A_∞-algebras in representation theory and homological algebra

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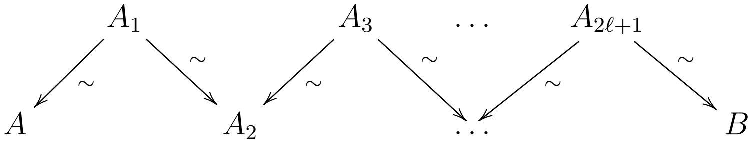
$\rightsquigarrow$ What is missing?

More precisely, we say that two dg algebras (A, d_A) and (B, d_B) are *quasi-isomorphic* if there is a diagram



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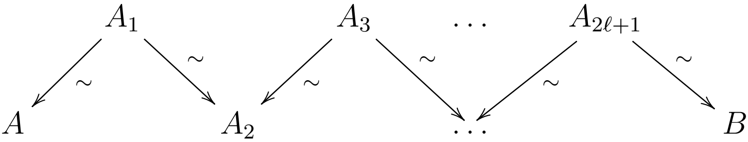

The question now reads:

What structure should we impose to $H(A)$ and $H(B)$ so that:

$H(A)$ and $H(B)$ are 'equivalent' $\Leftrightarrow$ A and B are quasi-isomorphic?

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Let $\Lambda(n) = k[x]/(x^n)$ ($n > 2$), $P_\bullet(n) \to k$ a min. proj. res. and define $A(n) = \mathcal{E}nd_{\Lambda(n)}(P(n))$.

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Consequence: the algebra structure of cohomology is not enough!

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An *A_∞-algebra* (J. Stasheff, '63) is a graded vector space $A = \oplus_{i \in \mathbb{Z}} A^i$ with maps

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$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+1+t} \circ (\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0, \forall n \in \mathbb{N} \quad \text{(SI(n))}$$

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(2) $\mathrm{SI}(2)$ says that m_1 is a derivation for m_2;

(3) $\mathrm{SI}(3)$ means that m_2 is associative up to the homotopy m_3:

$$m_2 \circ (\mathrm{id} \otimes m_2) - m_2 \circ (m_2 \otimes \mathrm{id}) = \delta(m_3)$$

where δ is the differential of $\mathcal{H}om(A^{\otimes 3}, A)$ induced by m_1.

A *morphism* (A. Clark, '65) of A_∞-algebras $f_\bullet : (A, m_\bullet^A) \to (B, m_\bullet^B)$ is a collection of maps

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(2) $\mathrm{MI}(2)$ says that f_2 commutes with multiplications up to the homotopy f_2:

$$f_1 \circ m^A_2 - m^B_2 \circ (f_1 \otimes f_1) = \delta'(f_2)$$

where δ' is the differential of $\mathcal{H}om(A^{\otimes 2}, B)$ induced by m^B_1 and m^A_1.

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Theorem 2 (B. Keller, '02).

Let A and B be two dg algebras. Then, A and B are quasi-isomorphic iff there is a quasi-isomorphism of A_∞-algebras from A to B.

More properties

A (left) A_∞-*module* over an A_∞-algebra A is a complex of vector spaces $(M = \oplus_{i \in \mathbb{Z}} M^i, d)$ with a morphism of A_∞-algebras $A \to \mathcal{E}nd(M)$.

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Theorem 3 (B. Keller, '02).

(a) *Let (A, d_A) be a dg algebra, $\mathcal{C}_{\mathrm{dg}}(A)$ be the category of dg modules with morphisms of dg modules and $\mathcal{D}_{\mathrm{dg}}(A)$ be its derived category.*

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(b) *Let $f_\bullet : A \to B$ be a quasi-isomorphism of A_∞-algebras. Then, the induced functor $f^* : \mathcal{D}_\infty(B) \to \mathcal{D}_\infty(A)$ is an equivalence of triangulated categories sending B to A.*

The dual notions

There is also the dual notion of A_∞-*coalgebra* $C = \oplus_{i \in \mathbb{Z}} C_i$, with a loc. finite collection of maps

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$$f_n : C \to D^{\otimes n}, \forall n \in \mathbb{N}, \text{ of degree } |f_n| = n - 1,$$

that fulfil equalities analogous to $\mathrm{MI}(n)$.

Motivation

If A is a dg algebra and C is an A_∞-coalgebra, then $\mathcal{H} = \mathcal{H}om(C, A)$ has an explicit structure of A_∞-algebra!

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$$m_{p,q}\big(\phi_1 \otimes \cdots \otimes \phi_p \otimes (m \otimes c) \otimes \psi_1 \otimes \cdots \otimes \psi_q\big) \\ = \pm \, \big(\phi_1(c_{(q+2)}) \dots \phi_p(c_{(q+p+1)})\big).m.\big(\psi_1(c_{(1)}) \dots \psi_q(c_{(q)})\big) \otimes c_{(q+1)}, \tag{1}$$

where $\Delta_{p+q+1}^C(c) = c_{(1)} \otimes \cdots \otimes c_{(p+q+1)}$.

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Observation: These formulas also apply to A_∞-bimodules. Hence, if N is an A_∞-bimodule over H then we obtain an A_∞-bimodule N^a over H^a.

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Let A be the algebra $\Lambda(n) = k[x]/(x^n)$. Then $\mathcal{E}xt^\bullet_A(k,k) \simeq k[X,Y]/(X^2)$ (as graded vector spaces!).

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Let A be the algebra $\Lambda(n) = k[x]/(x^n)$. Then $\mathcal{E}xt^\bullet_A(k,k) \simeq k[X,Y]/(X^2)$. Define a basis $\{Z_j : j \in \mathbb{N}_0\}$ of it by

$$Z_j = Y^{j/2} \text{ if } j \text{ is even, and } Z_j = XY^{(j-1)/2} \text{ else.}$$

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$$m_n(Z_{j_1}, \dots, Z_{j_n}) = Z_{j_1+\cdots+j_n-n+2} \text{ if all } j_p \text{ are odd, and zero else.}$$

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In this case, taking graded dual we obtain an A_∞-coalgebra C and the map $\tau : C \to A$ sending $X^{\#}$ to x and the other monomials to zero is a twisting cochain satisfying condition (ii).

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Corollary 6.

We directly obtain the formulas for the cup product of Hochschild cohomology for Koszul algebras given by R. Buchweitz, E. Green, N. Snashall and Ø. Solberg, '08, and for N-Koszul algebras by Y. Xu and H. Xiang, '11.