as you can confirm by differentiating the first equation and substituting the result in the second. ▲

We shall follow this example further in succeeding sections, and we shall show in Section 6.4 how *any* higher order differential equation can actually be expressed as a system of first order equations.

Except for *linear* equations *with constant coefficients*, which we shall study at length in Chapter 7,

> *there are very* few *systems of differential equations that can be solved explicitly.*

We shall study in this chapter three systems of differential equations that can nevertheless be analyzed very profitably:

in 6.3, the sharks and sardines equation;
in 6.5, the equation of motion of a particle with one degree of freedom;
in 6.7, the central force problem.

Still, each of these examples is a particular equation, and the methods used do *not* generalize to any substantial class.

The qualitative behavior of solutions of differential equations in $\mathbb{R}^n$, for $n > 1$, is enormously more complicated than the behavior that we examined in Part I , as we shall see in Section 6.1.

In higher dimensions, solutions have ever so much more space in which to get tangled up, and they definitely take advantage of this opportunity. For example, in $\mathbb{R}^2$ a curve can separate the plane into two parts; not so in $\mathbb{R}^3$! Even a closed curve in $\mathbb{R}^3$ encloses nothing—other curves can sneak through and around it.

In Part I of this text we examined a single differential equation in $\mathbb{R}^1$, with graphs of solutions in $\mathbb{R}^2$, the tx-plane. The next simplest system is in $\mathbb{R}^2$, where we seek functions x and y both dependent on t; this will require graphs in $\mathbb{R}^3$, or txy-space . We shall explore those in Section 6.1, and most of our examples will be of systems in $\mathbb{R}^2$ throughout this chapter.

But when we consider systems in $\mathbb{R}^3$, there is virtually no theory anymore, and our understanding of differential equations of dimension n greater than 3 is practically nil (except, we repeat, for the special case of linear equations with constant coefficients, Chapter 7).

So it may come as a pleasant surprise that the numerical methods of Chapter 3 and the theory of Chapter 4 (including the Fundamental Inequality, existence, and uniqueness theorems) remain virtually unchanged when generalized to systems of differential equations, regardless of the dimension n: the statements, the proofs, and all the formulas require nothing more than an arrow over all the vector quantities to become correct. We shall revisit all of these in Section 6.2.

Throughout this chapter the general theory is in $\mathbb{R}^n$, although most of the specific examples are in $\mathbb{R}^2$.

6.1 Graphical Representation of Systems

What kind of drawings can we get for systems of differential equations? For an equation $x' = f(t, x)$ in $\mathbb{R}^1$, the direction field is in $\mathbb{R}^2$; in general, for a differential equation in $\mathbb{R}^n$, the direction field will be in $\mathbb{R}^{n+1}$, which is considerably more difficult to draw and to visualize solutions within, even for $n = 2$. Let us begin with the simplest case beyond a differential equation in $\mathbb{R}^1$.

Differential Equations in $\mathbb{R}^2$; Representation in $\mathbb{R}^3$

Suppose

$$\begin{aligned} dx/dt &= f(t, x, y) \\ dy/dt &= g(t, x, y), \end{aligned} \tag{3}$$

with solutions

$$\begin{aligned} x &= u(t) \\ y &= v(t). \end{aligned} \tag{4}$$

A system (3) of differential equations in $\mathbb{R}^2$ gives a direction field in $\mathbb{R}^3$.

Example 6.1.1. Consider the nonlinear, nonautonomous system

$$\begin{aligned} x' &= y \\ y' &= x^2 - t, \end{aligned}$$

A drawing of some solutions in txy-space looks like Figure 6.1.1:

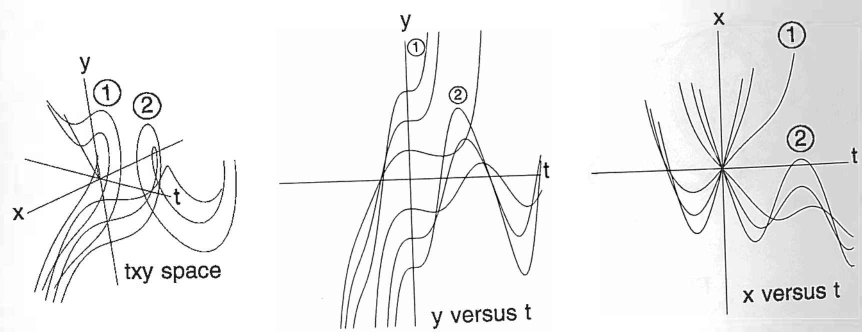

FIGURE 6.1.1. Some solutions to $x' = y$, $y' = x^2 - t$.

As you can see on the left of Figure 6.1.1, the solutions in txy-space appear to form a tangle of curves in $\mathbb{R}^3$, especially, for example, when projected onto the plane of a paper or computer screen. It is a real problem to interpret which parts of the picture are in the foreground and which are in the background. ▲