \mathcal{Y}$, $\{\|h(P_{h,z}^s - \imath Q_{h,z}^s)^{-1} f\|_{L^2} :\ h < h_0\}$ is bounded. Indeed, let $v_h = h(P_{h,z}^s - \imath Q_{h,z}^s)^{-1} f$, so we need to show that v_h is bounded. Suppose first that hv_h is not bounded, so consider a sequence h_j with $h_j \|v_{h_j}\|_{L^2} \geq 1$. Then let $u_h = h^{-2} v_h / \|v_h\|_{L^2}$, $h \in \{h_j :\ j \in \mathbb{N}\}$, so $h^2 u_h$ is bounded in L^2, so u_h is in particular polynomially bounded in L^2. Also, $(P_{h,z}^s - \imath Q_{h,z}^s)u_h = h^{-1} f / \|v_h\|_{L^2}$ is bounded in L^2 as $\|v_h\| \geq h^{-1}$. Thus, by (4.9), hu_h is bounded in L^2, i.e. $h^{-1} v_h / \|v_h\|_{L^2}$ is bounded, which is a contradiction, showing that hv_h is bounded. Thus, introducing a new u_h, namely $u_h = h^{-1} v_h$, u_h is polynomially bounded, and $(P_{h,z}^s - \imath Q_{h,z}^s)u_h = f$ is bounded, so, by (4.9), $hu_h = v_h$ is bounded as claimed.

Thus, by the uniform boundedness principle, $j \circ h(P_{h,z}^s - \imath Q_{h,z}^s)^{-1}$ is equicontinuous. Undoing the transformation, we deduce that

$$\|(P_{h,z} - \imath Q_{h,z})^{-1} f\|_{H_\hbar^s} \leq C h^{-1} \|f\|_{H_\hbar^{s-1}},$$

which is exactly the high energy estimate we were after.

Our arguments were under the assumption of semiclassical non-trapping. As discussed in Subsections 3.4 and 3.5, this always holds in sectors $\delta|\operatorname{Re}\sigma| < \operatorname{Im}\sigma < \delta_0|\operatorname{Re}\sigma|$ (with Q_σ supported in $\mu < 0$!) since $P_{h,z} - \imath Q_{h,z}$ is actually semiclassically elliptic then. In particular this gives the meromorphy of $P_\sigma - \imath Q_\sigma$ by giving invertibility of large σ in such a sector. Rephrasing in the large parameter notation, using σ instead of h,

Theorem 4.7. *Let P_σ, Q_σ, $\mathbb{C}_s$ be as above, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in (4.7). Then, for $\sigma \in \mathbb{C}_s$,*

$$P_\sigma - \imath Q_\sigma : \mathcal{X}^s \to \mathcal{Y}^s$$

has a meromorphic inverse

$$R(\sigma) : \mathcal{Y}^s \to \mathcal{X}^s.$$

Moreover, there is $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ there is $\sigma_0 > 0$ such that $R(\sigma)$ is invertible in

$$\{\sigma : \ \delta|\operatorname{Re}\sigma| < \operatorname{Im}\sigma < \delta_0|\operatorname{Re}\sigma|, \ |\operatorname{Re}\sigma| > \sigma_0\},$$

and non-trapping estimates *hold:*

$$\|R(\sigma)f\|_{H^s_{|\sigma|^{-1}}} \leq C'|\sigma|^{-1}\|f\|_{H^{s-1}_{|\sigma|^{-1}}}.$$

If the metric g_0 is non-trapping then $p_{\hbar,z} - \imath q_{\hbar,z}$ and its complex conjugate are semiclassically non-trapping by Subsection 3.4, so the high energy estimates are then applicable in half-planes $\operatorname{Im}\sigma < -C$, i.e. half-planes $\operatorname{Im} z \geq -Ch$. The same holds for trapping g_0 provided that we add a complex absorbing operator near the trapping, as discussed in Subsection 3.5.

Translated into the classical setting this gives

Theorem 4.8. *Let P_σ, Q_σ, $\mathbb{C}_s$, $\delta_0 > 0$ be as above, in particular semiclassically non-trapping, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in (4.7). Let $C > 0$. Then there exists σ_0 such that*

$$R(\sigma) : \mathcal{Y}^s \to \mathcal{X}^s,$$

is holomorphic in $\{\sigma : \ -C < \operatorname{Im}\sigma < \delta_0|\operatorname{Re}\sigma|, \ |\operatorname{Re}\sigma| > \sigma_0\}$, assumed to be a subset of $\mathbb{C}_s$, and non-trapping estimates

$$\|R(\sigma)f\|_{H^s_{|\sigma|^{-1}}} \leq C'|\sigma|^{-1}\|f\|_{H^{s-1}_{|\sigma|^{-1}}}$$

hold. For $s = 1$ this states that for $|\operatorname{Re}\sigma| > \sigma_0$, $\operatorname{Im}\sigma > -C$,

$$\|R(\sigma)f\|^2_{L^2} + |\sigma|^{-2}\|dR(\sigma)\|^2_{L^2} \leq C''|\sigma|^{-2}\|f\|^2_{L^2}.$$

While we stated just the global results here, one also has microlocal estimates for the solution. In particular we have the following, stated in the semiclassical language, as immediate from the estimates used to derive from the Fredholm property:

Theorem 4.9. *Let P_σ, Q_σ, $\mathbb{C}_s$ be as above, in particular semiclassically non-trapping, and $\mathcal{X}^s$, $\mathcal{Y}^s$ as in (4.7).*

For $\operatorname{Re} z > 0$ and $s' > s$, the resolvent $R_{h,z}$ is semiclassically outgoing with a loss of h^{-1} *in the sense that if $\alpha \in \overline{T}^* X \cap \Sigma_{\hbar,\pm}$, and if for the backward $(-)$, resp. forward $(+)$, bicharacteristic $\gamma_\mp$, from α, $\mathrm{WF}_\hbar^{s'-1,0}(f) \cap \overline{\gamma_\mp} = \emptyset$ then $\alpha \notin \mathrm{WF}_\hbar^{s',-1}(R_{h,z}f)$.*

In fact, for any $s' \in \mathbb{R}$, the resolvent $R_{h,z}$ extends to $f \in H_h^{s'}(X)$, with non-trapping bounds, provided that $\mathrm{WF}_\hbar^{s,0}(f) \cap (L_+ \cup L_-) = \emptyset$. The semiclassically outgoing with a loss of h^{-1} result holds for such f and s' as well.

Proof. The only part that is not immediate by what has been discussed is the last claim. This follows immediately, however, by microlocal solvability in arbitrary ordered Sobolev spaces away from the radial points (i.e. solvability modulo $\mathcal{C}^\infty$, with semiclassical estimates), combined with our preceding results to deal with this smooth remainder plus the contribution near $L_+ \cup L_-$, which are assumed to be in $H_h^s(X)$. □

This result is needed for gluing constructions as in [4], namely polynomially bounded trapping with appropriate microlocal geometry can be glued to our resolvent. Furthermore, it gives non-trapping estimates microlocally away from the trapped set provided the overall (trapped) resolvent is polynomially bounded as shown by Datchev and Vasy [5].

5. Results in the conformally compact setting

We now state our results in the original conformally compact setting. Without the non-trapping estimate, these are a special case of a result of Mazzeo and Melrose [11], with improvements by Guillarmou [9], with 'special' meaning that evenness is assumed. If the space is asymptotic to actual hyperbolic space, the non-trapping estimate is a slightly stronger version of the estimate of [13], where it is shown by a parametrix construction; here conformal infinity can have arbitrary geometry. The point is thus that first, we do not need the machinery of the zero calculus here, second, we do have non-trapping high energy estimates in general (and without a parametrix construction), and third, we add the semiclassically outgoing property which is useful for resolvent gluing, including for proving non-trapping bounds microlocally away from trapping, provided the latter is mild, as shown by Datchev and Vasy [4, 5].

Theorem 5.1. *Suppose that (X_0, g_0) is an n-dimensional manifold with boundary with an even conformally compact metric and boundary defining function x. Let $X_{0,\mathrm{even}}$ denote the even version of X_0, i.e. with the boundary defining function replaced by its square with respect to a decomposition in which g_0 is even. Then the inverse of*

$$\Delta_{g_0} - \left(\frac{n-1}{2}\right)^2 - \sigma^2,$$

written as $\mathcal{R}(\sigma) : L^2 \to L^2$, has a meromorphic continuation from $\operatorname{Im}\sigma \gg 0$ to $\mathbb{C}$,

$$\mathcal{R}(\sigma) : \dot{\mathcal{C}}^\infty(X_0) \to \mathcal{C}^{-\infty}(X_0),$$

with poles with finite rank residues. If in addition (X_0, g_0) is non-trapping, then, with ϕ as in Subsection 3.1, and for suitable $\delta_0 > 0$, non-trapping estimates hold in every region $-C < \operatorname{Im}\sigma < \delta_0|\operatorname{Re}\sigma|$, $|\operatorname{Re}\sigma| \gg 0$: for $s > \frac{1}{2} + C$,

(5.1)
$$\|x^{-(n-1)/2} e^{\imath\sigma\phi}\mathcal{R}(\sigma) f\|_{H^s_{|\sigma|^{-1}}(X_{0,\mathrm{even}})} \leq \tilde{C}|\sigma|^{-1}\|x^{-(n+3)/2}e^{\imath\sigma\phi} f\|_{H^{s-1}_{|\sigma|^{-1}}(X_{0,\mathrm{even}})}.$$

If f is supported in X_0°, the $s-1$ norm on f can be replaced by the $s-2$ norm.

Furthermore, for $\operatorname{Re} z > 0$, $\operatorname{Im} z = \mathcal{O}(h)$, the resolvent $\mathcal{R}(h^{-1}z)$ is semiclassically outgoing with a loss of h^{-1} *in the sense that if f has compact support in X_0°,*

$\alpha \in T^*X$ is in the semiclassical characteristic set and if $\mathrm{WF}_\hbar^{s-1,0}(f)$ is disjoint from the backward bicharacteristic from α, then $\alpha \notin \mathrm{WF}_\hbar^{s,-1}(\mathcal{R}(h^{-1}z)f)$.

We remark that although in order to go through without changes, our methods require the evenness property, it is not hard to deduce more restricted results without this. Essentially one would have operators with coefficients that have a conormal singularity at the event horizon; as long as this is sufficiently mild relative to what is required for the analysis, it does not affect the results. The problems arise for the analytic continuation, when one needs strong function spaces (H^s with s large); these are not preserved when one multiplies by the singular coefficients.

Proof. All of the results of Section 4 apply.

By self-adjointness and positivity of Δ_{g_0} and as $\dot{\mathcal{C}}^\infty(X_0)$ is in its domain,

$$\left(\Delta_{g_0} - \sigma^2 - \left(\frac{n-1}{2}\right)^2\right) u = f \in \dot{\mathcal{C}}^\infty(X_0)$$

has a unique solution $u = \mathcal{R}(\sigma)f \in L^2(X_0, |dg_0|)$ when when $\operatorname{Im}\sigma \gg 0$. On the other hand, let ϕ be as in Subsection 3.1, so $e^\phi = \mu^{1/2}(1+\mu)^{-1/4}$ near $\mu = 0$ (so $e^\phi \sim x$ there), $\tilde{f}_0 = e^{\imath\sigma\phi}x^{-(n+1)/2}x^{-1}f$ in $\mu \geq 0$, and $\tilde{f}_0$ still vanishes to infinite order at $\mu = 0$. Let $\tilde{f}$ be an arbitrary smooth extension of $\tilde{f}_0$ to the compact manifold X on which $P_\sigma - \imath Q_\sigma$ is defined. Let $\tilde{u} = (P_\sigma - \imath Q_\sigma)^{-1}\tilde{f}$, with $(P_\sigma - \imath Q_\sigma)^{-1}$ given by our results in Section 4; this satisfies $(P_\sigma - \imath Q_\sigma)\tilde{u} = \tilde{f}$ and $\tilde{u} \in \mathcal{C}^\infty(X)$. Thus, $u' = e^{-\imath\sigma\phi}x^{(n+1)/2}x^{-1}\tilde{u}|_{\mu>0}$ satisfies $u' \in x^{(n-1)/2}e^{-\imath\sigma\phi}\mathcal{C}^\infty(X_0)$, and

$$\left(\Delta_{g_0} - \sigma^2 - \left(\frac{n-1}{2}\right)^2\right) u' = f$$

by (3.5) and (3.14) (as Q_σ is supported in $\mu < 0$). Since $u' \in L^2(X_0, |dg_0|)$ for $\operatorname{Im}\sigma > 0$, by the aforementioned uniqueness, $u = u'$.

To make the extension from $X_{0,\mathrm{even}}$ to X more systematic, let $E_s : H^s(X_{0,\mathrm{even}}) \to H^s(X)$ be a continuous extension operator, $R_s : H^s(X) \to H^s(X_{0,\mathrm{even}})$ the restriction map. Then, as we have just seen, for $f \in \dot{\mathcal{C}}^\infty(X_0)$,

$$\mathcal{R}(\sigma)f = e^{-\imath\sigma\phi}x^{(n+1)/2}x^{-1}R_s(P_\sigma - \imath Q_\sigma)^{-1}E_{s-1}e^{\imath\sigma\phi}x^{-(n+1)/2}x^{-1}f. \tag{5.2}$$

While, for the sake of simplicity, Q_σ is constructed in Subsection 3.5 in such a manner that it is not holomorphic in all of $\operatorname{Im}\sigma > -C$ due to a cut in the upper half plane, this cut can be moved outside any fixed compact subset, so taking into account that $\mathcal{R}(\sigma)$ is independent of the choice of Q_σ, the theorem follows immediately from the results of Section 4. □

Our argument proves that every pole of $\mathcal{R}(\sigma)$ is a pole of $(P_\sigma - \imath Q_\sigma)^{-1}$ (for otherwise (5.2) would show $\mathcal{R}(\sigma)$ does not have a pole either), but it is possible for $(P_\sigma - \imath Q_\sigma)^{-1}$ to have poles which are not poles of $\mathcal{R}(\sigma)$. However, in the latter case, the Laurent coefficients of $(P_\sigma - \imath Q_\sigma)^{-1}$ would be annihilated by multiplication by R_s from the left, i.e. the resonant states (which are smooth) would be supported in $\mu \leq 0$, in particular vanish to infinite order at $\mu = 0$.

In fact, a stronger statement can be made: by a calculation completely analogous to what we just performed, we can easily see that in $\mu < 0$, P_σ is a conjugate (times a power of μ) of a Klein-Gordon-type operator on n-dimensional de Sitter space with $\mu = 0$ being the boundary (i.e. where time goes to infinity). Thus, if σ is not a pole of $\mathcal{R}(\sigma)$ and $(P_\sigma - \imath Q_\sigma)\tilde{u} = 0$ then one would have a solution u of this Klein-Gordon-type equation near $\mu = 0$, i.e. infinity, that rapidly vanishes at infinity. It is shown in [20, Proposition 5.3] by a Carleman-type estimate that this cannot happen; although there $\sigma^2 \in \mathbb{R}$ is assumed, the argument given there goes through almost verbatim in general. Thus, if Q_σ is supported in $\mu < c$, $c < 0$, then $\tilde{u}$ is also supported in $\mu < c$. This argument can be iterated for Laurent coefficients of higher order poles; their range (which is finite dimensional) contains only functions supported in $\mu < c$.

Remark 5.2. We now return to our previous remarks regarding the fact that our solution disallows the conormal singularities $(\mu \pm i0)^{\imath\sigma}$ from the perspective of conformally compact spaces of dimension n. Recalling that $\mu = x^2$, the two indicial roots on these spaces correspond to the asymptotics $\mu^{\pm\imath\sigma/2+(n-1)/4}$ in $\mu > 0$. Thus for the operator

$$\mu^{-1/2}\mu^{\imath\sigma/2-(n+1)/4}(\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2)\mu^{-\imath\sigma/2+(n+1)/4}\mu^{-1/2},$$

or indeed P_σ, they correspond to

$$\left(\mu^{-\imath\sigma/2+(n+1)/4}\mu^{-1/2}\right)^{-1}\mu^{\pm\imath\sigma/2+(n-1)/4} = \mu^{\imath\sigma/2\pm\imath\sigma/2}.$$

Here the indicial root $\mu^0 = 1$ corresponds to the smooth solutions we construct for P_σ, while $\mu^{\imath\sigma}$ corresponds to the conormal behavior we rule out. Back to the original Laplacian, thus, $\mu^{-\imath\sigma/2+(n-1)/4}$ is the allowed asymptotics and $\mu^{\imath\sigma/2+(n-1)/4}$ is the disallowed one. Notice that $\operatorname{Re} \imath\sigma = -\operatorname{Im}\sigma$, so the disallowed solution is growing at $\mu = 0$ relative to the allowed one, as expected in the physical half plane, and the behavior reverses when $\operatorname{Im}\sigma < 0$. Thus, in the original asymptotically hyperbolic picture one has to distinguish two different rates of growths, whose relative size changes. On the other hand, in our approach, we rule out the singular solution and allow the non-singular (smooth one), so there is no change in behavior at all for the analytic continuation.

Remark 5.3. For *even* asymptotically de Sitter metrics on an n-dimensional manifold X_0' with boundary, the methods for asymptotically hyperbolic spaces work, except $P_\sigma - \imath Q_\sigma$ and $P_\sigma^* + \imath Q_\sigma^*$ switch roles, which does not affect Fredholm properties, see Remark 4.4. Again, evenness means that we may choose a product decomposition near the boundary such that

$$g_0 = \frac{dx^2 - h}{x^2} \tag{5.3}$$

there, where h is an even family of Riemannian metrics; as above, we take x to be a globally defined boundary defining function. Then with $\tilde{\mu} = x^2$, so $\tilde{\mu} > 0$ is the Lorentzian region, $\overline{\sigma}$ in place of σ (recalling that our aim is to get to $P_\sigma^* + \imath Q_\sigma^*$) the above calculations for $\Box_{g_0} - \frac{(n-1)^2}{4} - \overline{\sigma}^2$ in place of $\Delta_{g_0} - \frac{(n-1)^2}{4} - \sigma^2$ leading to (3.4) all go through with μ replaced by $\tilde{\mu}$, σ replaced by $\overline{\sigma}$ and Δ_h replaced by $-\Delta_h$. Letting $\mu = -\tilde{\mu}$, and conjugating by $(1+\mu)^{\imath\overline{\sigma}/4}$ as above, yields

$$-4\mu D_\mu^2 + 4\overline{\sigma}D_\mu + \overline{\sigma}^2 - \Delta_h + 4\imath D_\mu + 2\imath\gamma(\mu D_\mu - \overline{\sigma}/2 - \imath(n-1)/4), \tag{5.4}$$

modulo terms that can be absorbed into the error terms in operators in the class (3.5), i.e. this is indeed of the form $P_\sigma^* + \imath Q_\sigma^*$ in the framework of Subsection 3.5, at least near $\tilde{\mu} = 0$. If now X_0' is extended to a manifold without boundary in such a way that in $\tilde{\mu} < 0$, i.e. $\mu > 0$, one has a classically elliptic, semiclassically non-trapping problem, then all the results of Section 4 are applicable.

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