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Nodal sets of eigenfunctions on Riemannian manifolds

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1. Introduction

Let M^n be a compact connected manifold, with C^∞ Riemannian metric. The Laplacian Δ of M is a negative definite, self-adjoint, elliptic operator. Suppose that F is a real eigenfunction of Δ with eigenvalue λ, $\Delta F = -\lambda F$. The nodal set N of F is defined to be the set of points $x \in M$ where $F(x) = 0$.

The unique continuation theorem [1] states that F never vanishes to infinite order. This places strong restrictions on the zeroes of F. By developing the machinery of Aronszajn [1], we establish a number of quantitative results concerning the nodal set. These theorems seem most interesting for large λ.

One of our main conclusions is

Theorem 1.1. *The eigenfunction F vanishes at most to order $c\sqrt{\lambda}$, for any point in M.*

When M is two dimensional, it follows from the work of Cheng [5], that F vanishes at most to order $c\lambda$. Using spherical harmonics on S^n, one obtains sequences of eigenfunctions which vanish to order $\sqrt{\lambda}$. Theorem 1.1 is a consequence of more explicit estimates for the growth of F near its zero set. The constants appearing in these estimates depend only upon the curvature and diameter of M.

Now suppose the M is a real analytic manifold with real analytic metric. The theory of analytic sets implies that N has finite $n-1$ dimensional Hausdorff measure, denoted $\mathscr{H}^{n-1}(N)$. We establish upper and lower bounds:

Theorem 1.2. $c_1\sqrt{\lambda} \leqq \mathscr{H}^{n-1}(N) \leqq c_2\sqrt{\lambda}$.

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Brüning [3] proved the lower bound for C^∞ metrics on surfaces, $n=2$. Yau has conjectured that Theorem 1.2 holds for C^∞ metrics in any dimension. This seems to be a difficult problem. A related concern is to optimize the geometric dependence of the constants c_1 and c_2. Eigenfunctions of form $\sin(k_1 x_1)\sin(k_2 x_2)\ldots\sin(k_n x_n)$ on the torus $T^n = S^1\times S^1\times\ldots\times S^1$ show that $\sqrt{\lambda}$ is the correct order of magnitude for $\mathscr{H}^{n-1}(N)$.

We plan to treat manifolds with boundary in a subsequent paper.

It may be helpful to provide some motivation for the arguments presented in this work. The remainder of the introduction serves only to orient the reader. Logically, one may omit this discussion and proceed directly to the main body of the text.

Consider the functions $F_k(z)=\mathrm{Re}(z^k)$ defined in R^2. Each F_k is harmonic, $\Delta F_k=0$, but F_k can vanish to arbitrarily high order at the origin. One sees that Theorem 1.1 will not be established by purely local arguments. However, as this example suggests, there are local constraints which relate the growth of eigenfunctions on large balls to their order of vanishing on small balls.

Suppose F is an eigenfunction of Δ, $\Delta F=-\lambda F$, defined on some geodesic ball $B(p,h_0)$ in a Riemannian manifold. Let $h<h_0$ be sufficiently small. Assume $\lambda\geqq 0$,

$$\beta > a_1\sqrt{\lambda}+a_2 \quad\text{and}\quad \beta > a_3\log\left(\max_{r\leqq h}|F| \Big/ \max_{h/10\leqq r\leqq h/5}|F|\right) \tag{1.3}$$

where r is the distance from p.

In Section 3, we will prove that (1.3) yields

$$\max_{r\leqq\delta}|F| \geqq (C_{13}\delta)^{D_{13}\beta}\max_{h/10\leqq r\leqq h/5}|F| \tag{1.4}$$

Thus, if F vanishes to high order at the center of $B(p,h_0)$ then either λ is large or $|F|$ grows rapidly on concentric balls of scale h. Related arguments give control on the ratios of the size of $|F|$ on three commensurable balls centered at p. If $|F|\leqq 1$ in $r\leqq h$ and $\max_{r\leqq h/5}|F|\geqq \exp(-D_{15}\sqrt{\lambda}-C_{14})$ then one has

$$\max_{r\leqq h/10}|F|\geqq \exp(-D_{16}\sqrt{\lambda}-C_{15}) \tag{1.5}$$

Given (1.4) and (1.5), which are strictly local results, Theorem 1.1 follows by an elementary global argument, using the compactness of M. One multiplies F by a constant to achieve $|F|\leqq 1$ and $F(x_0)=1$, for some $x_0\in M$. Recall that M is connected. If $x\in M$ is arbitrary, we join x_0 to x by an overlapping chain of balls, with radius $h/5$, whose centers are separated by a distance at most $h/10$. Using (1.5) inductively, and the analogous statements for h replaced by a fraction of h, we see that, for any $x\in M$,

$$\max_{B(x,h/200)}|F|\geqq \exp(-C_4\sqrt{\lambda}-C_5)$$

We may now use (1.4) to deduce the conclusion of Theorem 1.1. The point is that the hypothesis (1.3) has been established for $\beta > a_4\sqrt{\lambda}+a_5$.

It remains to comment on the proof of the local result (1.4). This rests upon a Carleman inequality, as does Aronszajn's proof [1] of unique continuation. Suppose u is a smooth function having compact support in $\delta/2<r<h$. If

$\beta > a_1\sqrt{\lambda} + a_2$, then the basic Carleman estimate is

$$\iint \bar{r}^{2(2-\beta)} |(\Delta+\lambda)u|^2 r^{-1}\, dr\, dt \geqq B_9 \beta^2 \iint \bar{r}^{2-2\beta} u^2 r^{-1}\, dr\, dt \tag{1.6}$$

Here $\bar{r}$ is a carefully chosen weight function, comparable to the geodesic distance r from p, in $B(p, h_0)$. Section 2 is devoted to the proof of a stronger version of (1.6). One works in geodesic polar coordinates and does repeated partial integrations in the radial and spherical variables. This is similar to the approach of Aronszajn [1]. However, we must give special attention to the dependence upon the parameter λ. To apply (1.6), let θ be a suitable cut off function supported in an annulus. If F is our eigenfunction, we may substitute $u = \theta F$ in (1.6). Applying standard elliptic theory to bound L^∞ norms by L^2 norms, one deduces

$$\begin{gathered} D_1 \beta^3 \delta^{-2\beta} \max_{(1-\frac{1}{\beta})\delta \leqq \bar{r} \leqq (1+\frac{1}{\beta})\delta} |F|^2 + (D_2\lambda + D_3) \\ \left(\frac{h}{2}\right)^{2(2-\beta)} \max_{h/4 \leqq \bar{r} \leqq 3h/4} |F|^2 \geqq (D_4\lambda + D_5)^{-n/2} \left(\frac{h}{3}\right)^{2(2-\beta)} \\ \beta^2 \max_{h/12 \leqq \bar{r} \leqq h/4} |F|^2 \end{gathered} \tag{1.7}$$

The hypothesis (1.3) permits the absorption of the second term on the left hand side of (1.7) into the right side of (1.7). Elementary arguments now give (1.4).

We proceed to motivate the proof of Theorem 1.2, valid for real analytic metrics. A main theme of this paper is that a solution of $\Delta F = -\lambda F$, on a real analytic manifold, behaves like a polynomial of degree $c_3\sqrt{\lambda}$. In fact, we will prove that F continues analytically, from a small coordinate patch $|x| < 1$ in R^n, to the complex ball $|z| < 1$ in C^n, and satisfies the growth condition

$$\max_{|z|<1} |F(z)| \leqq e^{c_4\sqrt{\lambda}} \max_{|x|<1/5} |F(x)| \tag{1.8}$$

Note that (1.8) holds for polynomials of degree $c_3\sqrt{\lambda}$. Conversely, motivated by Nevanlinna theory, we expect that (1.8) forces strong restrictions on the zero set of F. For purposes of studying the nodal set, one anticipates that F will share many common properties with polynomials.

Before discussing the nodal set in more detail, we first sketch the proof of (1.8). We may assume that our Riemannian metric continues analytically into the complex ball $|z| < 2$. The Laplacian is elliptic with analytic coefficients, so we know that F continues analytically to some neighborhood of the origin. By carefully examining the proof of analyticity [8], we see that F continues to $|z| < 1$. Moreover, one obtains the estimate

$$\max_{|z|<1} |F(z)| \leqq e^{c_5\sqrt{\lambda}} \max_{|x|<2} |F(x)| \tag{1.9}$$

Note that (1.9) is the natural estimate for solutions of $\Delta F = -\lambda F$, as one guesses from the simple one dimensional example $F(x) = \cos(\sqrt{\lambda}x)$. In itself, the inequality (1.9) does not place strong restrictions on the growth of $|F(z)|$. However, by invoking Theorem 1.1 and its proof, we obtain

$$\max_{|x|<2} |F(x)| \leqq e^{c_6\sqrt{\lambda}} \max_{|x|<1/5} |F(x)| \tag{1.10}$$

Combining the estimates (1.9) and (1.10), yields the powerful inequality (1.8).

Let us indicate the idea for obtaining the upper bound, $\mathscr{H}^{n-1}(N) \leqq c_2 \sqrt{\lambda}$, of Theorem 1.2. First suppose that $P(x)$ is a non-zero polynomial of degree $c_3 \sqrt{\lambda}$, defined for $x \in R^n$. Let $V = \{|x| < 1 \mid P(x) = 0\}$. If $\mathscr{L}$ denotes the set of lines in R^n that intersect $|x| < 1$, then integral geometry gives

$$\mathscr{H}^{n-1}(V) \leqq \int_{\mathscr{L}} |L \cap V| \, d\mu(L)$$

Here $L \in \mathscr{L}$ and $d\mu$ is a measure on $\mathscr{L}$. Moreover, $|L \cap V|$ denotes the cardinality of $L \cap V$. Clearly, $|L \cap V| \leqq c_3 \sqrt{\lambda}$ almost everywhere. So $\mathscr{H}^{n-1}(V)$ is bounded by a multiple of $\sqrt{\lambda}$. Our eigenfunction $F(x)$ need not be a polynomial, but it does extend to an analytic function satisfying (1.8). We shall show that integral geometry methods carry over to prove the required upper bound $\mathscr{H}^{n-1}(N) \leqq c_2 \sqrt{\lambda}$. Of course, the proof is considerably more difficult. Full details appear in Section 6.

Finally, we turn our attention to the lower bound $\mathscr{H}^{n-1}(N) \geqq c_1 \sqrt{\lambda}$. A maximum principle argument [3], [6] shows that every ball of radius $d_1/\sqrt{\lambda}$ contains a zero of F. Consequently, we obtain a family of pairwise disjoint balls $B_\nu = B(x_\nu, d_2/\sqrt{\lambda})$, covering a fixed portion of the volume of M, with F vanishing at the centers x_ν. The number of B_ν is at least of magnitude $d_3 \lambda^{n/2}$. Using (1.8), we shall prove that $\mathscr{H}^{n-1}(B_\nu \cap N) \geqq d_5 \lambda^{-(n-1)/2}$, the natural expectation from scaling considerations, for at least half of the balls B_ν. The desired estimate $\mathscr{H}^{n-1}(N) \geqq c_1 \sqrt{\lambda}$ follows immediately.

It remains to explicate the lower bounds $\mathscr{H}^{n-1}(B_\nu \cap N) \geqq d_5 \lambda^{-(n-1)/2}$, for half of the B_ν. We begin with the model problem of a harmonic function F on a ball $B \subset R^n$, where F vanishes at the center of B. The mean value property of harmonic functions implies that F integrates to zero over B. Consequently,

$$\int_{B_+} |F| = \int_{B_-} |F| = \tfrac{1}{2} \int_B |F|$$

where B_+ denotes the set of points where F is positive and $B_- = B - B_+$. There are three possibilities

(i) $\operatorname{Vol} B_+$ is commensurable to $\operatorname{Vol} B_-$

(ii) $\operatorname{Vol} B_+ \ll \operatorname{Vol} B_-$, but F is strongly peaked on B_+

(iii) $\operatorname{Vol} B_- \ll \operatorname{Vol} B_+$, but F is strongly peaked on B_-

In case (i), we can apply the isoperimetric inequality [7], $\mathscr{H}^{n-1}(B \cap N) \geqq d_6 \min(\operatorname{Vol} B_+, \operatorname{Vol} B_-)^{n-1/n}$, to obtain the desired lower bound $\mathscr{H}^{n-1}(B \cap N) \geqq d_7 (\operatorname{Vol} B)^{n-1/n}$. Unfortunately, cases (ii) and (iii) may sometimes occur. Examples can be constructed using Runge's approximation theorem. However, suppose one has the additional growth condition

$$\int_Q F^2 \leqq c_7 \int_B F^2 \tag{1.11}$$

where Q is a cube containing the double of B. We show that (1.11) excludes the cases (ii) and (iii). By standard elliptic theory the L^∞ norm of F on B is bounded by the L^2 norm of F on Q. If (1.11) holds, then the L^∞ norm of F on B is actually bounded by the L^2 norm of F on B itself. This allows one to bound the L^2 norm of F on B by using the L^1 norm of F. Let $E \subset B$ be any measurable set. The Cauchy-

Schwartz inequality now gives

$$\frac{\int_E |F|}{\int_B |F|} \leqq c_8 \left(\frac{\operatorname{Vol} E}{\operatorname{Vol} B}\right)^{\frac{1}{2}}$$

Taking $E=B_+$ or $E=B_-$ shows that (1.11) does indeed force case (i).

Although we have been discussing harmonic functions on R^n, similar arguments can be applied to solutions of $\Delta F = -\lambda F$ on $B_\nu = B(x_\nu, d_2/\sqrt{\lambda})$. The point is that B_ν is sufficiently small, relative to the operator $\Delta + \lambda$. Thus one has $\mathscr{H}^{n-1}(B_\nu \cap N) \geqq d_5 \lambda^{-(n-1)/2}$ provided

$$\int_{Q_\nu} |F|^2 \leqq c_9 \int_{B_\nu} |F|^2 \tag{1.12}$$

Here Q_ν is a cube, defined in a suitable coordinate system, containing the double of B_ν.

To establish (1.12) for at least half the B_ν, we need the substantial Proposition 5.11, concerning analytic functions G which satisfy a growth estimate like (1.8). One assumes that $G(x)$ is real and non-negative for real x, lying in a standard cube Q, centered at the origin. The conclusion of Proposition 5.11 is that, for $x \in Q_\nu - S$,

$$\left|\log G(x) - \log \operatorname*{Av}_{Q_\nu} G(x)\right| < d_8$$

where the set $S \subset Q$ has measure less than ε. Applying this with $G = F^2$ easily gives (1.12) for at least half of the Q_ν. The proof of Proposition 5.11 again involves reduction to the case where G is a polynomial. One proceeds by induction on the dimension n. Curiously, the one dimensional case seems deeper than the induction step. The weak type (1,1) inequality for the Hilbert transform, a basic result of Fourier analysis, lies at the heart of our argument.

This completes our guide to the proofs of Theorems 1.1 and 1.2. We now turn to the complete proofs, with all the technical details.

2. Quantitative Aronszajn inequalities

The basic tool for proving unique continuation [1] is an integral inequality of Carleman type. Our purpose here is to provide a similar estimate with better dependence upon the parameter $\lambda > 0$ and the geometry of M. This result is fundamental for our later investigations.

Let M^n be a C^∞ Riemannian manifold. Suppose $p \in M$ and the exponential map $\exp: T_p M \to M$ is a diffeomorphism up to distance h_0 from p. Then one has geodesic coordinates on the ball $B(p, h_0)$. Choose a coordinate system $t_1, t_2, \ldots, t_{n-1}$ on the standard unit sphere. In geodesic polar coordinates, we may write the metric and volume element as

$$ds^2 = dr^2 + r^2 \gamma_{ij}\, dt_i\, dt_j$$
$$d\mathrm{vol} = r^{n-1} \sqrt{\gamma}\, dr\, dt$$

Here $\gamma = \det(\gamma_{ij})$. In Euclidean space, the γ_{ij} are independent of r.

We now introduce a local conformal change in the metric and volume element. The new volume element may be different from the volume element naturally associated to the new metric. Let v be a positive constant. One may define the metric $\bar{g}_{ij}=\exp(-2vr^2)g_{ij}$. The geodesic lines starting from p coincide for the two metrics and one has

$$\bar{r}=\int_0^r e^{-vs^2}\,ds$$

In particular $\bar{r}=r+0(r^3)$ near the origin. The modified volume element is obtained by multiplying the volume element of $\bar{g}$ by $\psi(\bar{r})=\exp(2/3\,v(n-2)\bar{r}^2)$.

Consider geodesic polar coordinates for the metric $\bar{g}$. One has

$$d\bar{s}^2=d\bar{r}^2+\bar{r}^2\,\bar{\gamma}_{ij}\,dt_i\,dt_j$$

$$\overline{d\mathrm{vol}}=\bar{r}^{n-1}\sqrt{\bar{\gamma}}\,\psi(\bar{r})\,d\bar{r}\,dt$$

The inverse matrix of $\bar{\gamma}_{ij}$ will be denoted by $\bar{\gamma}^{ij}$. Also we may define $\omega=\partial/\partial\bar{r}\log(\psi\sqrt{\bar{\gamma}})$. The metric and volume element were changed to achieve:

Lemma 2.1. *If $h<h_0$ is sufficiently small and v is suitably large, then on $B(p,h)$,*

(i) $\dfrac{\partial\bar{\gamma}^{ij}}{\partial\bar{r}}\geqq(v\bar{r}-\omega)\,\bar{\gamma}^{ij}$

(ii) $\omega\leqq -v\bar{r}$.

Proof. Aronszajn [1, p. 241] calculated

$$\frac{\partial\bar{\gamma}^{ij}}{\partial\bar{r}}=\frac{\partial\gamma^{ij}}{\partial r}+\frac{8}{3}\,v\bar{r}\,\bar{\gamma}^{ij}+0(r^2)$$

which implies

$$\omega=\frac{\partial}{\partial r}\log\sqrt{\gamma}-\frac{4}{3}\,v\bar{r}+0(r^2)$$

The lemma follows from the theory of Jacobi fields, [2, pp. 250–257].

The primary step is to establish a Carleman estimate for the operator $\bar{\Delta}+\bar{\lambda}$. Here $\bar{\Delta}$ is the Laplacian of $\bar{g}$ and $\bar{\lambda}=\lambda\exp(2vr^2)$. Let $u\in C_0^\infty(B(p,h))$ and suppose that u vanishes on a neighborhood of the origin p. Suppose α is a positive constant. We want to derive a lower bound for the integral

$$I=\iint\bar{r}^{-2\alpha}\,|(\bar{\Delta}+\bar{\lambda})u|^2\,\bar{r}^{n-1}\sqrt{\bar{\gamma}}\,\psi\,d\bar{r}\,dt \tag{2.2}$$

In geodesic polar coordinates, the Laplacian may be written as

$$\bar{\Delta}u=\frac{\partial^2u}{\partial\bar{r}^2}+\left(\frac{n-1}{\bar{r}}+\frac{\partial\log\sqrt{\bar{\gamma}}}{\partial\bar{r}}\right)\frac{\partial u}{\partial\bar{r}}+\frac{1}{\bar{r}^2\sqrt{\bar{\gamma}}}\,\frac{\partial}{\partial t_i}\left(\sqrt{\bar{\gamma}}\,\bar{\gamma}^{ij}\,\frac{\partial u}{\partial t_j}\right)$$

We substitute this expression in I and make a change of variable, $\bar{r}=e^{-\varrho}$. Define $\beta=\alpha-\dfrac{n}{2}+2$ and $u=e^{-\beta\varrho}w$. One has, with $w'=\partial w/\partial\varrho$,

$$I=\iint|\,w''-(n-2+2\beta-\theta)\,w'+\beta(\beta+n-2-\theta)\,w+\Delta_\varrho w+\bar{\lambda}e^{-2\varrho}w\,|^2\sqrt{\bar{\gamma}}\,\psi\,d\varrho\,dt$$

In the above integrand, $\theta=\partial/\partial\varrho(\log\sqrt{\bar{\gamma}})$ and $\Delta_\varrho w=\dfrac{1}{\sqrt{\bar{\gamma}}}\,\dfrac{\partial}{\partial t_i}\left(\sqrt{\bar{\gamma}}\,\bar{\gamma}^{ij}\,\dfrac{\partial w}{\partial t_j}\right)$.

For later reference, we change variables from r to ϱ in Lemma 2.1. Define $\mu = \partial/\partial\varrho \log(\psi\sqrt{\bar{\gamma}})$ and $\chi = -\log h$. We may write

Lemma 2.3. *If* $\varrho > \chi$, *then*

(i) $-(\bar{\gamma}^{ij})' \geqq (ve^{-2\varrho} + \mu)\bar{\gamma}^{ij}$

(ii) $\mu \geqq ve^{-2\varrho}$.

Removing the θ terms from I gives

$$I_0 = \iint |w'' - (n-2+2\beta)w' + \beta(\beta+n-2)w + \Delta_\varrho w + \bar{\lambda}e^{-2\varrho}w|^2 \sqrt{\bar{\gamma}}\,\psi\, dt\, d\varrho$$

Clearly, by the triangle inequality,

$$I \geqq \tfrac{1}{2} I_0 - I_1$$

with

$$I_1 = \iint \theta^2 |w' - \beta w|^2 \sqrt{\bar{\gamma}}\,\psi\, dt\, d\varrho$$

We proceed to derive a lower bound for I_0. The term I_1 will be absorbed later. If f is any function, then elementary algebra gives

$$I_0 = I_2 + I_3 + I_4 + I_5$$

with

$$\begin{aligned} I_2 = \iint [&(w'' + \beta(\beta+n-2)w + \Delta_\varrho w + \bar{\lambda}e^{-2\varrho}w)^2 \\ &+ (2\beta+n-2)^2 f^2 w^2 + 2(\beta(\beta+n-2)w + \bar{\lambda}e^{-2\varrho}w) \\ &\cdot (2\beta+n-2)fw]\,\psi\sqrt{\bar{\gamma}}\, dt\, d\varrho \\ I_3 = (2\beta&+n-2)^2 \iint (w' + fw)^2\, \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho \\ I_4 = -2(2&\beta+n-2)\iint w'[w'' + \beta(\beta+n-2)w \\ &+ \Delta_\varrho w + \bar{\lambda}e^{-2\varrho}w + (2\beta+n-2)fw]\,\psi\sqrt{\bar{\gamma}}\, dt\, d\varrho \\ I_5 = -2(2&\beta+n-2)\iint fw[\beta(\beta+n-2)w + \bar{\lambda}e^{-2\varrho}w \\ &+ (2\beta+n-2)fw]\,\psi\sqrt{\bar{\gamma}}\, dt\, d\varrho\,. \end{aligned}$$

Suppose that $\beta > a_1\sqrt{\lambda}$ for a sufficiently large constant a_1. We choose

$$f = \frac{1}{2}\frac{\partial}{\partial\varrho}\log\left(1 + \frac{\bar{\lambda}}{\beta(\beta+n-2)}e^{-2\varrho}\right) + \frac{1}{2}\mu$$

By Lemma 2.3, we have $f > 0$. The positivity of f insures that $I_2 \geqq 0$.

Integration by parts in ϱ gives

$$I_4 + I_5 = J_4 + J_5$$

with

$$J_4 = -2(2\beta+n-2)\iint w'[w'' + \Delta_\varrho w + (2\beta+n-2)fw]\,\psi\sqrt{\bar{\gamma}}\, dt\, d\varrho$$

$$J_5 = -2(2\beta+n-2)^2 \iint f^2 w^2\, \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho$$

Partial integration in t gives

$$J_4 = -2(2\beta+n-2)\iint \left[w'w'' - \bar{\gamma}^{ij} \frac{\partial w'}{\partial t_i}\frac{\partial w}{\partial t_j} + (2\beta+n-2) f w w' \right] \psi \sqrt{\bar{\gamma}}\, dt\, d\varrho$$

For this calculation, it is crucial that ψ is a function only of ϱ, independent of t.

We now integrate by parts in ϱ to yield

$$J_4 = (2\beta+n-2)\iint \left[\mu(w')^2 + (-(\bar{\gamma}^{ij})' - \mu\bar{\gamma}^{ij}) \frac{\partial w}{\partial t_i}\frac{\partial w}{\partial t_j} + (2\beta+n-2)(f'+f\mu) w^2 \right] \psi \sqrt{\bar{\gamma}}\, dt\, d\varrho$$

Using Lemma 2.3 (i), one finds that

$$J_4 \geqq (2\beta+n-2)\iint [\mu(w')^2 + (2\beta+n-2)(f'+f\mu)w^2]\psi\sqrt{\bar{\gamma}}\, dt\, d\varrho$$

Moreover, the definitions of f and μ imply, for $\beta > a_2$,

$$J_4 \geqq -C_1\beta^2 \iint e^{-2\varrho}[(w')^2 + w^2]\psi\sqrt{\bar{\gamma}}\, dt\, d\varrho$$

Similarly

$$J_5 \geqq -C_2\beta^2 \iint e^{-2\varrho} w^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho$$

and

$$I_3 \geqq \beta^2 \iint (w')^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho - C_3\beta^2 \iint e^{-2\varrho} w^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho$$

Combining these estimates, we may write

$$I_0 \geqq \beta^2 \iint (w')^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho - C_4\beta^2 \iint e^{-2\varrho}[(w')^2 + w^2]\psi\sqrt{\bar{\gamma}}\, dt\, d\varrho$$

If χ is sufficiently large, then since $\varrho > \chi$,

$$I_0 \geqq B_1\beta^2 \iint (w')^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho - C_4\beta^2 \iint e^{-2\varrho} w^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho$$

Integration by parts and the Schwartz inequality force

$$\iint e^{-2\varrho} w^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho \leqq C_6 e^{-2\chi} \iint (w')^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho \tag{2.4}$$

So, for large χ, we have

$$I_0 \geqq B_2\beta^2 \iint (w')^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho$$

The definition of θ implies

$$I_1 \leqq C_7\beta^2 \iint [(w')^2 + w^2] e^{-2\varrho} \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho$$

Using (2.4) again, we absorb I_1 in part of I_0 to deduce

$$I \geqq B_3\beta^2 \iint (w')^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho \tag{2.5}$$

Suppose that the support of w is contained in $\bar{r} > \delta$. Integrating by parts and applying the Schwartz inequality gives

$$\delta \iint e^{\varrho} w^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho \leqq C_8 \iint (w')^2 \psi\sqrt{\bar{\gamma}}\, dt\, d\varrho \tag{2.6}$$

From (2.4), (2.5), and (2.6) one finds that

$$I \geqq B_4 \beta^2 \iint e^{-2\varrho} w^2 \psi \sqrt{\bar{\gamma}}\, dt\, d\varrho + B_5 \delta \beta^2 \iint e^{\varrho} w^2 \psi \sqrt{\bar{\gamma}}\, dt\, d\varrho$$

We now change variables from ϱ and w back to $\bar{r}$ and u. Our results thus far are summarized in

Proposition 2.7. *Suppose that u has support in $\delta < \bar{r} < h$, where $h < h_0$ is sufficiently small. Assume that $\alpha > a_1 \sqrt{\lambda} + a_2$ with suitably large constants a_1, a_2. Then the integral I given in* (2.2) *satisfies*

$$I \geqq B_4 \beta^2 \iint \bar{r}^{-2\alpha-2} u^2 \psi \sqrt{\bar{\gamma}}\, \bar{r}^{n-1}\, dt\, d\bar{r} + B_5 \delta \beta^2 \iint \bar{r}^{-2\alpha-5} u^2 \psi \sqrt{\bar{\gamma}}\, \bar{r}^{n-1}\, dt\, d\bar{r}.$$

Thus far we have dealt solely with the Laplacian $\bar{\Delta}$. Of course, our basic interest concerns the Laplacian Δ of the original metric. Recall that $\bar{g} = \phi g$ with conformal factor $\phi = \exp(-2\nu r^2)$. A calculation in geodesic polar coordinates gives

$$\bar{\Delta} u = \phi^{-1} \Delta u - (2n-4)\, \phi^{-1} \nu r \frac{\partial u}{\partial r} \tag{2.8}$$

Define

$$K = \iint \bar{r}^{-2\alpha} |(\Delta + \lambda) u|^2 \bar{r}^{n-1} \sqrt{\bar{\gamma}}\, \psi\, d\bar{r}\, dt$$

Using (2.8) and the triangle inequality, we have, for h sufficiently small,

$$K \geqq B_6 I - B_7 \iint \bar{r}^{-2\alpha} \left(\bar{r} \frac{\partial u}{\partial \bar{r}}\right)^2 \bar{r}^{n-1} \sqrt{\bar{\gamma}}\, d\bar{r}\, dt$$

However

$$\iint \bar{r}^{-2\alpha} \left(\bar{r} \frac{\partial u}{\partial \bar{r}}\right)^2 \bar{r}^{n-1} \sqrt{\bar{\gamma}}\, \psi\, d\bar{r}\, dt = \iint |w' - \beta w|^2 e^{-4\varrho}\, d\varrho\, dt$$

Using (2.4) and (2.5) as was done above gives

$$K \geqq B_8 I \tag{2.9}$$

for χ sufficiently large.

Recall the definition of $\beta = \alpha - \frac{n}{2} + 2$. Using Proposition 2.7 and (2.9) we deduce the main result of this section:

Proposition 2.10. *Suppose that u has support in $\delta/2 < r < h$, where $h < h_0$ is suitably small. Assume that $\beta > a_1 \sqrt{\lambda} + a_2$ with sufficiently large constants a_1 and a_2. Then*

$$\iint \bar{r}^{2(2-\beta)} |(\Delta + \lambda) u|^2 r^{-1}\, dr\, dt$$
$$\geqq B_9 \beta^2 \iint \bar{r}^{2-2\beta} u^2 r^{-1}\, dr\, dt + C_9 \delta \beta^2 \iint \bar{r}^{-1-2\beta} u^2 r^{-1}\, dr\, dt.$$

Let C_{10} be an upper bound for the absolute values of the sectional curvatures in $B(p, h_0)$. The theory of Jacobi fields implies that the constants appearing in Proposition 2.10 depend only upon C_{10} and h_0 [2, pp. 250–257]. Recall that Jacobi's equation is a second order homogeneous linear differential equation with coefficients depending upon the curvature tensor. The solution space consists of

Jacobi fields, which arise as the transverse vector fields to one parameter families of geodesics. In geodesic polar coordinates, the coefficients γ_{ij} of the metric tensor are given by the inner product of suitable Jacobi fields, defined along radial geodesics starting at the origin. Elementary calculations and standard comparison theorems allow one to estimate the γ_{ij}, the first two radial derivatives of the γ_{ij}, other related quantities such as γ^{ij} and γ, and their analogous derivatives, in terms of C_{10} and h_0. At each step of our derivation of Proposition 2.10, only such information about the metric was demanded. This involved some modification of Aronszajn's original approach [1]. In particular, the weight function ψ was introduced to achieve good geometric dependence.

3. Local properties of eigenfunctions

We continue in the framework of our previous section. Let F be an eigenfunction defined on $B(p, h_0)$. That is, $\Delta F = -\lambda F$, for some $\lambda > 0$. The idea is to substitute $u = \theta F$ into Proposition 2.10, where θ is an appropriate cut-off function. This gives interesting relations between the order of vanishing of F at p and the rate of growth of F on a neighborhood of p.

Suppose that $h < h_0$ is sufficiently small. Recall the estimate $\bar{r} = r + 0(r^3)$ near the origin p. Therefore, we may construct a smooth function $\theta(r)$ satisfying the following conditions:

i) $\theta = 0$ $\qquad \bar{r} < \delta\left(1 - \frac{1}{10\beta}\right)$

ii) $|\nabla\theta| \leqq C_1 \beta\delta^{-1}$ $\qquad \left(1 - \frac{1}{10\beta}\right)\delta < \bar{r} < \delta$

$|\Delta\theta| \leqq C_2 \beta^2 \delta^{-2}$

iii) $\theta = 1$ $\qquad \delta < \bar{r} < h/2$

iv) $|\nabla\theta| \leqq C_3$ $\qquad h/2 < \bar{r} < \frac{2}{3}h$

$|\Delta\theta| \leqq C_4$

v) $\theta = 0$ $\qquad \bar{r} > \frac{2}{3}h$

Of course, the constants appearing in iv) depend upon h. However, we want to emphasize the dependence on the parameters β and δ.

Define $u = \theta F$. Since F is an eigenfunction with eigenvalue $-\lambda$, a computation gives $\Delta u + \lambda u = F\Delta\theta + 2\nabla\theta \cdot \nabla F$. Standard elliptic theory bounds $|\nabla F|$ on a ball using $\max|F|$ on a larger ball. It is easy to deduce:

i) $\Delta u + \lambda u = 0$ $\qquad \bar{r} < \delta\left(1 - \frac{1}{10\beta}\right)$

ii) $|\Delta u + \lambda u| \leqq C_5 \beta^2 \delta^{-2} \max\limits_{(1-\frac{1}{5})\delta \leqq \bar{r} \leqq (1+\frac{1}{5})\delta} |F|$ $\qquad \left(1 - \frac{1}{10\beta}\right)\delta < \bar{r} < \delta$

iii) $\Delta u + \lambda u = 0$ $\qquad \delta < \bar{r} < h/2$

iv) $|\Delta u + \lambda u| \leqq (C_6 \lambda^{\frac{1}{2}} + C_7) \max\limits_{h/4 \leqq \bar{r} \leqq 3h/4} |F|$ $\qquad h/2 < \bar{r} < 2h/3$

v) $\Delta u + \lambda u = 0$ $\qquad \bar{r} > 2/3h$

The eigenvalue λ does not appear explicitly in ii) since $\beta > a_1\sqrt{\lambda} + a_2$. We designed our quantitative Aronszajn inequalities so that iii) could be exploited.

One now substitutes $u = \theta F$ in Proposition 2.10. For δ less than a suitable multiple of h:

$$\begin{aligned} &D_1\beta^3\delta^{-2\beta} \max_{(1-\frac{1}{\beta})\delta \leqq \bar r \leqq (1+\frac{1}{\beta})\delta} |F|^2 + (D_2\lambda + D_3) \\ &\left(\frac{h}{2}\right)^{2(2-\beta)} \max_{h/4 \leqq \bar r \leqq 3h/4} |F|^2 \geqq (D_4\lambda + D_5)^{-n/2}\left(\frac{h}{3}\right)^{2(2-\beta)} \\ &\beta^2 \max_{h/12 \leqq \bar r \leqq h/4} |F|^2 + D_6\beta^2\delta \iint_{\delta < \bar r < h/2} \bar r^{-1-2\beta} F^2 r^{-1}\, dr\, dt \end{aligned} \tag{3.1}$$

In the first term on the right hand side, we used standard elliptic theory to bound the L^∞ norm of F by a multiple of its L^2 norm.

It is now straightforward to deduce the central result of this section:

Proposition 3.2. *Suppose that $\beta > a_1\sqrt{\lambda} + a_2$ for sufficiently large constants a_1 and a_2. In addition, assume one has the lower bound*

$$\beta > a_3 \log\left(\max_{h/4 \leqq \bar r \leqq 3h/4} |F| \Big/ \max_{h/12 \leqq \bar r \leqq h/4} |F|\right).$$

Then, we may write

$$\begin{aligned} &D_1\beta^3\delta^{-2\beta} \max_{(1-\frac{1}{\beta})\delta \leqq \bar r \leqq (1+\frac{1}{\beta})\delta} |F|^2 \\ &\geqq \tfrac{1}{2}(D_4\lambda + D_5)^{-n/2}\left(\frac{h}{3}\right)^{2(2-\beta)} \beta^2 \max_{h/12 \leqq \bar r \leqq h/4} |F|^2 \\ &+ D_6\beta^2\delta \iint_{\delta \leqq \bar r \leqq h/2} \bar r^{-1-2\beta} F^2 r^{-1}\, dr\, dt. \end{aligned}$$

Proof. The additional hypothesis on β allows one to absorb the last term on the left hand side of (3.1) into the first term on the right hand side.

The remainder of this section is devoted to developing various corollaries of Proposition 3.2. We assume that the given hypotheses on β are satisfied throughout.

To begin, one has

Corollary 3.3.

$$\max_{\frac{2}{3}\delta \leqq \bar r \leqq 2\delta} |F| \geqq D_7 (D_8\delta)^{C_8\beta} \max_{h/12 \leqq \bar r \leqq h/4} |F|.$$

Proof. This follows by dropping the second term on the right hand side of 3.2 and applying elementary estimates.

Retaining only the integral term on the right hand side of 3.2 yields

$$\max_{\frac{2}{3}\delta \leqq \bar r \leqq 2\delta} |F|^2 \geqq C_9 e^{-D_9\beta} \iint_{\frac{4}{3}\delta \leqq \bar r \leqq 10\delta} F^2 r^{-1}\, dr\, dt \tag{3.4}$$

Moreover, we may write

Corollary 3.5.

(i) $\max_{\frac{2}{3}\delta \leqq \bar r \leqq 2\delta} |F| \geqq C_{10} e^{-D_{10}\beta} \max_{\frac{3}{2}\delta \leqq \bar r \leqq \frac{9}{2}\delta} |F|$

(ii) $\max_{\bar r \leqq 2\delta} |F| \geqq C_{10} e^{-D_{10}\beta} \max_{\bar r \leqq \frac{9}{2}\delta} |F|.$

Proof. Part (i) follows from (3.4) and the standard elliptic theory bounding L^∞ norm by L^2 norm. The λ dependence does not appear explicitly since $\beta > a_1\sqrt{\lambda} + a_2$. Part (ii) is a consequence of (i) via simple logic.

We return again to 3.2 and employ an alternative lower bound for the integral term on the right hand side. One finds that

$$\max_{(1-\frac{1}{\beta})\delta \leqq \bar{r} \leqq (1+\frac{1}{\beta})\delta} F^2 \geqq C_{11}\beta^{-1} \iint_{\delta < \bar{r} < (1+\frac{10}{\beta})\delta} F^2 r^{-1}\, dr\, dt \tag{3.6}$$

From this we deduce

Corollary 3.7.

(i) $\max_{(1-\frac{1}{\beta})\delta \leqq \bar{r} \leqq (1+\frac{1}{\beta})\delta} |F| \geqq D_{11}\beta^{\frac{-n-1}{2}} \max_{(1+\frac{1}{\beta})\delta < \bar{r} < (1+\frac{1}{\beta})^2\delta/(1-\frac{1}{\beta})} |F|$

(ii) $\max_{\bar{r} \leqq \delta(1+\frac{1}{\beta})} |F| \geqq D_{11}\beta^{\frac{-n-1}{2}} \max_{\bar{r} < (1+\frac{1}{\beta})^2\delta/(1-\frac{1}{\beta})} |F|$.

Proof. Entirely analogous to the proof of Corollary 3.5.

One may apply Corollary 3.7 to obtain bounds for $|\nabla F|$. Replacing δ by $\delta/(1+\frac{1}{\beta})$ in (ii) gives

$$\max_{\bar{r} \leqq \delta} |F| \geqq D_{11}\beta^{\frac{-n-1}{2}} \max_{\bar{r} \leqq (1+\frac{1}{\beta})\delta} |F|$$

The theory of elliptic differential equations implies

$$\max_{\bar{r} \leqq \delta} |\nabla F| \leqq C_{12}\beta\delta^{-1} \max_{\bar{r} \leqq (1+\frac{1}{\beta})\delta} |F|$$

Thus one has

Corollary 3.8. $\max_{\bar{r} \leqq \delta} |\nabla F| \leqq D_{12}\beta^{\frac{n}{2}+\frac{3}{2}}\delta^{-1} \max_{\bar{r} \leqq \delta} |F|$.

It would be interesting to improve the power of β appearing in Corollary 3.8.

Since $r = \bar{r} + 0(r^3)$ it is straightforward to formulate these corollaries with the domains specified by bounds on r. We will do this explicitly for those results which will be quoted in subsequent sections. Let $h < h_0$ be sufficiently small. To satisfy the hypothesis of Proposition 3.2, it suffices to assume

$$\beta > a_1\sqrt{\lambda} + a_2 \quad \text{and} \quad \beta > a_3 \log\left(\max_{r \leqq h} |F| \Big/ \max_{h/10 \leqq r \leqq h/5} |F|\right) \tag{3.9}$$

Using Corollary 3.3 and Corollary 3.5 (ii), one deduces

Proposition 3.10. *If* (3.9) *holds, then*

(i) $\max_{r \leqq \delta} |F| \geqq (C_{13}\delta)^{D_{13}\beta} \max_{h/10 \leqq r \leqq h/5} |F|$

(ii) $\max_{r \leqq \delta} |F| \geqq e^{-D_{14}\beta} \max_{r \leqq 2\delta} |F|$.

We need to derive a logical refinement. Assume instead that

$$\beta > a_1\sqrt{\lambda} + a_2 + a_3 \log\left(\max_{r \leqq h} |F| \Big/ \max_{r \leqq h/5} |F|\right) \tag{3.11}$$

Then one has

Proposition 3.12. *If* (3.11) *holds, it implies*

$$\max_{r \leqq h/10} |F| \geqq e^{-D_{14}\beta} \max_{r \leqq h/5} |F|.$$

Proof. If $\max_{r \leqq h/10} |F| = \max_{r \leqq h/5} |F|$ the result is obvious. Otherwise, (3.9) holds and one employs Proposition 3.10 (ii).

A special case of Proposition 3.12 is

Corollary 3.13. *Assume* $|F| \leqq 1$ *in* $r \leqq h$ *and* $\max_{r \leqq h/5} |F| \geqq \exp(-D_{15}\sqrt{\lambda} - C_{14})$. *Then one has*

$$\max_{r \leqq h/10} |F| \geqq \exp(-D_{16}\sqrt{\lambda} - C_{15}).$$

Let C_{16} be an upper bound for the absolute value of the sectional curvature in $B(p, h_0)$. Using Jacobi fields and the geometric treatment of elliptic theory [4, pp. 16–18], we see that the constants of the above results depend only upon the chosen $h < h_0$ and C_{16}.

4. Eigenfunctions on compact manifolds

Let M be a compact C^∞ Riemannian manifold. Suppose that F is an eigenfunction of the Laplacian $\Delta F = -\lambda F$. Since Δ is negative and self-adjoint, one has $\lambda > 0$. We may normalize F in $L^\infty M$, $\|F\|_\infty = 1$. Our purpose is to extend the results of earlier sections using global considerations.

Suppose that C_1 is a positive upper bound for the absolute values of the sectional curvatures on M. We apply our earlier results with $h = b_1 C_1^{-\frac{1}{2}}$, where b_1 is a constant. If $x \in M$, let $B(x, h)$ denote a ball of radius h centered at x. Because M is compact, one has

Proposition 4.1. *For any* $x \in M$, *we have*

$$\max_{B(x, h/5)} |F| > \exp(-C_2\sqrt{\lambda} - C_3)$$

Here the constants depend only upon an upper bound for i) *the absolute values of the sectional curvatures of M and* ii) *the diameter of M.*

Proof. Our normalization $\|F\|_\infty = 1$ guarantees the existence of a point $x_0 \in M$ with $|F(x_0)| = 1$. Choose a finite sequence of points $x_0, x_1, x_2, \ldots, x_l = x$ with $x_{i+1} \in B(x_i, h/10)$. Of course, l is bounded above using the diameter of M and C_1.

Suppose by induction that

$$\max_{B(x_i, h/5)} |F| \geqq \exp(-D_i\sqrt{\lambda} - E_i)$$

where the constants D_i and E_i have the correct geometric dependence. Our choice of h guarantees that exp: $T_{x_i}M \to M$ is a local diffeomorphism. To avoid dependence upon the injectivity radius of M, we lift our metric and eigenfunction to a ball $\bar{B}(0, h/5) \subset T_{x_i}M$. This lift preserves L^∞ norm. Applying Corollary 3.13

and returning from $T_{x_i}M$ to M yields

$$\max_{B(x_i,h/10)} |F| \geqq \exp\left(-D_{i+1}\sqrt{\lambda}-E_{i+1}\right)$$

Since $B(x_i, h/10) \subset B(x_{i+1}, h/5)$ this completes the induction.

Replacing h by $h/20$ and applying the same argument gives, for any $x \in M$,

$$\max_{B(x,h/200)} |F| \geqq \exp\left(-C_4\sqrt{\lambda}-C_5\right)$$

Therefore $\beta > a_4\sqrt{\lambda}+a_5$ satisfies (3.9), for balls centered at any point $p \in M$. Applying Proposition 3.10 gives

Theorem 4.2. *For any $x \in M$, one has for $\delta < a_6 h$,*

(i) $\max_{B(x,\delta)} |F| \geqq (C_6\delta)^{C_7\sqrt{\lambda}+C_8} \max_{B(x,h/5)-B(x,h/10)} |F|$

(ii) $\max_{B(x,\delta)} |F| \geqq e^{-C_9\sqrt{\lambda}-C_{10}} \max_{B(x,2\delta)} |F|$

The constants appearing depend only upon an upper bound for i) *the absolute values of the sectional curvatures on M and* ii) *the diameter of M.*

Of course, Theorem 4.2 (i) contains Theorem 1.1 of the introduction. Theorem 4.2 (ii) will be a major tool in the remainder of this work.

5. Holomorphic functions – lower bound

Our goal in the rest of our paper is to establish upper and lower bounds for the nodal volume on real analytic Riemannian manifolds. Sections 5 and 6 are devoted to some preliminary results concerning the zero sets of holomorphic functions. These results will be applied in Section 7 to prove Theorem 1.2. The present section contains information relevant to the lower bound on the nodal volume.

We begin with one complex variable. The basic result is then

Proposition 5.1. *Suppose $F(z)$ is holomorphic on $|z|<3$ and $\max_{|z|\leqq 2}|F(z)| \leqq |F(0)|\exp(C_1 d)$. Assume $F(x)$ is real and non-negative for $|x| \leqq 1$. For d sufficiently large, cover $|x| \leqq 1$ by disjoint subintervals Q_ν of length C_2/d. Let $\varepsilon > 0$ be given. Then outside a set E of measure less than ε, we have $\left|\log F(x) - \log \operatorname{Av}_{Q_\nu} F\right| \leqq C_3$, $x \in Q_\nu - E$. The constant C_3 depends upon ε but not on d.*

The proof of Proposition 5.1 will be presented through a sequence of lemmas. We may assume that ε is sufficiently small and $F(0)=1$. Constants appearing below may depend upon ε.

Choose r so that $F(z) \neq 0$ for $|z|=r$. The Blaschke factor is defined by

$$B_r(z,\alpha) = \frac{(z-\alpha)/r}{1-\bar{\alpha}z/r^2}$$

We may write $F(z) = e^{G(z)} \prod_{\alpha} B_r(z, \alpha)$, $|z| \leqq r$. The product runs over the zeroes of F in $|z| < r$ and G is holomorphic. One has

Lemma 5.2. *F has at most $0(d)$ zeroes in $|z| < 3/2$.*

Proof. Choose r close to two and evaluate the corresponding Blaschke representation at zero.

Now fix r close to 3/2. The function G appearing in the product formula then satisfies:

Lemma 5.3.

(i) $\max_{|z|=r} \operatorname{Re} G \leqq C_3 d$

(ii) $\operatorname{Av}_{|z|=r} \operatorname{Re} G \geqq 0$

(iii) $\max_{|z| \leqq 1} |\nabla \operatorname{Re} G| \leqq C_4 d$

Proof. Part (i) follows immediately from the Blaschke formula. For (ii), one uses the mean value property of harmonic function. Part (iii) is deduced from (i), (ii) and the Poisson kernel representation of harmonic functions.

Define $f(z) = \sum_{\alpha} \log |z - \alpha|$. Using Lemma 5.3 (iii) and elementary arguments, we have for $x \in Q_v$,

$$\left| \log F(x) - \log \operatorname{Av}_{Q_v} F \right| < |f(x) - \log \operatorname{Av} e^f| + C_6$$

The main part of the proof is to bound the right hand side. To begin one has.

Lemma 5.4. *Outside a set E_1 of measure less than $a_1 \varepsilon$, we have $|f'| < C_7 d$.*

Proof. Since $|x - \alpha| = |x - \bar{\alpha}|$, we may assume $\operatorname{Im} \alpha \leqq 0$. If all $\operatorname{Im} \alpha < 0$, then the definition of the Hilbert transform [9, p. 130] H, gives

$$f' = H \sum_{\alpha} q_\alpha, \quad q_\alpha = -\operatorname{Im} \left(\frac{1}{z - \alpha} \right)$$

Clearly $\| q_\alpha \|_1 < C_8$, with C_8 independent of α. The weak type (1,1) property of H [9, p. 187] completes the proof for $\operatorname{Im} \alpha < 0$. Since these estimates are uniform in α, the result also holds for $\operatorname{Im} \alpha \leqq 0$.

Suppose $x, x_v \in Q_v$ and let A_v be the set of roots α with $\operatorname{Re} \alpha$ of distance less than $m(Q_v) = 0(1/d)$ from Q_v. We decompose

$$f(x) = \sum_{\alpha \in A_v} \log |x - \alpha| + \sum_{\alpha \notin A_v} \log |x - \alpha| = b_v(x) + g_v(x)$$

and estimate each of the two terms.

Define E_2 to be the union of those Q_v with A_v containing more than C_9 roots. By Lemma 5.2, we may require the measure $m(E_2) < a_2 \varepsilon$. If $Q_v \not\subset E_2$, then let $Q_{v\varepsilon} \subset Q_v$ be the subset of $x \in Q_v$ with $|x - \alpha| < C_{10}/d$ for some α. One may assume $m(Q_{v\varepsilon}) < a_3 \varepsilon / d < m(Q_v)/2$.

Lemma 5.5. *If $Q_\nu \not\subset E_2$ and $x \in Q_\nu - Q_{\nu\varepsilon}$, then*

(i) $|b_\nu(x) - \max_{Q_\nu} b_\nu(x)| < C_{11}$

(ii) $|b'_\nu(x)| < C_{12} d$.

Proof. If $x \in Q_\nu - Q_{\nu\varepsilon}$, then $\log|x-\alpha| \geqq \max_{x\in Q_\nu} \log|x-\alpha| - C_{13}$, for each $\alpha \in A_\nu$. Summing over α gives $b_\nu(x) \geqq \max_{x\in Q_\nu} b_\nu(x) - C_{14}$, which implies (i). Part (ii) is immediate from the definitions of E_2 and $Q_{\nu\varepsilon}$.

We now turn to g_ν. Since $g'_\nu = f' - b'_\nu$, the following is immediate from Lemma 5.4 and Lemma 5.5 (ii):

Lemma 5.6. *Suppose $Q_\nu \not\subset E_1 \cup E_2 \cup Q_{\nu\varepsilon}$, then there exists $x_\nu \in Q_\nu - Q_{\nu\varepsilon}$ with $|g'_\nu(x_\nu)| < C_{14} d$.*

It is also necessary to estimate the second derivative. There exists a union E_3 of intervals Q_ν with $m(E_3) < a_4 \varepsilon$ so that one has:

Lemma 5.7. *If $Q_\nu \not\subset E_3$, then* $\max_{x\in Q_\nu} |g''_\nu(x)| \leqq C_{15} d^2$.

Proof. Clearly $|g''_\nu(x)| \leqq C_{16} \sum_{\alpha \notin A_\nu} |x-\alpha|^{-2}$ and the right hand side has a constant order of magnitude for $x \in Q_\nu$. Thus

$$\sum_\nu \max_{Q_\nu} |g''_\nu(x)|\, |Q_\nu| \leqq C_{17} \sum_\alpha \int\limits_{\substack{|x|<1 \\ |x-\mathrm{Re}\,\alpha| > C_{18} d^{-1}}} |x-\alpha|^{-2} \leqq C_{19} d^2$$

This implies the existence of E_3 so that the conclusion of Lemma 5.7 holds.

It is now easy to deduce

Lemma 5.8. *Suppose $Q_\nu \not\subset E_1 \cup E_2 \cup E_3 \cup Q_{\nu\varepsilon}$ and $x_\nu \in Q_\nu$ is obtained from Lemma 5.6. Then for $x \in Q_\nu$ one has $|g_\nu(x) - g_\nu(x_\nu)| < C_{20}$.*

Proof. This follows immediately from Lemma 5.6, 5.7, and Taylor's formula with remainder.

Suppose $Q_\nu \not\subset E_1 \cup E_2 \cup E_3 \cup Q_{\nu\varepsilon}$ and $x \in Q_\nu - Q_{\nu\varepsilon}$. Using Lemma 5.5 (i), Lemma 5.8, and elementary arguments, we find that

$$\left| f(x) - \log \operatorname*{Av}_{Q_\nu} e^f \right| < C_{21}$$

This completes the proof of Proposition 5.1.

We now turn to several complex variables. It is straightforward to derive

Proposition 5.9. *Suppose $F(z)$ is holomorphic on $z \in C^n$, $|z| < 3$, and satisfies $\max_{|z|\leqq 2} |F(z)| \leqq |F(0)| \exp(B_1 d)$. Assume $F(x)$ is real and non-negative on the cube Q given by $|x_i| \leqq 1$, $1 \leqq i \leqq n$, in R^n. Additionally, suppose that $F(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 1$ on any hyperplane $x_i = 0$, $1 \leqq i \leqq n$. For d sufficiently large, subdivide Q into cubes Q_ν of side B_2/d.*

Given $\varepsilon > 0$, outside a set E of measure less than ε, we have

$$\left|\log F(x) - \log \operatorname*{Av}_{Q_\nu} F\right| \leqq B_3 \qquad x \in Q_\nu - E$$

The constant B_3 depends upon ε but not on d.

Proof. This follows from Proposition 5.1 by induction. One successively averages over each coordinate direction. The extra technical hypothesis that $F = 1$ on each coordinate hyperplane is crucial for this argument.

We proceed to remove the technical hypothesis of Proposition 5.9. Define maps T_j by

$$T_j(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_j, x_j x_{j+1}, \ldots, x_n) \qquad 1 \leqq j \leqq n-1$$
$$T_n(x_1, x_2, \ldots, x_n) = (x_n x_1, x_2, \ldots, x_j, \ldots, x_n)$$

Set $T = T_n T_{n-1} \ldots T_1$ and $W = T^2$. This mapping W then satisfies.

Lemma 5.10. *W maps every coordinate hyperplane $x_i = 0$ to the origin. The Jacobian determinant of W vanishes along the coordinate hyperplanes only. There exists an open set $U \subset Q$ so that W: $U \to W(U)$ is a diffeomorphism.*

Proof. The first two statements are verified by direct computation. The last assertion then follows from the inverse function theorem.

The main result of this section is:

Proposition 5.11. *Suppose k is a sufficiently large integer depending on n. Let $G(z)$ be holomorphic in $|z| < 3^k$, $z \in C^n$, and satisfy $\max_{|z| < 2^k} |G(z)| \leqq |G(0)| \exp(B_4 d)$. Assume that $G(x)$ is real and non-negative for real $x \in Q$, $|x_j| \leqq 1$. Suppose R is a suitable cube contained in Q. Subdivide R into cubes R_μ of sides having length B_5/d.*

Let $\varepsilon > 0$ be given and suppose d is sufficiently large. Outside a set E of measure less than ε,

$$\left|\log G(x) - \log \operatorname*{Av}_{R_\mu} G\right| < B_6 \qquad x \in R_\mu - E$$

Proof. We may assume $G(0) = 1$. Choose $R \subset W(U)$ where $W(U)$ is obtained from Lemma 5.10. The function $F = G \circ W$ satisfies the hypotheses of Proposition 5.9. The conclusion of Proposition 5.9 then implies that if $Q_\nu \subset U$, one has, outside a set of measure $a_5 \varepsilon$,

$$\left|\log G(x) - \log \operatorname*{Av}_{W(Q_\nu)} G(x)\right| \leqq B_5$$

since the Jacobian of W is bounded on U. For d sufficiently large, we may assume that each R_μ is contained in some $W(Q_\nu)$ except for a union of R_μ whose total measure is less than $a_6 \varepsilon$. Also, one may require that $B_6 \leqq |m(R_\mu)/m(Q_\nu)| \leqq B_7$. Proposition 5.11 then follows by elementary arguments.

6. Holomorphic functions – upper bound

We continue with certain results applicable to the upper bound for the nodal volume. Consider first the case of one complex variable. Let $F(z)$ be analytic in some open neighborhood of $|z| \leqq 1$, with $|F(z)| \leqq 1$. Denote by l the number of zeroes of F in $|z| \leqq \frac{1}{2}$. One has

Lemma 6.1. $l \leqq C_1 \left| \log \max_{|z| \leqq \frac{1}{2}} |F(z)| \right|$.

Proof. Let $a_1, a_2, \ldots, a_l$ be the zeroes of F in $|z| \leqq \frac{1}{2}$. One has the Blaschke representation

$$F(z) = G(z) \prod_{i=1}^{l} \frac{z - a_i}{1 - \bar{a}_i z}, \qquad |z| \leqq 1$$

The maximum principle gives $|G(z)| \leqq 1$. Therefore, for $|z| \leqq \frac{1}{2}$, there exists $C_2 < 1$ with $|F(z)| \leqq C_2^l$.

The basic result for a single variable is

Proposition 6.2. *For every integer $k \geqq 0$, we have*

$$l \leqq C_3 \left(k + \left| \log \left(\frac{1}{k!} |F^{(k)}(0)| \right) \right| \right).$$

Proof. Cauchy's integral formula gives $\left| \frac{1}{k!} F^{(k)}(0) \right| \leqq 2^k \max_{|z| \leqq \frac{1}{2}} |F(z)|$. Now apply Lemma 6.1.

We now turn to several complex variables, $z \in C^n$. Suppose $F(z)$ is holomorphic in a neighborhood of $|z| \leqq 11/10$ and satisfies $|F(z)| \leqq \frac{1}{2}$. Let $x \in R^n$ denote the corresponding real variable. Define

$$\mathcal{M}(x) = \inf_{k \geqq 0} k + \int_{\omega \in S^{n-1}} \left| \log \left(\frac{1}{k!} |\partial_\omega^k F(x)| \right) \right| d\omega$$

One has

Lemma 6.3. *Let N_F denote the set of points x with $|x| < 1/20$ and $F(x) = 0$. Then*

$$\mathcal{H}^{n-1}(N_F) \leqq C_4 \int_{|x| < 1/10} \mathcal{M}(x)\, dx.$$

Proof. By the theory of analytic sets [7, p. 337] the singular points of N_F have $n-1$ dimensional Hausdorff measure equal to zero. Therefore, it suffices to consider the regular manifold points of N.

Let $\mathcal{L}(x, \omega)$ be the number of points of intersection of N with the line through $x \in R^n$ having direction $\omega \in S^{n-1}$. Integral geometry [7, p. 2] gives

$$\mathcal{H}^{n-1}(N_F) \leqq C_5 \int_{|x| < 1/10} \int_{S^{n-1}} \mathcal{L}(x, \omega)\, d\omega\, dx$$

For fixed x, look at $f(t) = F(x + t\omega)$ defined for complex $t \in C$, $|t| < 1$. The lemma follows by applying Proposition 6.2 to f.

To estimate $\mathcal{M}$ we employ a fact from calculus:

Lemma 6.4. *Let P be a polynomial of degree j on R^n. Suppose* $\max |P(\omega)| = 1$, *for* $|\omega| = 1$. *Then*

$$\int_{|\omega|=1} |\log |P(\omega)|| \, d\omega \leqq C_6 j.$$

Proof. Choose spherical coordinates (ϕ, θ) where $-\pi \leqq \phi < \pi$ and θ is a coordinate on the hemisphere S_+^{n-2}. Assume $|P| = 1$ at the north pole $\phi = 0$. For fixed θ, we may write the restriction of P to S^{n-1} as $P = P_1(\cos\phi) + \sin\phi \, P_2(\cos\phi)$. Set $\bar{P} = P_1(\cos\phi) - \sin\phi \, P_2(\cos\phi)$, and $Q = P\bar{P}$.

For fixed θ, we may write $Q_\theta(\cos\phi) = \pm \prod_{\nu=1}^{j_\theta} (\cos\phi - \alpha_\nu)/(1-\alpha_\nu)$, with $j_\theta \leqq 2j$. Here we used the fact that $Q_\theta(1) = 1$ and the α_ν are complex numbers. Thus

$$\int_0^\pi \log |Q_\theta(\cos\phi)| \sin^{n-2}\phi \, d\phi = \int_0^\pi \sum_{\nu=1}^{j_\theta} \log \left| \frac{\cos\phi - \alpha_\nu}{1-\alpha_\nu} \right| \sin^{n-2}\phi \, d\phi \geqq C_7 j$$

Integrating over $\theta \in S_+^{n-2}$ and recalling the definition of Q gives Lemma 6.4.

We will apply Lemma 6.4 with $P(\omega)$ equal to a multiple of $\partial_\omega^k F(x)$, for suitable k. Let $d = |\log \max |F(x)|| > \log 2$, with $|x| < 1/5$. One has

Lemma 6.5. *For some positive constants B_2, B_3, independent of F,*

$$\max_{|\omega|=1} \left| \frac{1}{k!} \partial_\omega^k F(0) \right| \geqq B_2^d$$

for some $0 \leqq k \leqq B_3 d$.

Proof. We argue by contradiction. If the claim fails then for $|\omega| = 1$ and $|t| \leqq \frac{1}{2}$,

$$\left| \sum_{k=0}^{B_3 d} \frac{1}{k!} \partial_\omega^k F(0) \, (t\omega)^k \right| \leqq 2B_2^d$$

Also, the analyticity of $f(t) = F(t\omega)$, $t \in C$, $|t| < 1$, yields, by Cauchy's integral formula $|\partial_\omega^k F(0)| \leqq k!$, for all k. Thus

$$\left| \sum_{k \geqq B_3 d} \frac{1}{k!} \partial_\omega^k F(0) \, (t\omega)^k \right| \leqq 2(\tfrac{1}{2})^{B_3 d}$$

Adding, we get for $|\omega| = 1$, $|t| < \frac{1}{2}$, $|F(t\omega)| \leqq 2B_2^d + 2(\frac{1}{2})^{B_3 d}$. This contradicts the definition of d.

Combining the last two results gives

Lemma 6.6. $\mathcal{M}(0) \leqq C_8 d$.

Proof. Let $k \leqq B_3 d$ satisfy $\max_{|\omega|=1} \left| \frac{1}{k!} \partial_\omega^k F(0) \right| = A \geqq B_2^d$. We apply Lemma 6.4 to the polynomial $P(\omega) = 1/k! \, \partial_\omega^k F(0)/A$, of degree k:

$$\int_{|\omega|=1} \left| \log \left| \frac{1}{k!} \partial_\omega^k F(0) \right| - \log A \right| \leqq C_9 d$$

Since $B_2^d \leqq A \leqq 1$, $|\log A| \leqq B_4 d$. The result now follows from the triangle inequality.

We are prepared to prove the main result of this section. Suppose that $H(z)$ is holomorphic in $|z| \leqq 2$, $z \in C^n$. Assume that $\alpha > 1$ and

$$\max_{B(x,1/5)} |H| \geqq e^{-B_5 \alpha} \max_{|z| \leqq 2} |H(z)|$$

for $x \in R^n$, $|x| < 1/10$. Here $B(x, 1/5) \subset R^n$ is a ball of radius 1/5 centered at x. Under these hypotheses we may conclude.

Proposition 6.7. *$\mathscr{H}^{n-1}(N_H) \leqq C_{10}\,\alpha$. Here N_H is the set of $|x| < 1/20$, $x \in R^n$, with $H(x) = 0$.*

Proof. Set $\gamma = 2 \max_{|z| \leqq \frac{3}{2}} |H(z)|$. Applying Lemma 6.6 to the translated function $F_x(z) = H(x+z)/\gamma$, $|x| \leqq 1/10$ gives $\mathscr{M}(x) \leqq C_{11}\,\alpha$. Here $\mathscr{M}(x)$ is the $\mathscr{M}$ corresponding to $F_0(z) = H(z)/\gamma$. The proposition now follows by using Lemma 6.3 for F_0.

7. Volumes of nodal sets on compact manifolds

Suppose that M is a compact real analytic manifold with analytic metric. Let F be an eigenfunction of Δ with eigenvalue λ. Our purpose is to present proofs of the upper and lower bounds in Theorem 1.2. We may assume λ is sufficiently large.

Let U be a sufficiently small coordinate neighborhood on M, where the metric can be expanded in power series. We identify U with a ball $B(0, \varrho_0)$ about the origin in $R^n \subset C^n$. One has

Lemma 7.1. *The eigenfunction F extends to a holomorphic function on $|z| < \varrho_1 < \varrho_0$, $z \in C^n$.* Moreover, if $x \in R^n$,

$$\sup_{|z| < \varrho_1} |F(z)| \leqq e^{E_1 \sqrt{\lambda}} \sup_{|x| < \varrho_0} |F(x)|\,.$$

Proof. The fundamental estimate proving analytic hypoellipticity [8, p. 178] gives:

$$\left|\frac{D^\alpha u(0)}{\alpha!}\right| \leqq C_1^{|\alpha|}\, \lambda^{|\alpha|/2}\, \|u\|_\infty$$

for eigenfunctions u on ball $B(0, C_2 \lambda^{-\frac{1}{2}})$. The point is that an operator with bounded coefficients is obtained after rescaling to balls of radius one. Summing a geometric series gives a holomorphic extension of u with

$$\sup_{|z| \leqq C_3 \lambda^{-\frac{1}{2}}} |u(z)| < C_4 \sup_{|x| \leqq C_2 \lambda^{-\frac{1}{2}}} |u(x)|$$

The lemma follows by applying this with u equal to a translate of F and iterating $\lambda^{\frac{1}{2}}$ times.

The next result is fundamental in the proofs of both the upper and lower bounds:

Lemma 7.2. *For any* $\varrho_2 < \varrho_0$

$$\sup_{|z|<\varrho_1} |F(z)| < e^{E_2\sqrt{\lambda}} \sup_{|x|<\varrho_2} |F(x)|$$

The constant E_2 *depends upon* ϱ_2.

Proof. The assertion is an immediate consequence of Lemma 7.1 and Theorem 4.2 (ii).

First, we prove the lower bound of Theorem 1.2. It suffices to consider the nodal points contained in a single coordinate patch U. A standard argument [3], [6] shows that there is at least one nodal point inside every ball of radius $a_1\lambda^{-\frac{1}{2}}$. Cover U by cubes Q_ν of side $a_2\lambda^{-\frac{1}{2}}$, $a_2 > a_1$, so that there exists a nodal point x_ν in the middle tenth of Q_ν. Choose a_3 so that $B_\nu = B(x_\nu, a_3\lambda^{-\frac{1}{2}})$ is completely contained in the middle $\frac{1}{2}$ of Q_ν.

One uses Proposition 5.11, for the non-negative function F^2. The required hypotheses are guaranteed by Lemma 7.2. This gives

Lemma 7.3. *There exists a fixed cube* $R \subset Q$, *so that given* $\varepsilon > 0$, *sufficiently small,*

$$\left|\log F^2(x) - \log \operatorname*{Av}_{Q_\nu} F^2\right| < C_5 \qquad x \in R \cap Q_\nu$$

for x outside a set of measure less than ε. *The constant* C_5 *depends upon* ε.

Let $R_\nu \subset Q_\nu$ be the set of x where the inequality of Lemma 7.3 is satisfied. A logical corollary of that lemma is

Lemma 7.4. *At least half of the* Q_ν *satisfy* $m(R_\nu) \geqq (1 - a_4\varepsilon)\, m(Q_\nu)$. *Here m denotes the measure.*

The symbol $\mathscr{S}$ will denote the set of those Q_ν satisfying Lemma 7.4. Fix $\varepsilon > 0$ sufficiently small and consider only those $Q_\nu \in \mathscr{S}$. Clearly, one has

$$\operatorname*{Av}_{B_\nu} F^2 \geqq e^{-C_6} \operatorname*{Av}_{Q_\nu} F^2 \tag{7.5}$$

One deduces

Lemma 7.6.

(i) $\|F\|_{L^\infty(B_\nu)} \leqq E_3 \left(\dfrac{1}{m(B_\nu)} \displaystyle\int_{B_\nu} F^2\right)^{\frac{1}{2}}$

(ii) $\left(\dfrac{1}{m(B_\nu)} \displaystyle\int_{B_\nu} F^2\right)^{\frac{1}{2}} \leqq E_4 \dfrac{1}{m(B_\nu)} \displaystyle\int_{B_\nu} |F|$.

Proof. Standard elliptic theory gives

$$\|F\|_{L^\infty(B_\nu)} \leqq E_5 \left(\frac{1}{m(Q_\nu)} \int_{Q_\nu} F^2\right)^{\frac{1}{2}}$$

Part (i) then follows from (7.5). Elementary arguments show that (i) implies (ii).

Using Lemma 7.6 (ii) and the Cauchy Schwartz inequality we find

Lemma 7.7. *If* $G_\nu \subset B_\nu$ *is a measurable set then*

$$\int_{G_\nu} |F| \leqq E_6 \left(\frac{m(G_\nu)}{m(B_\nu)}\right)^{\frac{1}{2}} \int_{B_\nu} |F|.$$

If a_3 is sufficiently small, then we can solve the Dirichlet problem for $\Delta + \lambda$ on balls $B(x_\nu, r)$, $0 < r < a_3 \lambda^{-\frac{1}{2}}$. Rescaling to balls of radius 1, one has a small perturbation of the Dirichlet problem for the Euclidean Laplacian on the unit ball. Thus, we may write

$$0 = F(x_\nu) = \int_{|x_\nu - x| = r} \phi(x)\, F(x)\, d\theta \tag{7.8}$$

where $0 < C_7 < \phi(x) < E_7$. Also, $d\theta$ denotes the volume element on the standard unit sphere S^{n-1}.

Multiplying (7.8) by r^{n-1} and integrating in r, we find that

$$0 = \int_{B_\nu} \phi F$$

Let $G_\nu^+ \subset B_\nu$ be the set where $F > 0$ and G_ν^- the set where $F < 0$. From the bounds on ϕ, we deduce

$$\min\left(\int_{G_\nu^+} |F|,\ \int_{G_\nu^-} |F|\right) \geqq C_8 \int_{B_\nu} |F|$$

Lemma 7.7 then gives

$$\min\,(m(G_\nu^+),\ m(G_\nu^-)) \geqq E_8\, m(B_\nu) \tag{7.9}$$

One now invokes the isoperimetric inequality [7, p. 476] to give a lower bound for the nodal volume inside B_ν. Here we recall that the nodal points N form an analytic set with finite $n-1$ dimensional Hausdorff measure. Thus

$$\mathscr{H}^{n-1}(N \cap B_\nu) \geqq C_9 (\lambda^{-\frac{1}{2}})^{n-1}$$

Summing over those $Q_\nu \in \mathscr{S}$ preferred by Lemma 7.4 gives

$$\mathscr{H}^{n-1}(N) \geqq \sum_{Q_\nu \in \mathscr{S}} \mathscr{H}^{n-1}(N \cap B_\nu) \geqq E_9 \lambda^{\frac{1}{2}}$$

This completes the proof of the lower bound in Theorem 1.2.

Using (7.9) and the previous discussion one has

Corollary 7.10. *Let M^+, M^- be the set of points in M where $F > 0$, $F < 0$. Then*

$$\min\,(\operatorname{vol} M^+,\ \operatorname{vol} M^-) \geqq C_{10} \operatorname{vol} M.$$

It remains to establish the upper bound on the nodal volume. Let U be a coordinate patch where the conclusion of Lemma 7.2 holds. Proposition 6.7 yields $V \subset U$, a patch having the same center point, with

$$\mathscr{H}^{n-1}(N \cap V) \leqq C_{11} \lambda^{\frac{1}{2}}$$

The V are independent of λ. By compactness, we can cover M by a finite number of such V. The upper bound

$$\mathscr{H}^{n-1}(N) \leqq E_{11} \lambda^{\frac{1}{2}}$$

follows immediately. The proof of Theorem 1.2 is complete.

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