A NATURAL LOWER BOUND FOR THE SIZE OF NODAL SETS

HAMID HEZARI AND CHRISTOPHER D. SOGGE

The purpose of this brief note is to prove a natural lower bound for the $(n-1)$-dimensional Hausdorff measure of nodal sets of eigenfunctions. To wit:

Theorem 1. *Let (M,g) be a compact manifold of dimension n and e_λ an eigenfunction satisfying*

$$-\Delta_g e_\lambda = \lambda e_\lambda, \ \text{and} \ \int_M |e_\lambda|^2\, dV_g = 1.$$

Then if $Z_\lambda = \{x \in M : e_\lambda(x) = 0\}$ is the nodal set and $|Z_\lambda|$ its $(n-1)$-dimensional Hausdorff measure, we have

$$\lambda^{\frac12}\left(\int_M |e_\lambda|\, dV_g\right)^2 \le C|Z_\lambda|, \quad \lambda \ge 1, \tag{1}$$

for some uniform constant C. Consequently,

$$\lambda^{\frac{3-n}{4}} \lesssim |Z_\lambda|, \quad \lambda \ge 1. \tag{2}$$

Inequality (2) follows from (1) and the lower bounds in [14]

$$\lambda^{\frac{1-n}{8}} \lesssim \int_M |e_\lambda|\, dV_g. \tag{3}$$

The lower bound (2) is due to Colding and Minicozzi [3]. Yau [17] conjectured that $\lambda^{\frac12} \approx |Z_\lambda|$. This lower bound $\lambda^{\frac12} \lesssim |Z_\lambda|$ was verified in the 2-dimensional case by Brüning [2] and independently by Yau (unpublished). The bounds in (2) seem to be the best known ones for higher dimensions, although Donnelly and Fefferman [5]-[6] showed that, as conjectured, $|Z_\lambda| \approx \lambda^{\frac12}$, if (M,g) is assumed to be real analytic.

The first "polynomial type" lower bounds appear to be due to to Colding and Minicozzi [3] and Zelditch and the second author [14] (see also [9]). As we shall point out inequality (1) cannot be improved and it to some extent unifies the approaches in [3] and [14]. As was shown in [14], the L^1-lower bounds in (3) follow from Hölder's inequality and the L^p eigenfunction estimates of the second author [11] for the range where $2 < p \le \frac{2(n+1)}{n-1}$. These too cannot be improved, but it is thought better L^p-bounds hold for a typical eigenfunction or if one makes geometric assumptions such as negative curvature (cf. [15]-[16]). Thus, it is natural to expect to be able to improve (3) and hence the lower bounds (2) for all eigenfunctions on manifolds with negative curvature, or for "typical" eigenfunctions on any manifold. Of course, Yau's conjecture that $|Z_\lambda| \approx \lambda^{\frac12}$ would be the ultimate goal, but understanding when (3) can be improved is a related problem of independent interest.

The authors were supported in part by NSF grants DMS-0969745 and DMS-1069175.

Let us now turn to the proof of Theorem 1. We shall use an identity from the recent work of the second author and Zelditch [14]:

$$\int_M |e_\lambda|\,(\Delta_g+\lambda)f\,dV_g = 2\int_{Z_\lambda} |\nabla_g e_\lambda|\,f\,dS_g, \tag{4}$$

Here dS_g is the Riemannian surface measure on Z_λ, and ∇_g is the gradient coming from the metric and $|\nabla_g u|$ is the norm coming from the metric, meaning that in local coordinates

$$|\nabla_g u|_g^2 = \sum_{jk=1}^n g_{jk}(x)\partial_j u \partial_k u. \tag{5}$$

Identity (4) follows from the Gauss-Green formula and a related earlier identity was proved by Dong [4].

As in [8], if we take $f\equiv 1$ and apply Schwarz's inequality we get

$$\lambda\int_M |e_\lambda|\,dV_g \le 2|Z_\lambda|^{1/2}\left(\int_{Z_\lambda}|\nabla_g e_\lambda|^2\,dS_g\right)^{1/2}. \tag{6}$$

Thus we would have (1) if we could prove that the energy of e_λ on its nodal set satisfies the natural bounds

$$\int_{Z_\lambda}|\nabla_g e_\lambda|^2\,dS_g \lesssim \lambda^{\frac32}. \tag{7}$$

We shall do this by choosing a different auxiliary function f. This time we want to use

$$f=\big(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|_g^2\big)^{\frac12}. \tag{8}$$

If we plug this into (4) we get that

$$2\int_{Z_\lambda}|\nabla_g e_\lambda|_g^2 dS_g \le \int_M |e_\lambda|\,(\Delta_g+\lambda)\big(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\big)^{\frac12}\,dV_g.$$

Since we have the L^2-Sobolev bounds

$$\|e_\lambda\|_{H^s(M)} = O(\lambda^{\frac{s}{2}}), \tag{9}$$

it is clear that

$$\lambda\int_M |e_\lambda|\,\big(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|_g^2\big)^{\frac12}\,dV_g = O(\lambda^{\frac32}),$$

and thus to prove (7), it suffices to show that

$$\int_M |e_\lambda|\,\Delta_g\big(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|_g^2\big)^{\frac12}\,dV_g = O(\lambda^{\frac32}). \tag{10}$$

To prove this we first note that

$$\partial_k\big(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|_g^2\big)^{\frac12} = \frac{\lambda e_\lambda\partial_k e_\lambda+\frac12\partial_k|\nabla_g e_\lambda|_g^2}{\big(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\big)^{\frac12}},$$

from this and (9) we deduce that

$$\int_M |e_\lambda|\,\Big|\,\nabla_g\big(1+\lambda e_\lambda^2+|\nabla_g e_\lambda|^2\big)^{\frac12}\,\Big|\,dV_g = O(\lambda).$$

This means that the contribution of the first order terms of the Laplace-Beltrami operator (written in local coordinates) to (10) are better than required, and so it suffices to show that in a compact subset K of a local coordinate patch we have

$$\int_K |e_\lambda| \left| \partial_j \partial_k \big(1 + \lambda e_\lambda^2 + |\nabla_g e_\lambda|^2 \big)^{\frac{1}{2}} \right| dV_g = O(\lambda^{\frac{3}{2}}). \tag{11}$$

A calculation shows that $\partial_j \partial_k \big(\lambda e_\lambda^2 + |\nabla_g e_\lambda|^2 \big)^{\frac{1}{2}}$ equals

$$-\frac{\big(\lambda e_\lambda \partial_j e_\lambda + \frac{1}{2}\partial_j |\nabla_g e_\lambda|_g^2\big)\big(\lambda e_\lambda \partial_k e_\lambda + \frac{1}{2}\partial_k |\nabla_g e_\lambda|_g^2\big)}{\big(1 + \lambda e_\lambda^2 + |\nabla_g e_\lambda|^2 \big)^{\frac{3}{2}}} + \frac{\lambda \partial_j e_\lambda \partial_k e_\lambda + \lambda e_\lambda \partial_j \partial_k e_\lambda + \frac{1}{2}\partial_j \partial_k |\nabla_g e_\lambda|_g^2}{\big(1 + \lambda e_\lambda^2 + |\nabla_g e_\lambda|^2 \big)^{\frac{1}{2}}}.$$

If $|D^m f| = \sum_{|\alpha|=m} |\partial^\alpha f|$, then by (5)

$$\partial_k |\nabla_g e_\lambda|^2 = O(|D^2 e_\lambda|\,|De_\lambda| + |De_\lambda|^2),$$

and

$$\partial_j \partial_k |\nabla_g e_\lambda|_g^2 = O(|D^3 e_\lambda|\,|De_\lambda| + |D^2 e_\lambda|^2 + |D^2 e_\lambda|\,|De_\lambda| + |De_\lambda|^2).$$

Therefore,

$$\partial_j \partial_k \big(\lambda e_\lambda^2 + |\nabla_g e_\lambda|^2 \big)^{\frac{1}{2}} = O\left(\frac{\lambda^2 |e_\lambda|^2 |De_\lambda|^2 + |D^2 e_\lambda|^2 |De_\lambda|^2 + |De_\lambda|^4}{\big(1 + \lambda e_\lambda^2 + |\nabla_g e_\lambda|^2 \big)^{\frac{3}{2}}} \right)$$
$$+ O\left(\frac{\lambda |De_\lambda|^2 + \lambda |e_\lambda|\,|D^2 e_\lambda| + |D^3 e_\lambda|\,|De_\lambda| + |D^2 e_\lambda|^2 + |D^2 e_\lambda|\,|De_\lambda| + |De_\lambda|^2}{\big(1 + \lambda e_\lambda^2 + |\nabla_g e_\lambda|^2 \big)^{\frac{1}{2}}} \right).$$

This implies that the integrand in the left side of (11) is dominated by

$$\Big(\lambda^{\frac{1}{2}} |De_\lambda|^2 + \lambda^{-\frac{1}{2}} |D^2 e_\lambda|^2 + |De_\lambda|^2 \Big)$$
$$+ \Big(\lambda^{\frac{1}{2}} |De_\lambda|^2 + \lambda^{\frac{1}{2}} |e_\lambda|\,|D^2 e_\lambda| + |e_\lambda|\,|D^3 e_\lambda| + \lambda^{-\frac{1}{2}} |D^2 e_\lambda|^2 + |D^2 e_\lambda|\,|e_\lambda| + |De_\lambda|\,|e_\lambda| \Big),$$

leading to (11) after applying (9). □

Remarks:

- We could also have taken f to be $(\lambda + \lambda e_\lambda^2 + |\nabla_g e_\lambda|^2)^{\frac{1}{2}}$ and obtained the same upper bounds, but there does not seem to be any advantage to doing this.
- Inequality (1) cannot be improved. There are many cases when the L^1 and L^2-norms of eigenfunctions are comparable. For instance, for the sphere the zonal functions have this property and it is easy to check that their nodal sets satisfy $|Z_\lambda| \approx \lambda^{\frac{1}{2}}$, which means that for zonal functions (1) cannot be improved.
- There are many cases where inequality (1) can be improved. For instance, the L^2-normalized highest weight spherical harmonics Q_k have eigenvalues $\lambda = \lambda_k \approx k^2$, and L^1-norms $\approx k^{-\frac{n-1}{4}}$ (see e.g., [10]). This means that for the highest weight spherical harmonics the left side is proportional to $\lambda^{\frac{3-n}{4}}$ even though here too $|Z_\lambda| \approx \lambda^{\frac{1}{2}}$. Similarly, the highest weight spherical harmonics saturate (7). It is

because of functions like the highest weight spherical harmonics that the current techniques only seem to yield (2). Note that inequality (2) gives the correct lower bound in the trivial case where the dimension n is one. As the dimension increases, the bound gets worse and worse due to the fact that (3) is saturated by functions like the highest weight spherical harmonics ("Gaussian beams") whose mass is supported on a $\lambda^{-\frac{1}{4}}$ neighborhood of a geodesic and the volume of such a tube decreases geometrically as n increases. (See [1] and [13] for related work on this phenomena.)

Acknowledgments The authors wish to thank W. Minicozzi and S. Zelditch for several helpful and interesting discussions.

References

[1] J. Bourgain: *Geodesic restrictions and L^p-estimates for eigenfunctions of Riemannian surfaces*, Linear and complex analysis, 27–35, Amer. Math. Soc. Tranl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.

[2] J. Brüning: *Über Knoten Eigenfunktionen des Laplace-Beltrami Operators*, Math. Z. 158 (1978), 15–21.

[3] T. Colding and W. P. Minicozzi II: *Lower bounds for nodal sets of eigenfunctions*, arXiv:1009.4156, to appear in Comm. Math. Phys.

[4] R. T. Dong: *Nodal sets of eigenfunctions on Riemann surfaces*, J. Differential Geom. 36 (1992), no. 2, 493–506.

[5] H. Donnelly and C. Fefferman: *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. 93 (1988), 161-183.

[6] H. Donnelly and C. Fefferman: *Nodal sets for eigenfunctions of the Laplacian on surfaces*, J. Amer. Math. Soc. 3(2) (1990), 332–353.

[7] Q. Han and F. H. Lin: *Nodal sets of solutions of Elliptic Differential Equations*, book in preparation (online at http://www.nd.edu/ qhan/nodal.pdf).

[8] H. Hezari and Z. Wang: *Lower bounds for volumes of nodal sets: an improvement of a result of Sogge-Zelditch*, arXiv:1107.0092.

[9] D. Mangoubi: *A remark on recent lower bounds for nodal sets*, arXiv 1010.4579.

[10] C. D. Sogge: *Oscillatory integrals and spherical harmonics*, Duke Math. J. 53 (1986), no. 1, 43–65.

[11] C. D. Sogge: *Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds*, J. Funct. Anal. 77 (1988), no. 1, 123–138.

[12] C. D. Sogge: *Fourier integrals in classical analysis*, Cambridge Tracts in Mathematics, 105. Cambridge University Press, Cambridge, 1993.

[13] C. D. Sogge: *Kakeya-Nikodym averages and L^p-norms of eigenfunctions*, arXiv:0907.4827, to appear Tohoku Math. J (centennial edition).

[14] C. D. Sogge and S. Zelditch: *Lower bounds on the Hausdorff measure of nodal sets*, Math. Res. Lett. 18 (2011), 25–37.

[15] C. D. Sogge and S. Zelditch: *Concerning the L^4 norms of typical eigenfunctions on compact surfaces*, arXiv:1011.0215.

[16] C. D. Sogge and S. Zeldtich: *On eigenfunction restriction estimates and L^4-bounds for compact surfaces with nonpositive curvature*, in preparation.

[17] S.T. Yau: *Survey on partial differential equations in differential geometry, Seminar on Differential Geometry*, pp. 371, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982.

Department of Mathematics, M.I.T., Cambridge, MA 02139

Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218