INVARIANT DISTRIBUTIONS AND TIME AVERAGES FOR HOROCYCLE FLOWS

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ABSTRACT. There are infinitely many obstructions to existence of smooth solutions of the cohomological equation $Uu = f$, where U is the vector field generating the horocycle flow on the unit tangent bundle SM of a Riemann M surface of finite area and f is a given function on SM. We study the Sobolev regularity of these obstructions, construct smooth solutions of the cohomological equation and derive asymptotics for the ergodic averages of horocycle flows.

1. INTRODUCTION

The classical horocycle flow is the flow on (compact) homogeneous spaces of the form $\Gamma\backslash PSL(2,\mathbb{R})$ given by right multiplication by the one-parameter subgroup $\left\{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right\}$. The ergodic properties of this flow have been an active subject of study for a long time since it has the intriguing characteristic of presenting at the same time some of properties of "orderly" ergodic systems, (zero entropy [21], minimality [23, 19, 57], unique ergodicity [14, 35, 36] etc.), and some of the properties which are more frequently associated with "chaotic" systems, (e.g. multiple strong mixing [44, 38, 37]). The works of M. Ratner in the 80-90's [48, 47, 50] have also put in evidence the tight relation of the ergodic properties of the horocycle flow with the geometry of the underlying space $\Gamma\backslash PSL(2,\mathbb{R})$.

Representation theory is a natural tool for the study of flows on homogeneous spaces ([16, 17, 41, 2], etc.); in particular, M. Ratner [49] and C. Moore [42] have obtained precise estimates for the mixing rate of the geodesic and horocycle flow and, more recently, M. Burger [5] has proved uniform upper bounds for the deviation of ergodic averages of sufficiently regular functions along the orbits of the horocycle flow on open complete surfaces with positive injectivity radius and on compact surfaces. Invariant distributions for geodesic flows on manifolds of constant curvatures were studied in [11].

In this article we establish precise asymptotics for the ergodic averages of sufficiently regular functions along the orbits of the horocycle flow on compact surfaces, improving on Burger's results. For cuspidal horocycles on non-compact surfaces of finite area such asymptotics were obtained by P. Sarnak in [51], inspired by D. Zagier [58], by a method based on Eisenstein series. Our approach does not use automorphic forms and it is based on the study of the cohomological equation and of invariant distributions for the horocycle flow,

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via representation theory. Such a method yields sharp results in the compact case while in the non-compact (finite area) case we obtain a generalisation of Sarnak's result to arbitrary horocycle arcs. G. Margulis and D. Kleinbock [31] proved, for finite volume quotients of general semisimple Lie groups, exponential decay of the deviation from equidistribution for smooth measures supported on a compact subset of any horosphere under the action of diagonal subgroups. Our results yield precise asymptotics in the particular case of quotients of the semisimple group $SL(2, \mathbb{R})$.

We recall that for a flow $\{\phi^t\}$ we say that G is a *coboundary* if there exists a solution F (continuous , L^1, L^2, depending on the context) of the *cohomological equation*

$$\frac{d}{dt} F \circ \phi^t \bigg|_{t=0} = G. \tag{1}$$

The cohomological equation arises in several problems in dynamics e.g. in the study of the existence of invariant measures, in conjugacy problems, in the study of repararametrisations of flows etc. It is well understood in two opposite dynamical setups:

- linear toral flows
- Anosov flows

In the first case, provided that a Diophantine condition on the rotation number is satisfied (and generically this is the case), the only obstruction to the solution of the cohomological equation is given by the unique invariant measure: given a sufficiently regular G of mean zero there exist a solution F of the equation (1) with a loss of regularity controlled by the Diophantine condition. Hence, for any sufficiently regular function G, ergodic averages converge to the mean with an error bounded by Const.$/T$. The Denjoy-Koksma inequality [32] yields estimates for functions which are only of bounded variation.

In the Anosov case the celebrated Livshitz Theorem [33, 6, 20, 7] states that a Hölder function G is a coboundary iff G integrates to zero along every periodic orbit, or, equivalently, if it has mean zero with respect to all invariant measures. Thus, even in this case the only obstructions to the solution of the cohomological equation are represented by measures. In contrast, the strong ergodic properties of these flows are reflected in the fact that ergodic averages converge to the mean as stochastically independent processes do, i.e. the central limit Theorem holds [46].

The horocycle flow, differently from the cases above, has invariant distributions which are not signed measures [29]. In the spirit of [12, 13] we find that these distributions control the asymptotics of ergodic averages. Furthermore the exponents of the asymptotic expansion coincide with the Sobolev order of the invariant distributions.

1.1. **Statement of the results.** The group $PSL(2, \mathbb{R})$ acts (on the left) by isometries and holomorphically on Poincaré upper half plane

$$H = (\{z \in \mathbb{C} \mid \Im z > 0\}, |dz|^2/(\Im z)^2).$$

The quotient space $M = \Gamma\backslash H$, by a discrete subgroup of $PSL(2, \mathbb{R})$ acting without fixed points, is a Riemannian manifold of constant curvature -1 and also a complex curve (a Riemann surface).

The isometric action of $PSL(2, \mathbb{R})$ on H induces a free transitive action on the unit tangent bundle SH of H and by fixing (i, i) as an origin in SH we identify $PSL(2, \mathbb{R})$

to SH. By means of this identification elements of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ of $PSL(2,\mathbb{R})$ are identified with the generators of some flows on SH which project to flows on the quotient space $SM = \Gamma\backslash SH \approx \Gamma\backslash PSL(2,\mathbb{R})$. The $\mathfrak{sl}(2,\mathbb{R})$ matrices

$$U = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \qquad X = \left\{ \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right\}$$

define respectively the generators of the (stable) horocycle flow $\{\phi_t^U\}$ and of the geodesic flow $\{\phi_t^X\}$ on the unit tangent bundle SM of an hyperbolic surface $M := \Gamma\backslash H$.

For all hyperbolic surfaces of finite area, the spectrum $\sigma(\Delta_M)$ of the Laplace-Beltrami operator Δ_M on M has 0 as a simple eigenvalue and there is a "spectral gap", i.e. the bottom of the non-zero spectrum $\mu_0 := \inf \sigma(\Delta_M) \setminus \{0\}$ is strictly positive. If M is compact, by standard elliptic theory, its spectrum is pure point and discrete with eigenvalues of finite multiplicity and it satisfies the Weyl asymptotics. Examples of 'pinched' compact surfaces with arbitrarily small spectral gap $\mu_0 > 0$ were first constructed by B.Randol [45] (see [54], [9] for more recent sharper results).

If M is not compact, its spectrum can be described as follows (see for instance [53] and references therein, in particular D.Hejal [24]):

- Lebesgue spectrum on the interval $[1/4, \infty[$ with multiplicity equal to the number of cusps of M;
- possibly, finitely many eigenvalues of finite multiplicity in the interval $]0, 1/4[$;
- possibly, embedded eigenvalues of finite multiplicity in the interval $[1/4, \infty[$.

Examples of non-compact 'pinched' surfaces with an arbitrarily small spectral gap, hence with non-empty (pure point) spectrum in the open interval $]0, 1/4[$, were already constructed by A. Selberg [55] (see [9] for a more recent approach). In the non-compact case, the Weyl asymptotics for the pure point spectrum fails *conjecturally* for a generic subgroup in any given Teichmüller space with the exception of the Teichmüller space of the once-punctured torus [8], [52].

The picture is quite different for a relevant class of arithmetic subgroups. For all $N \in \mathbb{Z}^+$, let $\Gamma(N) := \{g \in PSL(2,\mathbb{Z}) \mid g = \text{id} \mod N\}$. The subgroups $\Gamma < PSL(2,\mathbb{R})$ such that

$$\Gamma(1) > \Gamma > \Gamma(N), \qquad N \in \mathbb{Z}^+,$$

are called *congruence subgroups*. For all such subgroups, Selberg's spectral gap conjecture [55] claims that there are no eigenvalues in $]0, 1/4[$, hence $\mu_0 = 1/4$. Selberg's conjecture is known to be true for all congruence subgroups $\Gamma > \Gamma(N)$, $N \leq 17$ [27], [28]. In the case of a general congruence subgroup, the classical Selberg's lower bound $\mu_0 \geq 3/16$ [55] was significantly improved by W. Luo, Z. Rudnick and P. Sarnak [34] who proved that $\mu_0 \geq 171/784 \approx 0.218\ldots$. Recently, H. Kim and P. Sarnak [30] have announced a further step in the direction of the Selberg's conjecture:

$$\mu_0 \geq \frac{975}{4096} \approx 0.238\ldots$$

For all congruence subgroups, the Weyl asymptotics holds for the pure point spectrum [26], [10].

As we shall see the (pure point) spectrum within the interval $]0, 1/4[$ is especially relevant to the asymptotics of ergodic averages of horocycles, both in the compact and the non-compact case. We shall denote by σ_{pp} the set of all eigenvalues of the Laplace operator Δ_M

and by $\mathcal{C}$ the set of cusps of M. If M is compact, the Laplacian Δ_M has pure point spectrum supported on the set σ_{pp} and $\mathcal{C} = \emptyset$.

Let $L^2(SM)$ be the space of all complex-valued square-integrable functions with respect to the $PSL(2, \mathbb{R})$-invariant volume on SM. Let $W^s(SM)$, $s \in \mathbb{R}$, be the *Sobolev spaces* of functions $f \in L^2(SM)$ with $\Delta^{s/2} f \in L^2(SM)$, where Δ is an *elliptic* second order element of the enveloping algebra of $\mathfrak{sl}(2, \mathbb{R})$ (a Laplacian). The dual space $(W^s(SM))'$ of linear bounded functionals on $W^s(SM)$ is isomorphic to the space $W^{-s}(SM)$, according to the standard theory of Sobolev spaces [1].

Let $\mathcal{E}'(SM)$ be the dual space of the space $C^\infty(SM)$ of infinitely differentiable functions on SM. The space

$$\mathcal{I}(SM) := \left\{ \mathcal{D} \in \mathcal{E}'(SM) \mid \mathcal{L}_U \mathcal{D} = 0 \right\}$$

of U-invariant distributions is completely determined by the spectrum of the Laplacian Δ_M and by the genus of M, as follows.

Theorem 1.1. *The space $\mathcal{I}(SM)$ has infinite countable dimension. There is a decomposition*

$$\mathcal{I}(SM) = \bigoplus_{\mu \in \sigma_{pp}} \mathcal{I}_\mu \oplus \bigoplus_{n \in \mathbb{Z}^+} \mathcal{I}_n \oplus \bigoplus_{c \in \mathcal{C}} \mathcal{I}_c \tag{2}$$

where

1. *for $\mu = 0$, the space $\mathcal{I}_0$ is spanned by the $PSL(2, \mathbb{R})$-invariant volume;*
2. *for $0 < \mu < 1/4$, there is a splitting $\mathcal{I}_\mu = \mathcal{I}^+_\mu \oplus \mathcal{I}^-_\mu$, where $\mathcal{I}^\pm_\mu \subset W^{-s}(SM)$, iff $s > \frac{1 \pm \sqrt{1-4\mu}}{2}$, and each subspace has dimension equal to the multiplicity of $\mu \in \sigma_{pp}$;*
3. *for $\mu \geq 1/4$, the space $\mathcal{I}_\mu \subset W^{-s}(SM)$, iff $s > 1/2$, and it has dimension equal to twice the multiplicity of $\mu \in \sigma_{pp}$;*
4. *for $n \in \mathbb{Z}^+$, the space $\mathcal{I}_n \subset W^{-s}(SM)$, iff $s > n$ and it has dimension equal to twice the rank of the space of holomorphic sections of the n-th power of the canonical line bundle over M;*
5. *for $c \in \mathcal{C}$, the space $\mathcal{I}_c \subset W^{-s}(SM)$, iff $s > 1/2$, and it has infinite countable dimension.*

The Sobolev regularity of U-invariant distributions can be summarised as follows. The *Sobolev order* of a distribution $\mathcal{D} \in \mathcal{E}'(SM)$, is the extended real number

$$S_{\mathcal{D}} := \inf \left\{ s \in \mathbb{R}^+ \mid \mathcal{D} \in W^{-s}(SM) \right\} .$$

By Theorem 1.1, we have:

$$S_{\mathcal{D}} = \begin{cases} \frac{1 \pm \Re\sqrt{1-4\mu}}{2} & \text{if } \mathcal{D} \in \mathcal{I}^\pm_\mu \,, \quad \text{for } \mu > 0 \,; \\ n & \text{if } \mathcal{D} \in \mathcal{I}_n \,, \quad \text{for } n \in \mathbb{Z}^+ \,; \\ \frac{1}{2} & \text{if } \mathcal{D} \in \mathcal{I}_c \,, \quad \text{for } c \in \mathcal{C} \,; \end{cases} \tag{3}$$

Let $\mu_0 > 0$ be the bottom of the non-zero spectrum of Δ_M. Let $\nu_0 \in [0, 1[$ be defined as

$$\nu_0 := \begin{cases} \sqrt{1 - 4\mu_0}, & \text{if } \mu_0 < \frac{1}{4}; \\ 0, & \text{if } \mu_0 \geq \frac{1}{4}. \end{cases} \tag{4}$$

A complete set of obstructions to the existence of smooth solutions of the *cohomological equation* $Uf = g$, for functions $g \in W^s(SM)$, is given by the following space of invariant distributions:

$$\mathcal{I}^s(SM) := \left\{ \mathcal{D} \in W^{-s}(SM) \mid \mathcal{L}_U \mathcal{D} = 0\,. \right\}\,,$$

The space $\mathcal{I}^s(SM)$ is completely determined by Theorem 1.1.

Theorem 1.2. *For all $s > \frac{1+\nu_0}{2}$ and all $t \in \mathbb{R}$, there exists a constant $C := C(\nu_0, s, t) > 0$ such that, for all $g \in W^s(SM)$,*

- *if $t < -\frac{1+\nu_0}{2}$ and g has zero average on SM, or*
- *if $t < s-1$ and $\mathcal{D}(g) = 0$, for all $\mathcal{D} \in \mathcal{I}^s(SM)$,*

then the cohomological equation $Uf = g$ has a solution $f \in W^t(SM)$ which satisfies the Sobolev estimate

$$\| f \|_t \le C \| g \|_s .$$

A solution $f \in W^t(SM)$ of the cohomological equation $Uf = g$ is unique up to additive constants if and only if $t \ge -\frac{1-\nu_0}{2}$.

By definition, invariant distributions of order $S > 0$ are obstructions to the existence of solutions of the cohomological equation of (Sobolev) regularity $S+1$. However, it turns out that they obstruct also the existence of solutions of lower regularity. In fact, Theorem 1.2 is sharp, in the following sense:

Theorem 1.3. *Let $g \in W^s(SM)$, $s > \frac{1+\nu_0}{2}$, and let $\mathcal{D} \in \mathcal{I}(SM)$ be an U-invariant distribution of order $S_{\mathcal{D}} < s$. If the equation $Uf = g$ has a solution $f \in W^t(SM)$ for any $t \ge S_{\mathcal{D}} - 1$, then $\mathcal{D}(g) = 0$.*

The following remarkable result holds:

Theorem 1.4. *The action of the geodesic one-parameter group $\{\phi_t^X\}$ on the distributional space $\mathcal{I}(SM)$ has a spectral representation, if $1/4 \notin \sigma_{pp}$, and it has a generalized spectral representation with a finite number of 2×2 Jordan blocks, if $1/4 \in \sigma_{pp}$. The splitting (2) of $\mathcal{I}(SM)$ is $\{\phi_t^X\}$-invariant, hence the subspaces*

$$\mathcal{I}_{pp} := \bigoplus_{\mu\in\sigma_{pp}} \mathcal{I}_\mu\,, \quad \mathcal{I}_d := \mathcal{I}_{pp} \oplus \bigoplus_{n\in\mathbb{Z}^+} \mathcal{I}_n\,, \quad \mathcal{I}_{\mathcal{C}} := \bigoplus_{c\in\mathcal{C}} \mathcal{I}_c\,. \tag{5}$$

are $\{\phi_t^X\}$-invariant. The spectrum of $\{\phi_t^X\}$ on $\mathcal{I}_d$ is discrete, while the spectrum on $\mathcal{I}_{\mathcal{C}}$ is Lebesgue with finite multiplicity equal to the number of cusps.

The discrete spectrum of $\{\phi_t^X\}$ on $\mathcal{I}_d$ can be described as follows. For $\mu = 0$, the subspace $\mathcal{I}_0$ is generated by the $PSL(2,\mathbb{R})$-invariant, hence $\{\phi_t^X\}$-invariant, volume form. For all $\mu \in \sigma_{pp} \setminus \{0\}$, there is a splitting $\mathcal{I}_\mu = \mathcal{I}_\mu^+ \oplus \mathcal{I}_\mu^-$ into subspaces of equal dimension, which coincides for $0 < \mu < 1/4$ with the splitting induced by the Sobolev order (see Theorem 1.1, (2)), such that the following holds. For all $\mu \ne 1/4$, the subspaces $\mathcal{I}_\mu^\pm$ are eigenspaces for $\{\phi_t^X\}$, in fact

$$\phi_t^X \mid \mathcal{I}_\mu^\pm \;=\; \exp\Big(\frac{1 \pm \sqrt{1-4\mu}}{2}\, t\Big)\, I\,. \tag{6}$$

If $1/4 \in \sigma_{pp}$, the subspace $\mathcal{I}_{1/4} \neq \{0\}$ and $\{\phi_t^X\}\,|\,\mathcal{I}_{1/4}$ has a Jordan normal form. In fact, $\mathcal{I}_{1/4}$ is generated by pairs $\{\mathcal{D}^+, \mathcal{D}^-\}$ such that $\mathcal{D}^\pm \in \mathcal{I}^\pm_{1/4}$ and

$$\phi_t^X \begin{pmatrix} \mathcal{D}^+ \\ \mathcal{D}^- \end{pmatrix} = e^{-\frac{t}{2}} \begin{pmatrix} 1 & 0 \\ -\frac{t}{2} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{D}^+ \\ \mathcal{D}^- \end{pmatrix} , \tag{7}$$

For all $n \in \mathbb{Z}^+$, the subspace $\mathcal{I}_n$ is an eigenspace for $\{\phi_t^X\}$. In fact,

$$\phi_t^X \,|\, \mathcal{I}_n \;=\; e^{-nt}\, I \;. \tag{8}$$

The Lebesgue spectrum of the operator ϕ_t^X on $\mathcal{I}_\mathcal{C}$ is supported on the circle of radius $e^{-t/2}$ in $\mathbb{C}$, for all $t \in \mathbb{R}$.

It follows from Theorem 1.4 that the action of the geodesic flow $\{\phi_t^X\}$ on the infinite dimensional space $\mathcal{I}(SM)$ has a well-defined Lyapunov spectrum and an Oseledec's decomposition. It turns out by comparing the formulae in Theorem 1.4 with the Sobolev orders (3) of invariant distributions, that the Lyapunov exponent of any generalized eigenvector $\mathcal{D} \in \mathcal{I}_d$ of the geodesic flow and of any $\mathcal{D} \in \mathcal{I}_\mathcal{C}$ is equal to the negative of its Sobolev order.

Theorems 1.1-1.4 are derived from abstract results on unitary representations of the Lie group $PSL(2,\mathbb{R})$ (Theorems 3.2, 4.1 and 4.6). Theorem 3.2 states that the invariant distributions for the "horocycle vector field" in each irreducible representation of parameter $\mu \neq 1/4$ are generated by eigendistributions of the "geodesic vector field". Such eigendistributions correspond to the *conical distributions* for the Lie group $SL(2,\mathbb{R})$ in the sense of S. Helgason [25], §2. In the exceptional case of irreducible representations with parameter $\mu = 1/4$, the subspace of conical distributions is one-dimensional, while the subspace of horocycle-invariant distributions has dimension 2. Theorem 1.1 is based on a direct analysis of the Sobolev regularity of such (conical) distributions and Theorem 1.4 on the explicit computation of the action of the geodesic flow on invariant distributions.

The abstract Theorem 4.1, on existence of smooth solutions of the cohomological equation, holds under the only condition that the Casimir operator associated with the unitary representation has a 'spectral gap'. Theorem 1.2 on solutions of the cohomological equation on hyperbolic surfaces of finite area follows immediately, since the gap property holds. Theorem 4.6 is a converse result which proves that, in each irreducible component, invariant distributions are obstructions to the existence of solutions of smoothness lower than expected. This result shows that, in a sense, our result on existence of solutions of the cohomological equation is optimal as far as the loss of Sobolev regularity is concerned. Theorem 1.3, the corresponding statement for hyperbolic surfaces, plays a non-trivial role in obtaining L^2 lower bounds for ergodic averages.

Let $\mathcal{B} \subset \mathcal{I}_d$ be a basis of (generalized) eigenvectors for $\{\phi_t^X\}$ on $\mathcal{I}_d$ with the following properties. For all $\mu \in \sigma_{pp} \setminus \{1/4\}$ and all $n \in \mathbb{Z}^+$, the sets $\mathcal{B}_\mu := \mathcal{B} \cap \mathcal{I}_\mu$, $\mathcal{B}_n := \mathcal{B} \cap \mathcal{I}_n$ are basis of eigenvectors for $\{\phi_t^X\}\,|\,\mathcal{I}_\mu$ and $\{\phi_t^X\}\,|\,\mathcal{I}_n$. If $1/4 \in \sigma_{pp}$, the set $\mathcal{B}^+_{1/4} := \mathcal{B} \cap \mathcal{I}^+_{1/4}$ is a basis of eigenvectors for $\{\phi_t^X\}\,|\,\mathcal{I}^+_{1/4}$, the set $\mathcal{B}^-_{1/4} := \mathcal{B} \cap \mathcal{I}^-_{1/4}$ is a basis of generalized eigenvectors (of order 2) for $\{\phi_t^X\}\,|\,\mathcal{I}^-_{1/4}$ and $\mathcal{B}_{1/4} := \mathcal{B}^+_{1/4} \cup \mathcal{B}^-_{1/4}$ is a union of pairs $\{\mathcal{D}^+, \mathcal{D}^-\}$ such that $\mathcal{D}^\pm \in \mathcal{B}^\pm_{1/4}$ and formula (7) holds.

Let $\mathcal{I}^1_+(SM)$ be the complement of the line generated by the invariant volume form in $\mathcal{I}^1(SM) \subset W^{-1}(SM)$. The subset $\mathcal{B}^1_+ := \mathcal{B} \cap \mathcal{I}^1_+(SM)$ is a basis of (generalized) eigenvectors for $\{\phi^X_t\} \,|\, \mathcal{I}^1_+(SM)$. The basis $\mathcal{B}$, $\mathcal{B}^1_+$ have the following decompositions:

$$\mathcal{B} = \bigcup_{\mu \in \sigma_{pp}} \mathcal{B}_\mu \cup \bigcup_{n \in \mathbb{Z}^+} \mathcal{B}_n \,, \quad \mathcal{B}^1_+ = \bigcup_{\mu \in \sigma_{pp} \setminus \{0\}} \mathcal{B}_\mu \,. \tag{9}$$

In the *compact* case, Theorems 1.2, 1.3 and 1.4 imply the following quantitative unique ergodicity result.

Theorem 1.5. *The horocycle flow $\{\phi^U_t\}$ on the unit tangent bundle SM of a compact hyperbolic surface M has a deviation spectrum in the following sense. For any $s > 3$ and for all $(x,T) \in SM \times \mathbb{R}^+$, there exist a sequence of real-valued functions $\{c^s_{\mathcal{D}}(x,T)\}_{\mathcal{D} \in \mathcal{B}^1_+}$ and distributional functions $\mathcal{D}^s_1(x,T) \in \mathcal{I}_1$, $\mathcal{R}^s(x,T) \in W^{-s}(SM)$ such that, for all $f \in W^s(SM)$ and all $T \geq 0$,*

$$\begin{aligned} \frac{1}{T}\int_0^T f(\phi^U_t(x))dt = \int_{SM} f\, dvol \,+ \sum_{\mathcal{D} \in \mathcal{B}^1_+ \setminus \mathcal{B}^+_{1/4}} c^s_{\mathcal{D}}(x,T)\mathcal{D}(f)\, T^{-S_{\mathcal{D}}} + \\ + \sum_{\mathcal{D} \in \mathcal{B}^+_{1/4}} c^s_{\mathcal{D}}(x,T)\mathcal{D}(f)\, T^{-S_{\mathcal{D}}} \log^+ T \,+\, \frac{\mathcal{D}^s_1(x,T)(f) \log^+ T + \mathcal{R}^s(x,T)(f)}{T} \,. \end{aligned} \tag{10}$$

The functions $c^s_{\mathcal{D}}$, $\mathcal{D}^s_1$ and $\mathcal{R}^s$ satisfy the following uniform estimates. There exists $C(s) > 0$, such that, for all $(x,T) \in SM \times \mathbb{R}^+$,

$$\begin{aligned} \sum_{\mathcal{D} \in \mathcal{B}^1_+} |c^s_{\mathcal{D}}(x,T)|^2 &\leq C(s)\,; \\ \|\, \mathcal{D}^s_1(x,T) \,\|_{-s} &\leq C(s)\,; \\ \|\, \mathcal{R}^s(x,T) \,\|_{-s} &\leq C(s)\,. \end{aligned}$$

For every invariant distribution $\mathcal{D} \in \mathcal{B}^1_+$, there exists $C(\mathcal{D}, s) > 0$ such that, for sufficiently large $T > 0$,

$$\|\, c^s_{\mathcal{D}}(\cdot, T) \,\|_0 \geq C(\mathcal{D}, s)\,.$$

Theorem 1.5 implies that the Central Limit Theorem does not hold for the horocycle flow on compact hyperbolic surfaces.

Corollary 1.6. *There exist zero average functions $f \in C^\infty(SM)$ such that any weak limit (as $T \to \infty$) of the probability distributions of the functions*

$$\frac{\int_0^T f(\phi^U_t(\cdot))\, dt}{\|\, \int_0^T f(\phi^U_t(\cdot))\, dt \,\|_0}$$

has compact support $\neq \{0\}$.

In the *non-compact* case, we obtain by the same methods the following generalisation of a result proved by P. Sarnak [51] for closed cuspidal horocycle arcs.

Let $\mathcal{I}^{1/2}_+(SM)$ be the orthogonal complement of the line generated by the invariant volume in the space $\mathcal{I}^{1/2}(SM) \subset W^{-1/2}(SM)$. The subset $\mathcal{B}^{1/2}_+ := \mathcal{B} \cap \mathcal{I}^{1/2}_+(SM)$ is a basis of eigenvectors for $\{\phi^X_t\} \,|\, \mathcal{I}^{1/2}_+(SM)$. Let $\sigma_{pp}(0,1/4) := \sigma_{pp} \cap]0,1/4[$. By Theorem 1.1,

$$\mathcal{B}^{1/2}_+ = \bigcup_{\mu \in \sigma_{pp}(0,1/4)} \mathcal{B}_\mu \,. \tag{11}$$

It follows that $\mathcal{B}^{1/2}_+$ has finite cardinality equal to the total multiplicity of the eigenvalues of the Laplacian Δ_M within the interval $]0,1/4[$. In particular, it is empty for hyperbolic surfaces satisfying the Selberg's conjecture.

Let $\gamma_{x,T}$ be the uniformly distributed probability measure along the unstable horocycle arc in SM of initial point $x \in SM$ and length $T > 0$. For all $t \in \mathbb{R}^+$, let $\phi^X_t(\gamma_{x,T})$ be the push-forward of $\gamma_{x,T}$ under the action of the geodesic flow. The measure $\phi^X_t(\gamma_{x,T})$ is equal to the uniformly distributed probability measure along the horocycle arc of initial point $\phi^X_t(x)$ and length $T_t = e^t T$. The following result is derived from Theorem 5.14, which contains more precise asymptotics for the push-forward of a horocycle arc depending on the rate of escape in the cusps of its endpoints.

Theorem 1.7. *For any $s > 3$, there exist bounded coefficients $\{c^s_{\mathcal{D}}(x,T,t)\}_{\mathcal{D}\in\mathcal{B}^{1/2}_+}$ and a bounded distributional coefficient $\mathcal{R}^s(x,T,t) \in W^{-s}(SM)$ such that, for all $f \in W^s(SM)$ and all $t \geq 0$,*

$$\phi^X_t(\gamma_{x,T})(f) = \int_{SM} f\, dvol \;+ \sum_{\mathcal{D}\in\mathcal{B}^{1/2}_+} c^s_{\mathcal{D}}(x,T,t)\, \mathcal{D}(f)\, T_t^{-S_{\mathcal{D}}} + \\ + \; \mathcal{R}^s(x,T,t)(f)\, T_t^{-1/2} \log^2 T_t \;. \tag{12}$$

There exists a continuous function $C := C_s : SM \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \geq 0$,

$$\sum_{\mathcal{D}\in\mathcal{B}^{1/2}_+} |c^s_{\mathcal{D}}(x,T,t)|^2 \leq C(x,T)\,;$$

$$\| \, \mathcal{R}^s(x,T,t) \, \|_{-s} \leq C(x,T)\,.$$

For every invariant distribution $\mathcal{D} \in \mathcal{B}^{1/2}_+$, there exists a constant $C(\mathcal{D},s) > 0$ such that, if $T > 0$ is sufficiently large, then for all $t \geq 0$,

$$C(\mathcal{D},s)^{-1} \leq \; \| \, c^s_{\mathcal{D}}(\cdot,T,t) \, \|_0 \leq C(\mathcal{D},s)\,.$$

In the particular case that $\gamma_{x,T}$ is supported on a closed cuspidal horocycle, we prove by the same methods that all the coefficients $c^s_{\mathcal{D}}(x,T,t)$ are constant functions of $t \geq 0$ and that the remainder term is $O(T_t^{-1/2} \log T_t)$ (Proposition 5.15). Hence, we obtain an asymptotic formula with remainder purely by methods based on representation theory. We remark that P. Sarnak [51] obtained a sharper asymptotics with remainder $o(T_t^{1/2})$ by Eisenstein series methods (Rankin-Selberg method), taking into account the explicit spectral decomposition of cuspidal horocycles and the absolute continuity of the continuous part of the spectrum. In the particular case of the modular group, the remainder term for cuspidal horocycles is $O(T_t^{-3/4+\epsilon})$, for all $\epsilon > 0$, iff the Riemann hypothesis holds [58], [51].

2. FOURIER ANALYSIS

2.1. **Sobolev spaces.** We choose as generators for $\mathfrak{sl}(2,\mathbb{R})$ the elements

$$X = \left\{ \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \right\}, \quad Y = \left\{ \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix} \right\}, \quad \Theta = \left\{ \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix} \right\}.$$

Recall the commutations rules:

$$[X,Y] = -\Theta, \quad [\Theta, X] = Y, \quad [\Theta, Y] = -X. \tag{13}$$

The basis element $X \in \mathfrak{sl}(2,\mathbb{R})$ is the "geodesic vector field", in the sense that it corresponds to the generator of the geodesic flow on SH. The elements

$$V = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} = -Y - \Theta, \quad U = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} = -Y + \Theta$$

are respectively the unstable and the stable "horocycle vector fields", in the sense that they correspond to the generators of the unstable and the stable horocycle flows on SH. In fact, the following commutation relations hold:

$$[X,V] = -V, \qquad [X,U] = U\,. \tag{14}$$

The *Laplacian* determined by the basis $\{X, Y, \Theta\}$ is the *elliptic* element of the enveloping algebra of $\mathfrak{sl}(2,\mathbb{R})$ defined as

$$\Delta := -(X^2 + Y^2 + \Theta^2)\,. \tag{15}$$

Let $\mathcal{H}$ be the (Hilbert) space of a unitary representation of $PSL(2,\mathbb{R})$. Any element of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ acts on $\mathcal{H}$ as an essentially skew-adjoint operator and the Laplacian Δ acts as an essentially self-adjoint operator [43]. The *Sobolev space* of order $s \in \mathbb{R}^+$ is the maximal domain $W^s(\mathcal{H}) \subset \mathcal{H}$ of the operator $(I+\Delta)^{s/2}$ endowed with the inner product

$$\langle f, g \rangle_s := \langle (I+\Delta)^s f, g \rangle_{\mathcal{H}}\,. \tag{16}$$

The spaces $W^s(\mathcal{H})$ are Hilbert spaces which coincide with the completion of the subspace $C^\infty(\mathcal{H}) \subset \mathcal{H}$ of *infinitely differentiable* vectors with respect to the norm $\| f \|_s := \| (I+\Delta)^{s/2} f \|_{\mathcal{H}}$, induced by the inner product (16). The subspace $C^\infty(\mathcal{H})$ coincides with the intersection of the spaces $W^s(\mathcal{H})$ for all $s \geq 0$. The Sobolev spaces with negative exponent $W^{-s}(\mathcal{H})$, $s > 0$, defined as the Hilbert space duals of the spaces $W^s(\mathcal{H})$, are subspaces of the space $\mathcal{E}'(\mathcal{H})$ of distributions, defined as the dual space of $C^\infty(\mathcal{H})$.

The *Sobolev order* of a distribution $\mathcal{D} \in \mathcal{E}'(\mathcal{H})$ is the extended real number

$$S_{\mathcal{D}} := \inf\{ s \in \mathbb{R} \,|\, \mathcal{D} \in W^{-s}(\mathcal{H}) \}\,. \tag{17}$$

2.2. **Direct decompositions.** The *Casimir* operator, which generates the centre of the enveloping algebra of $\mathfrak{sl}(2,\mathbb{R})$, is the element

$$\Box := -X^2 - Y^2 + \Theta^2\,. \tag{18}$$

The Casimir operator $\Box$ acts as a constant $\mu \in \mathbb{R}^+ \cup \{-n^2+n \,|\, n \in \mathbb{Z}^+\}$ on the Hilbert space of each *irreducible* unitary representation and its value classifies all non-trivial irreducible unitary representations according to three different types. Let $\mathcal{H}_\mu$ be the Hilbert space of an irreducible unitary representation on which the Casimir operator takes the value $\mu \in \mathbb{R}^+ \cup \{-n^2 + n \,|\, n \in \mathbb{Z}^+\}$. The representation is said to belong to the *principal series* if $\mu \geq 1/4$, to the *complementary series* if $0 < \mu < 1/4$ and to the *discrete series* if $\mu \leq 0$ [15, 3].

Let $\mathcal{H}$ be the Hilbert space of any non-trivial unitary representation of $PSL(2,\mathbb{R})$. Since the Casimir operator $\Box$ is in the centre of the enveloping algebra of $\mathfrak{sl}(2,\mathbb{R})$ and acts on $\mathcal{H}$ as an essentially self-adjoint operator, there exists a $PSL(2,\mathbb{R})$-invariant *direct integral decomposition* [39, 40, 18],

$$\mathcal{H} = \int_{\oplus} \mathcal{H}_\mu \tag{19}$$

with respect to a positive Stieltjes measure $ds(\mu)$ over the spectrum $\sigma(\Box)$. The Casimir operator acts as the constant $\mu \in \sigma(\Box)$ on every Hilbert space $\mathcal{H}_\mu$. The representations induced on $\mathcal{H}_\mu$ do not need to be irreducible. In fact, $\mathcal{H}_\mu$ is in general the direct sum of an (at most countable) number of unitary representations equal to the spectral multiplicity of $\mu \in \sigma(\Box)$.

All the operators in the enveloping algebra are *decomposable* with respect to the direct integral decomposition (19). Hence there exists for all $s \in \mathbb{R}$ an induced direct decomposition of the Sobolev spaces:

$$W^s(\mathcal{H}) = \int_{\oplus} W^s(\mathcal{H}_\mu) \tag{20}$$

with respect to the measure $ds(\mu)$. The existence of the direct integral decompositions (19), (20) allows us to reduce our analysis of invariant distributions and of the cohomological equation for the horocycle vector field to irreducible unitary representations.

2.3. **Hyperbolic surfaces.** If M is a hyperbolic surface of finite area, the spectrum $\sigma(\Delta_M)$ of the Laplacian on M has a pure point discrete component of finite multiplicity and an absolutely continuous component on the interval $[1/4, \infty[$ with finite multiplicity equal to the number of (standard) cusps of M [24]. Hence, the Laplacian Δ_M has a "spectral gap", in the sense that $\sigma(\Delta_M) \setminus \{0\}$ has a lower bound $\mu_0 > 0$. Examples of 'pinched' surfaces with $\mu_0 < 1/4$ were constructed in [54, 45, 9]. We shall denote σ_{pp} the pure point spectrum of Δ_M and $\mathcal{C}$ will denote the (finite) set of cusps of M. If M is compact, then the Laplacian has pure-point spectrum supported on the set σ_{pp} and $\mathcal{C} = \emptyset$.

There is a standard unitary representation of $PSL(2,\mathbb{R})$ on the Hilbert space $L^2(SM)$. The corresponding Sobolev spaces $W^s(SM)$ have the following splitting as a direct integral of irreducible representations (see (19) and (20)):

$$\begin{aligned} W^s(SM) = \bigoplus_{\mu \in \sigma_{pp}} W^s(\mathcal{H}_\mu) \oplus \\ \oplus \bigoplus_{n \in \mathbb{Z}^+} \left(W^s(\mathcal{H}_{+n}) \oplus W^s(\mathcal{H}_{-n})\right) \oplus \bigoplus_{c \in \mathcal{C}} W^s(\mathcal{H}_c)\,. \end{aligned} \tag{21}$$

For $\mu = 0$, the sub-representation $\mathcal{H}_\mu$ is the trivial representation, which appears with multiplicity 1, otherwise the sub-representations $\mathcal{H}_\mu$, $\mathcal{H}_{\pm n}$ and $\mathcal{H}_c$ are not yet (in general) irreducible.

Let $m_\mu \in \mathbb{Z}^+$ denote the multiplicity of an eigenvalue $\mu \in \sigma_{pp} \setminus \{0\}$ and $m_n \in \mathbb{Z}^+$ denote the dimension of the space of holomorphic (anti-holomorphic) sections of the n-th power of the canonical line bundle, which can be computed by the Riemann-Roch Theorem. We have:

$$
\begin{aligned}
W^s(\mathcal{H}_\mu) &= \bigoplus_{i=1}^{m_\mu} W^s(\mathcal{H}_\mu^{(i)})\,; \\
W^s(\mathcal{H}_{\pm n}) &= \bigoplus_{i=1}^{m_n} W^s(\mathcal{H}_{\pm n}^{(i)})\,; \qquad (22)\\
W^s(\mathcal{H}_c) &= \int_\oplus W^s\left(\mathcal{H}_c(\mu)\right)\, ds_c(\mu)\,.
\end{aligned}
$$

The sub-representations $\mathcal{H}_\mu^{(i)}$ are irreducible with Casimir parameter $\mu > 0$, hence they belong to either the principal series ($\mu \geq 1/4$) or to the complementary series ($\mu_0 \leq \mu < 1/4$); the sub-representations $\mathcal{H}_{+n}^{(i)}$, $\mathcal{H}_{-n}^{(i)}$ are irreducible with Casimir parameter $-n^2 + n$, $n \in \mathbb{Z}^+$, hence they belong to the holomorphic, respectively to the anti-holomorphic, discrete series; the sub-representations $\mathcal{H}_c(\mu)$ are irreducible with Casimir parameter $\mu \geq 1/4$, hence they belong to the principal series, and, for all $c \in \mathcal{C}$, the Stieltjes measures $ds_c(\mu)$ are supported on $[1/4, +\infty[$ and are absolutely continuous.

2.4. **Orthogonal basis.** Let $\mathcal{H}_\mu$ be the Hilbert space of an irreducible unitary representation of Casimir parameter $\mu \in \mathbb{R}^+ \cup \{-n^2 + n \,|\, n \in \mathbb{Z}^+\}$. In order to construct a convenient orthogonal basis of $\mathcal{H}_\mu$, we consider the following elements of the enveloping algebra

$$\eta_+ = X - iY, \quad \eta_- = X + iY\,,$$

which have the property of raising and lowering eigenvalues of $-i\Theta$. In fact, from

$$[-i\Theta, \eta_\pm] = \pm\eta_\pm$$

it follows that, if $-i\Theta u = ku$, then

$$-i\Theta(\eta_\pm u) = \eta_\pm(-i\Theta u) + [-i\Theta, \eta_\pm]u = k\eta_\pm u \pm \eta_\pm u = (k \pm 1)(\eta_\pm u)\,.$$

If $\mu \geq 0$ (i.e. if $\mathcal{H}_\mu$ belongs to the complementary or principal series), an orthogonal basis of $\mathcal{H}_\mu$ will be

$$\dots, {\eta_-}^k v_0, \dots, {\eta_-}^2 v_0, \eta_- v_0, v_0, \eta_+ v_0, {\eta_+}^2 v_0, \dots, {\eta_+}^k v_0, \dots$$

where v_0 is a Θ-invariant vector ($\Theta v_0 = 0$).

If $\mu = -n^2 + n$ and $\mathcal{H}_\mu$ belongs to the holomorphic discrete series, an orthogonal basis of $\mathcal{H}_\mu$ will be

$$v_n, \eta_+ v_n, {\eta_+}^2 v_n, \dots, {\eta_+}^k v_n, \dots$$

where $\Theta v_n = inv_n$.

The anti-holomorphic case is similar. In fact, there is a complex anti-linear isomorphism between holomorphic and anti-holomorphic irreducible representations of the discrete series of identical Casimir parameter. It is therefore not restrictive to explicitly consider only the holomorphic case.

In all cases such a basis is formed by analytic vectors, since they are by definition eigenvectors of the operator Θ, hence eigenvectors of the Laplacian $\Delta = \Box - 2\Theta^2$.

Rather then dealing with the basis $\{v_k\}$, we introduce an adapted orthogonal basis $\{u_k\}$ of $\mathcal{H}_\mu$. Let ν be a complex solution of the equation

$$1-\nu^2 = 4\mu\,. \tag{23}$$

We remark that

- if $\mathcal{H}_\mu$ belongs to the principal series, then $\mu \geq 1/4$ and therefore ν is purely imaginary;
- if $\mathcal{H}_\mu$ belongs to the complementary series, then $\mu \in]0,1/4[$ and ν is real belonging to $]-1,1[\setminus\{0\}$;
- if $\mathcal{H}_\mu$ belongs to the discrete series, then $\mu = -n^2+n$ and $\nu = \pm(2n-1)$, n a positive integer.

The basis $\{u_k\}$ is defined as follows:

$$\begin{aligned} u_k &= c_k(\eta_+ u_{k-1}), \quad c_k = \frac{2}{2k-1+\nu} \quad \text{for } k>0,\\ u_k &= c_k(\eta_- u_{k+1}), \quad c_k = \frac{2}{-2k-1+\nu} \quad \text{for } k<0, \end{aligned} \tag{24}$$

where the initial definition is $u_0 = v_0$ ($\|u_0\| = 1$), in the case $\mu > 0$ while $u_n = v_n$ ($\|u_n\| = 1$), for $\mu = -n^2+n$.

We remark that, for an irreducible representation of the discrete series with $\mu = -n^2+n$, the basis $\{u_k\}$ is well defined only for $\nu = 2n-1$.

The basis $\{u_k\}$ is an orthogonal, but not orthonormal, basis of eigenvectors of the operator Θ, hence of the Laplacian operator $\Delta = \Box - 2\Theta^2$. In fact, for all $k\in\mathbb{Z}$ ($k \geq n$),

$$\Theta u_k = iku_k\,, \qquad \Delta u_k = (\mu + 2k^2)u_k\,. \tag{25}$$

The norms of its vectors are given, for all $k \neq 0$ (or $k \neq n$) by the formula:

$$\|u_k\|^2 = \begin{cases} \|u_{k-1}\|^2\,, & \text{if } \nu\in i\mathbb{R},\ \text{i.e. if } \mu \geq 1/4\,,\\ \frac{2|k|-1-\nu}{2|k|-1+\nu}\|u_{k-1}\|^2 & \text{if } \nu \in \mathbb{R}\,,\ \text{i.e. if } \mu < 1/4\,. \end{cases} \tag{26}$$

In fact, since the adjoint ${\eta_+}^* = -\eta_-$ and

$$\eta_+\eta_- = -\Box - i\Theta + \Theta^2, \qquad \eta_-\eta_+ = -\Box + i\Theta + \Theta^2\,, \tag{27}$$

we obtain that for $k>0$

$$\begin{aligned} \|u_k\|^2 &= \|c_k\|^2\langle \eta_+ u_{k-1}, \eta_+ u_{k-1}\rangle = -\|c_k\|^2\langle \eta_-\eta_+ u_{k-1}, u_{k-1}\rangle = \\ &= \|c_k\|^2\langle (\Box - i\Theta - \Theta^2)u_{k-1}, u_{k-1}\rangle = \frac{2k-1-\nu}{2k-1+\bar\nu}\|u_{k-1}\|^2 \end{aligned}$$

and a similar computation holds for $k<0$.

We introduce for convenience the sequence

$$\Pi_{\nu,k} = \prod_{i=i_\nu+1}^{k} \frac{2i-1-\nu}{2i-1+\nu} \quad \text{for any integer } k \geq i_\nu = \left[\frac{1+\Re(\nu)}{2}\right] \tag{28}$$

(empty products are set equal to 1, hence, if $k = i_\nu$, then $\Pi_{\nu,k} = 1$ in all cases).

By (26) we have

$$\|u_k\|^2 = \left|\Pi_{\nu,|k|}\right|\,. \tag{29}$$

The behaviour of the norms $\| u_k \|$ is described by the following Lemma whose proof is postponed to the Appendix.

Lemma 2.1. *If $\nu \in i\mathbb{R}$, for all $k \geq i_\nu = 0$,*

$$|\Pi_{\nu,k}| = 1. \tag{30}$$

There exists $C > 0$ such that, if $\nu \in]-1,1[\setminus \{0\}$, for all $k > i_\nu = 0$, we have

$$C^{-1}\left(\frac{1-\nu}{1+\nu}\right)(1+k)^{-\nu} \leq \Pi_{\nu,k} \leq C\left(\frac{1-\nu}{1+\nu}\right)(1+k)^{-\nu}\,; \tag{31}$$

if $\nu = 2n-1$, for all $k \geq \ell \geq i_\nu = n$, we have

$$C^{-1}\left(\frac{k-n+1}{\ell-n+1}\right)^{-\nu} \leq \frac{\Pi_{\nu,k}}{\Pi_{\nu,\ell}} \leq C\left(\frac{k-n+\nu}{\ell-n+\nu}\right)^{-\nu}\,. \tag{32}$$

The Sobolev norms of the vectors of the orthogonal basis $\{u_k\}$ are given by the identities

$$\| u_k \|_s^2 = \langle (I+\Delta)^s u_k, u_k\rangle = (1+\mu+2k^2)^s \, \| u_k \|^2 \tag{33}$$

By (33)and (29), a vector $f = \sum_{-\infty}^{\infty} f_k u_k \in \mathcal{H}_\mu$ belongs to $W^s(\mathcal{H}_\mu)$ $(s > 0)$ iff the Sobolev norm

$$\| f \|_s := \left(\sum_{-\infty}^{\infty}(1+\mu+2k^2)^s|\Pi_{\nu,k}||f_k|^2\right)^{\frac{1}{2}} < \infty\,. \tag{34}$$

By Lemma 2.1,

$$\| u_k \|_s^2 \approx (1+|k|)^{2s-\Re(\nu)}\,. \tag{35}$$

It follows that

$$\| f \|_s \approx \left(\sum_{-\infty}^{\infty}(1+|k|)^{2s-\Re(\nu)}\,|f_k|^2\right)^{\frac{1}{2}}\,. \tag{36}$$

We recall that if $\mathcal{H}_\mu$ belongs to the *principal series* then ν is purely imaginary; if $\mathcal{H}_\mu$ belongs to the *complementary series* then $\nu \in]-1,1[\setminus \{0\}$; and $\nu = 2n-1$ if $\mathcal{H}_\mu$ belongs to the *discrete series* ($\mu = -n^2+n$, $n \in \mathbb{Z}^+$).

3. Invariant distributions

For any unitary representation of $PSL(2,\mathbb{R})$ on a Hilbert space $\mathcal{H}$, let

$$\mathcal{I}(\mathcal{H}) := \{\mathcal{D} \in \mathcal{E}'(\mathcal{H}) \,|\, \mathcal{L}_U\mathcal{D} = 0\}$$

be the space of all U-invariant distributions in $\mathcal{E}'(\mathcal{H})$. The analysis of $\mathcal{I}(\mathcal{H})$ and of its subspaces $\mathcal{I}^s(\mathcal{H})$ of U-invariant distributions of order $\leq s$ can be reduced to the case of *irreducible* unitary representations. In fact, we have:

Lemma 3.1. *Let $\mathcal{H}$ be a direct integral of unitary representations $\mathcal{H}_\lambda$ of $PSL(2,\mathbb{R})$ with respect to a Stieltjes measure ds. The spaces $\mathcal{I}(\mathcal{H})$, $\mathcal{I}^s(\mathcal{H})$ of U-invariant distributions have direct integral decompositions:*

$$\mathcal{I}(\mathcal{H}) = \int_{\oplus} \mathcal{I}(\mathcal{H})\, ds(\lambda)\,,$$
$$\mathcal{I}^s(\mathcal{H}) = \int_{\oplus} \mathcal{I}^s(\mathcal{H})\, ds(\lambda)\,.$$

Proof. The space $C^\infty(\mathcal{H})$, its dual space $\mathcal{E}'(\mathcal{H})$, all Sobolev spaces $W^s(\mathcal{H})$ are decomposable and the horocycle vector field U, as a densely defined essentially skew-adjoint operator on $\mathcal{H}$, is also decomposable. Hence the Lemma holds. □

Invariant distributions for irreducible unitary representations are described by the following

Theorem 3.2. *Let $\mathcal{H}_\mu$ be an irreducible unitary representation of $PSL(2,\mathbb{R})$ of Casimir parameter μ. Then*

- *if $\mathcal{H}_\mu$ belongs to the principal or complementary series, the space $\mathcal{I}(\mathcal{H}_\mu)$ has dimension 2 and it is generated by two eigenvectors $\mathcal{D}^\pm_\mu$ of the "geodesic vector field" X of eigenvalues $-\frac{1\pm\nu}{2}$ and Sobolev order $\frac{1\pm\Re(\nu)}{2}$ respectively.*
- *if $\mathcal{H}_\mu$ belongs to the discrete series and $\mu = -n^2+n$, the space $\mathcal{I}(\mathcal{H}_\mu)$ has dimension 1 and it is generated by an eigenvector $\mathcal{D}^+_\mu$ of X of eigenvalue $-n$ and Sobolev order n.*

Theorems 1.1 and 1.4 can be immediately derived from Lemma 3.1 and Theorem 3.2, by the decompositions (21), (22) of Sobolev spaces into irreducible sub-representations.

3.1. **Formal invariant distributions.** Any distribution $\mathcal{D} \in \mathcal{E}'(\mathcal{H}_\mu)$ is uniquely determined by its Fourier coefficients $d_k = \mathcal{D}(u_k)$, $k \in \mathbb{Z}$ (principal and complementary series) or $k \geq n$ (discrete series).

Since $\mathcal{L}_U\mathcal{D} = 0$ iff

$$\mathcal{D}(Uf) = 0, \text{ for all } f \in C^\infty(\mathcal{H}_\mu),$$

a distribution $\mathcal{D}$ is U-invariant iff $\mathcal{D}(Uu_k) = 0$, for all $k \in \mathbb{Z}$ or all $k \geq n$. We need therefore to compute the action of U on the vectors of the basis $\{u_k\}$. The resulting formula (37) below, as well as the similar formula (45) in §3.3 for the action of the 'geodesic vector field', can be found in the literature on $SL(2,\mathbb{R})$ harmonic analysis (see for instance [4], formula (4.4)).

Lemma 3.3. *Let $I_\nu = \mathbb{Z}$ if μ parametrises the principal or complementary series or $I_\nu = [n,\infty[\subset \mathbb{Z}$ if $\mu = -n^2+n$ parametrises the holomorphic discrete series. Then*

$$Uu_k = -\frac{i(2k+1+\nu)}{4}u_{k+1} + iku_k - \frac{i(2k-1-\nu)}{4}u_{k-1}, \quad \textit{for all } k \in I_\nu \tag{37}$$

(for $\nu = 2n-1$ and $k = n$ the above equation must be read as $Uu_n = -inu_{n+1} + inu_n$).

Proof. Since

$$Uu_k = (-Y+\Theta)u_k = (\Theta - \frac{i}{2}(\eta_+ - \eta_-))u_k$$

for the principal ot the complementary series we have

$$Uu_{i_\nu} = Uu_0 = -\frac{i}{2}\left(\eta_+ u_0 - \eta_- u_0\right) = \\ = -\frac{i}{2}\left(\frac{u_1}{c_1} - \frac{u_{-1}}{c_{-1}}\right) = -\frac{i(\nu+1)}{4}u_1 + \frac{i(\nu+1)}{4}u_{-1},$$

while for the holomorphic discrete series with parameter $\nu = 2n-1$, $n \in \mathbb{Z}^+$, we have

$$Uu_{i_\nu} = Uu_n = (\Theta - \frac{i}{2}\eta_+)u_n = inu_n - \frac{i}{2}\frac{u_{n+1}}{c_{n+1}} = -inu_{n+1} + inu_n$$

Thus the Lemma is true in these particular cases. For any $k > i_\nu$ we have instead

$$\begin{aligned} Uu_k = iku_k - \frac{i}{2}\eta_+ u_k + \frac{i}{2}\eta_-(c_k\eta_+ u_{k-1}) \\ = iku_k - \frac{i}{2c_{k+1}}u_{k+1} + \frac{ic_k}{2}\eta_-\eta_+ u_{k-1} \\ = -\frac{i}{2c_{k+1}}u_{k+1} + iku_k + \frac{ic_k}{2}(-\Box + i\Theta + \Theta^2)u_{k-1}\,. \end{aligned}$$

A straightforward computation, based on the values of c_{k-1}, c_k given in (24) and on (25), yields (37) for all $k > i_\nu$. A similar computation shows that, for the principal or the complementary series, the equation (37) is also valid for all $k < 0$. □

Let L_ν be the linear difference operator that to a sequence $\hat{d} = (d_k)_{k \in I_\nu}$ assigns the sequence $L_\nu \hat{d}$ defined by

$$(L_\nu \hat{d})_k = -\frac{i(2k+1+\nu)}{4}d_{k+1} + ikd_k - \frac{i(2k-1-\nu)}{4}d_{k-1}\ , \quad k \in I_\nu. \tag{38}$$

We remark that, if $\nu = 2n-1$ (discrete series) and $k = n$, formula (38) should be read as

$$(L_\nu \hat{d})_n = -ind_{n+1} + ind_n\,.$$

Lemma 3.3 immediately implies that the sequence $\hat{d}$ of the Fourier coefficients $d_k = \mathcal{D}(u_k)$ of an U-invariant distribution $\mathcal{D} \in \mathcal{E}'(\mathcal{H}_\mu)$ must satisfy the difference equation

$$L_\nu \hat{d} = 0\,. \tag{39}$$

It is immediate to check that

$$d_k^+ = 1, \quad \text{for all } k \in I_\nu, \tag{40}$$

is a solution of the equation (39) for any value of ν.

In the case where $\nu = 2n-1$ parametrises the discrete series there are no other linearly independent solutions. In fact the identity $(L_\nu \hat{d})_n = -ind_{n+1} + ind_n = 0$ implies $d_{n+1} = d_n$, hence, by (38) and (39), $d_k = d_n$ for all $k \in I_\nu$.

If $\nu \neq 0$ parametrises the principal or the complementary series, another linearly independent solution can be obtained as follows. Let u_k^- be the vectors obtained by replacing the parameter ν by $-\nu$ in the definition (24) of the adapted Fourier basis. The difference equation for the Fourier coefficients of U-invariant distributions with respect to the basis $\{u_k^-\}$

becomes $L_{-\nu}\hat{d} = 0$. For $\nu \neq 0$, by writing the solution $d_k^+ = 1$ for the basis $\{u_k^-\}$ in terms of the basis $\{u_k\}$, we obtain as a second solution

$$d_k^- = \prod_{i=1}^{|k|} \frac{2i-1-\nu}{2i-1+\nu} = \Pi_{\nu,|k|} \quad \text{for } k \neq 0\,. \tag{41}$$

For $\nu = 0$ ($\mu = 1/4$), a second solution can be found by directly solving the difference equation:

$$d_0^- = 0, \quad d_k^- = \sum_{i=1}^{|k|} \frac{1}{2i-1} \quad \text{for } k \neq 0\,. \tag{42}$$

We have thus shown that the space $\mathcal{I}(\mathcal{H}_\mu) \subset \mathcal{E}'(\mathcal{H}_\mu)$ of invariant distributions is at most two dimensional if $\mathcal{H}_\mu$ belongs to the principal or the complementary series, and it is at most one-dimensional if $\mathcal{H}_\mu$ belongs to the discrete series.

3.2. **Sobolev order.** The linear functional

$$\mathcal{D}_\mu^+ : f = \sum_{-\infty}^{\infty} f_k u_k \mapsto \sum_{-\infty}^{\infty} f_k$$

is defined, for any $t > 1$, on the set of $f \in \mathcal{H}_\mu$ for which the series $\sum_{-\infty}^{\infty} (1+|k|)^t \, |f_k|^2$ converges. In fact, there exists a constant $C_{\nu,t} > 0$ such that

$$|\mathcal{D}_\mu^+(f)| \leq C_{\nu,t} \left(\sum_{-\infty}^{\infty} (1+|k|)^t \, |f_k|^2 \right)^{1/2} . \tag{43}$$

By (36) we conclude that $\mathcal{D}_\mu^+$ defines a continuous linear functional on $W^s(\mathcal{H}_\mu)$ iff $s > \frac{1+\Re(\nu)}{2}$.

If $\nu \neq 0$, the linear functional

$$\mathcal{D}_\mu^- : f = \sum_{-\infty}^{\infty} f_k u_k \mapsto \sum_{-\infty}^{\infty} f_k \Pi_{\nu,|k|}$$

is defined, for any $t > 1 - 2\Re(\nu)$, on the set of $f \in \mathcal{H}_\mu$ for which the series $\sum_{-\infty}^{\infty} (1 + |k|)^t \, |f_k|^2$ converges. In fact, by Lemma 2.1, there exists a constant $C_{\nu,t} > 0$ such that

$$|\mathcal{D}_\mu^-(f)| \leq C_{\nu,t} \left(\sum_{-\infty}^{\infty} (1+|k|)^t \, |f_k|^2 \right)^{1/2} . \tag{44}$$

By (36) it follows that $\mathcal{D}_\mu^+$ defines a continuous linear functional on $W^s(\mathcal{H}_\mu)$ iff $s > \frac{1-\Re(\nu)}{2}$.

If $\nu = 0$, it is immediate to check that the linear functional

$$\mathcal{D}_\mu^- : f = \sum_{-\infty}^{\infty} f_k u_k \mapsto \sum_{-\infty}^{\infty} f_k \left(\sum_{i=1}^{|k|} \frac{1}{2i-1} \right)$$

is defined, for any $t > 1$, on the set of $f \in \mathcal{H}_\mu$ for which the series $\sum_{-\infty}^{\infty}(1+|k|)^t\,|f_k|^2$ converges. It follows that $\mathcal{D}^-_\mu$ defines a continuous linear functional on $W^s(\mathcal{H}_\mu)$ iff $s > \frac{1}{2} = \frac{1-\Re(\nu)}{2}$.

3.3. **Eigenvectors.** The generators $\{\mathcal{D}^+_\mu, \mathcal{D}^-_\mu\}$ of the space $\mathcal{I}(\mathcal{H}_\mu)$ of invariant distributions for the "horocycle vector field" are for all $\mu \neq 1/4$ distributional eigenvectors for the Lie derivative operator $\mathcal{L}_X$ with respect to the "geodesic vector field" X. In other terms, they are exactly the *conical distributions* for the Lie group $PSL(2,\mathbb{R})$, in the sense of S. Helgason [25]. In the special case $\mu = 1/4$, the distribution $\mathcal{D}^+_\mu$ is still an eigenvector (a conical distribution), but $\mathcal{D}^-_\mu$ is not. In this case, the matrix of the operator $\mathcal{L}_X$ with respect to the basis $\{\mathcal{D}^+_{1/4}, \mathcal{D}^-_{1/4}\}$ is a 2×2 Jordan block.

Lemma 3.4. *Let $I_\nu = \mathbb{Z}$ if μ parametrises the principal or complementary series or $I_\nu = [n,\infty[\subset \mathbb{Z}$ if $\mu = -n^2+n$ parametrises the holomorphic discrete series. Then*

$$Xu_k = \frac{2k+1+\nu}{4}u_{k+1} - \frac{2k-1-\nu}{4}u_{k-1} \quad \textit{for all } k \in I_\nu \tag{45}$$

(for $\nu = 2n-1$, and $k = n$ the above equation must be read as $Xu_n = nu_{n+1}$).

Proof. We have $[\Theta, Y] = -X$ and $U = -Y + \Theta$, hence $X = [\Theta, U]$. By Lemma 3.3, since $\Theta u_k = i\,k\,u_k$ for all $k \in I_\nu$, an immediate calculation yields the above formula for Xu_k. □

Lemma 3.5. *If $\mu > 0$ (principal or complementary series) and $\mu \neq 1/4$,*

$$\mathcal{L}_X \mathcal{D}^\pm_\mu = -\frac{1\pm\nu}{2}\mathcal{D}^\pm_\mu\,. \tag{46}$$

If $\mu = 1/4$ ($\nu = 0$),

$$\mathcal{L}_X \begin{pmatrix}\mathcal{D}^+_\mu \\ \mathcal{D}^-_\mu\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}\mathcal{D}^+_\mu \\ \mathcal{D}^-_\mu\end{pmatrix}\,. \tag{47}$$

If $\mu = -n^2+n$, $n \in \mathbb{Z}^+$ (discrete series),

$$\mathcal{L}_X \mathcal{D}^+_\mu = -\frac{1+\nu}{2}\mathcal{D}^+_\mu = -n\mathcal{D}^+_\mu.$$

Proof. The distribution $\mathcal{D}^+_\mu$ is determined by $\mathcal{D}^+_\mu(u_k) = 1$ for all $k \in I_\nu$. Hence, by Lemma 3.4,

$$\mathcal{L}_X\mathcal{D}^+_\mu(u_k) = -\mathcal{D}^+_\mu(Xu_k) = -\left(\frac{2k+1+\nu}{4} - \frac{2k-1-\nu}{4}\right) = -\frac{1+\nu}{2}\mathcal{D}^+_\mu(u_k). \tag{48}$$

The distribution $\mathcal{D}^-_\mu$ is determined, if $\nu \neq 0$, by $\mathcal{D}^-_\mu(u_k) = \Pi_{\nu,|k|}$. Hence, by Lemma 3.4, for all $k \in \mathbb{Z}$,

$$\begin{aligned}\mathcal{L}_X\mathcal{D}^-_\mu(u_k) = -\mathcal{D}^-_\mu(Xu_k) = -\left(\frac{2k+1+\nu}{4}\Pi_{\nu,|k+1|} - \frac{2k-1-\nu}{4}\Pi_{\nu,|k-1|}\right) = \\ = -\left(\frac{2k+1-\nu}{4} - \frac{2k-1+\nu}{4}\right)\Pi_{\nu,|k|} = -\frac{1-\nu}{2}\mathcal{D}^-_\mu(u_k).\end{aligned}$$

If $\nu = 0$, the distribution $\mathcal{D}^-_\mu$ is determined by

$$\mathcal{D}^-_\mu(u_0) = 0, \quad \mathcal{D}^-_\mu(u_k) = \sum_{i=1}^{|k|} \frac{1}{2i-1} \quad \text{for } k \neq 0. \tag{49}$$

Hence, for all $k \in \mathbb{Z}$,

$$\mathcal{L}_X \mathcal{D}^-_\mu(u_k) = -\left(\frac{2k+1}{4} \sum_{i=1}^{|k+1|} \frac{1}{2i-1} - \frac{2k-1}{4} \sum_{i=1}^{|k-1|} \frac{1}{2i-1} \right) =$$

$$= -\frac{1}{2} - \left(\frac{2k+1}{4} - \frac{2k-1}{4} \right) \sum_{i=1}^{|k|} \frac{1}{2i-1} = -\frac{1}{2}\mathcal{D}^+_\mu(u_k) - \frac{1}{2}\mathcal{D}^-_\mu(u_k).$$

□

Theorem 3.2 is therefore completely proved.

4. The cohomological equation

Let $\mathcal{H}$ be a unitary representation of $PSL(2,\mathbb{R})$. We prove that, if the Casimir operator $\Box$ on $\mathcal{H}$ has a 'spectral gap' then the only obstructions to the existence of a smooth solution $f \in \mathcal{H}$ of the equation $Uf = g$, for any smooth vector $g \in \mathcal{H}$, are given by U-invariant distributions.

Theorem 4.1. *If there exists $\mu_0 > 0$ such that $\sigma(\Box) \cap]0, \mu_0[= \emptyset$, then the following holds. Let ν_0 be defined as in (4). Let $s > \frac{1+\nu_0}{2}$ and $t \in \mathbb{R}$. Then there exists a constant $C := C(\nu_0, s, t) > 0$ such that, for all $g \in W^s(\mathcal{H})$,*

- *if either $t < -\frac{1+\nu_0}{2}$ and g has no component on the trivial sub-representation of $\mathcal{H}$, or*
- *if $t < s-1$ and $\mathcal{D}(g) = 0$, for all $\mathcal{D} \in \mathcal{I}^s(\mathcal{H})$,*

then the equation $Uf = g$ has a solution $f \in W^t(\mathcal{H})$ which satisfies the Sobolev estimate

$$\| f \|_t \leq C \| g \|_s .$$

A solution $f \in W^t(\mathcal{H})$ of the equation $Uf = g$ is unique modulo the trivial sub-representation if and only if $t \geq -\frac{1-\nu_0}{2}$.

Theorem 4.1 immediately implies Theorem 1.2.

4.1. Formal Green operators.

Let $\mathcal{H}_\mu$ be any irreducible unitary representation of Casimir parameter μ and let $f, g \in \mathcal{E}'(\mathcal{H}_\mu)$ be distributions satisfying the equation

$$Uf = g\,. \tag{50}$$

Let $f = \sum_k f_k u_k$ and $g = \sum_k g_k u_k$ be the Fourier expansions of the distributions f, g with respect to the adapted basis (24) of $\mathcal{H}_\mu$. Let $\hat f = (f_k)_{k \in I_\nu}$, $\hat g = (g_k)_{k \in I_\nu}$ denote the sequences of the Fourier coefficients of f and g. The equation (50) is equivalent to the difference equation:

$$L^*_\nu \hat f = \hat g \tag{51}$$

where L_ν^* is the operator transposed of L_ν. More explicitly we have, for all $k \in I_\nu$,

$$(L_\nu^* \hat{f})_k = -i\frac{2k+1-\nu}{4} f_{k+1} + ikf_k - i\frac{2k-1+\nu}{4} f_{k-1} = g_k \,. \tag{52}$$

If $\nu = 2n-1$ (discrete series) and $k = n$, equation (52) should be read as

$$inf_n - \frac{i}{2} f_{n+1} = g_n.$$

We remark that when $\nu \in i\,\mathbb{R} \cup \,]0,1[$ parametrises the principal or the complementary series we have $L_\nu^* = L_{-\nu}$; this equality holds formally also for the discrete series.

Formal solutions of the homogeneous equation $Uf = 0$ are (formal) vectors $f \in \mathcal{H}_\mu$ representing invariant distributions with respect to the $\mathcal{H}_\mu$ inner product. It follows that, if $\mathcal{D} \in \mathcal{I}(\mathcal{H}_\mu)$, the sequence $\hat{f} = (f_k)_{k \in I_\nu}$ given by

$$f_k = \frac{\mathcal{D}(u_k)}{\| u_k \|^2} = \frac{\mathcal{D}(u_k)}{\Pi_{\nu,|k|}}$$

is a solution of the homogeneous equation $L_\nu^* f = 0$. By (29), (40) and (41), we obtain the following formulae for a basis of the kernel of the operator L_ν^*. For the *principal* or the *complementary series*:

$$f_k^{(1)} = \Pi_{\nu,|k|}^{-1} = \Pi_{-\nu,|k|} \tag{53}$$

and, if $\nu \neq 0$,

$$f_k^{(2)} = 1 \quad \text{for all } k \in \mathbb{Z}. \tag{54}$$

If $\nu = 0$, a second independent solution is

$$f_0^{(2)} = 0, \quad f_k^{(2)} = \sum_{i=1}^{|k|} \frac{1}{2i-1} \text{ for all } k \in \mathbb{Z} \setminus \{0\}. \tag{55}$$

Such formulae can also be deduced from (40), (41) by the identity $L_\nu^* = L_{-\nu}$.

For the *discrete series*:

$$f_k^{(1)} = \Pi_{\nu,k}^{-1}\,, \quad \text{for all } k \in I_\nu. \tag{56}$$

A standard construction based on the solutions $f^{(1)}$, $f^{(2)}$ yields the *Green operator* for L_ν^*. For the *principal and complementary series*, if $\nu \neq 0$ we have

$$G_\nu(k,\ell) = \begin{cases} \frac{2i}{\nu}\left(\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1\right) & k > \ell \\ 0 & k \leq \ell \end{cases} \tag{57}$$

As $\nu \to 0$ formula (57) converges to the Green operator for $\nu = 0$:

$$G_0(\ell,k) = \begin{cases} 4i\left(\sum_{i=1}^{|\ell|} \frac{1}{2i-1} - \sum_{i=1}^{|k|} \frac{1}{2i-1}\right) & k > \ell \\ 0 & k \leq \ell \end{cases} \tag{58}$$

We remark that the Green operator is chosen so that if the sequence $\hat{g}$ of the Fourier coefficients of the function $g \in \mathcal{H}_\mu$ has finite support and furthermore $\mathcal{D}_\mu^+(g) = \mathcal{D}_\mu^-(g) = 0$, then the sequence $\hat{f} = G_\nu(\hat{g})$ has also finite support and therefore yields a bona-fide solution $f \in C^\infty(\mathcal{H}_\mu)$.

The Green operator in the case of the (holomorphic) *discrete series* can be taken as

$$G_\nu(k,\ell) = \begin{cases} -\frac{2i}{\nu}\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} & i_\nu \le k < \ell \\ -\frac{2i}{\nu} & i_\nu \le \ell \le k \end{cases} \tag{59}$$

which has the property that if the sequence $\hat g$ of the Fourier coefficients of $g \in \mathcal{H}_\mu$ has finite support and $\mathcal{D}^+_\mu(g) = 0$, then $\hat f$ has also finite support and therefore yields a bona-fide solution $f \in C^\infty(\mathcal{H}_\mu)$.

In all cases it is straightforward to check that the sequences $(\hat G)_k = G_\nu(k,\ell)$ satisfy the equation

$$(L^*_\nu \hat G)_k = \delta_{\ell k}\,,$$

hence the Green operators (57), (58) and (59) are well-defined.

4.2. **Sobolev estimates.** We prove Sobolev estimates for the Green operator G_ν in each irreducible component $\mathcal{H}_\mu$. The dependence of the estimates on the Casimir parameter μ is studied explicitly, since it will be crucial in the construction of solutions of the equation $Uf = g$ for general unitary representations by 'gluing' the solutions obtained in each irreducible 'component' of a direct integral decomposition.

Sharp estimates for the *principal* or the *complementary series* are based on the following Lemma proven in the Appendix.

Lemma 4.2. *There exists $C > 0$ such that, if $\nu \in i\mathbb{R}$ or $\nu \in\,]-1,1[\setminus\{0\}$, for all $k \ge \ell \ge 0$,*

$$|\frac{\Pi_{\nu,\ell}}{\Pi_{\nu,k}} - 1| \le C|\nu| \max\{1, |\frac{\Pi_{\nu,\ell}}{\Pi_{\nu,k}}|\}\Big(1 + \log(\frac{1+k}{1+\ell})\Big).$$

Lemma 4.3 (Principal series)**.** *For all $s > 1/2$, $t < s-1$, there exists $C_{s,t} > 0$ such that, for all $\nu \in i\mathbb{R}$ ($\mu \ge 1/4$) and for all $g \in W^s(\mathcal{H}_\mu)$ we have:*

(a) *if $t < -1/2$ or*
(b) *if $\mathcal{D}^+_\mu(g) = \mathcal{D}^-_\mu(g) = 0$,*

then $G_\nu g \in W^t(\mathcal{H}_\mu)$ and

$$\| G_\nu g \|_t \le C_{s,t} \| g \|_s .$$

Proof. In the first case, the Lemma is equivalent to saying that, if $s > 1/2$ and $t < -1/2$, the operator

$$\bar g = (\bar g_\ell)_{\ell\in\mathbb{Z}} \overset{\overline{G}_\nu}{\longmapsto} \bar f = \left(\sum_{\ell\in\mathbb{Z}} G_\nu(k,\ell)\frac{\| u_k \|_t}{\| u_\ell \|_s}\bar g_\ell\right)_{k\in\mathbb{Z}}$$

is a bounded operator with uniformly bounded norm from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$. We claim that, in fact, it is a Hilbert-Schmidt operator with Hilbert-Schmidt norm $\| \overline{G}_\nu \|_{HS}$ bounded uniformly with respect to $\mu \ge 1/4$.

If $\nu \ne 0$,

$$\| \overline{G}_\nu \|^2_{HS} := \sum_{k,\ell\in\mathbb{Z}} |G_\nu(k,\ell)|^2 \frac{\| u_k \|^2_t}{\| u_\ell \|^2_s} = \frac{4}{|\nu|^2}\sum_{k\in\mathbb{Z}}\sum_{\ell<k} |\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1|^2 \frac{(1+\mu+2k^2)^t}{(1+\mu+2\ell^2)^s}\,.$$

By the estimate (30) in Lemma 2.1 and by Lemma 4.2, we have

$$|\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1| \le C|\nu|\,[1 + \log(1+|k|) + \log(1+|\ell|)]\,. \tag{60}$$

Hence, if $s > 1/2$, by the integral inequality,

$$\sum_{\ell<k} |\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1|^2 \, (1+\mu+2\ell^2)^{-s} \ \le\ C_s \, |\nu|^2 \, \frac{\log^2(1+\mu)}{(1+\mu)^{s-1/2}} \, [1+\log^2(1+|k|)] \, ,$$

and, if $t < -1/2$,

$$\sum_{k\in\mathbb{Z}} (1+\mu+2k^2)^t \, [1+\log^2(1+|k|)] \ \le\ C_t \, (1+\mu)^{t+1/2} \, \log^2(1+\mu) \, .$$

It follows that

$$\| \overline{G}_\nu \|^2_{HS} \le 4\, C_s \, C_t \, \frac{\log^4(1+\mu)}{(1+\mu)^{s-t-1}} \le C^2_{s,t} \, . \tag{61}$$

The estimate (61) is uniform as $\nu \to 0$ ($\mu \to 1/4$). Hence the estimate in the case $\nu = 0$ follows immediately. The first case of the Lemma is therefore proved.

Let $g \in W^s(\mathcal{H}_\mu)$, $s > 1/2$, with $\mathcal{D}^+_\mu(g) = \mathcal{D}^-_\mu(g) = 0$ and let $f := G_\nu g$. If $\nu \neq 0$, we have

$$f_k = \frac{2i}{\nu} \sum_{\ell<k} (\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1) \, g_\ell = -\frac{2i}{\nu} \sum_{\ell\ge k} (\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1) \, g_\ell \ .$$

It follows that

$$\| f \|^2_t \le \frac{4}{|\nu|^2} \, \| g \|^2_s \sum_{k\in\mathbb{Z}} \sum_{|\ell|\ge|k|} |\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1|^2 \frac{(1+\mu+2k^2)^t}{(1+\mu+2\ell^2)^s} \ .$$

Since, if $t < s-1$,

$$I_{s,t} := \int \int_{x\ge y} \frac{(1+y^2)^t}{(1+x^2)^s} \, [1+\log^2(1+x^2)] \, dx \, dy < +\infty \ ,$$

by (60) and by the integral inequality, there exists $C' > 0$ such that

$$\| f \|^2_t \le C' \, I_{s,t} \, \frac{\log^2(1+\mu)}{(1+\mu)^{s-t-1}} \, \| g \|^2_s \le C_{s,t} \, \| g \|^2_s \ .$$

Hence the second case of the Lemma is proved for $\nu \neq 0$.

If $\nu = 0$, we have

$$f_k = 4i \sum_{\ell\in\mathbb{Z}, \ell<k} (d^-_\ell - d^-_k) \, g_\ell = -4i \sum_{\ell\in\mathbb{Z}, \ell\ge k} (d^-_\ell - d^-_k) \, g_\ell \ ,$$

where (see (42))

$$d^-_0 = 0, \quad d^-_k = \sum_{i=1}^{|k|} \frac{1}{2i-1} \text{ for } k \neq 0 \, .$$

It follows that, if $t < s-1$,

$$\| f \|^2_t \le C \, \| g \|^2_s \sum_{k\in\mathbb{Z}} \sum_{|\ell|\ge|k|} \frac{(1+k^2)^t}{(1+\ell^2)^s} \, [1+\log(1+\ell^2)] \le C_{s,t} \, \| g \|^2_s \ .$$

□

Lemma 4.4 (Complementary series)**.** *For all* $\nu \in]0,1[$ *(*$0 < \mu < 1/4$*),* $s > \frac{1+\nu}{2}$ *and* $g \in W^s(\mathcal{H}_\mu)$ *we have*

(a) *if* $t < -\frac{1+\nu}{2}$ *or*
(b) *if* $t < -\frac{1-\nu}{2}$ *and* $\mathcal{D}^-_\mu(g) = 0$*, or*
(c) *if* $t < s-1$ *and* $\mathcal{D}^+_\mu(g) = \mathcal{D}^-_\mu(g) = 0$*,*

then $G_\nu g \in W^t(\mathcal{H}_\mu)$*. Furthermore there exists a constant* $C_{s,t} > 0$ *such that the following estimate holds. Setting*

$$C_{s,t,\nu} := \begin{cases} C_{s,t} \max\{1, [2s-(1+\nu)]^{-1/2}[-2t+1+\nu]^{-1/2}\} \text{ in case (a)}, \\ \frac{C_{s,t}}{\nu} \max\{1, [2s-(1+\nu)]^{-1/2}[-2t+1-\nu]^{-1/2}\} \text{ in case (b)}, \\ C_{s,t} \max\{1, [2s-(1+\nu)]^{-1/2}\} \text{ in case (c)}; \end{cases}$$

we have

$$\| G_\nu g \|_t \le \frac{C_{s,t,\nu}}{\sqrt{1-\nu}} \| g \|_s ,$$

Proof. We claim that, if $s > \frac{1+\nu}{2}$ and $t < -\frac{1+\nu}{2}$, the operator

$$\bar g = (\bar g_\ell)_{\ell\in\mathbb{Z}} \overset{\overline{G}_\nu}{\longmapsto} \bar f = \left(\sum_{\ell\in\mathbb{Z}} G_\nu(k,\ell) \frac{\| u_k \|_t}{\| u_\ell \|_s} \bar g_\ell \right)_{k\in\mathbb{Z}}$$

is a Hilbert-Schmidt operator from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$ with Hilbert-Schmidt norm

$$\| \overline{G}_\nu \|_{HS} \le \frac{C_{s,t,\nu}}{\sqrt{1-\nu}} \ . \tag{62}$$

The claim immediately implies the Lemma in case (a).

We have

$$\| \overline{G}_\nu \|^2_{HS} := \sum_{k,\ell\in\mathbb{Z}} |G_\nu(k,\ell)|^2 \frac{\| u_k \|^2_t}{\| u_\ell \|^2_s} = \frac{4}{\nu^2} \sum_{k\in\mathbb{Z}} \sum_{\ell<k} |\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1|^2 \frac{\Pi_{\nu,|k|}}{\Pi_{\nu,|\ell|}} \frac{(1+\mu+2k^2)^t}{(1+\mu+2\ell^2)^s}.$$

Since $0 < \mu < 1/4$, for all $t \in \mathbb{R}$ there exists a constant $C_t > 0$ such that, for all $k \in \mathbb{Z}$,

$$C_t^{-1}(1+|k|)^{2t} \le (1+\mu+2k^2)^t \le C_t (1+|k|)^{2t} . \tag{63}$$

If $\nu \in]0,1[$, by Lemma 4.2, for all k, $\ell \in \mathbb{Z}$,

$$\frac{4}{\nu^2} |\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1|^2 \frac{\Pi_{\nu,|k|}}{\Pi_{\nu,|\ell|}} \le C \max\{\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}}, \frac{\Pi_{\nu,|k|}}{\Pi_{\nu,|\ell|}}\} \times \\ \times [1 + \log(1+|k|) + \log(1+|\ell|)] . \tag{64}$$

Hence, by the estimate (31) in Lemma 2.1,

$$\| \overline{G}_\nu \|^2_{HS} \le \frac{C_{s,t}}{1-\nu} \sum_{k\in\mathbb{Z}} \{ \sum_{|\ell|\ge|k|} \frac{(1+|k|)^{2t-\nu}}{(1+|\ell|)^{2s-\nu}} [1+\log(1+|\ell|)] + \\ + \sum_{|\ell|<|k|} \frac{(1+|k|)^{2t+\nu}}{(1+|\ell|)^{2s+\nu}} [1+\log(1+|k|)] \}.$$

The estimate (62) then follows by the integral inequality.

Let $g \in W^s(\mathcal{H}_\mu)$, $s > \frac{1+\nu}{2}$, with $\mathcal{D}^-_\mu(g) = 0$ and let $f := G_\nu g$. We have

$$f_k = \frac{2i}{\nu} \sum_{\ell<k} (\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1) g_\ell = -\frac{2i}{\nu} \left(\sum_{\ell \geq k} \frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} g_\ell + \sum_{\ell<k} g_\ell \right).$$

Hence

$$|f_k| \leq \frac{2}{\nu} \left(\sum_{|\ell| \geq |k|} \frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} |g_\ell| + \sum_{\ell<k} |g_\ell| \right).$$

It follows by Lemma 2.1 that there exists a constant $C_{s,t} > 0$ such that

$$\| f \|^2_t \leq \frac{C_{s,t}}{\nu^2(1-\nu)} \| g \|^2_s \left(\sum_{|\ell| \geq |k|} \frac{(1+|k|)^{2t+\nu}}{(1+|\ell|)^{2s+\nu}} + \sum_{\ell<k} \frac{(1+|k|)^{2t-\nu}}{(1+|\ell|)^{2s-\nu}} \right).$$

The integral inequality then implies that, if $t < s-1$,

$$\sum_{|\ell| \geq |k|} \frac{(1+|k|)^{2t+\nu}}{(1+|\ell|)^{2s+\nu}} < C_s[2s - (1+\nu)]^{-1}$$

and, if $s > \frac{1+\nu}{2}$ and $t < -\frac{1-\nu}{2}$,

$$\sum_{\ell<k} \frac{(1+|k|)^{2t-\nu}}{(1+|\ell|)^{2s-\nu}} < C_{s,t}[2s - (1+\nu)]^{-1}[-2t+1-\nu]^{-1}.$$

The Lemma is therefore proved in case (b).

Let $g \in W^s(\mathcal{H}_\mu)$, $s > \frac{1+\nu}{2}$, with $\mathcal{D}^+_\mu(g) = \mathcal{D}^-_\mu(g) = 0$ and let $f := G_\nu g$. We have

$$f_k = \frac{2i}{\nu} \sum_{\ell<k} (\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1) g_\ell = -\frac{2i}{\nu} \sum_{\ell \geq k} (\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1) g_\ell.$$

It follows by (63), (64) and Lemma 2.1 that

$$\| f \|^2_t \leq \frac{4}{\nu^2} \| g \|^2_s \sum_{k \in \mathbb{Z}} \sum_{|\ell| \geq |k|} |\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1|^2 \frac{\Pi_{\nu,|k|}}{\Pi_{\nu,|\ell|}} \frac{(1+\mu+2k^2)^t}{(1+\mu+2\ell^2)^s} \leq \tag{65}$$
$$\leq \frac{C_{s,t}}{1-\nu} \| g \|^2_s \sum_{k \in \mathbb{Z}} \sum_{|\ell| \geq |k|} \frac{(1+|k|)^{2t-\nu}}{(1+|\ell|)^{2s-\nu}} [1 + \log(1+|\ell|)].$$

The estimate in case (c) then follows by the integral inequality. □

Lemma 4.5 (Discrete series). *For all $n \in \mathbb{Z}^+$ ($\nu = 2n-1$, $\mu = -n^2+n$),*

(a) *if $s > 1-n$, $t < \min(s-1, n-1)$ and $g \in W^s(\mathcal{H}_\mu)$ or*
(b) *if $s > n$, $t < s-1$, and $g \in W^s(\mathcal{H}_\mu)$ satisfy $\mathcal{D}^+_\mu(g) = 0$ $g \in W^s(\mathcal{H}_\mu)$,*

then $G_\nu g \in W^t(\mathcal{H}_\mu)$ and there exists a constant $C_{s,t} > 0$ such that the following estimates hold. Let

$$C_{s,t,\nu} := \begin{cases} C_{s,t} \max\{1, (s-n)^{-1/2}(n-1-t)^{-1/2}\} & \textit{in case (a), if } s > n; \\ C_{s,t} \max\{1, (n-1-t)^{-1/2}\} & \textit{in case (a), if } s = n; \\ C_{s,t} \max\{1, (s+n-1)^{-1/2}, (n-s)^{-1/2}\} & \textit{in case (a), if } s < n; \\ C_{s,t} \max\{1, (s-n)^{-1/2}\} & \textit{in case (b)}; \end{cases}$$

then

$$\| G_\nu g \|_t \le \frac{C_{s,t,\nu}}{\nu^{s-t}} \| g \|_s .$$

Proof. We claim that, if $s > 1-n$ and $t < \min(s-1, n-1)$, the operator

$$\bar g = (\bar g_\ell)_{\ell \in I_\nu} \overset{\overline{G}_\nu}{\longmapsto} \bar f = \left(\sum_{\ell \ge n} G_\nu(k,\ell) \frac{\| u_k \|_t}{\| u_\ell \|_s} \bar g_\ell \right)_{k \in I_\nu}$$

is a Hilbert-Schmidt operator from $\ell^2(I_\nu)$ to $\ell^2(I_\nu)$ with Hilbert-Schmidt norm

$$\| \overline{G}_\nu \|_{HS} \le \frac{C_{s,t,\nu}}{\nu^{s-t}} \,. \tag{66}$$

We have

$$\| \overline{G}_\nu \|^2_{HS} = \frac{4}{\nu^2} \sum_{k \ge n} \{ \sum_{n \le \ell \le k} \frac{\Pi_{\nu,k}}{\Pi_{\nu,\ell}} \frac{(1+\mu+2k^2)^t}{(1+\mu+2\ell^2)^s} + \sum_{\ell > k} \frac{\Pi_{\nu,\ell}}{\Pi_{\nu,k}} \frac{(1+\mu+2k^2)^t}{(1+\mu+2\ell^2)^s} \} \,.$$

Since $\nu = 2n-1 \ge 1$ and $\mu = -n^2+n$, there exists a constant $C_1 > 0$ such that, for all $n \in \mathbb{Z}^+$ and all $k \ge n$,

$$C_1^{-1}[\nu + (k-n)]^2 \le 1 + \mu + 2k^2 \le C_1 [\nu + (k-n)]^2 \tag{67}$$

and, by the upper bound (32) in Lemma 2.1, for all $k \ge \ell \ge n$,

$$\frac{\Pi_{\nu,k}}{\Pi_{\nu,\ell}} \le C_1 \left(\frac{k-n+\nu}{\ell-n+\nu} \right)^{-\nu} . \tag{68}$$

Hence, there exists a constant $C_{s,t} > 0$ such that

$$\| \overline{G}_\nu \|^2_{HS} \le C_{s,t} \nu^{2t-2s-2} \sum_{k \ge 0} \{ \sum_{0 \le \ell \le k} \frac{[1+(k/\nu)]^{2t-\nu}}{[1+(\ell/\nu)]^{2s-\nu}} + \sum_{\ell > k} \frac{[1+(k/\nu)]^{2t+\nu}}{[1+(\ell/\nu)]^{2s+\nu}} \}.$$

By the integral inequality, there exists a constants $C_2 > 0$, $C_3 > 0$ such that

$$\sum_{0 \le \ell \le k} \frac{1}{\nu} [1 + (\ell/\nu)]^{-2s+\nu} \le \begin{cases} C_2 \max\{1, (s-n)^{-1}\} & \text{if } s > n; \\ C_2 \log(1 + k/\nu) & \text{if } s = n; \\ C_2 \max\{1, (n-s)^{-1}\} (1 + k/\nu)^{-2s+\nu+1} & \text{if } s < n; \end{cases}$$

hence, if $t < \min(s-1, n-1)$,

$$\sum_{k \ge 0} \sum_{0 \le \ell \le k} \frac{1}{\nu^2} \frac{[1+(k/\nu)]^{2t-\nu}}{[1+(\ell/\nu)]^{2s-\nu}} \le \begin{cases} C_3 \max\{1, (s-n)^{-1}(n-1-t)^{-1}\} & \text{if } s > n; \\ C_3 \max\{1, (n-t-1)^{-1}\} & \text{if } s = n; \\ C_3 \max\{1, (n-s)^{-1}(s-1-t)^{-1}\} & \text{if } s < n; \end{cases}$$

and, if $s > 1-n$,

$$\sum_{\ell>k} \frac{1}{\nu}[1+(\ell/\nu)]^{-2s-\nu} \le C_2(s+n-1)^{-1}[1+(k/\nu)]^{-2s-\nu+1};$$

hence, if $t < s-1$,

$$\sum_{k\ge 0}\sum_{\ell>k} \frac{1}{\nu^2}\frac{[1+(k/\nu)]^{2t+\nu}}{[1+(\ell/\nu)]^{2s+\nu}} \le C_3(s+n-1)^{-1}(s-1-t)^{-1}. \tag{69}$$

It follows that there exists a constant $C_{s,t} > 0$ such that (66) holds and such an upper bound for the Hilbert-Schmidt norm of the operator $\overline{G}_\nu$ implies the Lemma in case (a).

Let $g \in W^s(\mathcal{H}_\mu)$, $s > n$, with $\mathcal{D}^+_\mu(g) = 0$ and let $f := G_\nu g$. For all $k \ge n$, let $f^{(\alpha)}$, $f^{(\beta)}$ be defined by

$$f^{(\alpha)}_k := -\frac{2i}{\nu}\sum_{n\le\ell\le k} g_\ell\,;$$

$$f^{(\beta)}_k := \frac{2i}{\nu}\sum_{\ell>k}^{\infty} g_\ell \frac{\Pi_{\nu,\ell}}{\Pi_{\nu,k}}$$

Since $f = f^{(\alpha)} + f^{(\beta)}$, estimates on the Sobolev norms of $f^{(\alpha)}$, $f^{(\beta)}$ imply corresponding estimates for f by the triangular inequality.

We have that

$$\| f^{(\beta)} \|_t^2 \le \frac{4}{\nu^2}\| g \|_s^2 \sum_{k\ge n}\sum_{\ell>k} \frac{\Pi_{\nu,\ell}}{\Pi_{\nu,k}}\frac{(1+\mu+2k^2)^t}{(1+\mu+2\ell^2)^s},$$

hence, if $s > n$ and $t < s-1$, by the estimates (67), 68 and (69), there exists a constant $C_{s,t} > 0$ such that

$$\| f^{(\beta)} \|_t^2 \le \frac{C^2_{s,t,\nu}}{\nu^{2s-2t}}\| g \|_s^2 .$$

Since $\mathcal{D}^+_\mu(g) = 0$,

$$f^{(\alpha)}_k = -\frac{2i}{\nu}\sum_{n\le\ell\le k} g_\ell = \frac{2i}{\nu}\sum_{\ell>k} g_\ell.$$

It follows that

$$|f^{(\alpha)}_k|^2 \le \frac{4}{\nu^2}\| g \|_s^2 \sum_{n\le\ell\le k} \Pi^{-1}_{\nu,\ell}(1+\mu+2\ell^2)^{-s};$$

$$|f^{(\alpha)}_k|^2 \le \frac{4}{\nu^2}\| g \|_s^2 \sum_{\ell>k} \Pi^{-1}_{\nu,\ell}(1+\mu+2\ell^2)^{-s}.$$

Let

$$\Sigma^{(1)}_{s,t,\nu} := \frac{4}{\nu^2}\sum_{k=n}^{n+\nu}\sum_{\ell=n}^{k} \frac{\Pi_{\nu,k}}{\Pi_{\nu,\ell}}\frac{(1+\mu+2k^2)^t}{(1+\mu+2\ell^2)^s};$$

$$\Sigma^{(2)}_{s,t,\nu} := \frac{4}{\nu^2}\sum_{k=n+\nu+1}^{\infty}\sum_{\ell=k+1}^{\infty} \frac{\Pi_{\nu,k}}{\Pi_{\nu,\ell}}\frac{(1+\mu+2k^2)^t}{(1+\mu+2\ell^2)^s}.$$

We then have

$$\| f^{(\alpha)} \|_t^2 \le \left(\Sigma^{(1)}_{s,t,\nu} + \Sigma^{(2)}_{s,t,\nu} \right) \| g \|_s^2 .$$

The function $\Sigma^{(1)}_{s,t,\nu}$ can be estimated as above. In fact, there exists a constant $C_{s,t} > 0$ such that

$$\Sigma^{(1)}_{s,t,\nu} \le C_{s,t}\, \nu^{2t-2s-2} \sum_{k=0}^{\nu} \sum_{\ell=0}^{k} \frac{[1+(k/\nu)]^{2t-\nu}}{[1+(\ell/\nu)]^{2s-\nu}} .$$

Since, by the integral inequality ,if $s > n$,

$$\sum_{\ell=0}^{k} \frac{1}{\nu} [1+(\ell/\nu)]^{-2s+\nu} \le C_1 \max\{1, (s-n)^{-1}\};$$

and

$$\sum_{k=0}^{\nu} \frac{1}{\nu} [1+(k/\nu)]^{2t-\nu} \le 2^t \,,$$

there exists a constant $C_{s,t} > 0$ such that

$$\Sigma^{(1)}_{s,t,\nu} \le C^2_{s,t} \max\{1, (s-n)^{-1}\}\, \nu^{2t-2s} \,.$$

The estimate of the function $\Sigma^{(2)}_{s,t,\nu}$, can be carried out as follows. By the lower bound (32) in Lemma 2.1, there exists a constant $C_4 > 0$ such that, if $\ell > k$,

$$\frac{\Pi_{\nu,k}}{\Pi_{\nu,\ell}} \le C_4 \left(\frac{\ell - n + 1}{k - n + 1} \right)^{\nu} .$$

It follows that there exists a constant $C_{s,t} > 0$ such that

$$\Sigma^{(2)}_{s,t,\nu} \le C_{s,t}\, \nu^{2t-2s-2} \sum_{k=\nu+1}^{\infty} \sum_{\ell=k+1}^{\infty} \frac{[1+(k/\nu)]^{2t}}{[1+(\ell/\nu)]^{2s}} \left(\frac{1+\ell}{1+k} \right)^{\nu} .$$

Since, for $k \ge \nu$, we have $[1+(k/\nu)]^{2t} \le 2^{2t}(k/\nu)^{2t}$ and since

$$\frac{1+\ell}{1+k} \le \frac{1+\ell/\nu}{k/\nu} \,,$$

it follows that

$$\Sigma^{(2)}_{s,t,\nu} \le C'_{s,t}\, \nu^{2t-2s-2} \sum_{k=\nu+1}^{\infty} \sum_{\ell=k+1}^{\infty} \left(\frac{k}{\nu} \right)^{2t-\nu} [1+(\ell/\nu)]^{-2s+\nu} .$$

By the integral inequality, if $s > n$ (hence $-2s + \nu + 1 < 0$) ,

$$\sum_{\ell=k+1}^{\infty} \frac{1}{\nu} [1+(\ell/\nu)]^{-2s+\nu} \le (s-n)^{-1} \left(\frac{k}{\nu} \right)^{-2s+\nu+1} ,$$

and, if $t < s - 1$,

$$\sum_{k=\nu+1}^{\infty} \frac{1}{\nu} \left(\frac{k}{\nu} \right)^{2t-2s+1} \le (s-1-t)^{-1}.$$

Hence, there exists a constant $C_{s,t} > 0$ such that

$$\Sigma^{(2)}_{s,t,\nu} \le C^2_{s,t} (s-n)^{-1} \nu^{2t-2s}.$$

It follows that

$$\| f^{(\alpha)} \|_t^2 \leq C_{s,t}^2 \max\{1, (s-n)^{-1}\}\, \nu^{2t-2s}.$$

□

Proof of Theorem 4.1. Let $\mathcal{H}$ be a direct integral of non-trivial unitary representations $\mathcal{H}_\lambda$ of $PSL(2,\mathbb{R})$ with respect to a Stieltjes measure ds. Every vector $g \in W^s(\mathcal{H})$ has a decomposition

$$g = \int g_\lambda \, ds(\lambda)$$

with $g_\lambda \in W^s(\mathcal{H}_\lambda)$. We claim that:

(a) $\mathcal{D}(g) = 0$ for all $\mathcal{D} \in \mathcal{I}^s(\mathcal{H})$ iff, for ds-almost all $\lambda \in \mathbb{R}$, $\mathcal{D}_\lambda(g) = 0$ for all $\mathcal{D}_\lambda \in \mathcal{I}^s(\mathcal{H}_\lambda)$;

(b) if there exists a constant $C(s,t) > 0$ such that, for ds-almost all $\lambda \in \mathbb{R}$, the equation $Uf_\lambda = g_\lambda$ has a solution $f_\lambda \in W^t(\mathcal{H}_\lambda)$ satisfying a *uniform* Sobolev estimate

$$\| f_\lambda \|_t \leq C(s,t) \, \| g_\lambda \|_s \,, \tag{70}$$

then the equation $Uf = g$ has a solution $f \in W^t(\mathcal{H})$ with the same Sobolev bound:

$$\| f \|_t \leq C(s,t) \, \| g \|_s \,.$$

The claim (a) follows immediately from Lemma 3.1.

The claim (b) can be proved as follows. Let

$$f := \int f_\lambda \, ds(\lambda)\,.$$

Since the operator U is decomposable and

$$\| f \|_t^2 = \int \| f_\lambda \|_t^2 \, ds(\lambda) \leq$$

$$\leq C(s,t)^2 \int \| g_\lambda \|_s^2 \, ds(\lambda) = C(s,t)^2 \, \| g \|_s^2 < +\infty\,,$$

the vector $f \in W^t(SM)$ yields a well-defined solution of the equation $Uf = g$ which satisfies the required Sobolev bound. The claim is therefore completely proved.

The existence part of the Theorem then follows from Lemmata 4.3, 4.4, 4.5 and the direct integral decompositions (19), (20). In fact, if the Casimir operator has a ‘spectral gap’, the uniform Sobolev estimates (70) holds for all irreducible unitary sub-representations.

The uniqueness part is proved as follows. A solution $f \in W^t(\mathcal{H})$ of the equation $Uf = g$ is *not* unique modulo the trivial sub-representation iff there exists a non-trivial invariant distribution $\mathcal{D} \in W^t(\mathcal{H})$. By Lemma 3.1 and Theorem 3.2, that is the case iff $t \geq -\frac{1-\nu_0}{2}$.
□

4.3. **Converse results.** Let $\mathcal{H}$ be the Hilbert space of a unitary representation of the group $PSL(2,\mathbb{R})$. It is immediate by the definition of an U-invariant distribution $\mathcal{D} \in \mathcal{I}^s(\mathcal{H})$ that, if $g \in \mathcal{H}$ is a vector such that the equation $Uf = g$ has a solution $f \in W^t(\mathcal{H})$ with $t \geq s+1$, then $\mathcal{D}(g) = 0$. We prove that $\mathcal{D}(g) = 0$ under weaker regularity assumptions on the solution f. In fact, it turns out that, if g is not in the kernel of U-invariant distributions, then Theorem 4.1 gives the optimal Sobolev regularity for the solution. Such converse result will also be a crucial tool in the proof of lower bounds on the deviations of ergodic averages of the horocycle flow.

Theorem 4.6. *Let $g \in W^s(\mathcal{H})$, $s > \frac{1+\nu_0}{2}$, and let $\mathcal{D} \in \mathcal{E}'(\mathcal{H})$ be a U-invariant distribution of Sobolev order $S_\mathcal{D} < s$. If the equation $Uf = g$ has a solution $f \in W^t(\mathcal{H})$ with $t \geq S_\mathcal{D} - 1$, then $\mathcal{D}(g) = 0$.*

Theorem 4.6 implies Theorem 1.3, for all finite area hyperbolic surfaces.

The proof of Theorem 4.6 can be reduced, by direct decomposition into irreducible subrepresentations, to the following Lemmata.

Lemma 4.7 (Principal series)**.** *Let $\nu \in i\mathbb{R}$ ($\mu \geq 1/4$). Let $g \in W^s(\mathcal{H}_\mu)$, $s > 1/2$, be any vector such that the cohomological equation $Uf = g$ has a solution $f \in W^t(\mathcal{H}_\mu)$. If $t \geq -1/2$, then $\mathcal{D}^+_\mu(g) = \mathcal{D}^-_\mu(g) = 0$.*

Lemma 4.8 (Complementary series)**.** *Let $\nu \in]0,1[$ ($0 < \mu < 1/4$). Let $g \in W^s(\mathcal{H}_\mu)$, $s > \frac{1+\nu}{2}$, be any vector such that the cohomological equation $Uf = g$ has a solution $f \in W^t(\mathcal{H}_\mu)$.*

(1) *If $t \geq -\frac{1+\nu}{2}$, then $\mathcal{D}^-_\mu(g) = 0$;*
(2) *if $t \geq -\frac{1-\nu}{2}$, then $\mathcal{D}^+_\mu(g) = \mathcal{D}^-_\mu(g) = 0$.*

Lemma 4.9 (Discrete Series)**.** *Let $\nu = 2n-1$ ($\mu = -n^2+n$), $n \in \mathbb{Z}^+$. Let $g \in W^s(\mathcal{H}_\mu)$, $s > n$, be any vector such that the cohomological equation $Uf = g$ has a solution $f \in W^t(\mathcal{H}_\mu)$. If $t \geq n-1$, then $\mathcal{D}^+_\mu(g) = 0$.*

We remark that the proof of Lemmata 4.7, 4.8 and 4.9 can be reduced to the case that the Fourier series $\hat{g}$ of $g \in \mathcal{H}_\mu$ with respect to the basis $\{u_k\}$ has *finite support*. In fact, for any $g \in W^s(\mathcal{H}_\mu)$, $s > \frac{1+\Re(\nu)}{2}$, there exists a vector $h \in \mathcal{H}_\mu$, with Fourier series $\hat{h}$ of finite support, such that $\mathcal{D}(h) = \mathcal{D}(g)$, for all $\mathcal{D} \in \mathcal{I}^s(\mathcal{H}_\mu)$. By Theorem 3.2, the sequence $\hat{h}$ can be determined as a solution of a linear system of two equations, in the case of the principal or the complementary series, or of a single equation, in the case of the discrete series. By Theorem 4.1, the equation $Uf = h - g$ has a solution which belongs to $W^t(\mathcal{H}_\mu)$ for all $t < s-1$. Hence, for any $t < s-1$, the equation $Uf = h$ has a solution which belongs to $W^t(\mathcal{H}_\mu)$ iff the equation $Uf = g$ does. It is therefore sufficient to consider the case of a vector $g \in \mathcal{H}_\mu$ with Fourier series of finite support.

Proof of Lemma 4.7. Let $\nu \neq 0$. If f is a distributional solution of the equation $Uf = g$, the sequence $\hat{f}$ of its Fourier coefficients satisfies the linear second order difference equation (52). The general solution can be written as

$$f_k = \frac{2i}{\nu} \sum_{\ell < k} (\frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}} - 1) g_\ell + c_1 f^{(1)}_k + c_2 f^{(2)}_k \,, \tag{71}$$

where $f^{(1)}_k = \Pi^{-1}_{\nu,|k|}$, $f^{(2)}_k = 1$, for all $k \in \mathbb{Z}$, and $c_1, c_2 \in \mathbb{C}$. If $\hat{g}$ has finite support, there exists $k^- \in \mathbb{Z}^-$ such that, for all $k < k^- < 0$,

$$f_k = c_1 f^{(1)}_k + c_2 f^{(2)}_k \,.$$

It follows that, if $f \in W^t(\mathcal{H}_\mu)$, $t \geq -1/2$, then $c_1 = c_2 = 0$. In fact, if $|c_1| \neq |c_2|$, since $|\Pi_{\nu,|k|}| = 1$, for all $k < k^-$, $|f_k| \geq ||c_1| - |c_2|| > 0$. If $|c_1| = |c_2|$, let $c_1 = e^{i\theta}c_2$ and $\Pi_{\nu,|k|} = e^{i\theta_k}$. Then $|f_k| \geq |c_2||e^{i(\theta-\theta_k)} + 1|$. By the definition of $\Pi_{\nu,|k|}$, if $\nu = is \neq 0$, we have

$$\theta_k = -2\sum_{i=1}^{|k|} \tan^{-1}(\frac{s}{2i-1})\,.$$

Let $I \subset S^1$ be a closed subinterval such that $\theta - \pi \notin I$. The set $\{k < k^- | \theta_k \in I\}$ contains infinitely many intervals $[k_j^-, k_j^+] \subset \mathbb{Z}^-$ such that there exists a constant $c > 1$ with $|k_j^-/k_j^+| \geq c$ for all $j \in \mathbb{N}$. Since I is closed, there exists a constant $d > 0$ such that $|\theta - \pi - \theta'| \geq d$ for all $\theta' \in I$. Hence, if $k \in [k_j^-, k_j^+]$, since $\theta_k \in I$, $|e^{i(\theta-\theta_k)} + 1| \geq d > 0$, for all $j \in \mathbb{N}$. In both cases it follows that, if $c_1 \neq 0$ or $c_2 \neq 0$, then

$$\sum_{k<k^-} \frac{|f_k|^2}{|k|} = +\infty\,,$$

hence $f \notin W^t(\mathcal{H}_\mu)$ for any $t \geq -1/2$.

If $c_1 = c_2 = 0$ and $\hat{g}$ has finite support, there exists $k^+ \in \mathbb{Z}^+$ such that, for all $k > k^+ > 0$,

$$f_k = \frac{2i}{\nu}\left(\mathcal{D}^-_\mu(g)\Pi^{-1}_{\nu,|k|} + \mathcal{D}^+_\mu(g)\right)\,.$$

The above argument implies that, if $f \in W^t(\mathcal{H}_\mu)$ with $t \geq -1/2$, then $\mathcal{D}^+_\mu(g) = \mathcal{D}^-_\mu(g) = 0$.

Let $\nu = 0$. The general solution of the difference equation (52) is

$$f_k = 4i\sum_{k<\ell}(d_\ell^- - d_k^-)g_\ell + c_1 f_k^{(1)} + c_2 f_k^{(2)}\,,$$

where $f_k^{(1)} = 1$, for all $k \in \mathbb{Z}$, c_1, $c_2 \in \mathbb{C}$ and

$$d_k^- = f_k^{(2)} = \sum_{i=1}^{|k|}\frac{1}{2i-1}\,.$$

If $\hat{g}$ has finite support, there exists $k^- \in \mathbb{Z}^-$ such that, for all $k < k^- < 0$,

$$f_k = c_1 f_k^{(1)} + c_2 f_k^{(2)}\,.$$

Since $f_k^{(2)}$ is unbounded, it is immediate that, if $f \in W^t(\mathcal{H}_\mu)$ with $t \geq -1/2$, then $c_1 = c_2 = 0$. Hence there exists $k^+ \in \mathbb{Z}^+$ such that, for all $k > k^+$,

$$f_k = 4i\left(\mathcal{D}^-_\mu(g) - \mathcal{D}^+_\mu(g)d_k^-\right)$$

and again, since $d_k^- = f_k^{(2)}$ is unbounded, if $f \in W^t(\mathcal{H}_\mu)$ with $t \geq -1/2$, then $\mathcal{D}^+_\mu(g) = \mathcal{D}^-_\mu(g) = 0$. □

Proof of Lemma 4.8. The sequence $\hat{f}$ of the Fourier coefficients can be written as in (71). If $\hat{g}$ has finite support, there exists $k^- \in \mathbb{Z}^-$ such that, for all $k < k^- < 0$,

$$f_k = c_1 f_k^{(1)} + c_2 f_k^{(2)} = c_1\Pi^{-1}_{\nu,|k|} + c_2.$$

By the estimates (31) in Lemma 2.1 and by (36), it follows that (1) if $t \geq -\frac{1+\nu}{2}$, then $c_1 = 0$; (2) if $t \geq -\frac{1-\nu}{2}$, then $c_1 = c_2 = 0$.

If $\hat{g}$ has finite support, there exists $k^+ \in \mathbb{Z}^+$ such that, for all $k > k^+ > 0$,

$$f_k = \frac{2i}{\nu}\left(\mathcal{D}^-_\mu(g)\Pi^{-1}_{\nu,|k|} + \mathcal{D}^+_\mu(g)\right) + c_1 f^{(1)}_k + c_2 f^{(2)}_k .$$

Again by Lemma 2.1, it follows that (1) if $t \geq -\frac{1+\nu}{2}$, since $c_1 = 0$, then $\mathcal{D}^-_\mu(g) = 0$; (2) if $t \geq -\frac{1-\nu}{2}$, since $c_1 = c_2 = 0$, then $\mathcal{D}^+_\mu(g) = \mathcal{D}^-_\mu(g) = 0$. □

Proof of Lemma 4.9. The sequence $\hat{f}$ of the Fourier coefficients of the solution f satisfies the linear difference equation (52) on the set $I_\nu = [n, +\infty] \subset \mathbb{Z}$. Since the space of solutions of the corresponding homogeneous equation is one-dimensional, the general solution can be written as

$$f_k = -\frac{2i}{\nu}\left(\sum_{\ell=n}^{k} g_\ell + \sum_{\ell=k+1}^{\infty} g_\ell \frac{\Pi_{\nu,|\ell|}}{\Pi_{\nu,|k|}}\right) + c_1 f^{(1)}_k ,$$

where $c_1 \in \mathbb{C}$ and $f^{(1)}_k = \Pi^{-1}_{\nu,k}$, for all $k \geq n$.

If $\hat{g}$ has finite support, there exists $k^+ > n$ such that, for all $k > k^+$,

$$f_k = -\frac{2i}{\nu}\mathcal{D}^+_\mu(g) + c_1 f^{(1)}_k .$$

Let $f \in W^t(\mathcal{H}_\mu)$. By the estimates (32) in Lemma 2.1, it follows that, if $t \geq -n$, then $c_1 = 0$ and, if $t \geq n-1$, then $c_1 = 0$ and $\mathcal{D}^+_\mu(g) = 0$. □

5. DEVIATION OF ERGODIC AVERAGES

5.1. **Spectral decomposition of horocycle orbits.** Since $\mathcal{I}^s(SM) \subset W^{-s}(SM)$ is closed, there is an orthogonal splitting

$$W^{-s}(SM) = \mathcal{I}^s(SM) \oplus^\perp \mathcal{I}^s(SM)^\perp . \tag{72}$$

Although the space $\mathcal{I}^s(SM)$ is $\{\phi^X_t\}$-invariant, the action of the geodesic one-parameter group $\{\phi^X_t\}$ on $W^s(SM)$ is *not* unitary and the orthogonal splitting (72) is *not* $\{\phi^X_t\}$-invariant.

According to Theorems 1.1 and 1.4, the one-parameter group $\{\phi^X_t\}$ has a (generalized) spectral representation on the space $\mathcal{I}^s(SM)$. In fact, for all $s > 0$, there is a $\{\phi^X_t\}$-invariant orthogonal splitting

$$\mathcal{I}^s(SM) = \mathcal{I}^s_d \oplus^\perp \mathcal{I}^s_{\mathcal{C}} \tag{73}$$

and the spectrum of ϕ^X_t is discrete on the subspace $\mathcal{I}^s_d := \mathcal{I}_d \cap \mathcal{I}^s(SM)$ and Lebesgue of finite multiplicity with spectral radius equal to $e^{-t/2}$ on $\mathcal{I}^s_{\mathcal{C}}$, for all $t \in \mathbb{R}$.

Let $\mathcal{B} \subset \mathcal{I}_d$ be a basis of (generalized) eigenvectors for $\{\phi^X_t\} \,|\, \mathcal{I}_d$ such that $\mathcal{B} \cap \big(\mathcal{I}_d \ominus \mathcal{I}_{1/4}\big)$ is a basis of eigenvectors for $\{\phi^X_t\} \,|\, \big(\mathcal{I}_d \ominus \mathcal{I}_{1/4}\big)$ and, if $1/4 \in \sigma_{pp}$, the set $\mathcal{B}_{1/4} := \mathcal{B} \cap \mathcal{I}_{1/4}$ is a basis which brings $\{\phi^X_t\} \,|\, \mathcal{I}_{1/4}$ into its Jordan normal form. For any $\mathcal{D} \in \mathcal{B} \setminus \mathcal{B}_{1/4}$ of Sobolev order $S_{\mathcal{D}} > 0$, there exists $\lambda_{\mathcal{D}} \in \mathbb{C}$ with $\Re(\lambda_{\mathcal{D}}) = -S_{\mathcal{D}} < 0$ such that, for all $t \in \mathbb{R}$,

$$\phi^X_t(\mathcal{D}) = e^{\lambda_{\mathcal{D}} t}\, \mathcal{D} ; \tag{74}$$

if $1/4 \in \sigma_{pp}$, the subset $\mathcal{B}_{1/4} \subset \mathcal{B}$ is the union of a finite number of pairs $\{\mathcal{D}^+, \mathcal{D}^-\}$ such that the distributions $\mathcal{D}^\pm \in \mathcal{B}^\pm_{1/4} = \mathcal{B} \cap \mathcal{I}^\pm_{1/4}$ have the same Sobolev order equal to $1/2$ and formula (7) holds:

$$\phi^X_t \begin{pmatrix} \mathcal{D}^+ \\ \mathcal{D}^- \end{pmatrix} = e^{-t/2} \begin{pmatrix} 1 & 0 \\ -\frac{t}{2} & 1 \end{pmatrix} \begin{pmatrix} \mathcal{D}^+ \\ \mathcal{D}^- \end{pmatrix} . \tag{75}$$

The set $\mathcal{B}^s := \mathcal{B} \cap \mathcal{I}^s_d$ is a basis of (generalized) eigenvectors for the action of $\{\phi^X_t\}$ on $\mathcal{I}^s_d$. By Theorem 1.1 and (9), for all $s > 1$, there is a decomposition

$$\mathcal{B}^s = \bigcup_{\mu \in \sigma_{pp}} \mathcal{B}_\mu \cup \bigcup_{1 \le n < s} \mathcal{B}_n . \tag{76}$$

The operator $\phi^X_t \,|\, \mathcal{I}^s_{\mathcal{C}}$ has Lebesgue spectrum of finite multiplicity supported on the circle of radius $e^{-t/2}$ in the complex plane, for all $t \in \mathbb{R}$. Its norm satisfies the following bound.

Lemma 5.1. *There exists a constant $C_1 := C_1(s) > 0$ such that, for all $t \in \mathbb{R}$,*

$$\| \, \phi^X_t \, \big| \, \mathcal{I}^s_{\mathcal{C}} \, \|_{-s} \le C_1 \, (1 + |t|) \, e^{-t/2} . \tag{77}$$

Proof. By §§3.1 and 3.3, if $\mathcal{H}_\mu$ is the Hilbert space of an irreducible representation of the principal series, then $\mathcal{I}^s(\mathcal{H}_\mu)$ has dimension 2 and it is generated by two complex conjugate invariant distributions $\mathcal{D}^\pm_\mu$. By Lemma 3.5, the distribution $\mathcal{D}^\pm_\mu$ is an eigenvector of the operator ϕ^X_t with eigenvalue the complex number $e^{-(1\pm\nu)t/2}$, where ν is the purely imaginary parameter defined by (23). The distortion of the basis $\{\mathcal{D}^+_\mu, \mathcal{D}^-_\mu\}$ is uniformly bounded as the parameter μ varies over subsets of the real line bounded away from $1/4$ (which corresponds to the imaginary parameter ν being bounded away from 0). Let $d^\pm_\mu$ be the sequences of the Fourier coefficients of the distributions $\mathcal{D}^\pm_\mu$, with respect to the basis (24) of $\mathcal{H}_\mu$. By the definition of $d^\pm_\mu$ in §3.1, an explicit calculation of the derivative of the coefficients d^-_μ of the distribution $\mathcal{D}^-_\mu$ yields

$$\lim_{\nu \to 0} \frac{d^+_\mu - d^-_\mu}{2\nu} = -\frac{1}{2} \frac{d}{d\nu} d^-_\nu \Big|_{\nu=0} = d^+_0 .$$

Since the Fourier basis (24) depends continuously on the parameter $\nu \in i\,\mathbb{R} \cup]-1,1[$, the basis $\{\Re \mathcal{D}^+_\mu, \Im \mathcal{D}^-_\mu / 2\nu\}$ of $\mathcal{I}^s(\mathcal{H}_\mu)$ has bounded distorsion, as $\mu \to 1/4$ ($\nu \to 0$). The 2×2 matrix of the operator ϕ^X_t with respect to this basis can be explicitly computed by Lemma 3.5. It follows that there exists a constant $C_1 > 0$ such that, for all $\mu \ge 1/4$, the norm of the operator ϕ^X_t on $\mathcal{I}^s(\mathcal{H}_\mu)$ is bounded by $C_1(1 + |t|)e^{-t/2}$. Since the space $\mathcal{I}^s_{\mathcal{C}}$ decomposes as a direct integral of the spaces $\mathcal{I}^s(\mathcal{H}_\mu)$ over the interval $[1/4, +\infty[$ and the operator ϕ^X_t is also decomposable, the desired estimates follows.

□

According to (72) and (73), every $\gamma \in W^{-s}(SM)$ can be written as

$$\gamma = \sum_{\mathcal{D} \in \mathcal{B}^s} c_{\mathcal{D}}(\gamma) \mathcal{D} + \mathcal{C}(\gamma) + \mathcal{R}(\gamma) \tag{78}$$

with $\mathcal{C}(\gamma) \in \mathcal{I}^s_{\mathcal{C}}$ and $\mathcal{R}(\gamma) \in \mathcal{I}^s(SM)^\perp$. The real number $c_{\mathcal{D}}(\gamma)$ will be called the $\mathcal{D}$-*component* of γ along $\mathcal{D} \in \mathcal{B}^s$ and the distribution $\mathcal{C}(\gamma)$ the (U-invariant) *continuous component* of γ. We recall that the continuous component vanishes for all $\gamma \in W^{-s}(SM)$ if

M is compact. The following Lemma tells us that bounds on the norms of distributions in $W^{-s}(SM)$ are equivalents to bounds on their coefficients.

Lemma 5.2. *There exists a constant $C_2 := C_2(s) > 0$ such that*

$$C_2^{-2} \| \gamma \|_{-s}^2 \le \sum_{\mathcal{D} \in \mathcal{B}^s} |c_{\mathcal{D}}(\gamma)|^2 + \| \mathcal{C}(\gamma) \|_{-s}^2 + \| \mathcal{R}(\gamma) \|_{-s}^2 \le C_2^2 \| \gamma \|_{-s}^2 . \tag{79}$$

Proof. The splittings (72) and (73) are orthogonal with respect to the Hilbert structure of $W^{-s}(SM)$. The basis $\mathcal{B}^s$ is not orthogonal, however we claim that its distortion is uniformly bounded. In fact, vectors of the basis supported on different irreducible representations are orthogonal; if $\mathcal{D}^+_\mu$, $\mathcal{D}^-_\mu \in \mathcal{B}^s$ are normalised eigenvectors supported on the same irreducible representation of Casimir parameter $\mu \in \mathbb{R}^+$ (principal or complementary series), a calculation shows that the function $\langle \mathcal{D}^+_\mu, \mathcal{D}^-_\mu \rangle_{-s}$ is continuous on the interval $]1/4, +\infty[$, it converges to 0 as $\mu \to +\infty$ and to 1 as $\mu \to 1/4$. Since $\mathcal{I}^s_d$ is contained in the pure point component of the the spectral representation of the Casimir operator, the angle between $\mathcal{D}^+_\mu$ and $\mathcal{D}^-_\mu$ has a strictly positive uniform lower bound as $\mu \in \sigma_{pp}$. □

5.1.1. *Horocycle orbits.* For $x \in SM$ and $T \in \mathbb{R}^+$, let $\gamma_{x,T}$ be the probability measure uniformly distributed on the horocycle orbit of length T starting at x. More precisely, for any continuous function f on SM, we define

$$\gamma_{x,T}(f) = \frac{1}{T} \int_0^T f(\phi^U_t(x))\, dt$$

By the Sobolev embedding Theorem (see [1]), for $s > 3/2$, the measures $\gamma_{x,T}$ are continuous functionals on $W^s(SM)$ (which depend weakly-continuously on $x \in SM$ and $T \in \mathbb{R}^+$). Thus the splitting (78) can be applied to horocycle orbits. We set

$$c_{\mathcal{D}}(x,T) := c_{\mathcal{D}}(\gamma_{x,T}), \qquad \mathcal{C}(x,T) := \mathcal{C}(\gamma_{x,T}) , \qquad \mathcal{R}(x,T) := \mathcal{R}(\gamma_{x,T}) .$$

so that

$$\gamma_{x,T} = \sum_{\mathcal{D} \in \mathcal{B}^s} c_{\mathcal{D}}(x,T)\mathcal{D} + \mathcal{C}(x,T) + \mathcal{R}(x,T) . \tag{80}$$

The proofs of Theorems 1.5 and 1.7 will reduce to estimates on the norms of the three parts of this splitting. We start by showing in next section that since the parts of this splitting invariant by the horocycle flow, namely $\sum_{\mathcal{D} \in \mathcal{B}^s} c_{\mathcal{D}}(x,T)\mathcal{D}$ and $\mathcal{C}(x,T)$, vanish on coboundaries, the remainder part $\mathcal{R}(x,T)$ must be of the order of $1/T$; furthermore the individual coefficients $c_{\mathcal{D}}(x,T)$ and the continuous component $\mathcal{C}(x,T)$ cannot be too small.

The uniform norm of functions on a compact manifold can be bounded in terms of a Sobolev norm by the Sobolev embedding theorem. In the case of a non-compact hyperbolic surfaces M of finite area, since the injectivity radius is not bounded away from zero, the Sobolev embedding theorem holds only locally. We therefore prove a version of the Sobolev embedding theorem on compact subsets of the unit tangent bundle SM, with an explicit bound on the constant.

We fix once for all a point $x_0 \in SM$ and let $d_0 : SM \to \mathbb{R}$ be the distance function from x_0. Constants in the following statements will implicitly depend (in a inessential way) on this choice.

Lemma 5.3. *There exists a constant $C_3 := C_3(x_0, M)$ such that for any function $F \in W^2(SM)$, we have that F is continuous and*

$$|F(x)| \leq C_3\, e^{d_0(x)/2} \, \| F \|_2 \,.$$

Proof. Recall that if G is a locally W^2 function on Poincaré's plane H then G is continuous and there exists $C(\epsilon) > 0$ such that

$$|G(x)|^2 < C(\epsilon) \int_{B(x,\epsilon)} (|G|^2 + |dG|^2 + |\Delta G|^2)\, dy$$

for any $x \in H$ ([22] page 63).

For x in SM denote by $\rho(x)$ the radius of injectivity of SM at x.

Let $\epsilon < \pi$ and set A_0 the open set of points $x \in SM$ where $\rho(x) > \epsilon$. By choosing ϵ sufficiently small we can assume that complement A_0^c consists of k connected components V_i each contained in SA_i, the tangent unit bundle of disjoint open cusps $A_i \approx S^1 \times R^+$ whose boundary horocycle has length 2ϵ.

By the Sobolev embedding theorem mentioned above there exists $C(\epsilon) > 0$ such that for any $x \in A_0$ we have

$$|F(x)|^2 < C(\epsilon) \int_{B(x,\epsilon)} (|F|^2 + |dF|^2 + |\Delta F|^2)\, dy.$$

For $x \in V_i$ let d be the distance of x from ∂A_i. It's easy to see that $2\epsilon e^{-d} \leq \rho(x) \leq 4\epsilon e^{-d}$. Let $\tilde{F}$ denote the lift of F to Poincare's half-plane H and let $\tilde{x}$ be a point SH projecting to x. Then, by the same embedding theorem,

$$|\tilde{F}(x)|^2 < C(\epsilon) \int_{B(\tilde{x},\epsilon)} (|\tilde{F}|^2 + |d\tilde{F}|^2 + |\Delta \tilde{F}|^2)\, dy$$

and, since, the ball $B(\tilde{x}, \epsilon) \subset SH$ covers the ball $B(x, \epsilon) \subset SM$ at most $[e^d/2] + 1$ times, we get

$$|F(x)|^2 < e^d C(\epsilon) \int_{B(x,\epsilon)} (|F|^2 + |dF|^2 + |\Delta F|^2)\, dy\,.$$

The proof is finished by observing that d_0 is bounded on A_0 and $d_0(x) < d + C'(\epsilon)$. □

Lemma 5.3 allows us to derive the following upper bound for the uniform norm of components and remainder terms of horocycle arcs.

Corollary 5.4. *For all $s \geq 2$, there exists a constant $C_4 := C_4(s, x_0) > 0$ such that, if the horocycle arc $\gamma_{x,T} \subset \overline{B(x_0, d)}$, then*

$$\sum_{\mathcal{D} \in \mathcal{B}^s} |c_{\mathcal{D}}(x,T)|^2 + \| \mathcal{C}(x,T) \|_{-s}^2 + \| \mathcal{R}(x,T) \|_{-s}^2 \leq C_4^2\, e^d \,. \tag{81}$$

Proof. Let $s \geq 2$. By Lemma 5.3, for any function $f \in W^s(SM)$, we have:

$$|\gamma_{x,T}(f)| \leq \max\{|f(x)| \,:\, x \in \overline{B(x_0, d)}\} \leq C_3 e^{d/2} \, \| f \|_s \,.$$

The estimate (81) then follows immediately from Lemma 5.2. □

5.2. **Coboundaries.** Let $\{\phi_t\}$ be a measure preserving ergodic flow on a probability space. We recall that a function g is a *coboundary* for $\{\phi_t\}$ if it is a derivative of a function f along this flow. The Gottschalk-Hedlund Theorem, or rather its proof, yields upper bounds for the uniform or the L^2 norm of ergodic averages of a coboundary g in terms of the uniform, or respectively, the L^2 norm of its primitive f. A key consequence is that the uniform bound for the remainder term $\mathcal{R}(x,T)$ proved in Corollary 5.4 can be significantly improved.

Lemma 5.5. *Let $x_0 \in SM$. For every $s > 3$, there exists a constant $C_5 := C_5(s, x_0)$ such that if the end-points of $\gamma_{x,T}$ are in $B(x_0, d)$, then*

$$\| \mathcal{R}(x,T) \|_{-s} \leq \frac{C_5}{T} e^{d/2} . \tag{82}$$

Proof. Let $\mathcal{I} := \mathcal{I}^s(SM)$. The orthogonal splitting (72) induces a dual orthogonal splitting

$$W^s(SM) = \text{Ann}(\mathcal{I}) \oplus \text{Ann}(\mathcal{I}^\perp) . \tag{83}$$

Hence, any function $g \in W^s(SM)$ has a unique (orthogonal) decomposition $g = g_1 + g_2$, where $g_1 \in \text{Ann}(\mathcal{I})$ and $g_2 \in \text{Ann}(\mathcal{I}^\perp)$. Since $\mathcal{R}(x,T) \in \mathcal{I}^\perp$, the function $g_2 \in \text{Ann}(\mathcal{I}^\perp)$ and $g_1 \in \text{Ann}(\mathcal{I})$, we have:

$$\mathcal{R}(x,T)(g) = \mathcal{R}(x,T)(g_1 + g_2) = \mathcal{R}(x,T)(g_1) = \gamma_{x,T}(g_1) . \tag{84}$$

The function g_1 is a coboundary for the horocycle flow. In fact, it belongs to the kernel of all U-invariant distributions of order $\leq s$; hence, by Theorem 1.2, there exists a function $f_1 \in W^t(SM)$, with $2 < t < s-1$, such that $Uf_1 = g_1$ and $\| f_1 \|_t \leq C \| g_1 \|_s$. Let $d > 0$ be such that $x, \phi^U_T(x) \in \overline{B}(x_0, d)$. By the Sobolev embedding Theorem, the function f_1 is continuous and by Lemma 5.3

$$\max\{|f_1(x)| \,:\, x \in \overline{B(x_0,d)}\} \leq C_3\, e^{d/2} \, \| f \|_t \leq C_3'\, e^{d/2} \| g_1 \|_s . \tag{85}$$

By the Gottschalk-Hedlund argument and the inequality (85),

$$|\gamma_{x,T}(g_1)| = \frac{1}{T} |f_1 \circ \phi^U_T(x) - f_1(x)| \leq \frac{2C_3'}{T} e^{d/2} \, \| g_1 \|_s . \tag{86}$$

Since the dual splitting (83) is orthogonal, by the estimates (84) and (86), we get

$$|\mathcal{R}(x,T)(g)| \leq 2C_3' \frac{e^{d/2}}{T} \, \| g_1 \|_s \leq \frac{2C'}{T} e^{d/2} \, \| g \|_s .$$

The lemma is therefore proved. □

A similar argument proves the following L^2 bound.

Lemma 5.6. *For every $s > 1$, there exists a constant $C_6 := C_6(s)$, such that, for all functions $g \in W^s(SM)$,*

$$\| \mathcal{R}(\cdot, T)(g) \|_0 \leq \frac{C_6}{T} \, \| g \|_s . \tag{87}$$

Proof. According to (84), since $g_1 = Uf_1$ is a coboundary and ϕ^X_t is volume preserving, by Theorem 1.3 and the orthogonality of the splitting (83), we have

$$\| \mathcal{R}(\cdot, T)(g) \|_0 = \| \gamma_{x,T}(g_1) \|_0 \leq \frac{2}{T} \| f_1 \|_0 \leq \frac{C_6}{T} \| g_1 \|_s \leq \frac{C_6}{T} \| g \|_s .$$

□

The following lemma is a widely known L^2 version for ergodic measurable flows. It will be a crucial tools in the proof of the L^2 lower bounds of Theorems 1.5 and 1.7.

Lemma 5.7 (Gottschalk-Hedlund Lemma)**.** *If an L^2-function F is a solution of the equation*

$$\left(\frac{dF \circ \phi_t}{dt}\right)_{t=0} = G\,, \tag{88}$$

then the one-parameter family of functions $\{G_T\}_{T\in\mathbb{R}}$ defined by

$$G_T(\cdot) := \int_0^T G\left(\phi_t(\cdot)\right) dt$$

is equibounded in L^2 by $2\,\|\,F\,\|$. Conversely if the family $\{G_T\}_{T\geq 0}$, is equibounded, then the equation (88) has an L^2 solution.

Proof. If there exists a solution F of the equation (88), then

$$\int_0^T G\left(\phi_t(x)\right) dt = F\left(\phi_T(x)\right) - F(x);$$

hence

$$\|\int_0^T G\left(\phi_t(\cdot)\right) dt\,\|_0 \leq 2\,\|\,F\,\|_0\,.$$

Conversely, if the family of functions $\{G_T\}_{T\geq 0}$ is equibounded in L^2, then the family of functions $\{F_T\}_{T\geq 0}$ defined by

$$F_T(\cdot) := -\frac{1}{T}\int_0^T \int_0^t G\left(\phi^s(\cdot)\right)\, ds\, dt$$

is equibounded in L^2. Since $\{\phi_t\}$ is ergodic, by Von Neumann L^2 ergodic Theorem, the function G has zero average and any weak limit $F \in L^2$ of $\{F_T\}$ is a (weak) L^2 solution of the equation (88). □

We have the following L^2 bounds for the components of horocycle arcs:

Lemma 5.8. *For every $\mathcal{D} \in \mathcal{B}^s$ there exists a constant $C_{\mathcal{D}} > 0$ such that, for all $T > 0$, the $\mathcal{D}$-component $c_{\mathcal{D}}(x,T)$ of $\gamma_{x,T}$ satisfies the L^2 upper bound*

$$\|\,c_{\mathcal{D}}(\cdot,T)\,\|_0 \leq C_{\mathcal{D}}\;. \tag{89}$$

If $\mathcal{D} \in \mathcal{I}^1(SM)$, the $\mathcal{D}$-component $c_{\mathcal{D}}(x,T)$ satisfy the L^2 lower bound

$$\sup_{T\in\mathbb{R}^+} T\,\|\,c_{\mathcal{D}}(\cdot,T)\,\|_0 = +\infty\;. \tag{90}$$

If the continuous spectrum is not empty, we also have

$$\sup_{T\in\mathbb{R}^+} T\;\|\,\mathcal{C}(\cdot,T)\,\|_{-s,0}\;=\;+\infty \tag{91}$$

for the continuous component $\mathcal{C}(x,T)$ of $\gamma_{x,T}$.

Proof. Let $\mathcal{D} \in \mathcal{B}^s$. There exists a unique function $g \in \text{Ann}(\mathcal{I}^s(SM)^\perp) \subset W^s(SM)$ such that $\mathcal{D}(g) = 1$, $\mathcal{D}'(g) = 0$, for all $\mathcal{D}' \in \mathcal{B}^s$ with $\mathcal{D}' \neq \mathcal{D}$ and all $\mathcal{D}' \in \mathcal{I}^s_{\mathcal{C}}$. By the splitting (80), for all $(x, T) \in SM \times \mathbb{R}^+$,

$$\gamma_{x,T}(g) = c_{\mathcal{D}}(x, T)\,.$$

It follows that, since the horocycle flow is volume preserving,

$$\| \, c(\cdot, T) \, \|_{\,0} \leq \; \| \, \frac{1}{T} \int_0^T g\left(\phi_t^U(x)\right) dt \, \|_{\,0} \leq \; \| \, g \, \|_{\,0} \; .$$

If the distribution $\mathcal{D} \in \mathcal{B}^s$ has Sobolev order $S \leq 1$, since $\mathcal{D}(g) \neq 0$, by Theorem 1.3 the equation $Uf = g$ has no solution $f \in L^2(SM)$. By the Gottschalk-Hedlund Lemma 5.7, it follows that the family of functions

$$T \, c_{\mathcal{D}}(x, T) = T \, \gamma_{x,T}(g) = \int_0^T g\left(\phi_t^U(x)\right) dt$$

is not equibounded in $L^2(SM)$. Hence (90) is proved.

The proof of (91) is completely analogous. Suppose that the continuous spectrum is not empty and let $g \in W^s(SM)$ be a function belonging to Ann $\left(\mathcal{I}^s(SM)^\perp\right) \cap$ Ann $(\mathcal{I}^s_d)$ and such that $\mathcal{D}(g) \neq 0$ for some $\mathcal{D} \in \mathcal{I}^s_{\mathcal{C}}$. Since any $\mathcal{D} \in \mathcal{I}^s_{\mathcal{C}}$ has Sobolev order equal to $1/2$, by Theorem 1.3, the equation $Uf = g$ has no solution $f \in L^2(SM)$. Then, as before, by the Gottschalk-Hedlund Lemma 5.7, the family of functions $\{T \, \mathcal{C}(\cdot, T)(g)\}_{T \in \mathbb{R}}$ is not equibounded in $L^2(SM)$. In fact,

$$T \, \mathcal{C}(x, T)(g) = T \, \gamma_{x,T}(g) = \int_0^T g\left(\phi_t^U(x)\right) dt \; .$$

Since

$$\| \, \mathcal{C}(\cdot, T)(g) \, \|_{\,0} \leq \; \| \, \mathcal{C}(\cdot, T) \, \|_{\,-s,0} \; \| \, g \, \|_{\,s} \; ,$$

the family of functions $\{T \, \| \, \mathcal{C}(\cdot, T) \, \|_{\,-s,0}\}_{T \in \mathbb{R}}$ is not equibounded in $L^2(SM)$, proving (91). □

5.3. **Iterative estimates.** By the commutation relations (14), the geodesic flow $\{\phi_t^X\}$ expands the orbits of unstable horocycle flow $\{\phi_s^V\}$ by a factor e^t and it contracts the orbits of stable horocycle flow $\{\phi_s^U\}$ by a factor e^{-t}:

$$\phi_t^X \circ \phi_s^U = \phi_{se^{-t}}^U \circ \phi_t^X \,, \qquad \phi_t^X \circ \phi_s^V = \phi_{se^t}^V \circ \phi_t^X \; . \tag{92}$$

It follows that, in the distributional sense,

$$\phi_t^X(\gamma_{x,T}) = \gamma_{\phi_{-t}^X(x), e^t \, T} \, . \tag{93}$$

Were the splitting (72), hence the splitting (80), invariant under the action geodesic flow, Theorems 1.5 and 1.7 would follow immediately. In fact, by the eigenvalue identities (74) on $\mathcal{I}^s_d$ and by the bound on the norm of ϕ_t^X on $\mathcal{I}^s_{\mathcal{C}}$ given by Lemma 5.1, the invariance of the splitting (80) would entail that, for all $\mathcal{D} \in \mathcal{B}^s \setminus \mathcal{B}_{1/4}$ and for the continuous component,

$$c_{\mathcal{D}}(\phi_{-t}^X(x), e^t) = c_{\mathcal{D}}(x, 1) \, e^{\lambda_{\mathcal{D}} t} \; ,$$

$$\| \, \mathcal{C}(\phi_{-t}^X(x), e^t) \, \|_{\,-s} \leq C_1(1 + |t|) \, e^{-t/2} \, \| \, \mathcal{C}(x, 1) \, \|_{\,-s}$$

and, if $1/4 \in \sigma_{pp}$, by identity (75) for all pairs $\{\mathcal{D}^+, \mathcal{D}^-\} \subset \mathcal{B}_{1/4}$,

$$
\begin{aligned}
c_{\mathcal{D}^+}(\phi^X_{-t}(x), e^t) \;&=\; [c_{\mathcal{D}^+}(x,1) - \frac{t}{2}\, c_{\mathcal{D}^-}(x,1)]\, e^{-t/2} \;, \\
c_{\mathcal{D}^-}(\phi^X_{-t}(x), e^t) \;&=\; c_{\mathcal{D}^-}(x,1)\, e^{-t/2} \;,
\end{aligned} \tag{94}
$$

allowing us to conclude.

The aim of this section is to prove that, in spite of the lack of invariance of the splitting (80), the projections onto $\mathcal{I}^s_{\mathcal{C}}$ and $\mathcal{D} \in \mathcal{B}$ of the geodesic push-forwards of the remainder term $\mathcal{R}(x,T)$ are negligible.

Let $x \in SM$, $T > 0$. It will be convenient to discretise the geodesic flow time $t \geq 1$ and to consider the push-forwards of the arc $\gamma_{x,T}$ by $\phi^X_{\ell h}$, where $h \in [1,2]$ and $\ell \in \mathbb{N}$. Then the distribution (measure) $\phi^X_{\ell h}(\gamma_{x,T})$ has a splitting

$$
\phi^X_{\ell h}(\gamma_{x,T}) = \sum_{\mathcal{D} \in \mathcal{B}^s} c_{\mathcal{D}}(x,T,\ell)\mathcal{D} \,+\, \mathcal{C}(x,T,\ell) \,+\, \mathcal{R}(x,T,\ell)\,. \tag{95}
$$

We will prove pointwise and L^2 bounds on the sequences of functions $c_{\mathcal{D}}(\cdot,T,\ell)$, $\mathcal{C}(\cdot,T,\ell)$ and $\mathcal{R}_(\cdot,T,\ell)$. By the identity (93) and the definition (80), we have:

$$
\begin{aligned}
c_{\mathcal{D}}(x,T,\ell) &= c_{\mathcal{D}}\left(\phi^X_{-\ell h}(x), e^{\ell h}\, T\right) \;, \\
\mathcal{C}(x,T,\ell) &= \mathcal{C}\left(\phi^X_{-\ell h}(x), e^{\ell h}\, T\right) \;, \\
\mathcal{R}(x,T,\ell) &= \mathcal{R}\left(\phi^X_{-\ell h}(x), e^{\ell h}\, T\right) \;.
\end{aligned} \tag{96}
$$

Uniform bounds and L^2 bounds on the functions $c_{\mathcal{D}}(\cdot,T,\ell)$, $\mathcal{C}(\cdot,T,\ell)$ and $\mathcal{R}(\cdot,T,\ell)$ are clearly equivalent to uniform and L^2 bounds on $c_{\mathcal{D}}(\cdot,e^{\ell h}\, T)$, $\mathcal{C}(\cdot,e^{\ell h}\, T)$ and $\mathcal{R}(\cdot,e^{\ell h}\, T)$ respectively. Let

$$
\begin{aligned}
r_{\mathcal{D}}(x,T,\ell) &:= c_{\mathcal{D}}\left(\phi^X_h \mathcal{R}(x,T,\ell)\right) \,\in \mathbb{R} \;, \\
\mathcal{R}_{\mathcal{C}}(x,T,\ell) &:= \mathcal{C}\left(\phi^X_h \mathcal{R}(x,T,\ell)\right) \,\in \mathcal{I}^s_{\mathcal{C}} \;.
\end{aligned} \tag{97}
$$

By the identity $\phi^X_{(\ell+1)h} = \phi^X_h \circ \phi^X_{\ell h}$, since the distributions $\mathcal{D} \in \mathcal{B}^s \backslash \mathcal{B}_{1/4}$ are eigenvectors of the geodesic flow $\{\phi^X_t\}$ (see (74)) and the space $\mathcal{I}^s_{\mathcal{C}}$ is $\{\phi^X_t\}$-invariant, we obtain by projecting on $\mathcal{D}$-components and on the continuous component:

$$
\begin{aligned}
c_{\mathcal{D}}(x,T,\ell+1) &= c_{\mathcal{D}}(x,T,\ell)\, e^{\lambda_{\mathcal{D}} h} \,+\, r_{\mathcal{D}}(x,T,\ell)\,; \\
\mathcal{C}(x,T,\ell+1) &= \phi^X_h \mathcal{C}(x,T,\ell) \,+\, \mathcal{R}_{\mathcal{C}}(x,T,\ell)\,.
\end{aligned} \tag{98}
$$

If $1/4 \in \sigma_{pp}$, for all pairs $\{\mathcal{D}^+, \mathcal{D}^-\} \subset \mathcal{B}_{1/4}$ we obtain by (75):

$$
\begin{aligned}
c_{\mathcal{D}^+}(x,T,\ell+1) &= [c_{\mathcal{D}^+}(x,T,\ell) - \frac{h}{2}\, c_{\mathcal{D}^-}(x,T,\ell)]\, e^{-h/2} \,+\, r_{\mathcal{D}^+}(x,T,\ell)\,; \\
c_{\mathcal{D}^-}(x,T,\ell+1) &= c_{\mathcal{D}^-}(x,T,\ell)\, e^{-h/2} \,+\, r_{\mathcal{D}^-}(x,T,\ell)\,.
\end{aligned} \tag{99}
$$

Bounds on the solutions of the difference equations (98) and (99) can be derived from the following trivial lemma.

Lemma 5.9. *Let $\Phi \in \mathcal{L}(E)$ be a bounded linear operator on a normed space E. Let $\{R_\ell\}$, $\ell \in \mathbb{N}$, be a sequence of elements of E. The solution $\{x_\ell\}$ of the following difference equation in E,*

$$x_{\ell+1} = \Phi(x_\ell) + R_\ell\,, \quad \ell \in \mathbb{N}\ , \tag{100}$$

has the form

$$x_\ell = \Phi^\ell(x_0) + \sum_{j=0}^{\ell-1} \Phi^{\ell-j-1} R_j\,. \tag{101}$$

By Lemma 5.9, the proof of Theorems 1.5 and 1.7 is essentially reduced to estimates on the 'remainder terms' $r_{\mathcal{D}}(x,T,\ell)$ and $\mathcal{R}_{\mathcal{C}}(x,T,\ell)$. Such estimates can be derived from Lemma 5.5. Let $x_0 \in SM$ be a reference point as in Lemma 5.5 and let $d_0 : SM \to \mathbb{R}$ be the distance function from x_0. For each $(x,T) \in SM \times \mathbb{R}^+$ and each $\ell \in \mathbb{N}$, let $d(x,T,\ell)$ be the the maximum distance of the endpoints of the horocycle arc $\phi^X_{\ell h}(\gamma_{x,T})$ from x_0:

$$d(x,T,\ell) := \max\{d_0\Big(\phi^X_{-\ell h}(x)\Big), d_0\Big(\phi^X_{-\ell h}\circ \phi^U_T(x)\Big)\}\ . \tag{102}$$

Lemma 5.10. *There exists a constant $C_7 := C_7(s)$ such that, for all $(x,T) \in SM \times \mathbb{R}^+$ and all $\ell \in \mathbb{N}$,*

$$\sum_{\mathcal{D}\in\mathcal{B}^s} |r_{\mathcal{D}}(x,T,\ell)|^2 + \|\,\mathcal{R}_{\mathcal{C}}(x,T,\ell)\,\|^2_{-s} \ \le\ \left(\frac{C_7}{T}\right)^2 \exp\{d(x,T,\ell) - 2\ell h\}\,. \tag{103}$$

Proof. Let $C_X(s) := \max_{h\in[1,2]} \|\,\phi^X_h\,\|_{-s}$. Then, by the definition (97) and Lemma 5.2, we have

$$\sum_{\mathcal{D}\in\mathcal{B}^s} |r_{\mathcal{D}}(x,T,\ell)|^2 + \|\,\mathcal{R}_{\mathcal{C}}(x,T,\ell)\,\|^2_{-s} \ \le C_2^2 C_X^2 \,\|\,\mathcal{R}(x,T,\ell)\,\|^2_{-s}\,.$$

But $\mathcal{R}(x,T,\ell)$ is the "$\mathcal{R}$" component of an arc of horocycle of length $e^{\ell h}T$ whose endpoints are at a distance $d(x,T,\ell)$ from the reference point x_0 (cf. (96), (93), (102)). Thus by Lemma 5.5 $\|\,\mathcal{R}(x,T,\ell)\,\|_{-s} < C_5 e^{d(x,T,\ell)/2}/e^{\ell h}T$ and the lemma follows. □

The difference equations (98) and (99) also yield estimates for the L^2 norms of the functions $c_{\mathcal{D}}(\cdot,T,\ell)$ and $\|\,\mathcal{C}(\cdot,T,\ell)\,\|_{-s}$ on SM. However, in the case of the continuous component, such L^2 bounds are not effective because of the lack of L^2 estimates on the remainder term $\|\,\mathcal{R}_{\mathcal{C}}(\cdot,T,\ell)\,\|_{-s}$. For the $\mathcal{D}$-components we can prove the following:

Lemma 5.11. *For each $\mathcal{D} \in \mathcal{B}^s$, there exists a constant $C_8 := C_8(\mathcal{D},s) > 0$ such that, for all $T > 0$ and all $\ell \in \mathbb{N}$,*

$$\|\,r_{\mathcal{D}}(\cdot,T,\ell)\,\|_0 \ \le\ \frac{C_8}{T}\, e^{-\ell h}\ . \tag{104}$$

Proof. Let $g := g_{\mathcal{D}} \in W^s(SM)$ be the unique function such that $\mathcal{D}(g) = 1$, $\mathcal{D}'(g) = 0$ for all $\mathcal{D}' \in \mathcal{B}^s$, $\mathcal{D}' \ne \mathcal{D}$, and $g \in \text{Ann}\,(\mathcal{I}^s_{\mathcal{C}}) \cap \text{Ann}(\mathcal{I}^s(SM)^\perp)$. Let $g_h := \phi^X_{-h}(g)$. By definition of the remainder term we have $r_{\mathcal{D}}(x,T,\ell) = \mathcal{R}(x,T,\ell)(g_h)$. Again by definition (see (96)) we have $\mathcal{R}(x,T,\ell) = \mathcal{R}\left(\phi^X_{-\ell h}(x), e^{\ell h}T\right)$ and we obtain, using Lemma 5.6:

$$\|\,r_{\mathcal{D}}(\cdot,T,\ell)\,\|_0 \ =\ \|\,\mathcal{R}(\cdot,T,\ell)(g_h)\,\|_0 \ \le\ \frac{C_6}{Te^{\ell h}}\,\|\,g_h\,\|_{-s} \le\ \frac{C_8}{T} e^{-\ell h}$$

with $C_8 := C_6 \max_{h\in[1,2]} \| g_h \|_s$. $\square$

5.4. **Bounds on the components: the cuspidal case.** In the cuspidal case, the precision of our asymptotics of geodesic push-forwards of a horocycle arc depends on the rate of escape into the cusps of its endpoints. Let $d_0 : SM \to \mathbb{R}^+$ be the distance function from a fixed point $x_0 \in SM$. For any given $\sigma \in [0,1]$ and $A \geq 0$ let

$$V_{A,\sigma} := \{x \in SM \mid d_0\left(\phi_t^X(x)\right) \leq A + \sigma\,|t|\,, \quad \text{for all} \quad t \leq 0\}\,.$$

$$V_\sigma := \bigcup_{A\geq 0} V_{A,\sigma}$$

The sets V_σ do not depend on the choice of the point $x_0 \in SM$ and they are measurable as they can be written as countable unions of closed sets (hence, they are F_σ sets). Since the geodesic flow has unit speed $V_1 = SM$. On the other hand, by the logarithmic law of geodesics, V_σ has full measure for any $\sigma > 0$ [56].

Lemma 5.12. *Let $s > 3$. For every $\mathcal{D} \in \mathcal{B}^s$ of order $S_{\mathcal{D}} > 0$, there exists a uniformly bounded sequence of positive bounded functions $\{K_{\mathcal{D}}(x,T,\ell)\}_{\ell\in\mathbb{Z}^+}$, $(x,T) \in V_{A,\sigma} \times \mathbb{R}^+$, such that the following estimates hold. For every horocycle arc $\gamma_{x,T}$ having endpoints x, $\phi_U^T(x) \in V_{A,\sigma}$ and for all $\ell \in \mathbb{Z}^+$ we have, if $\mathcal{D} \in \mathcal{B}^s \setminus \mathcal{B}^+_{1/4}$,*

$$|c_{\mathcal{D}}(x,T,\ell)| \leq \begin{cases} K_{\mathcal{D}}(x,T,\ell)\, e^{-S_{\mathcal{D}}\ell h}\,, & \text{if } S_{\mathcal{D}} < 1 - \frac{\sigma}{2}\,, \\ K_{\mathcal{D}}(x,T,\ell)\, \ell\, e^{-S_{\mathcal{D}}\ell h}\,, & \text{if } S_{\mathcal{D}} = 1 - \frac{\sigma}{2}\,, \\ K_{\mathcal{D}}(x,T,\ell)\, e^{-(1-\frac{\sigma}{2})\ell h}\,, & \text{if } S_{\mathcal{D}} > 1 - \frac{\sigma}{2}\,. \end{cases} \tag{105}$$

If $1/4 \in \sigma_{pp}$ and $\mathcal{D} \in \mathcal{B}^+_{1/4}$, we have

$$|c_{\mathcal{D}}(x,T,\ell)| \leq \begin{cases} K_{\mathcal{D}}(x,T,\ell)\, \ell\, e^{-\ell h/2}\,, & \text{if } \sigma < 1\,, \\ K_{\mathcal{D}}(x,T,\ell)\, \ell^2\, e^{-\ell h/2}\,, & \text{if } \sigma = 1\,. \end{cases} \tag{106}$$

There exists a positive constant $K_{\mathcal{C}} := K_{\mathcal{C}}(A,\sigma,s,T)$, such that the following estimates hold. For every horocycle arc $\gamma_{x,T}$ as above and for all $\ell \in \mathbb{Z}^+$, we have

$$\| \mathcal{C}(x,T,\ell) \|_{-s} \leq \begin{cases} K_{\mathcal{C}}\, \ell\, e^{-\ell h/2}\,, & \text{if } \sigma < 1\,, \\ K_{\mathcal{C}}\, \ell^2\, e^{-\ell h/2}\,, & \text{if } \sigma = 1\,. \end{cases} \tag{107}$$

In addition, there exists a positive constant $K := K(A,\sigma,T,s)$ such that, for all $\gamma_{x,T}$ with endpoints belonging to the set $V_{A,\sigma}$ and for all $\ell \in \mathbb{Z}^+$,

$$\sum_{\mathcal{D}\in\mathcal{B}^s} K^2_{\mathcal{D}}(x,T,\ell) \;+\; K^2_{\mathcal{C}} \;\leq\; K^2\,. \tag{108}$$

Proof. For all $\mathcal{D} \notin \mathcal{B}_{1/4}$, by the first difference equation in formula (98) and by Lemma 5.9 with $E := \mathbb{C}$ and Φ the multiplication operator by $e^{\lambda_{\mathcal{D}} h} \in \mathbb{C}$, we obtain

$$|c_{\mathcal{D}}(x,T,\ell)| \leq |c_{\mathcal{D}}(x,T,0)|\, e^{-S_{\mathcal{D}}\ell h} \;+\; \Sigma_{\mathcal{D}}(x,T,\ell)\,, \tag{109}$$

with

$$\Sigma_{\mathcal{D}}(x,T,\ell) := \sum_{j=0}^{\ell-1} |r_{\mathcal{D}}(x,T,j)| e^{-S_{\mathcal{D}} h(\ell-j-1)}\,.$$

We must therefore bound the terms $|c_{\mathcal{D}}(x,T,0)|$ and $e^{S_{\mathcal{D}}\ell h}\Sigma_{\mathcal{D}}(x,T,\ell)$ by $K_{\mathcal{D}}(x,T,\ell)$, $K_{\mathcal{D}}(x,T,\ell)\ell$ or $K_{\mathcal{D}}(x,T,\ell)e^{(S_{\mathcal{D}}-1+\frac{\sigma}{2})\ell h}$, according to the different values of $S_{\mathcal{D}}$, for some uniformly bounded sequence of functions $\{K_{\mathcal{D}}(x,T,\ell)\}_{\ell\in\mathbb{Z}^+}$ on $V_{A,\sigma}\times\mathbb{R}^+$.

It follows from Corollary 5.4, taking into account the fact that the endpoints of $\gamma_{x,T}$ belong to the set $V_{A,\sigma}\subset\overline{B(x_0,A)}$, that there exists a continuous function $C_s:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ (which can be chosen constant if M has a positive injectivity radius) such that

$$\sum_{\mathcal{D}\in\mathcal{B}^s}|c_{\mathcal{D}}(x,T,0)|^2\le C_s(A,T)^2\ ; \tag{110}$$

thus the term $|c_{\mathcal{D}}(x,T,0)|$ in formula (109) satisfies estimates finer than (105) and (108).

Using again the fact that the endpoints of $\gamma_{x,T}$ belong to the set $V_{A,\sigma}$ and the estimate (103), a calculation based on the Cauchy-Schwartz inequality yields the following bounds on the remainder terms $\Sigma_{\mathcal{D}}(x,T,\ell)$ for $\mathcal{D}\notin\mathcal{B}_{1/4}$. For all $S>0$, $A\ge 0$, $\sigma\in[0,1]$, there exists a constant $C':=C'(A,\sigma,S,s)>0$ such that

$$\begin{aligned}
&\sum_{\mathcal{D}:S_{\mathcal{D}}\le S}\Sigma^2_{\mathcal{D}}(x,T,\ell)\,e^{2S_{\mathcal{D}}\ell h}\le \frac{C'}{T^2}\,, && \text{if } S<1-\frac{\sigma}{2}\,;\\
&\sum_{\mathcal{D}:S_{\mathcal{D}}= S}\Sigma^2_{\mathcal{D}}(x,T,\ell)\,e^{2S_{\mathcal{D}}\ell h}\le \frac{C'\,\ell^2}{T^2}\,, && \text{if } S=1-\frac{\sigma}{2}\,;\\
&\sum_{\mathcal{D}:S_{\mathcal{D}}\ge S}\Sigma^2_{\mathcal{D}}(x,T,\ell)\le \frac{C'}{T^2}\,e^{-2(1-\frac{\sigma}{2})\ell h} && \text{if } S>1-\frac{\sigma}{2}\,.
\end{aligned} \tag{111}$$

It follows from (110) and (111) that, for each $\mathcal{D}\in\mathcal{B}^s$, the sequence of positive functions

$$K_{\mathcal{D}}(x,T,\ell):=\begin{cases}|c_{\mathcal{D}}(x,T,0)|+\Sigma_{\mathcal{D}}(x,T,\ell)e^{S_{\mathcal{D}}\ell h} & \text{if } S_{\mathcal{D}}<1-\frac{\sigma}{2}\,,\\ \Big(|c_{\mathcal{D}}(x,T,0)|+\Sigma_{\mathcal{D}}(x,T,\ell)e^{S_{\mathcal{D}}\ell h}\Big)\ell^{-1} & \text{if } S_{\mathcal{D}}=1-\frac{\sigma}{2}\,,\\ \Big(|c_{\mathcal{D}}(x,T,0)|+\Sigma_{\mathcal{D}}(x,T,\ell)e^{S_{\mathcal{D}}\ell h}\Big)e^{-(S_{\mathcal{D}}-1+\frac{\sigma}{2})\ell h}\,, & \text{if } S_{\mathcal{D}}>1-\frac{\sigma}{2}\,.\end{cases} \tag{112}$$

is uniformly bounded for all $\gamma_{x,T}$ with endpoints belonging to the set $V_{A,\sigma}$. In view of the bound (109), this proves the estimate (105) for each $\mathcal{D}\notin\mathcal{B}_{1/4}$. In addition, since the set of real numbers $\{S_{\mathcal{D}}\,|\,\mathcal{D}\in\mathcal{B}^s\}$ is finite, the inequalities (110) and (111) imply that, for all $\gamma_{x,T}$ with endpoints belonging to the set $V_{A,\sigma}$ and for all $\ell\in\mathbb{Z}^+$,

$$\sum_{\mathcal{D}\in\mathcal{B}^s\setminus\mathcal{B}_{1/4}}K^2_{\mathcal{D}}(x,T,\ell)\ \le\ C''\ , \tag{113}$$

for some constant $C'':=C''(A,\sigma,T,s)>0$, thereby proving the upper bound (108) over all $\mathcal{D}$-components with $\mathcal{D}\in\mathcal{B}^s\setminus\mathcal{B}_{1/4}$.

The proofs of the upper bounds for all pairs $\{\mathcal{D}^+,\mathcal{D}^-\}\subset\mathcal{B}_{1/4}$ (if $1/4\in\sigma_{pp}$) and for the continuous component are similar. In the first case, by formula (99) we can apply Lemma 5.9 with $E=\mathbb{R}^2$ and

$$\Phi:=e^{-h/2}\begin{pmatrix}1 & -h/2\\ 0 & 1\end{pmatrix}\ . \tag{114}$$

By formula (101), we obtain

$$\begin{aligned} |c_{\mathcal{D}^+}(x,T,\ell)| &\le |c_{\mathcal{D}^+}(x,T,0) - \frac{\ell h}{2}\, c_{\mathcal{D}^-}(x,T,0)|\, e^{-\ell h/2} + \Sigma_{\mathcal{D}^+}(x,T,\ell)\ , \\ |c_{\mathcal{D}^-}(x,T,\ell)| &\le |c_{\mathcal{D}^-}(x,T,0)|\, e^{-\ell h/2} + \Sigma_{\mathcal{D}^-}(x,T,\ell)\ , \end{aligned} \tag{115}$$

with

$$\begin{aligned} \Sigma_{\mathcal{D}^+}(x,T,\ell) &:= \sum_{j=0}^{\ell-1} |r_{\mathcal{D}^+}(x,T,j) - \frac{(\ell-j-1)h}{2}\, r_{\mathcal{D}^-}(x,T,j)|\, e^{-h(\ell-j-1)/2}\,, \\ \Sigma_{\mathcal{D}^-}(x,T,\ell) &:= \sum_{j=0}^{\ell-1} |r_{\mathcal{D}^-}(x,T,j)|\, e^{-h(\ell-j-1)/2}\,. \end{aligned} \tag{116}$$

Since the endpoints of the horocycle arc $\gamma_{x,T}$ belong to the set $V_{A,\sigma}$, by the estimates (110) and (103), the sequence of positive functions

$$K_{\mathcal{D}^+}(x,T,\ell) := \begin{cases} \left(|c_{\mathcal{D}^+} - \frac{\ell h}{2}\, c_{\mathcal{D}^-}|(x,T,0)\ +\ \Sigma_{\mathcal{D}^+}(x,T,\ell)\, e^{\ell h/2}\right)\ \ell^{-1}, & \text{if } \sigma < 1\,, \\ \left(|c_{\mathcal{D}^+} - \frac{\ell h}{2}\, c_{\mathcal{D}^-}|(x,T,0)\ +\ \Sigma_{\mathcal{D}^+}(x,T,\ell)\, e^{\ell h/2}\right)\ \ell^{-2}, & \text{if } \sigma = 1\,. \end{cases} \tag{117}$$

is uniformly bounded by a constant $K^+_{\mathcal{D}} := K^+_{\mathcal{D}}(A,\sigma,s,T) > 0$.

If $\mathcal{D} = \mathcal{D}^- \in \mathcal{B}^-_{1/4}$, it follows from the second lines in (115) and (116) that the sequence of positive functions $K_{\mathcal{D}}(x,T,\ell)$, defined as in (112) with $S_{\mathcal{D}} = 1/2 \le 1 - \frac{\sigma}{2}$, is uniformly bounded by a constant $K_{\mathcal{D}} := K_{\mathcal{D}}(A,\sigma,s,T) > 0$.

Since $\mathcal{B}_{1/4}$ is always a finite set, by the extimate (113) there exists a constant $C^{(3)} := C^{(3)}(A,s,T) > 0$ such that, for all $\gamma_{x,T}$ with endpoints belonging to the set $V_{A,\sigma}$ and for all $\ell \in \mathbb{Z}^+$,

$$\sum_{\mathcal{D}\in\mathcal{B}^s} K^2_{\mathcal{D}}(x,T,\ell)\ \le\ C^{(3)}\ . \tag{118}$$

For the continuous component, we apply Lemma 5.9, with $E = \mathcal{I}^s_{\mathcal{C}}$ and $\Phi = \phi^X_h$, to the second difference equation in formula (98). We obtain

$$\|\,\mathcal{C}(x,T,\ell)\,\|_{-s} \le\ \|\,\Phi^\ell\,\|_{-s}\,\|\,\mathcal{C}(x,T,0)\,\|_{-s}\ +\ \Sigma_{\mathcal{C}}(x,T,\ell) \tag{119}$$

with

$$\Sigma_{\mathcal{C}}(x,T,\ell) := \sum_{j=0}^{\ell-1} \|\,\Phi^{\ell-j-1}\mathcal{R}_{\mathcal{C}}(x,T,j)\,\|_{-s}\,.$$

By Lemma 5.1 the norm of the operator ϕ^X_t on $\mathcal{I}^s_{\mathcal{C}}$ is bounded by $C_1(1+|t|)e^{-t/2}$. Taking into account the fact that the endpoints of $\gamma_{x,T}$ belong to the set $V_{A,\sigma}$ we find: (1) by Lemmata 5.2 and 5.3 we obtain that there exists a constant $C^{(4)} := C^{(4)}(A,s,T)$ such that $\|\,\mathcal{C}(x,T,0)\,\|_{-s} \le C^{(4)}$; (2) using the estimate (103) for $\|\,\mathcal{R}_{\mathcal{C}}(x,T,j)\,\|_{-s}$ we find that the sequence of positive functions defined by:

$$K_{\mathcal{C}}(x,T,\ell) := \begin{cases} \left(\|\,\mathcal{C}(x,T,0)\,\|_{-s}\ +\ \Sigma_{\mathcal{C}}(x,T,\ell)\, e^{\ell h/2}\right)\ell^{-1}\,, & \text{if } \sigma < 1\,, \\ \left(\|\,\mathcal{C}(x,T,0)\,\|_{-s}\ +\ \Sigma_{\mathcal{C}}(x,T,\ell)\, e^{\ell h/2}\right)\ell^{-2}\,, & \text{if } \sigma = 1\,. \end{cases} \tag{120}$$

is uniformly bounded by a constant $K_{\mathcal{C}} := K_{\mathcal{C}}(A, \sigma, s, T) > 0$. □

Remark 1. In the case $\sigma = 1$, the set $V_\sigma = SM$ and the results of Lemma 5.12 hold for all $(x, T) \in SM \times \mathbb{R}^+$. In addition, for every bounded subset $F \subset SM$, there exists a constant $A := A_F > 0$ such that $F \subset V_{A,1}$. In fact, one can take $A_F := \max\{\text{dist}(x, x_0) \,|\, x \in F\}$, for a fixed $x_0 \in SM$.

For the $\mathcal{D}$-components we also have the following L^2 upper and lower bounds:

Lemma 5.13. *Let $s > 3$. For each $\mathcal{D} \in \mathcal{B}^s$, there exists a constant $K'_{\mathcal{D}} = K'(\mathcal{D}, s)$, such that for all $\ell \in \mathbb{Z}^+$:*

$$\| c_{\mathcal{D}}(\cdot, T, \ell) \|_0 \le \begin{cases} K'_{\mathcal{D}}\, e^{-S_{\mathcal{D}} \ell h}, & \text{if } \mathcal{D} \notin \mathcal{B}^+_{1/4} \;\; \& \;\; S_{\mathcal{D}} < 1\,; \\ K'_{\mathcal{D}}\, \ell\, e^{-S_{\mathcal{D}} \ell h}, & \text{if } \mathcal{D} \in \mathcal{B}^+_{1/4} \;\; \text{or} \;\; S_{\mathcal{D}} = 1\,; \\ K'_{\mathcal{D}}\, e^{-\ell h}, & \text{if } \mathcal{D} \in \mathcal{B}^s \;\; \& \;\; S_{\mathcal{D}} > 1\,. \end{cases} \tag{121}$$

If $S_{\mathcal{D}} < 1$, there exists $K''_{\mathcal{D}} := K''(\mathcal{D}, s) > 0$, $T_{\mathcal{D}} := T(\mathcal{D}, s) > 0$ and $\ell_{\mathcal{D}} := \ell(\mathcal{D}, s) \in \mathbb{Z}^+$ ($\ell_{\mathcal{D}} = 1$ if $\mathcal{D} \notin \mathcal{B}^+_{1/4}$) such that, for all $T \ge T_{\mathcal{D}}$ and all $\ell \ge \ell_{\mathcal{D}}$:

$$\| c_{\mathcal{D}}(\cdot, T, \ell) \|_0 \ge \begin{cases} K''_{\mathcal{D}}\, e^{-S_{\mathcal{D}} \ell h} & \text{if } \mathcal{D} \notin \mathcal{B}^+_{1/4}\,; \\ K''_{\mathcal{D}}\, \ell\, e^{-S_{\mathcal{D}} \ell h} & \text{if } \mathcal{D} \in \mathcal{B}^+_{1/4}\,. \end{cases} \tag{122}$$

Proof. The proof of L^2 upper bounds is similar to that of the pointwise upper bounds in Lemma 5.12. In case $\mathcal{D} \notin \mathcal{B}_{1/4}$, it is based on the difference equation for $\mathcal{D}$-components in formula (98), on the upper bounds (89) of Lemma 5.8, on Lemma 5.11 and on Lemma 5.9 for the normed space $E := L^2(SM)$ and the operator $\Phi \in \mathcal{L}(E)$ of multiplication by $e^{\lambda_{\mathcal{D}} h} \in \mathbb{C}$. In case $\mathcal{D} \in \mathcal{B}_{1/4}$, it is based on the vector-valued difference equation (99) and on Lemma 5.9 for the normed space $E := L^2(SM) \times L^2(SM)$ and the operator $\Phi \in \mathcal{L}(E)$ given by the natural action of the Jordan matrix (114) on E.

The proof of the lower bound (122) is as follows. In case $\mathcal{D} \notin \mathcal{B}_{1/4}$, by the difference equation for $\mathcal{D}$-components in formula (98) and by Lemma 5.9, we have:

$$\| c_{\mathcal{D}}(\cdot, T, \ell) \|_0 \ge \| c_{\mathcal{D}}(\cdot, T, 0) \|_0\, e^{-S_{\mathcal{D}} \ell h} - \Sigma_{\mathcal{D}}(T, \ell)$$

with

$$\Sigma_{\mathcal{D}}(T, \ell) := \sum_{j=0}^{\ell-1} \| r_{\mathcal{D}}(\cdot, T, j) \|_0\, e^{-S_{\mathcal{D}}(\ell - j - 1)h}\,.$$

If $\{\mathcal{D}^+, \mathcal{D}^-\} \subset \mathcal{B}_{1/4}$, by the vector-valued difference equation (99) and by Lemma 5.9, we have

$$\begin{aligned} &\| c_{\mathcal{D}^+}(\cdot, T, \ell) \|_0 \ge \| \frac{\ell h}{2} c_{\mathcal{D}^-}(\cdot, T, 0) - c_{\mathcal{D}^+}(\cdot, T, 0) \|_0 \; e^{-\ell h/2} - \Sigma_{\mathcal{D}^+}(T, \ell)\;, \\ &\| c_{\mathcal{D}^-}(\cdot, T, \ell) \|_0 \ge \| c_{\mathcal{D}^-}(\cdot, T, 0) \|_0 \; e^{-\ell h/2} - \Sigma_{\mathcal{D}^-}(T, \ell)\;, \end{aligned} \tag{123}$$

with

(124)
$$\Sigma_{\mathcal{D}^+}(T,\ell) := \sum_{j=0}^{\ell-1} \| \, r_{\mathcal{D}^+}(\cdot,T,j) - \frac{(\ell-j-1)h}{2} r_{\mathcal{D}^-}(\cdot,T,j) \, \|_0 \, e^{-h(\ell-j-1)/2} \,,$$
$$\Sigma_{\mathcal{D}^-}(T,\ell) := \sum_{j=0}^{\ell-1} \| \, r_{\mathcal{D}^-}(\cdot,T,j) \, \|_0 \, e^{-h(\ell-j-1)/2} \,.$$

From the upper bound (104) for $\| \, r_{\mathcal{D}}(\cdot,T,j) \, \|_0$ of Lemma 5.11, we obtain that, for each $\mathcal{D} \in \mathcal{B}^s$ with $S_{\mathcal{D}} < 1$, there exists a constant $C := C(\mathcal{D},s) > 0$ such that, for all $\ell \in \mathbb{Z}^+$,

(125)
$$\Sigma_{\mathcal{D}}(T,\ell) \, \leq \, \begin{cases} \frac{C}{T} \, e^{-S_{\mathcal{D}}\ell h} & \text{if } \mathcal{D} \notin \mathcal{B}^+_{1/4}\,; \\ \frac{C}{T} \, \frac{\ell h}{2} \, e^{-\ell h/2} & \text{if } \mathcal{D} \in \mathcal{B}^+_{1/4}\,. \end{cases}$$

The condition $S_{\mathcal{D}} < 1$ implies also, by Lemma 5.8, that there exists $T_{\mathcal{D}} := T(\mathcal{D},s) > 1$ such that, for all $T \geq T_{\mathcal{D}}$ and all $\ell \in \mathbb{Z}^+$,

$$\| \, c_{\mathcal{D}}(\cdot,T,0) \, \|_0 > 2\frac{C}{T} \,.$$

The L^2 lower bounds hence follow from the lower bounds (123) and for $\mathcal{D}^+ \in \mathcal{B}^+_{1/4}$ from the L^2 upper bound for the $\mathcal{D}^+$-component proved in Lemma 5.8. □

For all $t \geq 0$, the push-forward probability measure $\phi^X_t(\gamma_{x,T})$ is the uniformly distributed probability measure on a stable horocycle arc of length $T_t := e^t T$. The following quantitative equidistribution result holds. Let $\mathcal{I}^s_+(SM) \subset \mathcal{I}^s(SM)$ be the subspace of invariant distributions orthogonal to the volume form.

Theorem 5.14. *Let $s > 3$. Then there exists a constant $C_9 := C_9(A,\sigma,s,T)$ such that for any horocycle arc $\gamma_{x,T}$ with endpoints belonging to the set $V_{A,\sigma}$, for any $t \geq 1$ and for all $f \in W^s(SM)$, we have*

(126)
$$\phi^X_t(\gamma_{x,T})(f) = \int_{SM} f \, dvol \, + \sum_{\mathcal{D} \in \mathcal{B}^{1-\frac{\sigma}{2}}_+} c^s_{\mathcal{D}}(x,T,t) \, \mathcal{D}(f) \, T_t^{-S_{\mathcal{D}}} \, +$$
$$+ \, \mathcal{C}^s(x,T,t)(f) T_t^{-\frac{1}{2}} \log^{\alpha_\sigma} T_t \, + \, \mathcal{R}^s(x,T,t)(f) \, T_t^{\frac{\sigma}{2}-1} \log^{\beta_\sigma} T_t \,\, .$$

with $c^s_{\mathcal{D}}(x,T,t) \in \mathbb{C}$, $\mathcal{C}^s(x,T,t) \in \mathcal{I}^s_{\mathcal{C}}$ and $\mathcal{R}^s(x,T,t) \in W^{-s}(SM)$ satisfying the following upper bounds:

$$\sum_{\mathcal{D} \in \mathcal{B}^{1-\frac{\sigma}{2}}_+} |c^s_{\mathcal{D}}(x,T,t)|^2 \leq C_9 \,,$$
$$\| \, \mathcal{C}^s(x,T,t) \, \|_{-s} \leq C_9 \,,$$
$$\| \, \mathcal{R}^s(x,T,t) \, \|_{-s} \leq C_9 \,.$$

In the above asymptotics, the exponent α_σ is 1 if $\sigma < 1$ and equals 2 if $\sigma = 1$; the exponent β_σ is 0 if every $\mathcal{D} \in \mathcal{B}^s$ has Sobolev order $S_{\mathcal{D}} \neq 1 - \frac{\sigma}{2}$ and equals 1 otherwise.

In addition, for every invariant distribution $\mathcal{D} \in \mathcal{B}_+^{1-\frac{\sigma}{2}}$, there exists a constant $C_{\mathcal{D}} := C(\mathcal{D}, s) > 0$ such that, if $T > 0$ is sufficiently large, then for all $t \geq 0$,

$$C_{\mathcal{D}}^{-1} \leq \| c_{\mathcal{D}}(\cdot, T, t) \|_0 \leq C_{\mathcal{D}} .$$

Proof. Let $t \geq 1$. There exist $h \in [1, 2]$ and $\ell \in \mathbb{Z}^+$ such that $t = \ell h$. The distribution $\phi_t^X(\gamma_{x,T}) \in W^{-s}(SM)$ can be split as in (95), hence the expansion (126) follows. The pointwise upper bounds on the coefficients can be derived from Lemma 5.12 for the $\mathcal{D}$-components and the $\mathcal{C}$-component and, by its definition in (96), from Lemma 5.5 for the remainder term $\mathcal{R}(x, T, \ell)$ of the splitting (95). We remark that the term with coefficient $\mathcal{R}^s(x, T, t)(f)$ in (126) includes the contributions of all $\mathcal{D}$-components with $\mathcal{D} \notin \mathcal{B}^{1-\frac{\sigma}{2}}$ as well as the contribution of the remainder term $\mathcal{R}(x, T, \ell)$ of the splitting (95). Finally, the L^2 bounds for the $\mathcal{D}$-components follow from Lemma 5.13. In fact, all such estimates are uniform with respect to $h \in [1, 2]$. □

Theorem 1.7 follows from Theorem 5.14 (for $\sigma = 1$) and from Remark 1.

In the particular case that $\gamma_{x,T}$ is supported on a closed cuspidal horocycle, our methods yield a simple proof of a weaker version of a result of P. Sarnak [51]. Sarnak's proof is based on Eisenstein series (Rankin-Selberg method).

Proposition 5.15. *Let $s > 2$ and let $\gamma_{x,T}$ be a closed cuspidal horocycle. Then there exist coefficients $c^s_{\mathcal{D}} \in \mathbb{C}$ for $\mathcal{D} \in \mathcal{B}_+^{1/2}$ and a distribution-valued function $\mathcal{C}^s(t) \in \mathcal{I}_{\mathcal{C}}$, uniformly bounded with respect to $t \geq 0$, such that the following asymptotics holds. For all $t \geq 0$,*

$$\phi_t^X(\gamma_{x,T})(f) = \int_{SM} f\, dvol + \sum_{\mathcal{D} \in \mathcal{B}_+^{1/2}} c^s_{\mathcal{D}}\, \mathcal{D}(f)\, T_t^{-S_{\mathcal{D}}} + \mathcal{C}^s(t)(f) T_t^{-\frac{1}{2}} \log T_t . \tag{127}$$

Proof. A uniformly distributed probability measure supported on a closed cuspidal horocycle is a U-invariant distribution. In fact by the Sobolev embedding theorem, if $s > 2$, then $\gamma_{x,T} \in \mathcal{I}^s(SM)$. Hence, in the splitting of $\gamma_{x,T}$ according to (80), the remainder term $\mathcal{R}(x, T) = 0$. The result then follows immediately from the description of the spectral representation of the one-parameter group $\{\phi_t^X\}$ on the space $\mathcal{I}^s(SM)$ given by Theorem 1.4 and Lemma 5.1. □

5.5. **Bounds on the components: the compact case.** In the compact case, the spectrum of the Laplacian is purely discrete and therefore the splitting (80) of a horocycle arc $\gamma_{x,T}$ becomes:

$$\gamma_{x,T} = \sum_{\mathcal{D} \in \mathcal{B}^s} c_{\mathcal{D}}(x, T)\mathcal{D} + \mathcal{R}(x, T) .$$

Lemma 5.12 can be strengthened as follows:

Lemma 5.16. *Let $s > 3$ and let $T_0 \geq 1$. There exists a sequence of bounded positive functions $\{K_{\mathcal{D}}(x, T)\}_{\mathcal{D} \in \mathcal{B}^s}$, $(x, T) \in SM \times \mathbb{R}^+$, such that, for any horocycle arc $\gamma_{x,T}$ with $T > T_0$ we have*

(128)
$$|c_{\mathcal{D}}(x,T)| \le \begin{cases} K_{\mathcal{D}}(x,T)\, T^{-S_{\mathcal{D}}}\,, & \textit{if } \mathcal{D} \notin \mathcal{B}^+_{1/4} \ \ \& \ \ S_{\mathcal{D}} < 1\,; \\ K_{\mathcal{D}}(x,T)\, T^{-S_{\mathcal{D}}} \log T & \textit{if } \mathcal{D} \in \mathcal{B}^+_{1/4}\ (\Rightarrow S_{\mathcal{D}} = 1/2)\,; \\ K_{\mathcal{D}}(x,T)\, T^{-1} \log T\,, & \textit{if } \mathcal{D} \in \mathcal{B}^s \ \ \& \ \ S_{\mathcal{D}} = 1\,; \\ K_{\mathcal{D}}(x,T)\, T^{-1}\,, & \textit{if } \mathcal{D} \in \mathcal{B}^s \ \ \& \ \ S_{\mathcal{D}} \ge 2\,. \end{cases}$$

In addition, there exists $K := K(s,T_0) > 0$ such that, for all $(x,T) \in SM \times \mathbb{R}^+$, $T \ge T_0$,

(129)
$$\sum_{\mathcal{D}\in\mathcal{B}^s} K^2_{\mathcal{D}}(x,T) \ \le\ K^2\ .$$

On the other hand, for each $\mathcal{D} \in \mathcal{B}^1 \subset \mathcal{B}^s$, there exist $K'_{\mathcal{D}} := K'(\mathcal{D}, s) > 0$ and $T'_{\mathcal{D}} := T'(\mathcal{D}, s) > 0$ such that, for all $T \ge T'_{\mathcal{D}}$,

(130)
$$\|\, c_{\mathcal{D}}(\cdot,T)\,\|_0 \ge \begin{cases} K'_{\mathcal{D}}\, T^{-S_{\mathcal{D}}}\,, & \textit{if } \mathcal{D} \notin \mathcal{B}^+_{1/4}\,; \\ K'_{\mathcal{D}}\, T^{-S_{\mathcal{D}}} \log T\,, & \textit{if } \mathcal{D} \in \mathcal{B}^+_{1/4}\,. \end{cases}$$

Proof. Let $T_0 > 0$. For all $T \ge eT_0$, there exist $h \in [1,2]$ and $\ell \in \mathbb{Z}^+$ such that $T = e^{\ell h}T_0$. By the first identity in (96), it follows that

(131)
$$\begin{aligned} c_{\mathcal{D}}(x,T) &= c_{\mathcal{D}}\left(\phi^X_{\ell h}(x), T_0, \ell\right)\ , \\ \|\, c_{\mathcal{D}}(\cdot,T)\,\|_0 &= \|\, c_{\mathcal{D}}\left(\cdot, T_0, \ell\right)\,\|_0\ . \end{aligned}$$

Since M is compact, the unit tangent bundle SM is compact and has finite diameter. Hence, by Lemma 5.5 and by the argument of Lemma 5.12, there is a constant $K > 0$ and, for each $\mathcal{D} \in \mathcal{B}^s$, there exists a positive function $K_{\mathcal{D}}(x,T)$, $(x,T) \in SM \times \mathbb{R}^+$, such that the estimates in (105), (106) and (108) hold with $\sigma = 0$. The desired uniform upper bounds then follow. The L^2 lower bounds follow immediately from Lemma 5.13. □

Theorem 1.5 follows immediately from lemma 5.5 and lemma 5.16.

Lemma 5.16 implies the following pointwise lower bound on the deviation of ergodic averages.

Corollary 5.17. *If $\mathcal{D} \in \mathcal{B}^1 \subset \mathcal{B}^s$, there exist constants $K''_{\mathcal{D}} := K''(\mathcal{D}, s) > 0$, $\alpha_{\mathcal{D}} := \alpha(\mathcal{D}, s) > 0$ and $T''_{\mathcal{D}} := T''(\mathcal{D}, s) > 0$ such that the following holds. For all $T \ge T''_{\mathcal{D}}$ there exists a set $A_T \subset SM$ of measure at least $\alpha_{\mathcal{D}} > 0$ such that, for all $x \in A_T$,*

(132)
$$|c_{\mathcal{D}}(x,T)| \ge \begin{cases} K''_{\mathcal{D}}\, T^{-S_{\mathcal{D}}}\,, & \textit{if } \mathcal{D} \notin \mathcal{B}^+_{1/4}\,; \\ K''_{\mathcal{D}}\, T^{-S_{\mathcal{D}}} \log T\,, & \textit{if } \mathcal{D} \in \mathcal{B}^+_{1/4}\,. \end{cases}$$

*Hence, the lower bound (*132*) holds for almost all $x \in SM$ and for infinitely many $T > 0$.*

Proof. Let $\mathcal{D} \in \mathcal{B}^1$. Since $S_{\mathcal{D}} < 1$, by Lemma 5.16, there exist constants $K_{\mathcal{D}} \ge K'_{\mathcal{D}} > 0$ such that, for all $x \in SM$ and all $T > T'_{\mathcal{D}}$, we have

$$|c_{\mathcal{D}}(x,T)| \le \frac{K_{\mathcal{D}}}{K'_{\mathcal{D}}}\ \|\, c_{\mathcal{D}}(\cdot,T)\,\|_0\ .$$

Fix any constant $K \in\,]0,1[$ and let

$$A_T := \{x \in SM \mid |c_{\mathcal{D}}(x,T)| > K\,\|\, c_{\mathcal{D}}(\cdot,T)\,\|_0\}\ .$$

Then, for $T > T'_{\mathcal{D}}$, we have

$$\left(K^2 + \left(\frac{K_{\mathcal{D}}}{K'_{\mathcal{D}}}\right)^2 |A_T|\right) \| c_{\mathcal{D}}(\cdot, T) \|_0^2 \geq \int_{SM\setminus A_T} |c_{\mathcal{D}}(x,T)|^2 \, d\text{vol} + \\ + \int_{A_T} |c_{\mathcal{D}}(x,T)|^2 \, d\text{vol} = \| c_{\mathcal{D}}(\cdot, T) \|_0^2 . \tag{133}$$

It follows that the measure of the set A_T is uniformly bounded below for all $T > T'_{\mathcal{D}}$ by a positive constant. In fact, we have

$$|A_T| \geq (1 - K^2) \left(\frac{K'_{\mathcal{D}}}{K_{\mathcal{D}}}\right)^2 .$$

The desired pointwise lower bounds hold on A_T by its definition and by the L^2 lower bounds in Lemma 5.16.

By a standard Borel-Cantelli argument, the set A of $x \in SM$ such that $x \in A_T$ for infinitely many $T \geq T_0$, has positive measure. By the ergodicity of the horocycle flow, the set A has full measure.

□

We conclude by proving that the Central Limit Theorem does not hold for horocycle flows on compact hyperbolic surfaces.

Proof of Corollary 1.6. Let $\mathcal{D} \in \mathcal{B}^1 \subset \mathcal{B}^s$. Let $S_{\mathcal{D}} \in]0,1[$ be its Sobolev order. Let $f \in W^s(SM)$, $s > 3$, be any zero average function such that $\mathcal{D}(f) \neq 0$ and f belongs to the kernel of all U-invariant distributions of the basis $\mathcal{B}^s$ of Sobolev order $S \leq S_{\mathcal{D}}$.

By Theorem 1.5, there exists $K > 0$ such that, for $T > 0$ sufficiently large, the functions

$$F_T := \int_0^T f(\phi_t^U(\cdot)) \, dt$$

are uniformly bounded above by $K \| F_T \|_0$. It follows that the probability distribution ρ_T of $F_T / \| F_T \|_0$ is supported within the interval $[-K, K]$. By Theorem 1.3 and Corollary 5.17, there exists an interval around zero whose complement has ρ_T-measure larger than $\alpha_{\mathcal{D}} > 0$, for all sufficiently large $T > 0$. □

6. Appendix

Proof of Lemma 2.1. The case of the principal series ($\nu \in i\mathbb{R}$) is immediate.

In the case of the complementary series, if $\nu \in]0,1]$ we write, for $k \geq 1$,

$$\Pi_{\nu,k}^{-1} = \Pi_{-\nu,k} = \left(\frac{1+\nu}{1-\nu}\right) \prod_{i=2}^{k} \frac{2i-1+\nu}{2i-1-\nu}$$

(empty products are set equal to 1). We have

$$\log \prod_{i=2}^{k} \frac{2i-1+\nu}{2i-1-\nu} = \sum_{i=2}^{k} \log\left(1 + \frac{2\nu}{2i-1-\nu}\right) .$$

We then estimate the logarithms by the inequalities

$$\frac{x}{1+x} \leq \log(1+x) \leq x, \quad \text{for } x \in \mathbb{R}^+.$$

Finally we estimate from above and below the resulting series by the integral inequality.

In the case of the discrete series ($\nu = 2n-1$), we remark that, for all $k \geq n$,

$$\Pi_{\nu,k} = \prod_{i=n+1}^{k} \frac{2i-1-\nu}{2i-1+\nu} = \prod_{i=n+1}^{k} \frac{i-n}{i+n-1} = \frac{(k-n)!(2n-1)!}{(k+n-1)!}.$$

By the Stirling's formula,

$$\frac{(k-n+1)!}{(k+n-1)!} \approx e^{-2} \frac{(k-n+1)^{k-n+1}}{(k+n-1)^{k+n-1}} \left(\frac{k-n+1}{k+n-1}\right)^{\frac{1}{2}}.$$

where $\approx$ means that, for some constant $C > 1$ independent of n and k, the ratio of the left side to the right side is bounded above by C and below by C^{-1} for all $n \geq 1$ and $k \geq n$. Hence,

$$\Pi_{\nu,k} \approx e^{-2}\nu! \left(\frac{k-n+1}{k+n-1}\right)^{k-n+1} (k+n-1)^{-\nu+\frac{1}{2}}(k-n+1)^{-\frac{1}{2}} = \tag{134}$$

$$= e^{-2}\nu! \left(\frac{k-n+1}{k+n-1}\right)^{k+n-1} (k+n-1)^{-\frac{1}{2}}(k-n+1)^{-\nu+\frac{1}{2}}.$$

Since $k \geq \ell$ and the function $(1+x)^{\frac{1}{x}}$ is decreasing on the interval $x > -1$,

$$\left(\frac{\ell+n-1}{\ell-n+1}\right)^{\ell-n+1} = \left[(1+\frac{\nu-1}{\ell-n+1})^{\frac{\ell-n+1}{\nu-1}}\right]^{\nu-1} \leq \tag{135}$$

$$\leq \left[(1+\frac{\nu-1}{k-n+1})^{\frac{k-n+1}{\nu-1}}\right]^{\nu-1} = \left(\frac{k+n-1}{k-n+1}\right)^{k-n+1}$$

and

$$\left(\frac{\ell+n-1}{\ell-n+1}\right)^{\ell+n-1} = \left[(1-\frac{\nu-1}{\ell+n-1})^{-\frac{\ell+n-1}{\nu-1}}\right]^{\nu-1} \geq \tag{136}$$

$$\geq \left[(1-\frac{\nu-1}{k+n-1})^{-\frac{k+n-1}{\nu-1}}\right]^{\nu-1} = \left(\frac{k+n-1}{k-n+1}\right)^{k+n-1}.$$

Since $k \geq \ell$, the function

$$\frac{k-n+x}{\ell-n+x}$$

is non-increasing on $\mathbb{R}^+$. Hence

$$\frac{k+n-1}{\ell+n-1} = \frac{k-n+\nu}{\ell-n+\nu} \leq \frac{k-n+1}{\ell-n+1}.$$

It follows that

$$\frac{\Pi_{\nu,k}}{\Pi_{\nu,\ell}} \leq C \left(\frac{k-n+\nu}{\ell-n+\nu}\right)^{-\nu+\frac{1}{2}} \left(\frac{k-n+1}{\ell-n+1}\right)^{-\frac{1}{2}} \leq C \left(\frac{k-n+\nu}{\ell-n+\nu}\right)^{-\nu}$$

and

$$\frac{\Pi_{\nu,k}}{\Pi_{\nu,\ell}} \geq C^{-1}\left(\frac{k-n+1}{\ell-n+1}\right)^{-\nu+\frac{1}{2}}\left(\frac{k-n+\nu}{\ell-n+\nu}\right)^{-\frac{1}{2}} \geq C^{-1}\left(\frac{k-n+1}{\ell-n+1}\right)^{-\nu}.$$

□

Proof of Lemma 4.2. Since $|\Pi_{\nu,\ell}| \geq |\Pi_{\nu,k}|$ for all $k \geq \ell \geq 0$, if $|\nu| \geq 1/2$ the estimate is immediate by the triangular inequality. If $\nu \in i\mathbb{R}$ or $\nu \in]-1/2, 1/2[$, we have

$$\frac{d}{d\nu}\log\left(\frac{\Pi_{\nu,\ell}}{\Pi_{\nu,k}}\right) = \frac{d}{d\nu}\log\prod_{i=\ell+1}^{k}\frac{2i-1+\nu}{2i-1-\nu} = \sum_{i=\ell+1}^{k}\frac{2(2i-1)}{(2i-1)^2-\nu^2} > 0.$$

By the integral inequality,

$$|\frac{d}{d\nu}\left(\frac{\Pi_{\nu,\ell}}{\Pi_{\nu,k}}\right)| = |\frac{\Pi_{\nu,\ell}}{\Pi_{\nu,k}}||\frac{d}{d\nu}\log\left(\frac{\Pi_{\nu,\ell}}{\Pi_{\nu,k}}\right)| \leq C|\frac{\Pi_{\nu,\ell}}{\Pi_{\nu,k}}|\log\left(\frac{1+k}{1+\ell}\right).$$

Hence, if $|\nu| < 1/2$ the estimate follows from the intermediate value theorem. □

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