## Differential Equations in $\mathbb{R}^2$; Phase Plane Representation

The two-dimensional solution graphs for $x(t)$ and $y(t)$ are far more confusing than the two-dimensional drawings for one-dimensional differential equations we have used in Part I of this text, because these solutions can cross. Even in the comparatively well-organized pictures of Example 6.1.2, solutions are crossing in the $tx$- and $ty$-planes, and those pictures are extremely limited, to solutions with initial conditions $t = 0$, $x_0 = 0$, $y_0 = C$.

But there is another two-dimensional picture possible, in the $xy$-plane, called the *phase plane* (or sometimes the *state space* or *dynamical plane*) for a system in $\mathbb{R}^2$. In Examples 6.1.2 and 6.1.3 we shall see that the phase plane drawing consists of curves that do *not* cross, an important attribute for analyzing the behavior of the system.

**Example 6.1.3.** Consider again $x' = y$, $y' = -x$. Figure 6.1.5 shows the phase plane portrait (drawn by *MacMath*) of the solutions corresponding to those shown in Figures 6.1.3 and 6.1.4.

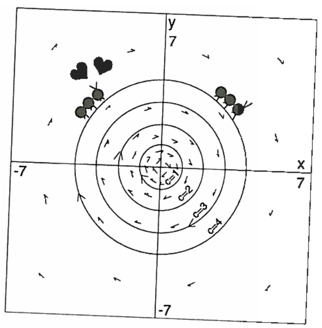


FIGURE 6.1.5. Trajectories of selected solutions to $x' = y$, $y' = -x$, with $x_0 = 0$, $y_0 = C$. ▲

A solution in $txy$-space (as in the left of Figures 6.1.2, 6.1.3, and 6.1.4) is *projected* onto the $xy$-phase plane. It is what you would see if you stood high on the $t$-axis looking down at the $xy$-plane. This projected curve does *not* correspond to the actual motion of a solution to the system, but rather it is a "track" or *trajectory* of the solution in the phase plane. If you compare Figures 6.1.3 and 6.1.5, you should see that the trajectories in the phase plane are indeed the "tracks" left by the solutions for $x$ and $y$ as functions of $t$, but

> *the trajectories alone give no information about how a point moves along a trajectory as a function of time.*



Nevertheless, the trajectories in the phase plane provide a very useful way to analyze the system, as we shall demonstrate in the examples of Sections 6.3, 6.5, and 6.7.

As you may have noticed, something especially simple about Examples 6.1.2 and 6.1.3 is the fact that this system of differential equations is *autonomous*, with no explicit dependence on $t$ in the functions for the derivatives. In fact,

> *it is only an autonomous system that will give a meaningful phase plane portrait,*

because then the solutions at different values of $t_0$ are just time translates of one another, so their projections pile up on the same trajectories.

Look back at Figure 6.1.5 and imagine an ant walking along a solution trajectory. He is always at some point $(x(t), y(t))$, and at $t = 0$ he is at $(x(0), y(0))$. Suppose a second ant starts half an hour later at the same point $(x(0), y(0))$; if the system is autonomous, she will follow the same solution curve, and her later position will always be at $(x(t-30), y(t-30))$. If, on the other hand, the system were nonautonomous, an ant starting at $(x(0), y(0))$ half an hour later than the first ant would feel completely different and would be "blown" along a different trajectory.

For a *non*autonomous system, phase plane trajectories cross over each other and project into an indecipherable mess.

**Example 6.1.4.** Return to the nonautonomous or time-dependent system $x' = y$, $y' = x^2 - t$ of Example 6.1.1. Some solutions were shown in Figure 6.1.1; the corresponding phase portrait is as follows:

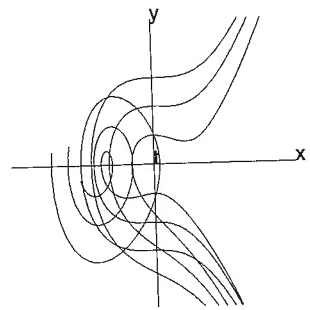


FIGURE 6.1.6. Phase plane projection for the nonautonomous system $x' = y$, $y' = x^2 - t$. ▲

We shall see in Section 6.2 that for autonomous systems, drawings in the $xy$-phase plane will not only be less of a jumble than Figure 6.1.6, but *will not cross*. As a result, the study of *autonomous* systems in $\mathbb{R}^2$ reduces to