Gröbner bases and Anick's resolution

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Abstract

This thesis mainly introduces non-commutative Gröbner bases and Anick's resolution. Mainly based on the works of Ufnarovskij, Varadarajan and Anick, firstly we explain how to find a Gröbner basis of a non-commutative algebra. Then we use the Gröbner basis to construct Anick's resolution. Finally, we specifically compute Anick's resolution of Fomin-Kirillov algebra with three generators.

Keywords: Gröbner bases, Anick's resolution, Koszul algebras, computations.

Contents

1 Introduction

A Gröbner basis is a particular generating set of an ideal in an associative algebra over a filed. The concept was first introduced in 1965, together with Buchberger's algorithm to compute them, by Bruno Buchberger in his Ph.D. thesis, whose English translation is in [5] by Michael P. Abramson. Buchberger's Gröbner bases theory mostly concerns commutative algebra. Bergman's diamond lemma generalizes the theory of Gröbner bases of commutative algebras to the case of non-commutative, which was first introduced in [4]. Gröbner bases theory tells us some problems, which are difficult to solve in terms of the ideal, are easier to solve with a Gröbner basis. Gröbner bases theory has many applications in computer algebra, algebraic geometry, invariant theory, graph theory, statistics and so on.

Let k be a field and A an associative augmented k-algebra. Anick's resolution is a projective resolution of k as an A-algebra, which was introduced by David J. Anick in [1]. We could use Anick's resolution to compute Ext groups, Tor groups and so on, while the bar resolution is often too large to compute in practice. Anick's resolution can also be used to prove Koszulness of an algebra which has a quadratic Gröbner basis.

Chapter 2 contains the definitions concerning Gröbner bases and how to find a Gröbner basis in a non-commutative associative algebra over a field.

Chapter 3 contains the construction of Anick's resolution, based on a Gröbner basis. We also introduce a little content about Koszulity.

In chapter 4, we partially do specific computations of a Gröbner basis, Anick's resolution and Ext groups of Fomin-Kirillov algebra with three generators. In order to find the general form of differentials in Anick's resolution, we need to compute a large number of lower order differentials, guess the general form of differentials and mainly use mathematical induction to prove the guess. Unfortunately, Chapter 4 does not give a complete general form.

2 Gröbner bases

We will present in this chapter the basic theory of non-commutative Gröbner bases. We will mainly follow Ufnarovskij [11] and Varadarajan [12].

2.1 Normal words and Gröbner bases

First, we introduce some famous concepts about order.

Definition 1. Let X be a non-empty set. A binary relation $\succeq$ on X is called a **total order** on X if the following statements hold for all $x, y, z \in X$:
(1) Antisymmetry: if $x \succeq y$ and $y \succeq x$, then $x = y$.
(2) Transitivity: if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.
(3) Connexity: $x \succeq y$ or $y \succeq x$.

Definition 2. Let $(X, \succeq)$ be a total ordered set. Then we define $x \succ y$ if $x \succeq y$ and $x \neq y$ for $x, y \in X$. We define $x \preceq y$ if $y \succeq x$ and define $x \prec y$ if $y \succ x$.

Definition 3. Let X be a non-empty set. A total order on X is called a **well order** on X if every non-empty subset of X has a least element in this ordering.

Remark 1. In a well ordered set $(X, \succeq)$, any sequence $x_1 \succeq x_2 \succeq x_3 \succeq \cdots$ stabilises, i.e. there exists $n \geqslant 1$ such that $x_m = x_n$ for all $m \geqslant n$.

Theorem 1. (Transfinite induction) Let $(X, \succeq)$ be a well ordered set and $x_0 \in X$ the least element of X. Let $P(x)$ be a property defined for all elements $x \in X$. Assume that the following conditions holds:
(1) $P(x_0)$ is true.
(2) Let $x \in X$. If $P(y)$ is true for any $x \succ y$, then $P(x)$ is true.
Then $P(x)$ is true for all $x \in X$.

Proof. Suppose that the set $A = \{x \in X | P(x) \text{ is not true}\}$ is non-empty. Then there is the least element $\alpha \in A$. We have $\alpha \neq x_0$. Then for any $y \in X$ with $\alpha \succ y$, $P(y)$ is true, which implies $P(\alpha)$ is true. Contradiction! □

Let X be a non-empty set whose elements are called **letters** and W the free (non-commutative) monoid with unit generated by X. Specifically, W is the set of all the finite sequences of zero or more elements from X with concatenation operation: $(x_1 \cdots x_m) \cdot (y_1 \cdots y_n) = x_1 \cdots x_m y_1 \cdots y_n$, where $x_i, y_j \in X$. The unique sequence of zero elements is the identitiy element $1 \in W$. The elements of W are called **words**. The **length** of a word $w \in W$ is the number of the letters inside w. The length of $1 \in W$ is zero.

Definition 4. For $w_1, w_2 \in W$, we say w_2 is a **subword** of w_1, denoted by $w_2 \subseteq w_1$, if $w_2 = 1$ or $w_1 = x_{i_1} x_{i_2} \cdots x_{i_m}$ and $w_2 = x_{i_l} x_{i_{l+1}} \cdots x_{i_r}$ for some $1 \leqslant l \leqslant r \leqslant m$ and $x_{i_j} \in X$. We define $w_2 \subsetneq w_1$ if $w_2 \subseteq w_1$ and $w_2 \neq w_1$.

We assume that the set X is well ordered. We may define an order $\succeq$ on W, called **homogeneous lexicographic order**, as follows: for $w_1, w_2 \in W$, if the length of w_1 is strictly larger than w_2, we define $w_1 \succ w_2$; if the length of w_1 equals to w_2, we sort them in lexicographic order induced by the well order on X. Then the homogeneous lexicographic order $\succeq$ is a well order on W.

Remark 2. The homogeneous lexicographic order on W has the following properties:
(1) For every $w \in W$ which is not 1, we have $w \succ 1$.
(2) For all $w_1, w_2, u, v \in W$, if $w_1 \succ w_2$, then $uw_1v \succ uw_2v$.

Let k be a field and $k\langle X\rangle$ the free (non-commutative) associative k-algebra generated by X. It's a fact that the algebra $k\langle X\rangle$ is a k-vector space spanned by all words. For any non-zero element x in $k\langle X\rangle$ with the form $x = \sum_{i=1}^{r} c_i w_i$ where $c_i \in k\backslash\{0\}, w_i \in W$ and $w_1 \succ w_2 \succ \cdots \succ w_r$, we call c_1w_1 the **leading term** of x, w_1 the **leading word** of x and c_1 the **leading coefficient** of x.

Let I be a non-zero two-sided ideal of $k\langle X\rangle$.

Definition 5. A word $w \in W$ is called **normal** with respect to I if w is not the leading term of any element in I.

Theorem 2. Let N be the k-vector space spanned by all normal words. Then we have a decomposition $k\langle X\rangle = N \oplus I$ as k-vector spaces.

Proof. Since $N \cap I = 0$, we only need to prove $k\langle X\rangle = N + I$. It is sufficient to prove that $W \subseteq N + I$. Let $w \in W$ be a word. If the word w is normal, we have $w = w + 0$. Otherwise, w is the leading term of an element y in I. Let $y = w + w_1$, where $w_1 \in k\langle X\rangle$. Let $w_1 = \sum_{i=1}^{r_1} a_i w_{i1}$, where $a_i \in k, w_{i1} \in W$ and $w \succ w_{i1}$. We have $w = -w_1 + y$. If all w_{i_1} are normal words, we obtain $w \in N + I$. Otherwise, say w_{11} is not normal, w_{11} is the leading term of an element $z \in I$. Let $z = w_{11} + w_2$, where $w_2 \in k\langle X\rangle$. Let $w_2 = \sum_{i=1}^{r_2} b_i w_{i2}$, where $b_i \in k, w_{i2} \in W$ and $w_{11} \succ w_{i2}$. If all w_{i2} are normal words, we have $w_{11} \in N + I$. Otherwise, say w_{12} is not normal, repeat the above process, then we get a sequence $w \succ w_{11} \succ w_{12} \succ \cdots$, which must be a finite sequence as W is well ordered. Finally we get $w \in N + I$. □

Definition 6. There is a natural projection map $p : k\langle X\rangle \to N$. For every $x \in k\langle X\rangle$, we call $p(x)$ the **normal form** of x.

Remark 3. Let $A = k\langle X\rangle/I$ be the quotient algebra. Then $A \cong N$ as k-vector spaces.

Now we introduce one of the most important definitions in this chapter.

Definition 7. A subset G of I is called a **Gröbner basis** of I in $k\langle X\rangle$ if the leading word of any non-zero element in I contains the leading word of some element in G as a subword. Moreover, if we require that no proper subset of G is a Gröbner basis, G is called a **minimal Gröbner basis**. A Gröbner basis G is called **reduced** if it is minimal and every element $x \in G$ has the form $w - p(w)$, where w is the leading term of x and the coefficient of w is 1.

Remark 4. (1) Many Gröbner bases exist. For example, the ideal I itself is a Gröbner basis of I.
(2) A Gröbner basis G is minimal if and only if $0 \notin G$ and the leading word of any element in G doesn't contain the leading word of any other element in G as a subword.

Proposition 1. Let G be a Gröbner basis of I in $k\langle X\rangle$. Then the set G generates the two-sided ideal I.

Proof. Since $G \subseteq I$, it is clear that $(G) \subseteq I$, where (G) is the two-sided ideal generated by G. We shall prove the converse. Let y be a non-zero element in I. We want to prove $y \in (G)$. Let $y = aw + z$, where $a \in k\backslash\{0\}, w \in W, z \in k\langle X\rangle$ and aw is the leading term of y. There exists $x \in G$ with the leading term x_1 and exist $c \in k\backslash\{0\}, u, v \in W$ such that $aw = cux_1v$. Then $y = cuxv + y_1$, where $cuxv \in (G)$, $y_1 = z - cu(x - x_1)v$. Let $y_1 = a_1w_1 + z_1$, where $a_1 \in k\backslash\{0\}, w_1 \in W, z_1 \in k\langle X\rangle$ and a_1w_1 is the leading term of y_1. We have $w \succ w_1$. By repeating the above process for $y_1, y_2, y_3 \cdots$, we get a sequence $w \succ w_1 \succ w_2 \succ \cdots$. As the set W is well ordered, the process will be terminated in a finite number of steps. Finally we obtain $y \in (G)$. □

Theorem 3. Let G be a Gröbner basis of I in $k\langle X\rangle$. Then a word $w \in W$ is normal if and only if w doesn't contain the leading word of any element in G as a subword.

Proof. ($\Rightarrow$) Let w be a normal word. Suppose there exist $x \in G$ with leading word x_1 and $w_1, w_2 \in W$ such that $w = w_1x_1w_2$. Then w is the leading word of $w_1xw_2 \in I$. Contradiction!
($\Leftarrow$) Suppose the word $w \in W$ is not normal. Then w is the leading word of some element in I. By the definition of Gröbner basis, w contains the leading word of some element in G as a subword. □

2.2 Bergman's diamond lemma

Now we introduce Bergman's diamond lemma according to Varadarajan [12], section 7.2 and give some examples from Ufnarovskij [11], section 2.6 about how to find a Gröbner basis.

Let k be a field, X a well ordered set and $A = k\langle X\rangle/I$ an associative k-algebra where I is a non-zero two-sided ideal of the free associative k-algebra $k\langle X\rangle$. The free monoid W generated by X is equipped with the homogeneous lexicographic order $\succeq$ induced by the well order on X. Suppose the ideal I is generated by the set

$$\{w_\sigma - f_\sigma \in I | w_\sigma \in W, f_\sigma \in k\langle X\rangle, \sigma \in \Sigma\}. \qquad (*)$$

We also assume that the following conditions hold:
(1) For any $\sigma \neq \tau$ in Σ, we have $w_\sigma \neq w_\tau$.
(2) For all $\sigma \in \Sigma$, we have $f_\sigma = 0$ or the leading word of f_σ is strictly less than w_σ in the homogeneous lexicographic order.

Definition 8. A word $w \in W$ is called **standard** with respect to $(*)$ if w doesn't contain any word w_σ $(\sigma \in \Sigma)$ as a subword.

Remark 5. Let S be the k-vector space spanned by all standard words with respect to $(*)$, N the k-vector space spanned by all normal words with respect to I. Then we have $N \subseteq S$.

Definition 9. For $\sigma \in \Sigma$ and $u, v \in W$, we define the **elementary reduction operator** as the k-linear map $R_{(u,w_\sigma,v)} : k\langle X\rangle \to k\langle X\rangle$ which maps the word $uw_\sigma v$ to $uf_\sigma v$, and fixes other words. The composition of finite elementary reduction operators is called a **reduction operator**.

Remark 6. $R_{(au,w_\sigma,vb)}(axb) = a(R_{(u,w_\sigma,v)}x)b$ for $a, b \in W$, $x \in k\langle X\rangle$. Moreover, for any reduction operator R and $a, b \in W$, there exists a reduction operator $\widetilde{R}$ such that $\widetilde{R}(axb) = a(Rx)b$ for any $x \in k\langle X\rangle$.

Remark 7. The k-vector space S is exactly the set of elements which are fixed by all reduction operators in $k\langle X\rangle$.

Lemma 1. For every $x \in k\langle X\rangle$ and reduction operator R, we have $x - Rx \in I$.

Proof. Let $R = R_n R_{n-1} \cdots R_1$ where R_i are elementary reduction operators. We will prove the lemma by induction on n. When $n = 1$, $R = R_{(u,w_\sigma,v)}$ is an elementary reduction operator, where $\sigma \in \Sigma, u, v \in W$, we write $x = cuw_\sigma v + x'$, where $c \in k$ and $x' \in k\langle X\rangle$ is a linear combination of words not equal to $uw_\sigma v$. Then we have $Rx = cuf_\sigma v + x'$, so $x - Rx = cu(w_\sigma - f_\sigma)v \in I$. Suppose that $x - Rx \in I$ for every $x \in k\langle X\rangle$ and every reduction operator R which can be written as a composition of $n-1$ elementary reduction operators. Then for $R = R_n R_{n-1} \cdots R_1$, we have $x - Rx = (x - R_1 x) + (R_1 x - R_n R_{n-1} \cdots R_2 R_1 x) \in I$. □

Definition 10. An element $x \in k\langle X\rangle$ is called **reduction finite** if for every sequence $\{R_i | i \geqslant 1\}$ of elementary reduction operators, the sequence

$$R_1 x, R_2 R_1 x, \cdots, R_i R_{i-1} \cdots R_1 x, \cdots$$

stabilizes, i.e. there exists $n \geqslant 1$, such that $R_m R_{m-1} \cdots R_1 x = R_n R_{n-1} \cdots R_1 x$ for all $m \geqslant n$. Let F be the set of all reduction finite elements.

Remark 8. An element $x \in k\langle X\rangle$ is reduction finite if and only if for every sequence $\{R_i | i \geqslant 1\}$ of reduction operators, the sequence $R_1 x, R_2 R_1 x, \cdots, R_i R_{i-1} \cdots R_1 x, \cdots$ stabilizes.

Lemma 2. We have $F = k\langle X\rangle$.

Proof. The set F is a k-vector space. It is sufficient to prove $W \subseteq F$. Suppose $W \backslash F \neq \emptyset$. Since W is well ordered, we can take a least element w in $W \backslash F$. Let $\{R_i | i \geqslant 1\}$ be a sequence of elementary reduction operators. Assume without loss of generality $R_1 w \neq w$, then $R_1 w$ is a linear combination of words strictly less than w in the homogeneous lexicographic order. Thus $R_1 w \in F$. This implies that the sequence $R_1 w, R_2 R_1 w, \cdots$ stabilizes. Hence $w \in F$. Contradiction! □

Remark 9. For every $x \in k\langle X\rangle$, there exists a reduction operator R such that $Rx \in S$.

Definition 11. We call Rx in last remark a **reduced form** of x. If all reduced forms of x are same, x is called **reduction unique**. Let U be the set of reduction unique elements.

Remark 10. (1) We have $S \subseteq U$ and U is a k-vector space which is stable under all reduction operators, i.e. for every reduction operator R, we have $R(U) \subseteq U$.
(2) We have a map $\mathcal{R} : U \to S$ which maps $x \in U$ to its reduced form. This is a k-linear map satisfying $\mathcal{R}(Rx) = \mathcal{R}(x)$ for every $x \in U$ and every reduction operator R. Moreover, $\mathcal{R}|_S = id_S$.

Definition 12. An **overlap ambiguity** is a triple (w_1, w_2, w_3), where $w_i \in W$ and there are $\sigma, \tau \in \Sigma$ such that $w_1 w_2 = w_\sigma, w_2 w_3 = w_\tau$. An **inclusion ambiguity** is a triple (w_1, w_2, w_3), where $w_i \in W$ and there are $\sigma, \tau \in \Sigma$ such that $w_2 = w_\sigma, w_1 w_2 w_3 = w_\tau$. An overlap ambiguity (w_1, w_2, w_3) is called **resolvable** if there are reduction operators R_1, R_2 such that $R_1(f_\sigma w_3) = R_2(w_1 f_\tau) \in S$. An inclusion ambiguity (w_1, w_2, w_3) is called **resolvable** if there are reduction operators R_1, R_2 such that $R_1(w_1 f_\sigma w_2) = R_2(f_\tau) \in S$.

Here is the main theorem in this chapter.

Theorem 4. (Bergman's diamond lemma) The following statements are equivalent:
(1) We have $S = N$.
(2) Every element in $k\langle X\rangle$ is reduction unique.
(3) All ambiguities are resolvable.
(4) The set $\{w_\sigma - f_\sigma | \sigma \in \Sigma\}$ is a Gröbner basis of I in $k\langle X\rangle$.

Proof. (1)⇒(2) Let x be an element in $k\langle X\rangle$ and $s = Rx \in S$ a reduced form of x, where R is a reduction operator. Then we have $x - s = x - Rx \in I$ by Lemma 1. Since $S = N$, we have $k\langle X\rangle = S \oplus I$ as k-vector spaces by Theorem 2. Then the decomposition $x = s + (x - s)$ is unique. This implies that the reduced form of x is unique.
(2)⇒(1) Suppose $U = k\langle X\rangle$. Then we have a k-linear map $\mathcal{R} : k\langle X\rangle \to S$. Let K be the kernel of $\mathcal{R}$. Since $\mathcal{R}|_S = id_S$, we can obtain $k\langle X\rangle = S \oplus K$ as k-vector spaces. We want to prove $I = K$. Let $x \in K$. Then $\mathcal{R}(x) = 0$. There is a reduction operator R such that $Rx = \mathcal{R}(x) = 0$. By Lemma 1, we have $x = x - Rx \in I$. Conversely, let $x \in I$. Then x is a linear combination of elements of the form $u(w_\sigma - f_\sigma)v$, where $\sigma \in \Sigma$ and $u, v \in W$. We have $\mathcal{R}(u(w_\sigma - f_\sigma)v) = \mathcal{R}(R_{(u,w_\sigma,v)}(u(w_\sigma - f_\sigma)v)) = \mathcal{R}(0) = 0$, which implies $u(w_\sigma - f_\sigma)v \in K$. Then $x \in K$. We obtain $k\langle X\rangle = S \oplus I$. Let $s \in S$. Then $s = s + 0 = n + y$ for some $n \in N$ and $y \in I$. As $N \subseteq S$, we get $s = n$. Hence $S = N$.
(2)⇒(3) Suppose (w_1, w_2, w_3) is an overlap ambiguity with $w_1 w_2 = w_\sigma,\ w_2 w_3 = w_\tau$, $\sigma, \tau \in \Sigma,\ w_1, w_2, w_3 \in W$. There are reduction operators R_1, R_2 such that $R_1(f_\sigma w_3) \in S,\ R_2(w_1 f_\tau) \in S$. The elements $R_1(f_\sigma w_3)$ and $R_2(w_1 f_\tau)$ are both reduced forms of the word $w_1 w_2 w_3$, hence they are same. By the same reason, all inclusion ambiguities are resolvable.
(3)⇒(2) We will prove that every word is reduction unique by transfinite induction. The least word 1 is reduction unique. Let $w \in W$. Suppose that all words strictly

less than w are reduction unique. We want to prove that w is reduction unique. Let R_1 and R_2 be two elementary reduction operators such that $R_1 w \neq w$ and $R_2 w \neq w$. It is sufficient to prove that $R_1 w$ and $R_2 w$ are reduction unique and they have the same reduced form. Let $R_1 = R_{(u_1, w_\sigma, v_1)}$ and $R_2 = R_{(u_2, w_\tau, v_2)}$, where $\sigma, \tau \in \Sigma$ and u_1, v_1, u_2, v_2 are words. There are three cases.

Assume first that $w = u w_1 w_2 w_3 v$, where $w_1 w_2 = w_\sigma$, $w_2 w_3 = w_\tau$, $u = u_1$, $w_3 v = v_1$, $u w_1 = u_2$, $v = v_2$ and $u, w_1, w_2, w_3, v \in W$. Since all overlap ambiguities are resolvable, there are reduction operators L_1 and L_2 such that $L_1(f_\sigma w_3) = L_2(w_1 f_\tau) \in S$. By Remark 6, there exist reduction operators $\widetilde{L_1}$ and $\widetilde{L_2}$ such that $\widetilde{L_1}(u f_\sigma w_3 v) = u L_1(f_\sigma w_3) v = u L_2(w_1 f_\tau) v = \widetilde{L_2}(u w_1 f_\tau v)$. The element $u f_\sigma w_3 v$ is a linear combination of words strictly less than w. Thus $u f_\sigma w_3 v \in U$. Similarly, $u w_1 f_\tau v \in U$. And $\mathcal{R}(u f_\sigma w_3 v) = \mathcal{R}(u w_1 f_\tau v)$. In other words, $R_1 w$ and $R_2 w$ are reduction unique and they have the same reduced form.

Assume now that $w = u_2 w_1 w_\sigma w_2 v_2$, where $w_1 w_\sigma w_2 = w_\tau$, $u_2 w_1 = u_1$, $w_2 v_2 = v_1$ and $w_1, w_2 \in W$. Since all inclusion ambiguities are resolvable, there are reduction operators L_1 and L_2 such that $L_1(w_1 f_\sigma w_2) = L_2 f_\tau \in S$. By Remark 6, there exist reduction operators $\widetilde{L_1}$ and $\widetilde{L_2}$ such that $\widetilde{L_1}(u_2 w_1 f_\sigma w_2 v_2) = u_2 L_1(w_1 f_\sigma w_2) v_2 = u_2 (L_2 f_\tau) v_2 = \widetilde{L_2}(u_2 f_\tau v_2)$. By induction hypothesis, $u_2 w_1 f_\sigma w_2 v_2$ and $u_2 f_\tau v_2$ are reduction unique and have the same reduced form.

Assume finally that $w = u_1 w_\sigma w w_\tau v_2$, where $w w_\tau v_2 = v_1$, $u_1 w_\sigma w = u_2$ and $w \in W$. Then we have $R_1 w = u_1 f_\sigma w w_\tau v_2$ and $R_2 w = u_1 w_\sigma w f_\tau v_2$. There exist reduction operators L_1 and L_2 such that $L_1 R_1 w = u_1 f_\sigma w f_\tau v_2 = L_2 R_2 w$. Then $R_1 w$ and $R_2 w$ are reduction unique and have the same reduced form.

(1)$\Rightarrow$(4) The leading word of any element in I is not normal, hence not standard by $S = N$, then contains w_σ as a subword for some $\sigma \in \Sigma$. Then $\{w_\sigma - f_\sigma | \sigma \in \Sigma\}$ is a Gröbner basis.

(4)$\Rightarrow$(1) If $\{w_\sigma - f_\sigma | \sigma \in \Sigma\}$ is a Gröbner basis, we have $N = S$ by Theorem 3 and that's all. □

Let G_0 be a generating set of the ideal I. In order to get a Gröbner basis starting from G_0, we apply a procedure consisting of the following steps. Assume that G' is an intermediate set with $G_0 \subseteq G' \subseteq I$.

Step 1. (*Normalization*) By multiplying a non-zero coefficient, the leading coefficient of every element in G' becomes 1. Then we get a new intermediate set G'.

Step 2. (*Reduction*) Take two normalized elements x and y in G' and two words $u, v \in W$. Let y_1 be the leading word of y. Compute $R_{(u,y_1,v)}(x)$. There are three cases: if $R_{(u,y_1,v)}(x) = 0$, we remove x from G'; if $R_{(u,y_1,v)}(x) \neq 0$ and $R_{(u,y_1,v)}(x) \neq x$, the leading word of $R_{(u,y_1,v)}(x) = R_{(u,y_1,v)}(x' - u y' v)$ is strictly less than the leading word of x in the homogeneous lexicographic order in which case we replace x by $R_{(u,y_1,v)}(x)$ in G'; if $R_{(u,y_1,v)}(x) = x$, we do nothing. This process is denoted by $x \to R_{(u,y_1,v)}(x)$.

Repeat Step 1 and Step 2 until G' does not change. Then we go to Step 3.

Step 3. (*Composition*) Take two normalized elements x and y in G' with the leading words x_1 and y_1 respectively. If there is a triple (w_1, w_2, w_3), where $w_i \in W$ such that $x_1 = w_1 w_2$, $y_1 = w_2 w_3$ and $w_2 \neq 1$, we compute $w_1 y - x w_3$. If $w_1 y - x w_3$

is not zero, it should be added to G'.

Repeat Step 1 to Step 3 until G' does not change, which may be an infinite number of repetitions. Finally we obtain a set G, which is a minimal Gröbner basis according to Bergman's diamond lemma. Here we give some examples about computing Gröbner bases.

Example 1. Let k be a field whose characteristic is not 2, L a Lie algebra over k with a basis $\{e_i | i \geqslant 1\}$. Then its defining relation will be $[e_i, e_j] = \sum_k c_{ij}^k e_k$. The universal enveloping algebra $U(L)$ of L is an associative k-algebra with the identical set of generators and the defining relations $e_i e_j - e_j e_i = \sum_k c_{ij}^k e_k$. We equip the set $\{e_i | i \geqslant 1\}$ with the order $e_i \succ e_j$ for $i > j$. Then the set $G = \{e_i e_j - e_j e_i - [e_i, e_j] | i > j\}$ is a Gröbner basis about $U(L)$. Indeed, for $i > j > k$ we have

$$\begin{aligned}
&(e_i e_j - e_j e_i - [e_i, e_j])e_k - e_i(e_j e_k - e_k e_j - [e_j, e_k]) \\
&= e_i e_k e_j + e_i[e_j, e_k] - e_j e_i e_k - [e_i, e_j]e_k \\
&\to e_k e_i e_j + [e_i, e_k]e_j + e_i[e_j, e_k] - e_j e_k e_i - e_j[e_i, e_k] - [e_i, e_j]e_k \\
&\to e_k e_j e_i + e_k[e_i, e_j] + [e_i, e_k]e_j + e_i[e_j, e_k] - e_k e_j e_i - [e_j, e_k]e_i - e_j[e_i, e_k] - [e_i, e_j]e_k \\
&\to [e_k, [e_i, e_j]] + [[e_i, e_k], e_j] + [e_i, [e_j, e_k]] \\
&= [e_k, [e_i, e_j]] + [e_j, [e_k, e_i]] + [e_i, [e_j, e_k]] \\
&= 0.
\end{aligned}$$

Hence all ambiguities are resolvable, which implies G is a Gröbner basis. Then a k-basis of $U(L)$ consists of the words of the form $e_{i_1} e_{i_2} \cdots e_{i_k}$ where $k \geqslant 0$ and $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$.

Example 2. Let $A = k\langle x, y\rangle/(x^2 - yx)$ be an algebra with the order $x \succ y$. In order to get a Gröbner basis, we start from the set $G' = \{x^2 - yx\}$. Applying Step 1 to Step 3, we have

$$(x, x, x) : (x^2 - yx)x - x(x^2 - yx) = xyx - yx^2 \to xyx - y^2x.$$

The element $xyx - y^2x$ should be added to the set G'. Then $G' = \{x^2 - yx, xyx - y^2x\}$.

$$(x, x, yx) : (x^2 - yx)yx - x(xyx - y^2x) = xy^2x - yxyx \to xy^2x - y^3x.$$

Then $G' = \{x^2 - yx, xyx - y^2x, xy^2x - y^3x\}$.

$$\begin{gathered}
(xy, x, x) : (xyx - y^2x)x - xy(x^2 - yx) = xy^2x - y^2x^2 \to xy^2x - y^3x \to 0; \\
(xy, x, yx) : (xyx - y^2x)yx - xy(xyx - y^2x) = xy^3x - y^2xyx \to xy^3x - y^4x.
\end{gathered}$$

Then $G' = \{x^2 - yx, xyx - y^2x, xy^2x - y^3x, xy^3x - y^4x\}$.

$$\begin{gathered}
(x, x, y^2x) : (x^2 - yx)y^2x - x(xy^2x - y^3x) = xy^3x - yxy^2x \to xy^3x - y^4x \to 0; \\
(xy^2, x, x) : (xy^2x - y^3x)x - xy^2(x^2 - yx) = xy^3x - y^3x^2 \to xy^3x - y^4x \to 0; \\
(xy, x, y^2x) : (xyx - y^2x)y^2x - xy(xy^2x - y^3x) = xy^4x - y^2xy^2x \to xy^4x - y^5x; \\
\cdots\cdots
\end{gathered}$$

Reasoning inductively, we claim that the set $G = \{xy^n x - y^{n+1}x | n \geqslant 0\}$ is a Gröbner basis of the ideal $(x^2 - yx)$ in $k\langle x, y\rangle$. Indeed, for $k, l \geqslant 0$ we have

$$xy^k \cdot x \cdot y^l x : (xy^k x - y^{k+1}x)y^l x - xy^k(xy^l x - y^{l+1}x) = xy^{k+l+1}x - y^{k+1}xy^l x \to \\ xy^{k+l+1}x - y^{k+l+2}x \to 0.$$

Hence all ambiguities are resolvable. The set $\{y^n, y^n xy^m | n, m \geqslant 0\}$ is a k-basis of A.

3 Anick's resolution

We will present in this chapter the construction by Anick in the article [1] of a projective resolution of the trivial module over an augmentation algebra.

3.1 Basic definitions in homological algebra

We first present some basic definitions in homological algebra. We refer the reader to the book [7] for the basics on module theory, and to the book [13] and [10] for more details on homological algebra.

Let R be a ring with unit, M a left R-module.

Definition 13. The R-module M is called **projective** if given a morphism $f : M \to Y$ of R-modules and a surjective morphism $u : X \to Y$ of R-modules, there exists a morphism $g : M \to X$ of R-modules such that $u \circ g = f$.

Remark 11. (1) A free R-module is projective.
(2) An R-module M is projective if and only if the functor $Hom_R(M, \cdot)$ is exact.

Definition 14. The sequence of R-modules and morphisms of R-modules

$$\cdots \to F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \xrightarrow{d_{-1}} 0$$

is called a **resolution** of M if the sequence is exact, i.e. $\mathrm{Im}(d_n) = \mathrm{Ker}(d_{n-1})$ for each $n \geqslant 0$. Moreover, the resolution is called a **free resolution** if each F_n is a free R-module. The resolution is called a **projective resolution** if each F_n is a projective R-module.

Remark 12. The free resolution of M always exists in the category of R-modules. Choosing a set of generators of M, we get an exact sequence

$$F_0 \xrightarrow{d_0} M \to 0,$$

where F_0 is a free R-module. Choosing a set of generators of $\mathrm{Ker}(d_0)$, we get a surjection $d_1 : F_1 \to Ker(d_0)$, then an exact sequence

$$F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0.$$

Lemma 3. Let

$$\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

be a complex (i.e. $d_{n-1}d_n = 0$) of R-modules with P_i projective. Let

$$\cdots \to Q_1 \xrightarrow{\delta_1} Q_0 \xrightarrow{\delta_0} N \to 0$$

be an exact sequence of R-modules. Then for any morphism $f = f_{-1} : M \to N$ of R-modules, there exist morphisms $f_n : P_n \to Q_n$ such that $f_{n-1}d_n = \delta_n f_n$ for all $n \geqslant 0$. Moreover, any two such series of morphisms $\{f_n | n \geqslant 0\}$ and $\{g_n | n \geqslant 0\}$ are homotopic, i.e. there are morphisms $s_n : P_n \to Q_{n+1}$, $n \geqslant 0$ such that $f_n - g_n = s_{n-1}d_n + \delta_{n+1}s_n$ for all $n \geqslant 0$, where $s_{-1} = 0$.

Proof. Since P_0 is projective, there is a morphism f_0 such that $\delta_0 f_0 = f d_0$. Then $\delta_0 f_0 d_1 = f d_0 d_1 = 0$ implies $\operatorname{Im}(f_0 d_1) \subseteq \operatorname{Ker}(\delta_0) = \operatorname{Im}(\delta_1)$. Since P_1 is projective, there is a morphism f_1 such that $f_0 d_1 = \delta_1 f_1$.

Let $\{f_n | n \geqslant 0\}$ and $\{g_n | n \geqslant 0\}$ be two such series of morphisms. There exists s_0 such that $f_0 - g_0 = \delta_1 s_0$ since $\delta_0(f_0 - g_0) = (f - f)d_0 = 0$ and P_0 is projective. We have $\delta_1(f_1 - g_1 - s_0 d_1) = 0$. Then there exists s_1 such that $\delta_2 s_1 = f_1 - g_1 - s_0 d_1$. □

Corollary 1. Let $\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ and $\cdots \to Q_1 \xrightarrow{\delta_1} Q_0 \xrightarrow{\delta_0} M \to 0$ be two projective resolutions of M. Then for any additive funcor $\mathcal{F}$, the homological groups of $\mathcal{F}(P_\bullet)$ and $\mathcal{F}(Q_\bullet)$ are isomorphic.

Proof. Let $f_n : P_n \to Q_n$ and $g_n : Q_n \to P_n$ be morphisms extending id_M. Then $\{f_n g_n\}$ and $\{id_{Q_n}\}$ are homotopic. Then $\{\mathcal{F}(f_n g_n)\}$ and $\{\mathcal{F}(id_{Q_n})\}$ are homotopic. Then $\{\mathcal{F}(f_n)\}$ induces isomorphisms on homological groups. □

Definition 15. Let M, N be left R-modules, $\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ a projective resolution of M. Then we get a sequence of abelian groups and morphisms of abelian groups

$$0 \xrightarrow{d_0^*=0} \operatorname{Hom}_R(P_0, N) \xrightarrow{d_1^*} \operatorname{Hom}_R(P_1, N) \xrightarrow{d_2^*} \operatorname{Hom}_R(P_2, N) \xrightarrow{d_3^*} \cdots,$$

where $d_n^* : \operatorname{Hom}_R(P_{n-1}, N) \to \operatorname{Hom}_R(P_n, N)$ for $n \geqslant 1$ is defined by $d_n^*(g) = g \circ d_n$ for $g \in \operatorname{Hom}_R(P_{n-1}, N)$. We have $d_n^* \circ d_{n-1}^* = 0$. For $n \geqslant 0$, the Ext **groups** are defined to be

$$\operatorname{Ext}_R^n(M, N) = H^n(\operatorname{Hom}_R(P_\bullet, N)) = \operatorname{Ker}(d_{n+1}^*)/\operatorname{Im}(d_n^*).$$

Note that $\operatorname{Ext}_R^0(M, N) \cong \operatorname{Hom}_R(M, N)$.

Remark 13. (1) The Ext groups are independent of the choice of the projective resolution by Corollary 1. Thus the Ext groups are well-defined.
(2) For any short exact sequence $0 \to N' \to N \to N'' \to 0$ of R-modules, there is a long exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_R(M, N') \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, N'') \\ &\to \operatorname{Ext}_R^1(M, N') \to \operatorname{Ext}_R^1(M, N) \to \operatorname{Ext}_R^1(M, N'') \\ &\to \operatorname{Ext}_R^2(M, N') \to \cdots . \end{aligned}$$

(3) The module M is projective if and only if $\operatorname{Ext}_R^n(M, N) = 0$ for all R-module N and all $n \geqslant 1$.

Definition 16. Let $n \geqslant 1$ and M, N be R-modules. An **n-extension** of M by N is an exact sequence of R-modules

$$\xi : \quad 0 \to N \to K_{n-1} \to \cdots \to K_0 \to M \to 0.$$

We let $E^n(M,N)$ denote the set of equivalence classes of n-extensions of M by N. The equivalence relation is generated by the relation that $\xi \sim \xi'$ if there is a diagram

$$\begin{array}{ccccccccccccc} \xi: & 0 & \longrightarrow & N & \longrightarrow & K_{n-1} & \longrightarrow & \cdots & \longrightarrow & K_0 & \longrightarrow & M & \longrightarrow 0 \\ & & & \downarrow{=} & & \downarrow & & & & \downarrow & & \downarrow{=} & \\ \xi': & 0 & \longrightarrow & N & \longrightarrow & K'_{n-1} & \longrightarrow & \cdots & \longrightarrow & K'_0 & \longrightarrow & M & \longrightarrow 0. \end{array}$$

Remark 14. (1) There is a bijection between $E^n(M,N)$ and $\operatorname{Ext}^n_R(M,N)$.
(2) Given an n-extension above and an m-extension $0 \to P \to L_{m-1} \to \cdots \to L_0 \to N \to 0$, we can get an $(n+m)$-extension $0 \to P \to L_{m-1} \to \cdots \to L_0 \to K_{n-1} \to \cdots \to K_0 \to M \to 0$.

3.2 Anick's resolution

Let k be a field, X a well ordered set and $A = k\langle X\rangle/I$ an associative k-algebra where I is a two-sided ideal of the free associative k-algebra $k\langle X\rangle$. The field k embeds in A via the ring homomorphism $\eta : k \hookrightarrow A$. Suppose that A has an augmentation, i.e. there is a ring homomorphism $\epsilon : A \to k$ such that $\epsilon \circ \eta = id_k$. Then k is an A-module via the map ϵ. Let W be the free monoid generated by X and equipped with the homogeneous lexicographic order $\succeq$ induced by the well order on X. Recall that the elements in X are called letters and the elements in W are called words.

Definition 17. A non-empty subset M of W provided with the homogeneous lexicographic order $\succeq$ is called an **order ideal** of $(W,\succeq)$ if $w_1 \in M$ and $w_2 \subseteq w_1$ imply $w_2 \in M$.

Remark 15. For an order ideal M, we always have $1 \in M$.

Lemma 4. Let $f : k\langle X\rangle \to A$ be the canonical surjection. Then the set

$$M = \{w \in W | f(w) \notin \operatorname{Span}(f(y)|y \prec w, y \in W)\}$$

is an order ideal of $(W,\succeq)$, where $\operatorname{Span}(f(y)|y \prec w, y \in W)$ is the k-vector space spanned by the elements $f(y)$ with $y \prec w$. And the elements $f(w)$ for $w \in M$ form a k-basis of the algebra A.

Proof. We have $1 \in M$. Suppose that M is not an order ideal. Then there exist $w \in M$ and $z \subseteq w$ with $z \notin M$. Hence

$$f(z) = \sum_{y \prec z, y \in W} c_y f(y),$$

where $c_y \in k$ such that $c_y = 0$ for almost all $y \prec z$, $y \in W$. Let $w = uzv$, where $u, v \in W$. Then

$$f(w) = \sum_{y \prec z, y \in W} c_y f(uyv).$$

We have $w \succ uyv$, which implies $w \in M$. Contradiction! Thus M is an order ideal.

We want to prove that the elements $f(w)$ for $w \in M$ are linearly independent over k in A. Suppose not, then there exist finite elements $w_i \in M$, $c_i \in k\backslash\{0\}$, $i = 1, \cdots, r$ such that $\sum_{i=1}^{r} c_i f(w_i) = 0$. Take the greatest element among w_i, say w_1, then $f(w_1) = -\sum_{i=2}^{r}(c_i/c_1)f(w_i)$ with $w_1 \succ w_i$ for $i \geqslant 2$, which implies $w_1 \notin M$. Contradiction!

Finally we will prove by transfinite induction that for every $u \in W$ we have $f(u) \in \mathrm{Span}(f(w)|w \in M)$. As $1 \in W$, the base case is satisfied. Let $u \in W$ and suppose that every element $v \in W$ with $v \prec u$ satisfies $f(v) \in \mathrm{Span}(f(w)|w \in M)$. If $u \in M$, the statement is true. If $u \notin M$, then

$$f(u) = \sum_{y \prec u, y \in W} c_y f(y),$$

where $c_y \in k$ such that $c_y = 0$ for almost all $y \prec u$, $y \in W$. Since $f(y) \in \mathrm{Span}(f(w)|w \in M)$ by induction hypothesis, we get $f(u) \in \mathrm{Span}(f(w)|w \in M)$. □

Remark 16. For the algebra $A = k\langle X \rangle / I$, the order ideal M in Lemma 4 is exactly the set of normal words. Indeed, a word w is not normal $\Leftrightarrow w - \sum_{y \prec w, y \in W} c_y y \in I$ for some $c_y \in k$ such that $c_y = 0$ for almost all $y \prec w, y \in W \Leftrightarrow f(w) = \sum_{y \prec w, y \in W} c_y f(y)$ for some $c_y \in k$ such that $c_y = 0$ for almost all $y \prec w, y \in W \Leftrightarrow w \notin M$.

Definition 18. Let M be an order ideal of $(W, \succeq)$. Then the set

$$C = C_M = \{w \in W\backslash M | u \subsetneq w \text{ implies } u \in M\}$$

is called the set of **obstructions** for M.

Remark 17. The set of obstructions C is an antichain in W, i.e. given $u, v \in W$ with $u \subsetneq v$, we have $u \notin C$ or $v \notin C$.

Lemma 5. Let M be an order ideal of $(W, \succeq)$ and $w \in W$. Then $w \notin M$ if and only if w contains some obstruction $u \in C_M$ as a subword.

Proof. ($\Leftarrow$) Note that $u \notin M$ and $u \subseteq w$ imply $w \notin M$.
($\Rightarrow$) Assume $w \notin M$. Let u be the subword of w of minimal length satisfying $u \notin M$. Then any proper subword of u is in M. Hence $u \in C_M$. □

Definition 19. Let C be an antichain in W with $1 \notin C$. Assume that the length of every element in C is not less than 2. For $n \geqslant 1$, a word $u = x_{i_1} \cdots x_{i_t}$, where $t \geqslant 2$, is called an **n-prechain** on C if there are integers a_j and b_j, $1 \leqslant j \leqslant n$, satisfying the following two conditions:
(a) $1 = a_1 < a_2 \leqslant b_1 < a_3 \leqslant b_2 < \cdots < a_n \leqslant b_{n-1} < b_n = t$;
(b) $x_{i_{a_j}} \cdots x_{i_{b_j}} \in C$ for $1 \leqslant j \leqslant n$.
Moreover, the n-prechain $u = x_{i_1} \cdots x_{i_t}$ is called an **n-chain** on C if the integers a_j and b_j may be chosen so as to satisfy
(c) $x_{i_1} \cdots x_{i_s}$ is not an m-prechain for any $s < b_m$ and $1 \leqslant m \leqslant n$.
The set of all n-chains is denoted by $C^{(n)}$.

Remark 18. (1) Note that an n-prechain has length strictly greater than n by (a).
(2) We have $C^{(1)} = C$. Conventionally, $C^{(0)} = X$, $b_0 = 1$ and $C^{(-1)} = \{1\}$, $b_{-1} = 0$.
(3) If $u = x_{i_1} \cdots x_{i_t}$ is an n-chain and the integers a_j and b_j satisfy (a)-(c), then $x_{i_1} \cdots x_{i_{b_m}}$ is an m-chain for any $1 \leqslant m \leqslant n$.
(4) Condition (c) guarantees that an n-chain cannot be an m-chain for any $m \neq n$.

Lemma 6. Let C be the antichain in Definition 19. Let $n \geqslant 1$ and $u = x_{i_1} \cdots x_{i_t}$ be an n-chain on C. Then the integers a_j and b_j satisfying (a)-(c) above are uniquely determined. In particular, there exists a unique $s < t$ such that $x_{i_1} \cdots x_{i_s}$ is an $(n-1)$-chain, in fact, $s = b_{n-1}$. For this s, the word $x_{i_{s+1}} \cdots x_{i_t}$ does not contain any element of C as a subword.

Proof. Suppose that the integers $\{a_j, b_j\}$ and $\{a'_j, b'_j\}$ both satisfy (a)-(c). From condition (c), we obtain $b_j = b'_j$ for any j. Since $x_{i_{a_j}} \cdots x_{i_{b_j}}$ and $x_{i_{a'_j}} \cdots x_{i_{b_j}}$ are both in the antichain C, we get $a_j = a'_j$ for any j. $\square$

Example 3. Let $C = \{x^3\}$, where $x \in X$. Then the set C is an antichain. x^3 is an 1-chain; x^4 is a 2-chain with $a_2 = 2$, $b_1 = 3$; x^5 is a 2-prechain with $a_2 = b_1 = 3$, but not a 2-chain as x^4 is a 2-chain, not a 3-chain as $a_3 = b_1$; x^6 is a 3-chain with $a_2 = 2$, $b_1 = 3$, $b_2 = a_3 = 4$; x^7 is a 4-chain with $a_2 = 2$, $b_1 = 3$, $b_2 = a_3 = 4$, $a_4 = 5$, $b_3 = 6$.

Let U be a subset of W. The notation Uk denotes the k-vector space spanned by the elements of U. Let M be the order ideal in Lemma 4. Then $Uk \otimes_k A$ has a k-basis $\hat{U} = \{u \otimes f(w) | u \in U, w \in M\}$. From now on we identify w with $f(w)$ for $w \in M$. So we write $\hat{U} = \{u \otimes w | u \in U, w \in M\}$.

Define an order on $\hat{U}$ by writing $u_1 \otimes w_1 \succ u_2 \otimes w_2$ if and only if $u_1 w_1 \succ u_2 w_2$. Next assume $X \subseteq M$ and we consider the case when $U = C_M^{(n)}$. Note that $u_1 w_1 = u_2 w_2$ for $u_1, u_2 \in U$, $w_1, w_2 \in M$ implies $u_1 = u_2$ and $w_1 = w_2$. Indeed, we may assume $u_1 \subseteq u_2$, then $u_1 = u_2$ as they are both n-chains. The order $\succeq$ is a total order on $\hat{U}$. Since every non-empty subset of $\hat{U}$ has a least element, $\succ$ is a well order on $\hat{U}$. For the element $y = \sum_{j=1}^{r} c_j (u_j \otimes w_j) \in Uk \otimes_k A$ with $c_j \in k \backslash \{0\}$, $u_j \in U$, $w_j \in M$, we say that $u_1 \otimes w_1$ is the **highest term** of y if $r = 1$ or $u_1 \otimes w_1 \succ u_j \otimes w_j$ for $2 \leqslant j \leqslant r$. Here we use the notation $HT(y) = u_1 w_1$.

Theorem 5. Let $f : k\langle X \rangle \to A$ be the canonical surjection and M the order ideal in Lemma 4. Assume $X \subseteq M$. Let C be the set of obstructions for M and $C^{(n)}$ the set of n-chains on C. Then there is a free resolution of k as (right) A-modules, called **Anick's resolution**, of the form

$$\cdots \xrightarrow{d_4} C^{(3)} k \otimes_k A \xrightarrow{d_3} C^{(2)} k \otimes_k A \xrightarrow{d_2} Ck \otimes_k A \xrightarrow{d_1} Xk \otimes_k A \xrightarrow{d_0} A \xrightarrow{\epsilon} k \to 0, \quad (1)$$

where

$$d_0(x \otimes 1) = f(x) - \eta\epsilon(f(x)) \quad (2)$$

for $x \in X$ and for $n \geqslant 1$, we have

$$d_n(x_{i_1} \cdots x_{i_{b_n}} \otimes 1) = x_{i_1} \cdots x_{i_{b_{n-1}}} \otimes x_{i_{b_{n-1}+1}} \cdots x_{i_{b_n}} + \omega, \quad (3)$$

where $x_{i_j} \in X$, $x_{i_1} \cdots x_{i_{b_n}} \in C^{(n)}$ and $x_{i_1} \cdots x_{i_{b_n}} \succ HT(\omega)$ if $\omega \neq 0$.

Proof. We need to define the morphism d_n and prove the sequence is exact. To see the exactness at A we have $\epsilon d_0 = 0$, $\mathrm{Ker}(\epsilon) = \{y - \eta\epsilon(y) \in A | y \in A\}$ and the fact that $\{f(w) - \eta\epsilon(f(w)) | w \in M\backslash\{1\}\}$ is a k-basis of $\mathrm{Ker}(\epsilon)$. For $y = f(x_{i_1} \cdots x_{i_t}) \in A$, where $x_{i_j} \in X$, $x_{i_1} \cdots x_{i_t} \in M\backslash\{1\}$, define

$$i_0(y - \eta\epsilon(y)) = x_{i_1} \otimes x_{i_2} \cdots x_{i_t} + \epsilon(x_{i_1})x_{i_2} \otimes x_{i_3} \cdots x_{i_t} + \cdots + \epsilon(x_{i_1} \cdots x_{i_{t-1}})x_{i_t} \otimes 1.$$

Then i_0 extends to a k-linear map $i_0 : \mathrm{Ker}(\epsilon) \to Xk \otimes_k A$. Compute

$$\begin{aligned} d_0 i_0(y - \eta\epsilon(y)) &= x_{i_1} \cdots x_{i_t} - \epsilon(x_{i_1})x_{i_2} \cdots x_{i_t} + \epsilon(x_{i_1})x_{i_2} \cdots x_{i_t} - \epsilon(x_{i_1}x_{i_2})x_{i_3} \cdots x_{i_t} \\ &\quad + \cdots + \epsilon(x_{i_1} \cdots x_{i_{t-1}})x_{i_t} - \eta\epsilon(x_{i_1} \cdots x_{i_t}) \\ &= y - \eta\epsilon(y). \end{aligned}$$

Then $d_0 i_0 = id_{\mathrm{Ker}(\epsilon)}$ implies $\mathrm{Im}(d_0) = \mathrm{Ker}(\epsilon)$.

We rewrite $A = C^{(-1)}k \otimes_k A$ and $\epsilon = d_{-1}$. Suppose inductively that for $n \geqslant 1$, the morphism d_j for $0 \leqslant j < n$ have been defined satisfying the formula (2) or (3) and that the sequence (1) has been proved exact to the right of $C^{(n-1)}k \otimes_k A$. Suppose further that there are k-linear maps $i_j : \mathrm{Ker}(d_{j-1}) \to C^{(j)}k \otimes_k A$ for $0 \leqslant j < n$ satisfying

$$d_j i_j = id_{\mathrm{Ker}(d_{j-1})} \tag{4}$$

and

$$HT(i_j(\omega)) = HT(\omega). \tag{5}$$

These properties hold for $n = 0$. We need to construct d_n.

For every $u \in C^{(n)}$, it can be uniquely written as $u = rs$ where $r \in C^{(n-1)}$ and $s \in M$. We write $u = x_{i_1} \cdots x_{i_{b_n}}$, then

$$\begin{aligned} d_{n-1}(r \otimes s) &= d_{n-1}(r)s \\ &= (x_{i_1} \cdots x_{i_{b_{n-2}}} \otimes x_{i_{b_{n-2}+1}} \cdots x_{i_{b_{n-1}}})s + \omega \\ &= x_{i_1} \cdots x_{i_{b_{n-2}}} \otimes f(x_{i_{b_{n-2}+1}} \cdots x_{i_{b_n}}) + \omega \end{aligned}$$

where $u \succ HT(\omega)$. As $x_{i_{b_{n-2}+1}} \cdots x_{i_{b_n}}$ has the obstruction $x_{i_{a_n}} \cdots x_{i_{b_n}}$ as a subword, we have $x_{i_{b_{n-2}+1}} \cdots x_{i_{b_n}} \notin M$, which implies $u \succ HT(d_{n-1}(r \otimes s))$ by Lemma 4. Let $\sigma = i_{n-1}d_{n-1}(r \otimes s)$. Then $HT(\sigma) = HT(d_{n-1}(r \otimes s))$ and $u \succ HT(\sigma)$. Define

$$d_n(u \otimes 1) = r \otimes s - \sigma.$$

Then the morphism d_n satisfies the formula (3) and $d_{n-1}d_n = 0$.

Next we construct the k-linear map $i_n : \mathrm{Ker}(d_{n-1}) \to C^{(n)}k \otimes_k A$. Let $\omega = \sum_{j=1}^{r} c_j(u_j \otimes v_j) \in \mathrm{Ker}(d_{n-1})$ where $c_j \in k\backslash\{0\}$, $u_j \in C^{(n-1)}$ and $v_j \in M$ such that $u_1 \otimes v_1$ is the highest term of ω. Suppose that $i_n(\omega')$ has been defined, is k-linear and satisfies the formula (4) and (5) for all $\omega' \in \mathrm{Ker}(d_{n-1})$ with $u_1v_1 \succ HT(\omega')$. Since $\omega \in \mathrm{Ker}(d_{n-1})$ we have $u_1v_1 \succ HT(d_{n-1}(\sum_{j=2}^{r} c_j(u_j \otimes v_j))) = HT(d_{n-1}(u_1 \otimes v_1))$. Writing $u_1 = x_{i_1} \cdots x_{i_{b_{n-1}}}$ and $v_1 = x_{i_{b_{n-1}+1}} \cdots x_{i_s}$, we have

$$u_1v_1 \succ HT(x_{i_1} \cdots x_{i_{b_{n-2}}} \otimes f(x_{i_{b_{n-2}+1}} \cdots x_{i_{b_{n-1}}} x_{i_{b_{n-1}+1}} \cdots x_{i_s})).$$

Then $x_{i_{b_{n-2}+1}} \cdots x_{i_{b_{n-1}}} x_{i_{b_{n-1}+1}} \cdots x_{i_s} \notin M$, hence contains some obstruction as a subword. Let $x_{i_{a_n}} \cdots x_{i_{b_n}}$ be the obstruction contained in $x_{i_{b_{n-2}+1}} \cdots x_{i_s}$ which starts furthest to the left. We have $b_{n-2} < a_n \leqslant b_{n-1} < b_n$. Then $x_{i_1} \cdots x_{i_{b_n}}$ is an n-chain. Let $\tau = x_{i_1} \cdots x_{i_{b_n}} \otimes y \in C^{(n)}k \otimes_k A$ where $y = x_{i_{b_n+1}} \cdots x_{i_s} \in M$. By formula (3), $u_1 \otimes v_1$ is the highest term of $d_n(\tau)$. Let $\omega' = \omega - c_1 d_n(\tau) \in \mathrm{Ker}(d_{n-1})$, then $u_1 v_1 \succ HT(\omega')$. Define

$$i_n(\omega) = c_1 \tau + i_n(\omega').$$

We have $HT(i_n(\omega)) = u_1 v_1 = HT(\omega)$ and $d_n i_n(\omega) = c_1 d_n(\tau) + \omega' = \omega$. And the map i_n defined above is k-linear. Then $d_n i_n = id_{\mathrm{Ker}(d_{n-1})}$ implies $\mathrm{Im}(d_n) = \mathrm{Ker}(d_{n-1})$. □

Definition 20. A k-algebra A with an augmentation $\epsilon : A \to k$ is called **connected graded** if there are k-vector subspaces $\{A_n | n \geqslant 0\}$ of A such that $A = \oplus_{n=0}^{\infty} A_n$ as k-vector spaces and $A_n \cdot A_m \subseteq A_{n+m}$ for all $m, n \geqslant 0$, where $A_0 = k$ and $\mathrm{Ker}(\epsilon) = \oplus_{n=1}^{\infty} A_n$. We denote $\oplus_{n=1}^{\infty} A_n$ by A_+.

Remark 19. For the connected graded augmented algebra A, the Anick's resolution can be made in the category of graded modules: modules are connected graded and differentials d_n preserve the grading.

The following example was first considered in [6].

Example 4. Let $A = k\langle x, y\rangle/(x^2 - yx)$ be a connected graded k-algebra with the order $x \succ y$. The set $G = \{xy^n x - y^{n+1}x | n \geqslant 0\}$ is a Gröbner basis. The set $M = \{y^n, y^n x y^m | n, m \geqslant 0\}$ is a k-basis of A. The set of obstructions for M is $C = \{xy^n x | n \geqslant 0\}$. The set of n-chains on C is $C^{(n)} = \{xy^{i_1} xy^{i_2} \cdots xy^{i_n} x | i_j \geqslant 0\}$. To compute Anick's resolution, we have

$$\begin{aligned}
d_0(x \otimes 1) &= x;\\
d_0(y \otimes 1) &= y;\\
d_1(xy^m x \otimes 1) &= x \otimes y^m x - i_0 d_0(x \otimes y^m x) = x \otimes y^m x - y \otimes y^m x;\\
d_2(xy^{i_1} xy^{i_2} x \otimes 1) &= xy^{i_1} x \otimes y^{i_2} x - i_1 d_1(xy^{i_1} x \otimes y^{i_2} x)\\
&= xy^{i_1} x \otimes y^{i_2} x - i_1[(x \otimes y^{i_1} x - y \otimes y^{i_1} x) y^{i_2} x]\\
&= xy^{i_1} x \otimes y^{i_2} x - i_1(x \otimes y^{i_1+i_2+1} x - y \otimes y^{i_1+i_2+1} x)\\
&= xy^{i_1} x \otimes y^{i_2} x - xy^{i_1+i_2+1} x \otimes 1\\
&\quad - i_1[x \otimes y^{i_1+i_2+1} x - y \otimes y^{i_1+i_2+1} x - d_1(xy^{i_1+i_2+1} x \otimes 1)]\\
&= xy^{i_1} x \otimes y^{i_2} x - xy^{i_1+i_2+1} x \otimes 1;\\
d_3(xy^{i_1} xy^{i_2} xy^{i_3} x \otimes 1) &= xy^{i_1} xy^{i_2} x \otimes y^{i_3} x - i_2 d_2(xy^{i_1} xy^{i_2} x \otimes y^{i_3} x)\\
&= xy^{i_1} xy^{i_2} x \otimes y^{i_3} x - i_2[(xy^{i_1} x \otimes y^{i_2} x - xy^{i_1+i_2+1} x \otimes 1) y^{i_3} x]\\
&= xy^{i_1} xy^{i_2} x \otimes y^{i_3} x - i_2(xy^{i_1} x \otimes y^{i_2+i_3+1} x - xy^{i_1+i_2+1} x \otimes y^{i_3} x)\\
&= xy^{i_1} xy^{i_2} x \otimes y^{i_3} x - xy^{i_1} xy^{i_2+i_3+1} x \otimes 1 - i_2[xy^{i_1} x \otimes y^{i_2+i_3+1} x\\
&\quad - xy^{i_1+i_2+1} x \otimes y^{i_3} x - d_2(xy^{i_1} xy^{i_2+i_3+1} x \otimes 1)]\\
&= xy^{i_1} xy^{i_2} x \otimes y^{i_3} x - xy^{i_1} xy^{i_2+i_3+1} x \otimes 1
\end{aligned}$$

$$+ i_2(xy^{i_1+i_2+1}x \otimes y^{i_3}x - xy^{i_1+i_2+i_3+2}x \otimes 1)$$
$$= xy^{i_1}xy^{i_2}x \otimes y^{i_3}x - xy^{i_1}xy^{i_2+i_3+1}x \otimes 1 + xy^{i_1+i_2+1}xy^{i_3}x \otimes 1;$$

Similarly, we could cmpute

$$d_4(xy^{i_1}xy^{i_2}xy^{i_3}xy^{i_4}x \otimes 1) = xy^{i_1}xy^{i_2}xy^{i_3}x \otimes y^{i_4}x - xy^{i_1}xy^{i_2}xy^{i_3+i_4+1}x \otimes 1 + xy^{i_1}xy^{i_2+i_3+1}xy^{i_4}x \otimes 1 - xy^{i_1+i_2+1}xy^{i_3}xy^{i_4}x.$$

We will prove by induction for $n \geqslant 2$

$$d_n(xy^{i_1}xy^{i_2}\cdots xy^{i_n}x \otimes 1) = xy^{i_1}xy^{i_2}\cdots xy^{i_{n-1}}x \otimes y^{i_n}x + \sum_{k=1}^{n-1}(-1)^{n-k}xy^{i_1}xy^{i_2}\cdots xy^{i_{k-1}}xy^{i_k+i_{k+1}+1}xy^{i_{k+2}}xy^{i_{k+3}}\cdots xy^{i_n}x \otimes 1. \qquad (6)$$

The base case $n = 2$ is satisfied. Suppose that (6) is true for $n-1$. First we prove by induction on m that we have

$$\begin{aligned} & d_n(xy^{i_1}xy^{i_2}\cdots xy^{i_n}x \otimes 1) = xy^{i_1}xy^{i_2}\cdots xy^{i_{n-1}}x \otimes y^{i_n}x \\ & + \sum_{k=m}^{n-1}(-1)^{n-k}xy^{i_1}xy^{i_2}\cdots xy^{i_{k-1}}xy^{i_k+i_{k+1}+1}xy^{i_{k+2}}xy^{i_{k+3}}\cdots xy^{i_n}x \otimes 1 \\ & + i_{n-1}[\sum_{k=1}^{m-1}(-1)^{n-k}xy^{i_1}xy^{i_2}\cdots xy^{i_{k-1}}xy^{i_k+i_{k+1}+1}xy^{i_{k+2}}xy^{i_{k+3}}\cdots xy^{i_{n-1}}x \otimes y^{i_n}x \\ & + \sum_{l=m+1}^{n-1}\sum_{k=1}^{m-1}(-1)^{l+k}xy^{i_1}\cdots xy^{i_{k-1}}xy^{i_k+i_{k+1}+1}xy^{i_{k+2}}\cdots xy^{i_{l-1}}xy^{i_l+i_{l+1}+1}xy^{i_{l+2}}\cdots \\ & \quad xy^{i_n}x \otimes 1 \\ & + \sum_{k=1}^{m-2}(-1)^{m+k}xy^{i_1}\cdots xy^{i_{k-1}}xy^{i_k+i_{k+1}+1}xy^{i_{k+2}}\cdots xy^{i_{m-1}}xy^{i_m+i_{m+1}+1}xy^{i_{m+2}}\cdots \\ & \quad xy^{i_n}x \otimes 1 \\ & - xy^{i_1}\cdots xy^{i_{m-2}}xy^{i_{m-1}+i_m+i_{m+1}+2}xy^{i_{m+2}}\cdots xy^{i_n}x \otimes 1] \end{aligned} \qquad (7)$$

for $2 \leqslant m \leqslant n-1$. Conventionally, the term $\sum_{k=a}^{b}\cdots$ is zero if $a > b$. We check the base case $m = n-1$.

$$\begin{aligned} & d_n(xy^{i_1}\cdots xy^{i_n}x \otimes 1) \\ &= xy^{i_1}\cdots xy^{i_{n-1}}x \otimes y^{i_n}x - i_{n-1}d_{n-1}(xy^{i_1}\cdots xy^{i_{n-1}}x \otimes y^{i_n}x) \\ &= xy^{i_1}\cdots xy^{i_{n-1}}x \otimes y^{i_n}x - i_{n-1}[xy^{i_1}\cdots xy^{i_{n-2}}x \otimes y^{i_{n-1}+i_n+1}x \\ & \quad + \sum_{k=1}^{n-2}(-1)^{n-1-k}xy^{i_1}\cdots xy^{i_{k-1}}xy^{i_k+i_{k+1}+1}xy^{i_{k+2}}\cdots xy^{i_{n-1}}x \otimes y^{i_n}x] \\ &= xy^{i_1}\cdots xy^{i_{n-1}}x \otimes y^{i_n}x - xy^{i_1}\cdots xy^{i_{n-2}}xy^{i_{n-1}+i_n+1}x \otimes 1 \end{aligned}$$

$$
\begin{aligned}
&- i_{n-1}[xy^{i_1}\cdots xy^{i_{n-2}}x\otimes y^{i_{n-1}+i_n+1}x\\
&+\sum_{k=1}^{n-2}(-1)^{n-1-k}xy^{i_1}\cdots xy^{i_{k-1}}xy^{i_k+i_{k+1}+1}xy^{i_{k+2}}\cdots xy^{i_{n-1}}x\otimes y^{i_n}x\\
&- d_{n-1}(xy^{i_1}\cdots xy^{i_{n-2}}xy^{i_{n-1}+i_n+1}x\otimes 1)]\\
=\ & xy^{i_1}\cdots xy^{i_{n-1}}x\otimes y^{i_n}x - xy^{i_1}\cdots xy^{i_{n-2}}xy^{i_{n-1}+i_n+1}x\otimes 1\\
&+ i_{n-1}[\sum_{k=1}^{n-2}(-1)^{n-k}xy^{i_1}\cdots xy^{i_{k-1}}xy^{i_k+i_{k+1}+1}xy^{i_{k+2}}\cdots xy^{i_{n-1}}x\otimes y^{i_n}x\\
&+\sum_{k=1}^{n-3}(-1)^{n-1-k}xy^{i_1}\cdots xy^{i_{k-1}}xy^{i_k+i_{k+1}+1}xy^{i_{k+2}}\cdots xy^{i_{n-2}}xy^{i_{n-1}+i_n+1}x\otimes 1\\
&- xy^{i_1}\cdots xy^{i_{n-3}}xy^{i_{n-2}+i_{n-1}+i_n+2}x\otimes 1].
\end{aligned}
$$

Suppose that (7) is true for m. We prove that it is true for $m-1$.

$$
\begin{aligned}
&d_n(xy^{i_1}\cdots xy^{i_n}x\otimes 1)\\
=\ & xy^{i_1}\cdots xy^{i_{n-1}}x\otimes y^{i_n}x + \sum_{k=m-1}^{n-1}(-1)^{n-k}xy^{i_1}\cdots xy^{i_k+i_{k+1}+1}\cdots xy^{i_n}x\otimes 1\\
&+ i_{n-1}[\sum_{k=1}^{m-1}(-1)^{n-k}xy^{i_1}\cdots xy^{i_k+i_{k+1}+1}\cdots xy^{i_{n-1}}x\otimes y^{i_n}x\\
&+\sum_{l=m+1}^{n-1}\sum_{k=1}^{m-1}(-1)^{l+k}xy^{i_1}\cdots xy^{i_k+i_{k+1}+1}\cdots xy^{i_l+i_{l+1}+1}\cdots xy^{i_n}x\otimes 1\\
&+\sum_{k=1}^{m-2}(-1)^{m+k}xy^{i_1}\cdots xy^{i_k+i_{k+1}+1}\cdots xy^{i_m+i_{m+1}+1}\cdots xy^{i_n}x\otimes 1\\
&- xy^{i_1}\cdots xy^{i_{m-1}+i_m+i_{m+1}+2}\cdots xy^{i_n}x\otimes 1\\
&- (-1)^{n-m+1}d_{n-1}(xy^{i_1}\cdots xy^{i_{m-1}+i_m+1}\cdots xy^{i_n}x\otimes 1)]\\
=\ & xy^{i_1}\cdots xy^{i_{n-1}}x\otimes y^{i_n}x + \sum_{k=m-1}^{n-1}(-1)^{n-k}xy^{i_1}\cdots xy^{i_k+i_{k+1}+1}\cdots xy^{i_n}x\otimes 1\\
&+ i_{n-1}[\sum_{k=1}^{m-2}(-1)^{n-k}xy^{i_1}\cdots xy^{i_k+i_{k+1}+1}\cdots xy^{i_{n-1}}x\otimes y^{i_n}x\\
&+\sum_{l=m}^{n-1}\sum_{k=1}^{m-2}(-1)^{l+k}xy^{i_1}\cdots xy^{i_k+i_{k+1}+1}\cdots xy^{i_l+i_{l+1}+1}\cdots xy^{i_n}x\otimes 1\\
&+\sum_{k=1}^{m-3}(-1)^{m-1+k}xy^{i_1}\cdots xy^{i_k+i_{k+1}+1}\cdots xy^{i_{m-1}+i_m+1}\cdots xy^{i_n}x\otimes 1\\
&- xy^{i_1}\cdots xy^{i_{m-2}+i_{m-1}+i_m+2}\cdots xy^{i_n}x\otimes 1].
\end{aligned}
$$

Now the formula (7) has been proved. When $m = 2$, we have

$$\begin{aligned}
&d_n(xy^{i_1}\cdots xy^{i_n}x\otimes 1)\\
=&\ xy^{i_1}\cdots xy^{i_{n-1}}x\otimes y^{i_n}x+\sum_{k=2}^{n-1}(-1)^{n-k}xy^{i_1}\cdots xy^{i_k+i_{k+1}+1}\cdots xy^{i_n}x\otimes 1\\
&+i_{n-1}[(-1)^{n-1}xy^{i_1+i_2+1}\cdots xy^{i_{n-1}}x\otimes y^{i_n}x\\
&+\sum_{l=3}^{n-1}(-1)^{l+1}xy^{i_1+i_2+1}\cdots xy^{i_l+i_{l+1}+1}\cdots xy^{i_n}x\otimes 1-xy^{i_1+i_2+i_3+2}\cdots xy^{i_n}x\otimes 1]\\
=&\ xy^{i_1}\cdots xy^{i_{n-1}}x\otimes y^{i_n}x+\sum_{k=1}^{n-1}(-1)^{n-k}xy^{i_1}\cdots xy^{i_k+i_{k+1}+1}\cdots xy^{i_n}x\otimes 1.
\end{aligned}$$

Then the formula (6) has been proved.

If we consider the order $y \succ x$, we can get that the set $G' = \{yx - x^2\}$ is a Gröbner basis. Then the set $M' = \{x^n y^m | n \geqslant 0, m \geqslant 0\}$ is a k-basis of A. The set of obstructions for M' is $C' = \{yx\}$. The set of n-chains on C' is empty when $n \geqslant 2$. Then Anick's resolution is given by $d_0(y \otimes 1) = y$, $d_0(x \otimes 1) = x$, $d_1(yx \otimes 1) = y \otimes x - x \otimes x$ and $d_n = 0$ for $n \geqslant 2$.

3.3 Minimal projective resolutions

First we present some basic definitions about graded algebras and graded modules referring to [8]. Then we introduce the minimal projective resolutions in the category of graded modules, referring to [2] and [8]. Let k be a field and A a connected graded k-algebra in Definition 20.

Definition 21. An A-module M is called **graded** if there are k-vector subspaces $\{M_n | n \in \mathbb{Z}\}$ of M such that $M = \oplus_{n\in\mathbb{Z}} M_n$ as k-vector spaces and $A_m \cdot M_n \subseteq M_{m+n}$ for all $m \geqslant 0$ and $n \in \mathbb{Z}$. The elements of M_n are called **homogeneous**. A graded A-module M is called **bounded below** if $M_n = 0$ for $n \ll 0$. A graded A-module M is called **locally finite dimensional** if every k-vector space M_n is finite dimensional over k. For a graded A-module M and $i \in \mathbb{Z}$, we denote by $M(i)$ the same module with shifted grading: $M(i)_n = M_{i+n}$.

Remark 20. The algebra A itself is a bounded below graded A-module.

Definition 22. In the category of graded A-modules, a surjection $d : P \to M$ is called **essential** if for any morphism $f : N \to P$, df a surjection implies that f is a surjection. If in addition P is a projective module, the pair (P, d) is called the **projective cover** of M.

Proposition 2. Let M be a graded A-module. Then the projective cover of M is unique up to isomorphism if exists.

Proof. Let (P, d) and (P', d') be two projective covers of M. Since P is projective and d' is surjective, there exists $g : P \to P'$ such that $d'g = d$. Since d' is essential and d is surjective, g is surjective. As P' is projecctive, there exists $s : P' \to P$ such that $gs = id_{P'}$. Then $ds = d'gs = d'$. Since d is essential and d' is surjective, s is surjective. Then s is an isomorphism. □

Lemma 7. (Nakayama's lemma) Let M be a bounded below graded A-module and $A_+M = M$. Then $M = 0$.

Proof. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be the decomposition. We prove $M_n = 0$ by induction on n. For $n \ll 0$, it is trivial. For any $x \in M_n$, we can write $x = \sum_{i+j=n,\ i>0} a_i m_j$, where $a_i \in A_i$ and $m_j \in M_j$. Then $m_j = 0$ by induction hypothesis. Hence $x = 0$. □

Remark 21. We have $M/A_+M \cong k \otimes_A M$ as graded A-modules.

Proposition 3. Let M be a bounded below graded A-module. Then M has the projective cover.

Proof. Take homogeneous elements $x_i \in M$, $i \in I$ such that the images of x_i in M/A_+M form a k-basis of M/A_+M. Let V be the k-vector subspace with the basis x_i, $i \in I$ and N the A-submodule of M generated by x_i, $i \in I$. Then $N + A_+M = M$. Note that $A_+(M/N) = (A_+M + N)/N = M/N$. By Nakayama's lemma, $M/N = 0$. Then $M = N$. So we have a surjection $d : A \otimes_k V \to M$. Let $f : K \to A \otimes_k V$ be a morphism of graded A-modules and f not surjective. Then $K \to A \otimes_k V \to M \to M/A_+M$ is not surjective. Then df is not surjective. This implies that d is essential. Then $(A \otimes_k V, d)$ is the projective cover of M. □

Corollary 2. Let M be a bounded below graded A-module. Then M is projective if and only if M is free.

Proof. Suppose M is projective and (P, d) is a projective cover constructed in the proof of Proposition 3. Then P is free. Since (M, id_M) is a projective cover of M, M and P are isomorphic. □

Definition 23. A projective resolution of graded A-module M

$$\cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

is called **minimal** if the induced map $id_k \otimes d_{i+1} : k \otimes_A P_{i+1} \to k \otimes_A P_i$ is zero for every $i \geqslant 0$.

Proposition 4. Every bounded below graded A-module M has a minimal projective resolution. A minimal bounded below graded projective resolution of M is unique up to isomorphism.

Proof. Let (P_0, d_0) be the projective cover of M in the proof of Proposition 3. Let $M_1 = \mathrm{Ker}(d_0)$ and V_1 be a k-vector subspace of M_1 such that $V_1 \cong M_1/A_+M_1$. Take $P_1 = A \otimes_k V_1$ and $d_1' : P_1 \to M_1$. Then (P_1, d_1') is the projective cover of M_1. Let d_1 be the composition $P_1 \to M_1 \to P_0$. Repeating above process, we get a minimal projective resolution of M. Take $x = \sum a_i \otimes x_i \in V_1 \subseteq M_1 \subseteq A \otimes_k V$, where $a_i \in A$ and $\{x_i\}$ the basis of V. $d(x) = 0$ implies $a_i \in A_+$. Then the image of

$1 \otimes 1 \otimes x \in k \otimes_A (A \otimes_k V_1)$ is $\sum 1 \otimes a_i \otimes x_i = 0 \in k \otimes_A (A \otimes_k V)$. Thus the projective resolution constructed above is exactly minimal.

Let $P_\bullet$ and $P'_\bullet$ be two minimal projective resolutions of M. By Lemma 3, there are morphisms $f_n : P_n \to P'_n$ and $g_n : P'_n \to P_n$ extending id_M. Moreover, $\{g_n f_n\}$ and $\{id_{P_n}\}$ are homotopic i.e. there exist morphisms $s_n : P_n \to P_{n+1}$ such that $g_n f_n - id_{P_n} = s_{n-1} d_n + d_{n+1} s_n$, where $d_n : P_n \to P_{n-1}$. Then $(id_k \otimes g_n)(id_k \otimes f_n) - id_{k \otimes_A P_n} = (id_k \otimes s_{n-1})(id_k \otimes d_n) + (id_k \otimes d_{n+1})(id_k \otimes s_n) = 0$. Hence $id_k \otimes f_n$ and $id_k \otimes g_n$ are isomorphisms between complexes. From Lemma 1.6 in [2], f_n and g_n are isomorphisms. □

Proposition 5. Let M be a bounded below graded A-module M. Let $P_\bullet$ and $P'_\bullet$ be two projective resolutions of M with $P_\bullet$ minimal. Suppose $f : P_\bullet \to P'_\bullet$ and $g : P'_\bullet \to P_\bullet$ be morphisms of complexes. Then f is injective, g is surjective and $P' = \mathrm{Im}(f) \oplus \mathrm{Ker}(g)$. Moreover, the homological groups of the complex $\mathrm{Ker}(g)$ are null.

Proof. The morphism gf is an isomorphism by Proposition 4. Thus f is injective and g is surjective. Let $f' = f(gf)^{-1}$. We have $\mathrm{Im}(f') = \mathrm{Im}(f)$. As $gf' = id_{P_\bullet}$, we have $P' = \mathrm{Im}(f') \oplus \mathrm{Ker}(g)$. Then $P' = \mathrm{Im}(f) \oplus \mathrm{Ker}(g)$. Consider the short exact sequence of complexes

$$0 \to \mathrm{Ker}(g) \to P'_\bullet \xrightarrow{g} P_\bullet \to 0.$$

By Corollary 1, g is a quasi-isomorphism. The long exact sequence of homological groups gives $H^n(\mathrm{Ker}(g)) = 0$. □

3.4 Koszul algebras

We close this chapter by giving a brief introduction to Koszul algebra, referring to [3] and [8].

Definition 24. Let V be a k-vector space. We define the **tensor algebra** generated by V to be $\mathbb{T}(V) = \oplus_{n=0}^{\infty} \mathbb{T}^n(V)$, where $\mathbb{T}^0(V) = k$ and $\mathbb{T}^n(V) = V^{\otimes n}$ for $n \geqslant 1$.

Definition 25. A connected graded k-algebra A is called **quadratic** if A is generated by A_1 as a k-algebra and the kernel J of the natural surjection $\mathbb{T}(A_1) \to A$ is generated as a two-sided ideal of $\mathbb{T}(A_1)$ by its subspace $J \cap \mathbb{T}^2(A_1)$.

Definition 26. Let A be a quadratic algebra. A graded A-module M is called **quadratic** if $M_n = 0$ for $n < 0$, M is generated by M_0 as an A-module, and the kernel J of the natural surjection $A \otimes_k M_0 \to M$ is generated as an A-submodule of $A \otimes_k M_0$ by the subspace $J \cap (A_1 \otimes_k M_0)$.

Example 5. Let $n \geqslant 2$. The Fomin-Kirillov algebra of order n, denoted as $FK(n)$, is a k-algebra generated by elements $[i, j]$ for $1 \leqslant i, j \leqslant n$ and $i \neq j$, subject to the following relations:
(1) $[i, j]^2 = 0$;
(2) $[i, j] = -[j, i]$;

(3) $[i,j][j,l] + [j,l][l,i] + [l,i][i,j] = 0$ for distinct i, j, l;
(4) $[i,j][l,m] = [l,m][i,j]$ for distinct i, j, l, m.
The Fomin-Kirillov algebras are quadratic algebras.

Definition 27. A quadratic algebra A is called **Koszul** if k has a graded projective resolution $(P_\bullet, d_\bullet)$ such that P_n as a graded A-module is generated by homogeneous elements of degree n.

Example 6. Roos proves in [9] that for $n \geqslant 3$, $FK(n)$ is not Koszul.

Now we assume our connected graded k-algebras to be locally finite dimensional. From Theorem 3.1, page 84 in [8], we have the following theorem:

Theorem 6. Let A be a quadratic algebra with a quadratic Gröbner basis. Then A is Koszul.

Theorem 7. Anick's resolution is never minimal for any quadratic non Koszul algebra.

Proof. Let $A = \mathbb{T}(V)/I$ be a quadratic non Koszul algebra, $R = I \cap \mathbb{T}^2(V)$ and $X = \{x_1, \cdots, x_m\}$ a k-basis of V. Then any Gröbner basis G of A has an homogeneous element of degree $d \geqslant 3$. Let C be the set of leading words of elements in G. Then Anick's resolution is

$$\cdots \to Ck \otimes_k A \to Xk \otimes_k A \to A \to k \to 0.$$

Referring to pages 709-710 in [3], the minimal projective resolution of k is

$$\cdots \to A \otimes_k R \xrightarrow{d_2} A \otimes_k V \to A \to k \to 0,$$

where d_2 is the restriction of the linear map $a \otimes (x \otimes y)$ to $ax \otimes y$ for $a \in A$ and $x, y \in V$. The two resolutions are not isomorphic, then Anick's resolution is not minimal. $\square$

Example 7. Anick's resolution of k is never minimal for $FK(n)$ with $n \geqslant 3$.

4 Computations of Anick's resolution of $FK(3)$

Here we will partially compute Anick's resolution and Ext groups for the Fomin-Kirillov algebra with three generators. We recall that k is a field, $A = FK(3) = k\langle X\rangle/I$ an associative k-algebra with unity, where the set $X = \{a, b, c\}$ is equipped an ordering by setting $c > b > a$ and I is a two-sided ideal generated by the words $a^2, b^2, c^2, ca+bc+ab, cb+ba+ac$. We associate to the free associative k-algebra $k\langle X\rangle$ the grading by length and $A = \oplus_{n=0}^{\infty} A_n$ inherits this gradation. Suppose that A has the augmentation $\epsilon : A \to k$, which satisfies $\epsilon\mid_{A_0=k}= id_k$ and $\epsilon\mid_{\oplus_{n=1}^{\infty} A_n}= 0$. We begin with computing a Gröbner basis.

4.1 Gröbner basis

In order to get a Gröbner basis of A, we start from the set $G' = \{a^2, b^2, c^2, ca + bc + ab, cb + ba + ac\}$.

$$(c, a, a) : (ca + bc + ab)a - ca^2 = bca + aba \to b(-bc - ab) + aba \to bab - aba.$$

The element $bab - aba$ should be adjoined to G'.

$$\begin{gathered}(c, b, b) : (cb + ba + ac)b - cb^2 = bab + acb \to bab + a(-ba - ac) \to bab - aba \to 0;\\ (c, c, a) : c(ca + bc + ab) - c^2a = cbc + cab \to (-ba - ac)c + (-bc - ab)b \to\\ bac + bcb \to bac + b(-ba - ac) \to 0;\\ (c, c, b) : c(cb + ba + ac) - c^2b = cba + cac \to (-ba - ac)a + (-bc - ab)c \to\\ aca + abc \to a(-bc - ab) + abc = -a^2b \to 0.\end{gathered}$$

Then check all ambiguities with respect to $G = \{a^2, b^2, c^2, ca + bc + ab, cb + ba + ac, bab - aba\}$ are resolvable.

$$\begin{gathered}(b, b, ab) : b^2ab - b(bab - aba) = baba \to (aba)a \to 0;\\ (ba, b, b) : bab^2 - (bab - aba)b = abab \to a(aba) \to 0;\\ (c, b, ab) : (cb + ba + ac)ab - c(bab - aba) = ba^2b + acab + caba \to a(-bc - ab)b +\\ (-bc - ab)ba \to ab(ba + ac) + b(ba + ac)a \to abac + ba(-bc - ab) \to abac - babc \to 0;\\ (ba, b, ab) : (bab - aba)ab - ba(bab - aba) = ba^2ba - aba^2b \to 0.\end{gathered}$$

By Bergman's diamond lemma, the set $G = \{a^2, b^2, c^2, ca+bc+ab, cb+ba+ac, bab-aba\}$ is a minimal Gröbner basis of A.

4.2 Anick's resolution

It can be calculated that the set of standard words (i.e. don't contain any of the words $a^2, b^2, c^2, ca, cb, bab$) is $M = \{1, a, b, c, ab, ac, ba, bc, aba, abc, bac, abac\}$. The set of obstructions for M is $C = \{c^2, cb, ca, b^2, a^2, bab\}$. Observe that every n-prechain on C is an n-chain. We get that the set of n-chains $C^{(n)}$ consists of words:

$$\begin{aligned} &c^i a^{n+1-i}, \quad 0 \leqslant i \leqslant n+1;\\ &c^l b^{j_1} a b^{j_2} \cdots a b^{j_s}, \quad 1 \leqslant j_1 \leqslant n+1, \quad 0 \leqslant l, j_2, \cdots, j_s \leqslant n, \quad l + j_1 + \cdots + j_s = n+1.\end{aligned}$$

Now we begin to construct the differentials d_n of Anick's resolution,

$$\cdots \xrightarrow{d_4} C^{(3)}k \otimes_k A \xrightarrow{d_3} C^{(2)}k \otimes_k A \xrightarrow{d_2} Ck \otimes_k A \xrightarrow{d_1} Xk \otimes_k A \xrightarrow{d_0} A \xrightarrow{\epsilon} k \to 0,$$

Let $f : k\langle X\rangle \to A$ be the canonical surjection and we identify the word x with the element $f(x)$ if $x \in M$.

Computation of d_0 As defined in Theorem 5, we have

$$d_0(a \otimes 1) = a,$$
$$d_0(b \otimes 1) = b,$$
$$d_0(c \otimes 1) = c.$$

Computation of d_1 For $cb \in C$, we may uniquely write $cb = c \cdot b$ where $c \in X$ and $b \in M$, so $d_1(cb \otimes 1) = c \otimes b - i_0 d_0(c \otimes b)$ where $i_0 : \mathrm{Ker}(\epsilon) \to Xk \otimes_k A$. Compute $i_0 d_0(c \otimes b) = i_0(f(cb)) = -i_0(ba + ac) = -b \otimes a - a \otimes c$. Hence we have $d_1(cb \otimes 1) = c \otimes b + b \otimes a + a \otimes c$. After similarly computating, we get the list:

$$d_1(c^2 \otimes 1) = c \otimes c,$$
$$d_1(b^2 \otimes 1) = b \otimes b,$$
$$d_1(a^2 \otimes 1) = a \otimes a,$$
$$d_1(bab \otimes 1) = b \otimes ab - a \otimes ba,$$
$$d_1(ca \otimes 1) = c \otimes a + b \otimes c + a \otimes b,$$
$$d_1(cb \otimes 1) = c \otimes b + b \otimes a + a \otimes c.$$

Computation of d_2 We have

$$C^{(2)} = \{c^3, c^2a, ca^2, a^3, c^2b, cb^2, cbab, b^3, b^2ab, bab^2, babab\}$$

and

$$\#C^{(2)} = 11.$$

For $c^2a \in C^{(2)}$, we may uniquely write $c^2a = c^2 \cdot a$ where $c^2 \in C$ and $a \in M$. Then $d_2(c^2a \otimes 1) = c^2 \otimes a - i_1 d_1(c^2 \otimes a) = c^2 \otimes a - i_1(c \otimes f(ca)) = c^2 \otimes a + i_1(c \otimes bc + c \otimes ab)$ where $i_1 : \mathrm{Ker}(d_0) \to Ck \otimes_k A$. The high term cbc contains an obstruction cb and $c \in M$, then $i_1(c \otimes bc + c \otimes ab) = cb \otimes c + i_1(c \otimes bc + c \otimes ab - d_1(cb \otimes c)) = cb \otimes c + i_1(c \otimes ab - b \otimes ac) = cb \otimes c + ca \otimes b + i_1(c \otimes ab - b \otimes ac - d_1(ca \otimes b)) = cb \otimes c + ca \otimes b + i_1(b \otimes ba) = cb \otimes c + ca \otimes b + b^2 \otimes a$. Hence $d_2(c^2a \otimes 1) = c^2 \otimes a + cb \otimes c + ca \otimes b + b^2 \otimes a$. The differential d_2 is given by

$$d_2(c^3 \otimes 1) = c^2 \otimes c,$$
$$d_2(b^3 \otimes 1) = b^2 \otimes b,$$
$$d_2(a^3 \otimes 1) = a^2 \otimes a,$$
$$d_2(bab^2 \otimes 1) = bab \otimes b + a^2 \otimes ba,$$
$$d_2(b^2ab \otimes 1) = b^2 \otimes ab - bab \otimes a,$$

$$d_2(babab \otimes 1) = bab \otimes ab,$$
$$d_2(c^2a \otimes 1) = c^2 \otimes a + cb \otimes c + ca \otimes b + b^2 \otimes a,$$
$$d_2(ca^2 \otimes 1) = ca \otimes a + b^2 \otimes c + bab \otimes 1,$$
$$d_2(c^2b \otimes 1) = c^2 \otimes b + cb \otimes a + ca \otimes c + a^2 \otimes b,$$
$$d_2(cb^2 \otimes 1) = cb \otimes b - bab \otimes 1 + a^2 \otimes c,$$
$$d_2(cbab \otimes 1) = cb \otimes ab - ca \otimes ba + bab \otimes c.$$

Similarly, we give the computation results of other differentials.

Computation of d_3

$$C^{(3)} = \{c^4, c^3a, c^2a^2, ca^3, a^4, c^3b, c^2b^2, c^2bab, cb^3, cb^2ab, cbab^2, cbabab, b^4, b^3ab, b^2ab^2, b^2abab, bab^3, bab^2ab, babab^2, bababab\},$$
$$\#C^{(3)} = 20.$$

$$d_3(c^4 \otimes 1) = c^3 \otimes c,$$
$$d_3(b^4 \otimes 1) = b^3 \otimes b,$$
$$d_3(a^4 \otimes 1) = a^3 \otimes a,$$
$$d_3(bab^3 \otimes 1) = bab^2 \otimes b - a^3 \otimes ba,$$
$$d_3(b^2ab^2 \otimes 1) = b^2ab \otimes b + babab \otimes 1,$$
$$d_3(babab^2 \otimes 1) = babab \otimes b,$$
$$d_3(b^3ab \otimes 1) = b^3 \otimes ab - b^2ab \otimes a,$$
$$d_3(bab^2ab \otimes 1) = bab^2 \otimes ab - babab \otimes a,$$
$$d_3(b^2abab \otimes 1) = b^2ab \otimes ab,$$
$$d_3(bababab \otimes 1) = babab \otimes ab,$$
$$d_3(c^3a \otimes 1) = c^3 \otimes a + c^2b \otimes c + c^2a \otimes b + cb^2 \otimes a - b^2ab \otimes 1 + a^3 \otimes b,$$
$$d_3(c^2a^2 \otimes 1) = c^2a \otimes a + cb^2 \otimes c + cbab \otimes 1,$$
$$d_3(ca^3 \otimes 1) = ca^2 \otimes a + b^3 \otimes c + b^2ab \otimes 1,$$
$$d_3(c^3b \otimes 1) = c^3 \otimes b + c^2b \otimes a + c^2a \otimes c + ca^2 \otimes b + b^3 \otimes a - bab^2 \otimes 1,$$
$$d_3(c^2b^2 \otimes 1) = c^2b \otimes b - cbab \otimes 1 + ca^2 \otimes c,$$
$$d_3(cb^3 \otimes 1) = cb^2 \otimes b + bab^2 \otimes 1 + a^3 \otimes c,$$
$$d_3(c^2bab \otimes 1) = c^2b \otimes ab - c^2a \otimes ba + cbab \otimes c + b^2ab \otimes a - a^3 \otimes ba,$$
$$d_3(cb^2ab \otimes 1) = cb^2 \otimes ab - cbab \otimes a - bab^2 \otimes c,$$
$$d_3(cbab^2 \otimes 1) = cbab \otimes b + ca^2 \otimes ba - b^2ab \otimes c,$$
$$d_3(cbabab \otimes 1) = cbab \otimes ab - bab^2 \otimes ac.$$

Computation of d_4

$$C^{(4)} = \{c^5, c^4a, c^3a^2, c^2a^3, ca^4, a^5, c^4b, c^3b^2, c^3bab, c^2b^3, c^2b^2ab, c^2bab^2, c^2babab,$$

$cb^4, cb^3ab, cb^2ab^2, cb^2abab, cbab^3, cbab^2ab, cbabab^2, cbababab, b^5, b^4ab,$
$b^3ab^2, b^3abab, b^2ab^3, b^2ab^2ab, b^2abab^2, b^2ababab, bab^4, bab^3ab, bab^2ab^2,$
$bab^2abab, babab^3, babab^2ab, bababab^2, babababab\},$

$\#C^{(4)} = 37.$

$$
\begin{aligned}
d_4(c^5 \otimes 1) &= c^4 \otimes c,\\
d_4(b^5 \otimes 1) &= b^4 \otimes b,\\
d_4(a^5 \otimes 1) &= a^4 \otimes a,\\
d_4(bab^4 \otimes 1) &= bab^3 \otimes b + a^4 \otimes ba,\\
d_4(b^2ab^3 \otimes 1) &= b^2ab^2 \otimes b - babab^2 \otimes 1,\\
d_4(babab^3 \otimes 1) &= babab^2 \otimes b,\\
d_4(b^3ab^2 \otimes 1) &= b^3ab \otimes b + b^2abab \otimes 1,\\
d_4(b^2abab^2 \otimes 1) &= b^2abab \otimes b,\\
d_4(bab^2ab^2 \otimes 1) &= bab^2ab \otimes b + bababab \otimes 1,\\
d_4(bababab^2 \otimes 1) &= bababab \otimes b,\\
d_4(b^4ab \otimes 1) &= b^4 \otimes ab - b^3ab \otimes a,\\
d_4(b^2ab^2ab \otimes 1) &= b^2ab^2 \otimes ab - b^2abab \otimes a - bababab \otimes 1,\\
d_4(bab^3ab \otimes 1) &= bab^3 \otimes ab - bab^2ab \otimes a,\\
d_4(babab^2ab \otimes 1) &= babab^2 \otimes ab - bababab \otimes a,\\
d_4(b^3abab \otimes 1) &= b^3ab \otimes ab,\\
d_4(b^2ababab \otimes 1) &= b^2abab \otimes ab,\\
d_4(bab^2abab \otimes 1) &= bab^2ab \otimes ab,\\
d_4(babababab \otimes 1) &= bababab \otimes ab,\\
d_4(c^4a \otimes 1) &= c^4 \otimes a + c^3b \otimes c + c^3a \otimes b + c^2b^2 \otimes a - cb^2ab \otimes 1 + ca^3 \otimes b\\
&\quad + b^4 \otimes a,\\
d_4(c^3a^2 \otimes 1) &= c^3a \otimes a + c^2b^2 \otimes c + c^2bab \otimes 1,\\
d_4(c^2a^3 \otimes 1) &= c^2a^2 \otimes a + cb^3 \otimes c + cb^2ab \otimes 1,\\
d_4(ca^4 \otimes 1) &= ca^3 \otimes a + b^4 \otimes c + b^3ab \otimes 1,\\
d_4(c^4b \otimes 1) &= c^4 \otimes b + c^3b \otimes a + c^3a \otimes c + c^2a^2 \otimes b + cb^3 \otimes a - cbab^2 \otimes 1\\
&\quad + a^4 \otimes b,\\
d_4(c^3b^2 \otimes 1) &= c^3b \otimes b - c^2bab \otimes 1 + c^2a^2 \otimes c - b^3ab \otimes 1 + bab^3 \otimes 1,\\
d_4(c^2b^3 \otimes 1) &= c^2b^2 \otimes b + cbab^2 \otimes 1 + ca^3 \otimes c,\\
d_4(cb^4 \otimes 1) &= cb^3 \otimes b - bab^3 \otimes 1 + a^4 \otimes c,\\
d_4(c^3bab \otimes 1) &= c^3b \otimes ab - c^3a \otimes ba + c^2bab \otimes c + cb^2ab \otimes a - ca^3 \otimes ba\\
&\quad + b^3ab \otimes c - bab^3 \otimes c,\\
d_4(c^2b^2ab \otimes 1) &= c^2b^2 \otimes ab - c^2bab \otimes a - cbab^2 \otimes c,\\
d_4(c^2bab^2 \otimes 1) &= c^2bab \otimes b + c^2a^2 \otimes ba - cb^2ab \otimes c - b^2abab \otimes 1 + a^4 \otimes ba,
\end{aligned}
$$

$$d_4(c^2babab \otimes 1) = c^2bab \otimes ab - cbab^2 \otimes ac,$$
$$d_4(cb^3ab \otimes 1) = cb^3 \otimes ab - cb^2ab \otimes a + bab^3 \otimes c,$$
$$d_4(cb^2ab^2 \otimes 1) = cb^2ab \otimes b + cbabab \otimes 1 - bab^3 \otimes a,$$
$$d_4(cb^2abab \otimes 1) = cb^2ab \otimes ab + bab^3 \otimes ac,$$
$$d_4(cbab^3 \otimes 1) = cbab^2 \otimes b - ca^3 \otimes ba + b^3ab \otimes c,$$
$$d_4(cbab^2ab \otimes 1) = cbab^2 \otimes ab - cbabab \otimes a + b^2ab^2 \otimes ac + bab^2ab \otimes c,$$
$$d_4(cbabab^2 \otimes 1) = cbabab \otimes b - bab^2ab \otimes a,$$
$$d_4(cbababab \otimes 1) = cbabab \otimes ab + bab^2ab \otimes ac.$$

Computation of d_5

$$C^{(5)} = \{c^6, c^5a, c^4a^2, c^3a^3, c^2a^4, ca^5, a^6, c^5b, c^4b^2, c^4bab, c^3b^3, c^3b^2ab, c^3bab^2, c^3babab, c^2b^4, c^2b^3ab, c^2b^2ab^2, c^2b^2abab, c^2bab^3, c^2bab^2ab, c^2babab^2, c^2bababab, cb^5, cb^4ab, cb^3ab^2, cb^3abab, cb^2ab^3, cb^2ab^2ab, cb^2abab^2, cb^2ababab, cbab^4, cbab^3ab, cbab^2ab^2, cbab^2abab, cbabab^3, cbabab^2ab, cbababab^2, cbabababab, b^6, b^5ab, b^4ab^2, b^4abab, b^3ab^3, b^3ab^2ab, b^3abab^2, b^3ababab, b^2ab^4, b^2ab^3ab, b^2ab^2ab^2, b^2ab^2abab, b^2abab^3, b^2abab^2ab, b^2ababab^2, b^2abababab, bab^5, bab^4ab, bab^3ab^2, bab^3abab, bab^2ab^3, bab^2ab^2ab, bab^2abab^2, bab^2ababab, babab^4, babab^3ab, babab^2ab^2, babab^2abab, bababab^3, bababab^2ab, babababab^2, bababababab\},$$
$$\#C^{(5)} = 70.$$

$$d_5(c^6 \otimes 1) = c^5 \otimes c,$$
$$d_5(b^6 \otimes 1) = b^5 \otimes b,$$
$$d_5(a^6 \otimes 1) = a^5 \otimes a,$$
$$d_5(bab^5 \otimes 1) = bab^4 \otimes b - a^5 \otimes ba,$$
$$d_5(b^4ab^2 \otimes 1) = b^4ab \otimes b + b^3abab \otimes 1,$$
$$d_5(b^3ab^3 \otimes 1) = b^3ab^2 \otimes b - b^2abab^2 \otimes 1,$$
$$d_5(b^2ab^4 \otimes 1) = b^2ab^3 \otimes b + babab^3 \otimes 1,$$
$$d_5(b^3abab^2 \otimes 1) = b^3abab \otimes b,$$
$$d_5(b^2abab^3 \otimes 1) = b^2abab^2 \otimes b,$$
$$d_5(b^2ab^2ab^2 \otimes 1) = b^2ab^2ab \otimes b + b^2ababab \otimes 1 + bababab^2 \otimes 1,$$
$$d_5(b^2ababab^2 \otimes 1) = b^2ababab \otimes b,$$
$$d_5(bab^3ab^2 \otimes 1) = bab^3ab \otimes b + bab^2abab \otimes 1,$$
$$d_5(bab^2ab^3 \otimes 1) = bab^2ab^2 \otimes b - bababab^2 \otimes 1,$$
$$d_5(bab^2abab^2 \otimes 1) = bab^2abab \otimes b,$$
$$d_5(babab^4 \otimes 1) = babab^3 \otimes b,$$
$$d_5(babab^2ab^2 \otimes 1) = babab^2ab \otimes b + babababab \otimes 1,$$
$$d_5(bababab^3 \otimes 1) = bababab^2 \otimes b,$$

$$
\begin{aligned}
d_5(bababab ab^2\otimes 1) &= babababab\otimes b,\\
d_5(b^5ab\otimes 1) &= b^5\otimes ab - b^4ab\otimes a,\\
d_5(b^3ab^2ab\otimes 1) &= b^3ab^2\otimes ab - b^3abab\otimes a - b^2ababab\otimes 1,\\
d_5(b^2ab^3ab\otimes 1) &= b^2ab^3\otimes ab - b^2ab^2ab\otimes a + babab^2ab\otimes 1,\\
d_5(b^2abab^2ab\otimes 1) &= b^2abab^2\otimes ab - b^2ababab\otimes a,\\
d_5(bab^4ab\otimes 1) &= bab^4\otimes ab - bab^3ab\otimes a,\\
d_5(bab^2ab^2ab\otimes 1) &= bab^2ab^2\otimes ab - bab^2abab\otimes a - babababab\otimes 1,\\
d_5(babab^3ab\otimes 1) &= babab^3\otimes ab - babab^2ab\otimes a,\\
d_5(bababab^2ab\otimes 1) &= bababab^2\otimes ab - babababab\otimes a,\\
d_5(b^4abab\otimes 1) &= b^4ab\otimes ab,\\
d_5(b^3ababab\otimes 1) &= b^3abab\otimes ab,\\
d_5(b^2ab^2abab\otimes 1) &= b^2ab^2ab\otimes ab + babababab\otimes 1,\\
d_5(b^2abababab\otimes 1) &= b^2ababab\otimes ab,\\
d_5(bab^3abab\otimes 1) &= bab^3ab\otimes ab,\\
d_5(bab^2ababab\otimes 1) &= bab^2abab\otimes ab,\\
d_5(babab^2abab\otimes 1) &= babab^2ab\otimes ab,\\
d_5(babababab ab\otimes 1) &= babababab\otimes ab,\\
d_5(c^5a\otimes 1) &= c^5\otimes a + c^4b\otimes c + c^4a\otimes b + c^3b^2\otimes a - c^2b^2ab\otimes 1 + c^2a^3\otimes b\\
&\quad + cb^4\otimes a - b^4ab\otimes 1 + a^5\otimes b,\\
d_5(c^4a^2\otimes 1) &= c^4a\otimes a + c^3b^2\otimes c + c^3bab\otimes 1,\\
d_5(c^3a^3\otimes 1) &= c^3a^2\otimes a + c^2b^3\otimes c + c^2b^2ab\otimes 1,\\
d_5(c^2a^4\otimes 1) &= c^2a^3\otimes a + cb^4\otimes c + cb^3ab\otimes 1,\\
d_5(ca^5\otimes 1) &= ca^4\otimes a + b^5\otimes c + b^4ab\otimes 1,\\
d_5(c^5b\otimes 1) &= c^5\otimes b + c^4b\otimes a + c^4a\otimes c + c^3a^2\otimes b + c^2b^3\otimes a - c^2bab^2\otimes 1\\
&\quad + ca^4\otimes b + b^5\otimes a - b^3ab^2\otimes 1,\\
d_5(c^4b^2\otimes 1) &= c^4b\otimes b - c^3bab\otimes 1 + c^3a^2\otimes c - cb^3ab\otimes 1 + cbab^3\otimes 1,\\
d_5(c^3b^3\otimes 1) &= c^3b^2\otimes b + c^2bab^2\otimes 1 + c^2a^3\otimes c + b^3ab^2\otimes 1 - bab^4\otimes 1,\\
d_5(c^2b^4\otimes 1) &= c^2b^3\otimes b - cbab^3\otimes 1 + ca^4\otimes c,\\
d_5(cb^5\otimes 1) &= cb^4\otimes b + bab^4\otimes 1 + a^5\otimes c,\\
d_5(c^4bab\otimes 1) &= c^4b\otimes ab - c^4a\otimes ba + c^3bab\otimes c + c^2b^2ab\otimes a - c^2a^3\otimes ba\\
&\quad + cb^3ab\otimes c - cbab^3\otimes c + b^4ab\otimes a - a^5\otimes ba,\\
d_5(c^3b^2ab\otimes 1) &= c^3b^2\otimes ab - c^3bab\otimes a - c^2bab^2\otimes c - b^3ab^2\otimes c + bab^4\otimes c,\\
d_5(c^3bab^2\otimes 1) &= c^3bab\otimes b + c^3a^2\otimes ba - c^2b^2ab\otimes c - cb^2abab\otimes 1 + ca^4\otimes ba\\
&\quad - b^4ab\otimes c - bab^4\otimes a,\\
d_5(c^3babab\otimes 1) &= c^3bab\otimes ab - c^2bab^2\otimes ac + cb^2abab\otimes c - b^3ab^2\otimes ac\\
&\quad + bab^4\otimes ac,\\
d_5(c^2b^3ab\otimes 1) &= c^2b^3\otimes ab - c^2b^2ab\otimes a + cbab^3\otimes c,
\end{aligned}
$$

$$
\begin{aligned}
d_5(c^2b^2ab^2 \otimes 1) &= c^2b^2ab \otimes b + c^2babab \otimes 1 - cbab^3 \otimes a, \\
d_5(c^2b^2abab \otimes 1) &= c^2b^2ab \otimes ab + cbab^3 \otimes ac, \\
d_5(c^2bab^3 \otimes 1) &= c^2bab^2 \otimes b - c^2a^3 \otimes ba + cb^3ab \otimes c + b^2abab^2 \otimes 1 - a^5 \otimes ba, \\
d_5(c^2bab^2ab \otimes 1) &= c^2bab^2 \otimes ab - c^2babab \otimes a + cb^2ab^2 \otimes ac + cbab^2ab \otimes c \\
&\quad + b^2ababab \otimes 1, \\
d_5(c^2babab^2 \otimes 1) &= c^2babab \otimes b - cbab^2ab \otimes a, \\
d_5(c^2bababab \otimes 1) &= c^2babab \otimes ab + cbab^2ab \otimes ac, \\
d_5(cb^4ab \otimes 1) &= cb^4 \otimes ab - cb^3ab \otimes a - bab^4 \otimes c, \\
d_5(cb^3ab^2 \otimes 1) &= cb^3ab \otimes b + cb^2abab \otimes 1 + bab^4 \otimes a, \\
d_5(cb^3abab \otimes 1) &= cb^3ab \otimes ab - bab^4 \otimes ac, \\
d_5(cb^2ab^3 \otimes 1) &= cb^2ab^2 \otimes b - cbabab^2 \otimes 1 + bab^3ab \otimes 1, \\
d_5(cb^2ab^2ab \otimes 1) &= cb^2ab^2 \otimes ab - cb^2abab \otimes a - cbababab \otimes 1 - bab^3ab \otimes c, \\
d_5(cb^2abab^2 \otimes 1) &= cb^2abab \otimes b + bab^3ab \otimes a, \\
d_5(cb^2ababab \otimes 1) &= cb^2abab \otimes ab - bab^3ab \otimes ac, \\
d_5(cbab^4 \otimes 1) &= cbab^3 \otimes b + ca^4 \otimes ba - b^4ab \otimes c, \\
d_5(cbab^3ab \otimes 1) &= cbab^3 \otimes ab - cbab^2ab \otimes a - b^3ab^2 \otimes ac - b^2ab^2ab \otimes c \\
&\quad - bab^2ab^2 \otimes c - bab^2abab \otimes 1, \\
d_5(cbab^2ab^2 \otimes 1) &= cbab^2ab \otimes b + cbababab \otimes 1 + b^2ab^2ab \otimes a + bab^2ab^2 \otimes a, \\
d_5(cbab^2abab \otimes 1) &= cbab^2ab \otimes ab - b^2ab^2ab \otimes ac - bab^2ab^2 \otimes ac, \\
d_5(cbabab^3 \otimes 1) &= cbabab^2 \otimes b + bab^2abab \otimes 1, \\
d_5(cbabab^2ab \otimes 1) &= cbabab^2 \otimes ab - cbababab \otimes a, \\
d_5(cbababab^2 \otimes 1) &= cbababab \otimes b + bab^2abab \otimes a, \\
d_5(cbabababab \otimes 1) &= cbababab \otimes ab - bab^2abab \otimes ac.
\end{aligned}
$$

Computation of Ext groups Now we could compute $\mathrm{Ext}^n_A(k,k)$ for $0 \leqslant n \leqslant 5$. The functor $\mathrm{Hom}_A(\cdot, k)$ acting on Anick's resolution, we get a complex

$$0 \to \mathrm{Hom}_A(A,k) \xrightarrow{d_0^*} \mathrm{Hom}_A(Xk \otimes_k A, k) \xrightarrow{d_1^*} \mathrm{Hom}_A(Ck \otimes_k A, k) \xrightarrow{d_2^*} \cdots,$$

which can be rewritten as a complex of k-linear vector spaces

$$0 \to k \xrightarrow{d_0^*} Xk \xrightarrow{d_1^*} Ck \xrightarrow{d_2^*} C^{(2)}k \xrightarrow{d_3^*} C^{(3)}k \xrightarrow{d_4^*} C^{(4)}k \xrightarrow{d_5^*} C^{(5)}k \to \cdots,$$

where $d_0^* = d_1^* = 0$. The k-linear map d_2^* is given by

$$d_2^*(bab) = ca^2$$

and vanish on other basis vectors. The k-linear map d_3^* is given by

$$
\begin{aligned}
d_3^*(babab) &= b^2ab^2, \\
d_3^*(b^2ab) &= ca^3 - c^3a,
\end{aligned}
$$

$$\begin{aligned} d_3^*(bab^2) &= cb^3 - c^3b, \\ d_3^*(cbab) &= c^2a^2 - c^2b^2 \end{aligned}$$

and vanish on other basis vectors. The k-linear map d_4^* is given by

$$\begin{aligned} d_4^*(b^2abab) &= b^3ab^2 - c^2bab^2, \\ d_4^*(babab^2) &= -b^2ab^3, \\ d_4^*(bababab) &= bab^2ab^2 - b^2ab^2ab, \\ d_4^*(b^3ab) &= ca^4 - c^3b^2, \\ d_4^*(bab^3) &= -cb^4, \\ d_4^*(cb^2ab) &= c^2a^3 - c^4a, \\ d_4^*(c^2bab) &= c^3a^2 - c^3b^2, \\ d_4^*(cbab^2) &= c^2b^3 - c^4b, \\ d_4^*(cbabab) &= cb^2ab^2 \end{aligned}$$

and vanish on other basis vectors. Then we could compute

$$\begin{aligned} \mathrm{Ext}_A^0(k,k) &= \mathrm{Ker}(d_0^*) = k, \\ \mathrm{Ext}_A^1(k,k) &= Xk \cong k^3, \\ \mathrm{Ext}_A^2(k,k) &= \mathrm{Span}(c^2, cb, ca, b^2, a^2) \cong k^5, \\ \mathrm{Ext}_A^3(k,k) &= \mathrm{Ker}(d_3^*)/\,\mathrm{Im}(d_2^*) \\ &= \mathrm{Span}(c^3, c^2a, ca^2, a^3, c^2b, cb^2, b^3)/\,\mathrm{Span}(ca^2) \\ &= \mathrm{Span}(c^3, c^2a, a^3, c^2b, cb^2, b^3) \\ &\cong k^6, \\ \mathrm{Ext}_A^4(k,k) &= \frac{\mathrm{Span}(c^4, c^3a, c^2a^2, ca^3, a^4, c^3b, c^2b^2, cb^3, b^4, b^2ab^2, bab^2ab)}{\mathrm{Span}(b^2ab^2, ca^3 - c^3a, cb^3 - c^3b, c^2a^2 - c^2b^2)} \\ &\cong k^7. \end{aligned}$$

Similarly, we could compute

$$\begin{aligned} \mathrm{Ext}_A^5(k,k) &= \frac{\mathrm{Span}\left(\substack{c^5, c^4a, c^3a^2, c^2a^3, ca^4, a^5, c^4b, c^3b^2, c^2b^3, cb^4, b^5, \\ b^2ab^3, c^2bab^2 - b^3ab^2, cb^2ab^2, cbab^2ab, cbabab^2 + bab^3ab, b^2ab^2ab, bab^2ab^2}\right)}{\mathrm{Span}\left(\substack{ca^4 - c^3b^2, cb^4, c^2a^3 - c^4a, c^3a^2 - c^3b^2, c^2b^3 - c^4b, \\ b^2ab^3, cb^2ab^2, b^3ab^2 - c^2bab^2, bab^2ab^2 - b^2ab^2ab}\right)} \\ &\cong k^9. \end{aligned}$$

Next, we partially present some general forms of differentials.

Proposition 6.

$$\begin{aligned} d_n(c^{n+1} \otimes 1) &= c^n \otimes c, \quad n \geqslant 1; \\ d_n(b^{n+1} \otimes 1) &= b^n \otimes b, \quad n \geqslant 1; \\ d_n(a^{n+1} \otimes 1) &= a^n \otimes a, \quad n \geqslant 1; \\ d_1(bab \otimes 1) &= b \otimes ab - a \otimes ba; \\ d_n(bab^n \otimes 1) &= bab^{n-1} \otimes b + (-1)^n a^n \otimes ba, \quad n \geqslant 2. \end{aligned}$$

Proof. We can prove the proposition by induction on n. It can be checked that the base cases are satisfied.

Suppose that $n \geqslant 2$ and the first formula is satisfied for $n-1$. Then

$$\begin{aligned} d_n(c^{n+1} \otimes 1) &= c^n \otimes c - i_{n-1} d_{n-1}(c^n \otimes c) \\ &= c^n \otimes c - i_{n-1}[(c^{n-1} \otimes c)c] \\ &= c^n \otimes c. \end{aligned}$$

Hence the first formula is true. By replacing c with b and a respectively, we can get that the second and third formula are correct.

Suppose that $n \geqslant 3$ and $d_{n-1}(bab^{n-1} \otimes 1) = bab^{n-2} \otimes b + (-1)^{n-1} a^{n-1} \otimes ba$. Then

$$\begin{aligned} d_n(bab^n \otimes 1) &= bab^{n-1} \otimes b - i_{n-1} d_{n-1}(bab^{n-1} \otimes b) \\ &= bab^{n-1} \otimes b - i_{n-1}[(bab^{n-2} \otimes b + (-1)^{n-1} a^{n-1} \otimes ba)b] \\ &= bab^{n-1} \otimes b - i_{n-1}((-1)^{n-1} a^{n-1} \otimes aba) \\ &= bab^{n-1} \otimes b + (-1)^n a^n \otimes ba. \end{aligned}$$

Hence the last formula is true. □

Proposition 7.

$$d_n(b^{j_1}ab^{j_2}\cdots ab^{j_s} \otimes 1) = b^{j_1}ab^{j_2}\cdots ab^{j_s-1} \otimes b$$
$$+ \sum_{\substack{1\leqslant k\leqslant s-1,\\ j_k\geqslant 2,\, j_{k+1}\geqslant 2}} (-1)^{j_{k+1}+j_{k+2}+\cdots+j_s} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_s} \otimes 1,$$
$2 \leqslant j_s \leqslant n-1,\ j_1+j_2+\cdots+j_s = n+1,\ s \geqslant 2,\ n \geqslant 3;$

$$d_n(b^{j_1}ab^{j_2}\cdots ab^{j_s}ab \otimes 1) = b^{j_1}ab^{j_2}\cdots ab^{j_s} \otimes ab - b^{j_1}ab^{j_2}\cdots ab^{j_s-1}ab \otimes a$$
$$+ \sum_{\substack{1\leqslant k\leqslant s-1,\\ j_k\geqslant 2,\, j_{k+1}\geqslant 2}} (-1)^{j_{k+1}+j_{k+2}+\cdots+j_s+1} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_s}ab \otimes 1,$$
$j_s \geqslant 2,\ j_1+j_2+\cdots+j_s = n,\ s \geqslant 1,\ n \geqslant 2;$

$$d_n(b^{j_1}ab^{j_2}\cdots ab^{j_s}abab \otimes 1) = b^{j_1}ab^{j_2}\cdots ab^{j_s}ab \otimes ab$$
$$+ \sum_{\substack{1\leqslant k\leqslant s-1,\\ j_k\geqslant 2,\, j_{k+1}\geqslant 2}} (-1)^{j_{k+1}+j_{k+2}+\cdots+j_s} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_s}abab \otimes 1,$$
$j_1+j_2+\cdots+j_s = n-1,\ s \geqslant 1,\ n \geqslant 2.$

Proof. We will prove the three formulas by induction on n. The base case is satisfied. Suppose that the formulas are true for $n-1$.

To get the first formula, we will prove

$$d_n(b^{j_1}ab^{j_2}\cdots ab^{j_s} \otimes 1) = b^{j_1}ab^{j_2}\cdots ab^{j_s-1} \otimes b$$
$$+ \sum_{k=m}^{s-1} (-1)^{j_{k+1}+j_{k+2}+\cdots+j_s} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_s} \otimes 1$$

$$+ i_{n-1}[\sum_{k=1}^{m-1}(-1)^{j_{k+1}+j_{k+2}+\cdots+j_s} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_{s-1}}ab^{j_s-1}\otimes b$$

$$+ \sum_{l=m}^{s-1}\sum_{k=1}^{m-1}(-1)^{j_{k+1}+j_{k+2}+\cdots+j_l} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_{l-1}}ab^{j_l-1}aba$$

$$b^{j_{l+1}-1}ab^{j_{l+2}}\cdots ab^{j_s}\otimes 1] \tag{1}$$

for $2 \leqslant m \leqslant s-1$ by induction on m. Conventionally, the term $\cdots ab^{j_k-1}abab^{j_{k+1}-1}a\cdots$ is zero in which a^2 appears. Firstly we check the base case $m = s-1$. For the case $j_s \geqslant 3$ or the case $j_s = 2$ and $j_{s-1} = 1$ we have

$$\begin{aligned}
&d_n(b^{j_1}\cdots ab^{j_s}\otimes 1)\\
&= b^{j_1}\cdots ab^{j_s-1}\otimes b - i_{n-1}d_{n-1}(b^{j_1}\cdots ab^{j_s-1}\otimes b)\\
&= b^{j_1}\cdots ab^{j_s-1}\otimes b - i_{n-1}[\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s-1} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s-1}\otimes b]\\
&= b^{j_1}\cdots ab^{j_s-1}\otimes b + (-1)^{j_s} b^{j_1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}\otimes 1\\
&\quad - i_{n-1}[\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s-1} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s-1}\otimes b\\
&\quad - (-1)^{j_s-1}d_{n-1}(b^{j_1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}\otimes 1)]\\
&= b^{j_1}\cdots ab^{j_s-1}\otimes b + (-1)^{j_s} b^{j_1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}\otimes 1\\
&\quad + i_{n-1}[\sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s-1}\otimes b\\
&\quad + \sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_{s-1}} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-2}}ab^{j_{s-1}-1}abab^{j_s-1}\otimes 1].
\end{aligned}$$

For the case $j_s = 2$ and $j_{s-1} \geqslant 2$, we have

$$\begin{aligned}
&d_n(b^{j_1}\cdots ab^{j_{s-1}}ab^2\otimes 1)\\
&= b^{j_1}\cdots ab^{j_{s-1}}ab\otimes b - i_{n-1}d_{n-1}(b^{j_1}\cdots ab^{j_{s-1}}ab\otimes b)\\
&= b^{j_1}\cdots ab^{j_{s-1}}ab\otimes b - i_{n-1}[-b_1^{j}\cdots ab^{j_{s-1}-1}ab\otimes ab\\
&\quad + \sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_{s-1}+1} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab\otimes b]\\
&= b^{j_1}\cdots ab^{j_{s-1}}ab\otimes b + b^{j_1}\cdots ab^{j_{s-1}-1}abab\otimes 1 - i_{n-1}[-b_1^{j}\cdots ab^{j_{s-1}-1}ab\otimes ab\\
&\quad + \sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_{s-1}+1} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab\otimes b\\
&\quad + d_{n-1}(b^{j_1}\cdots ab^{j_{s-1}-1}abab\otimes 1)]\\
&= b^{j_1}\cdots ab^{j_{s-1}}ab\otimes b + b^{j_1}\cdots ab^{j_{s-1}-1}abab\otimes 1
\end{aligned}$$

$$+ i_{n-1}[\sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_{s-1}}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab\otimes b$$

$$+\sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_{s-1}}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-2}}ab^{j_{s-1}-1}abab\otimes 1].$$

Suppose that the formula (1) is true for m. We will prove that it's true for $m-1$.

$$\begin{aligned}
&d_n(b^{j_1}ab^{j_2}\cdots ab^{j_s}\otimes 1)\\
&= b^{j_1}\cdots ab^{j_s-1}\otimes b+\sum_{k=m}^{s-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes 1\\
&\quad + i_{n-1}[\sum_{k=1}^{m-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab^{j_s-1}\otimes b\\
&\quad +\sum_{l=m}^{s-1}\sum_{k=1}^{m-1}(-1)^{j_{k+1}+\cdots+j_l}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}ab^{j_{l+2}}\cdots ab^{j_s}\otimes 1]\\
&= b^{j_1}\cdots ab^{j_s-1}\otimes b+\sum_{k=m-1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes 1\\
&\quad + i_{n-1}[\sum_{k=1}^{m-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab^{j_s-1}\otimes b\\
&\quad +\sum_{l=m}^{s-1}\sum_{k=1}^{m-1}(-1)^{j_{k+1}+\cdots+j_l}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}ab^{j_{l+2}}\cdots ab^{j_s}\otimes 1\\
&\quad -(-1)^{j_m+\cdots+j_s}d_{n-1}(b^{j_1}\cdots b^{j_{m-1}-1}abab^{j_m-1}ab^{j_{m+1}}\cdots ab^{j_s}\otimes 1)]\\
&= b^{j_1}\cdots ab^{j_s-1}\otimes b+\sum_{k=m-1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes 1\\
&\quad + i_{n-1}[\sum_{k=1}^{m-2}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab^{j_s-1}\otimes b\\
&\quad +\sum_{l=m}^{s-1}\sum_{k=1}^{m-2}(-1)^{j_{k+1}+\cdots+j_l}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}ab^{j_{l+2}}\cdots ab^{j_s}\otimes 1\\
&\quad +\sum_{k=1}^{m-2}(-1)^{j_{k+1}+\cdots+j_{m-1}}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots b^{j_{m-1}-1}abab^{j_m-1}ab^{j_{m+1}}\cdots ab^{j_s}\otimes 1].
\end{aligned}$$

Then the formula (1) has been proved. Consider $m=2$, we obtain

$$\begin{aligned}
&d_n(b^{j_1}\cdots ab^{j_s}\otimes 1)\\
&= b^{j_1}\cdots ab^{j_s-1}\otimes b+\sum_{k=2}^{s-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes 1
\end{aligned}$$

$$
\begin{aligned}
&+ i_{n-1}[(-1)^{j_2+\cdots+j_s} b^{j_1-1}abab^{j_2-1}\cdots ab^{j_{s-1}}ab^{j_s-1}\otimes b\\
&+ \sum_{l=2}^{s-1}(-1)^{j_2+\cdots+j_l} b^{j_1-1}abab^{j_2-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}ab^{j_{l+2}}\cdots ab^{j_s}\otimes 1]\\
= {} & b^{j_1}\cdots ab^{j_s-1}\otimes b + \sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes 1\\
&+ i_{n-1}[(-1)^{j_2+\cdots+j_s} b^{j_1-1}abab^{j_2-1}\cdots ab^{j_{s-1}}ab^{j_s-1}\otimes b\\
&+ \sum_{l=2}^{s-1}(-1)^{j_2+\cdots+j_l} b^{j_1-1}abab^{j_2-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}ab^{j_{l+2}}\cdots ab^{j_s}\otimes 1\\
&- (-1)^{j_2+\cdots+j_s} d_{n-1}(b^{j_1-1}abab^{j_2-1}ab^{j_3}\cdots ab^{j_s}\otimes 1)]\\
= {} & b^{j_1}\cdots ab^{j_s-1}\otimes b + \sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes 1.
\end{aligned}
$$

To get the second formula, we will prove

$$
\begin{aligned}
& d_n(b^{j_1}ab^{j_2}\cdots ab^{j_s}ab\otimes 1) = b^{j_1}ab^{j_2}\cdots ab^{j_s}\otimes ab - b^{j_1}ab^{j_2}\cdots ab^{j_{s-1}}ab^{j_s-1}ab\otimes a\\
&+ \sum_{k=m}^{s-1}(-1)^{j_{k+1}+j_{k+2}+\cdots+j_s+1} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_s}ab\otimes 1\\
&+ i_{n-1}[\sum_{k=1}^{m-1}(-1)^{j_{k+1}+j_{k+2}+\cdots+j_s+1} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_s}\otimes ab\\
&+ \sum_{k=1}^{m-1}(-1)^{j_{k+1}+j_{k+2}+\cdots+j_s} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_{s-1}}ab^{j_s-1}ab\otimes a\\
&+ \sum_{l=m}^{s-1}\sum_{k=1}^{m-1}(-1)^{j_{k+1}+j_{k+2}+\cdots+j_l} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_{l-1}}ab^{j_l-1}aba\\
&\qquad b^{j_{l+1}-1}ab^{j_{l+2}}\cdots ab^{j_s}ab\otimes 1] \qquad (2)
\end{aligned}
$$

for $2 \leqslant m \leqslant s-1$ by induction on m. Let us check the base case $m = s-1$.

$$
\begin{aligned}
& d_n(b^{j_1}\cdots ab^{j_s}ab\otimes 1)\\
= {} & b^{j_1}\cdots ab^{j_s}\otimes ab - i_{n-1}d_{n-1}(b^{j_1}\cdots ab^{j_s}\otimes ab)\\
= {} & b^{j_1}\cdots ab^{j_s}\otimes ab - i_{n-1}[b^{j_1}\cdots ab^{j_s-1}\otimes aba\\
&+ \sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes ab]\\
= {} & b^{j_1}\cdots ab^{j_s}\otimes ab - b^{j_1}\cdots ab^{j_s-1}ab\otimes a - i_{n-1}[b^{j_1}\cdots ab^{j_s-1}\otimes aba\\
&+ \sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes ab - d_{n-1}(b^{j_1}\cdots ab^{j_s-1}ab\otimes a)]\\
= {} & b^{j_1}\cdots ab^{j_s}\otimes ab - b^{j_1}\cdots ab^{j_s-1}ab\otimes a
\end{aligned}
$$

$$- i_{n-1}[\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes ab$$

$$-\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab^{j_s-1}ab\otimes a]$$

$$= b^{j_1}\cdots ab^{j_s}\otimes ab - b^{j_1}\cdots ab^{j_s-1}ab\otimes a + (-1)^{j_s+1} b^{j_1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}ab\otimes 1$$

$$- i_{n-1}[\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes ab$$

$$-\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab^{j_s-1}ab\otimes a$$

$$-(-1)^{j_s} d_{n-1}(b^{j_1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}ab\otimes 1)]$$

$$= b^{j_1}\cdots ab^{j_s}\otimes ab - b^{j_1}\cdots ab^{j_s-1}ab\otimes a + (-1)^{j_s+1} b^{j_1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}ab\otimes 1$$

$$+ i_{n-1}[\sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_s+1} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes ab$$

$$+\sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab^{j_s-1}ab\otimes a$$

$$+\sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_{s-1}} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}ab\otimes 1].$$

Suppose that that formula (2) is true for m. We will prove that it's true for $m-1$.

$$d_n(b^{j_1}\cdots ab^{j_s}ab\otimes 1)$$

$$= b^{j_1}\cdots ab^{j_s}\otimes ab - b^{j_1}\cdots ab^{j_s-1}ab\otimes a$$

$$+\sum_{k=m-1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s+1} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}ab\otimes 1$$

$$+ i_{n-1}[\sum_{k=1}^{m-1}(-1)^{j_{k+1}+\cdots+j_s+1} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes ab$$

$$+\sum_{k=1}^{m-1}(-1)^{j_{k+1}+\cdots+j_s} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab^{j_s-1}ab\otimes a$$

$$+\sum_{l=m}^{s-1}\sum_{k=1}^{m-1}(-1)^{j_{k+1}+\cdots+j_l} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}\cdots ab^{j_s}ab\otimes 1$$

$$-(-1)^{j_m+\cdots+j_s+1} d_{n-1}(b^{j_1}\cdots ab^{j_{m-1}-1}abab^{j_m-1}\cdots ab^{j_s}ab\otimes 1)]$$

$$= b^{j_1}\cdots ab^{j_s}\otimes ab - b^{j_1}\cdots ab^{j_s-1}ab\otimes a$$

$$+\sum_{k=m-1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s+1} b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}ab\otimes 1$$

$$+ i_{n-1}[\sum_{k=1}^{m-2}(-1)^{j_{k+1}+\cdots+j_s+1}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}\otimes ab$$
$$+\sum_{k=1}^{m-2}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}}ab^{j_s-1}ab\otimes a$$
$$+\sum_{l=m}^{s-1}\sum_{k=1}^{m-2}(-1)^{j_{k+1}+\cdots+j_l}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}\cdots ab^{j_s}ab\otimes 1$$
$$+\sum_{k=1}^{m-2}(-1)^{j_{k+1}+\cdots+j_{m-1}}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{m-1}-1}abab^{j_m-1}\cdots ab^{j_s}ab\otimes 1].$$

Then the formula (2) is proved. Consider $m = 2$, we get

$$\begin{aligned}
&d_n(b^{j_1}\cdots ab^{j_s}ab\otimes 1)\\
&= b^{j_1}\cdots ab^{j_s}\otimes ab - b^{j_1}\cdots ab^{j_s-1}ab\otimes a\\
&\quad+\sum_{k=2}^{s-1}(-1)^{j_{k+1}+\cdots+j_s+1}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}ab\otimes 1\\
&\quad+ i_{n-1}[(-1)^{j_2+\cdots+j_s+1}b^{j_1-1}abab^{j_2-1}\cdots ab^{j_s}\otimes ab\\
&\quad+(-1)^{j_2+\cdots+j_s}b^{j_1-1}abab^{j_2-1}\cdots ab^{j_{s-1}}ab^{j_s-1}ab\otimes a\\
&\quad+\sum_{l=2}^{s-1}(-1)^{j_2+\cdots+j_l}b^{j_1-1}abab^{j_2-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}\cdots ab^{j_s}ab\otimes 1]\\
&= b^{j_1}\cdots ab^{j_s}\otimes ab - b^{j_1}\cdots ab^{j_s-1}ab\otimes a\\
&\quad+\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s+1}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}ab\otimes 1\\
&\quad+ i_{n-1}[(-1)^{j_2+\cdots+j_s+1}b^{j_1-1}abab^{j_2-1}\cdots ab^{j_s}\otimes ab\\
&\quad+(-1)^{j_2+\cdots+j_s}b^{j_1-1}abab^{j_2-1}\cdots ab^{j_{s-1}}ab^{j_s-1}ab\otimes a\\
&\quad+\sum_{l=2}^{s-1}(-1)^{j_2+\cdots+j_l}b^{j_1-1}abab^{j_2-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}\cdots ab^{j_s}ab\otimes 1\\
&\quad-(-1)^{j_2+\cdots+j_s+1}d_{n-1}(b^{j_1-1}abab^{j_2-1}\cdots ab^{j_s}ab\otimes 1)]\\
&= b^{j_1}\cdots ab^{j_s}\otimes ab - b^{j_1}\cdots ab^{j_s-1}ab\otimes a\\
&\quad+\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s+1}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}ab\otimes 1.
\end{aligned}$$

To get the third formula,we will prove

$$d_n(b^{j_1}ab^{j_2}\cdots ab^{j_s}abab\otimes 1) = b^{j_1}ab^{j_2}\cdots ab^{j_s}ab\otimes ab$$
$$+\sum_{k=m}^{s-1}(-1)^{j_{k+1}+j_{k+2}+\cdots+j_s}b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_s}abab\otimes 1$$

$$+ i_{n-1}[\sum_{k=1}^{m-1}(-1)^{j_{k+1}+j_{k+2}+\cdots+j_s} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_s}ab\otimes ab$$

$$+ \sum_{l=m}^{s-1}\sum_{k=1}^{m-1}(-1)^{j_{k+1}+j_{k+2}+\cdots+j_l} b^{j_1}ab^{j_2}\cdots ab^{j_{k-1}}ab^{j_k-1}abab^{j_{k+1}-1}ab^{j_{k+2}}\cdots ab^{j_{l-1}}ab^{j_l-1}aba$$
$$b^{j_{l+1}-1}ab^{j_{l+2}}\cdots ab^{j_s}abab\otimes 1] \qquad (3)$$

for $2 \leqslant m \leqslant s-1$ by induction on m. Let us check the base case $m = s-1$.

$$\begin{aligned}
&d_n(b^{j_1}\cdots ab^{j_s}abab\otimes 1)\\
&= b^{j_1}\cdots ab^{j_s}ab\otimes ab - i_{n-1}d_{n-1}(b^{j_1}\cdots ab^{j_s}ab\otimes ab)\\
&= b^{j_1}\cdots ab^{j_s}ab\otimes ab - i_{n-1}[\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s+1}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}ab\otimes ab]\\
&= b^{j_1}\cdots ab^{j_s}ab\otimes ab + (-1)^{j_s}b^{j_1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}abab\otimes 1\\
&\quad + i_{n-1}[\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}ab\otimes ab\\
&\quad + (-1)^{j_s+1}d_{n-1}(b^{j_1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}abab\otimes 1)]\\
&= b^{j_1}\cdots ab^{j_s}ab\otimes ab + (-1)^{j_s}b^{j_1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}abab\otimes 1\\
&\quad + i_{n-1}[\sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}ab\otimes ab\\
&\quad + \sum_{k=1}^{s-2}(-1)^{j_{k+1}+\cdots+j_{s-1}}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{s-1}-1}abab^{j_s-1}abab\otimes 1].
\end{aligned}$$

Suppose that the formula (3) is true for m. We will prove that it's true for $m-1$.

$$\begin{aligned}
&d_n(b^{j_1}\cdots ab^{j_s}abab\otimes 1)\\
&= b^{j_1}\cdots ab^{j_s}ab\otimes ab + \sum_{k=m-1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}abab\otimes 1\\
&\quad + i_{n-1}[\sum_{k=1}^{m-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}ab\otimes ab\\
&\quad + \sum_{l=m}^{s-1}\sum_{k=1}^{m-1}(-1)^{j_{k+1}+\cdots+j_l}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}}\cdots ab^{j_l-1}abab^{j_{l+1}-1}\cdots ab^{j_s}abab\otimes 1\\
&\quad - (-1)^{j_m+\cdots+j_s}d_{n-1}(b^{j_1}\cdots ab^{j_{m-1}-1}abab^{j_m-1}\cdots ab^{j_s}abab\otimes 1)]\\
&= b^{j_1}\cdots ab^{j_s}ab\otimes ab + \sum_{k=m-1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}abab\otimes 1\\
&\quad + i_{n-1}[\sum_{k=1}^{m-2}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}ab\otimes ab
\end{aligned}$$

$$+\sum_{l=m}^{s-1}\sum_{k=1}^{m-2}(-1)^{j_{k+1}+\cdots+j_l}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}}\cdots ab^{j_l-1}abab^{j_{l+1}-1}\cdots ab^{j_s}abab\otimes 1$$
$$+\sum_{k=1}^{m-2}(-1)^{j_{k+1}\cdots+j_{m-1}}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_{m-1}-1}abab^{j_m-1}\cdots ab^{j_s}abab\otimes 1].$$

Hence the formula (3) is proved. Consider $m=2$, we obtain

$$\begin{aligned}
&d_n(b^{j_1}\cdots ab^{j_s}abab\otimes 1)\\
=&\, b^{j_1}\cdots ab^{j_s}ab\otimes ab+\sum_{k=2}^{s-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}abab\otimes 1\\
&+i_{n-1}[(-1)^{j_2+\cdots+j_s}b^{j_1-1}abab^{j_2-1}\cdots ab^{j_s}ab\otimes ab\\
&+\sum_{l=2}^{s-1}(-1)^{j_2+\cdots+j_l}b^{j_1-1}abab^{j_2-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}\cdots ab^{j_s}abab\otimes 1]\\
=&\, b^{j_1}\cdots ab^{j_s}ab\otimes ab+\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}abab\otimes 1\\
&+i_{n-1}[(-1)^{j_2+\cdots+j_s}b^{j_1-1}abab^{j_2-1}\cdots ab^{j_s}ab\otimes ab\\
&+\sum_{l=2}^{s-1}(-1)^{j_2+\cdots+j_l}b^{j_1-1}abab^{j_2-1}\cdots ab^{j_l-1}abab^{j_{l+1}-1}\cdots ab^{j_s}abab\otimes 1\\
&-(-1)^{j_2+\cdots+j_s}d_{n-1}(b^{j_1-1}abab^{j_2-1}\cdots ab^{j_s}abab\otimes 1)]\\
=&\, b^{j_1}\cdots ab^{j_s}ab\otimes ab+\sum_{k=1}^{s-1}(-1)^{j_{k+1}+\cdots+j_s}b^{j_1}\cdots ab^{j_k-1}abab^{j_{k+1}-1}\cdots ab^{j_s}abab\otimes 1.
\end{aligned}$$

Now the three formulas are true for n. By induction, the proof is finished. □

Proposition 6 and 7 give the expression of $d_n(x\otimes 1)$, where the chain x is generated by one letter or generated by a,b.

Proposition 8. For $n\geqslant 3$,

$$d_n(cb(ab)^{n-1}\otimes 1)=cb(ab)^{n-2}\otimes ab+(-1)^n bab^2(ab)^{n-3}\otimes ac.$$

Proof. We can prove the formula by induction on n. It is true for $n=3$. Suppose $n\geqslant 4$ and it is true for $n-1$. Then

$$\begin{aligned}
&d_n(cb(ab)^{n-1}\otimes 1)\\
=&\, cb(ab)^{n-2}\otimes ab-i_{n-1}d_{n-1}(cb(ab)^{n-2}\otimes ab)\\
=&\, cb(ab)^{n-2}\otimes ab-i_{n-1}[(cb(ab)^{n-3}\otimes ab+(-1)^{n-1}bab^2(ab)^{n-4}\otimes ac)ab]\\
=&\, cb(ab)^{n-2}\otimes ab-i_{n-1}((-1)^{n-1}bab^2(ab)^{n-4}\otimes abac)\\
=&\, cb(ab)^{n-2}\otimes ab+(-1)^n bab^2(ab)^{n-3}\otimes ac.
\end{aligned}$$

□

Proposition 9. For $j_1 + j_2 = n-2$ and $j_1 \geqslant 2$, $j_2 \geqslant 0$, we have

$$d_n(cb(ab)^{j_1}b(ab)^{j_2} \otimes 1) = \begin{cases} cb(ab)^{n-2} \otimes b + (-1)^{n-1}bab^2(ab)^{n-3} \otimes a & j_2 = 0 \\ cb(ab)^{n-3}b \otimes ab - cb(ab)^{n-2} \otimes a & j_2 = 1 \\ cb(ab)^{j_1}b(ab)^{j_2-1} \otimes ab & j_2 \geqslant 2 \end{cases}$$

Proof. When $j_2 = 0$, we use Proposition 8 to get

$$\begin{aligned} & d_n(cb(ab)^{n-2}b \otimes 1) \\ = {} & cb(ab)^{n-2} \otimes b - i_{n-1}d_{n-1}(cb(ab)^{n-2} \otimes b) \\ = {} & cb(ab)^{n-2} \otimes b - i_{n-1}[(cb(ab)^{n-3} \otimes ab + (-1)^{n-1}bab^2(ab)^{n-4} \otimes ac)b] \\ = {} & cb(ab)^{n-2} \otimes b - i_{n-1}((-1)^n bab^2(ab)^{n-4} \otimes aba) \\ = {} & cb(ab)^{n-2} \otimes b + (-1)^{n-1}bab^2(ab)^{n-3} \otimes a. \end{aligned}$$

Then when $j_2 = 1$, we have

$$\begin{aligned} & d_n(cb(ab)^{n-3}bab \otimes 1) \\ = {} & cb(ab)^{n-3}b \otimes ab - i_{n-1}d_{n-1}(cb(ab)^{n-3}b \otimes ab) \\ = {} & cb(ab)^{n-3}b \otimes ab - i_{n-1}[(cb(ab)^{n-3} \otimes b + (-1)^{n-2}bab^2(ab)^{n-4} \otimes a)ab] \\ = {} & cb(ab)^{n-3}b \otimes ab - i_{n-1}(cb(ab)^{n-3} \otimes aba) \\ = {} & cb(ab)^{n-3}b \otimes ab - cb(ab)^{n-2} \otimes a. \end{aligned}$$

For the case $j_2 \geqslant 2$, we use induction on j_2. Fix $j_1 \geqslant 2$ and $n = j_1 + j_2 + 2$. When $j_2 = 2$, we have

$$\begin{aligned} & d_n(cb(ab)^{j_1}babab \otimes 1) \\ = {} & cb(ab)^{j_1}bab \otimes ab - i_{n-1}d_{n-1}(cb(ab)^{j_1}bab \otimes ab) \\ = {} & cb(ab)^{j_1}bab \otimes ab - i_{n-1}[(cb(ab)^{j_1}b \otimes ab - cb(ab)^{j_1+1} \otimes a)ab] \\ = {} & cb(ab)^{j_1}bab \otimes ab. \end{aligned}$$

Suppose $j_2 \geqslant 3$ and the formula is true for $j_2 - 1$. Then

$$\begin{aligned} & d_n(cb(ab)^{j_1}b(ab)^{j_2} \otimes 1) \\ = {} & cb(ab)^{j_1}b(ab)^{j_2-1} \otimes ab - i_{n-1}d_{n-1}(cb(ab)^{j_1}b(ab)^{j_2-1} \otimes ab) \\ = {} & cb(ab)^{j_1}b(ab)^{j_2-1} \otimes ab - i_{n-1}[(cb(ab)^{j_1}b(ab)^{j_2-2} \otimes ab)ab] \\ = {} & cb(ab)^{j_1}b(ab)^{j_2-1} \otimes ab. \end{aligned}$$

□

Proposition 10.

$$\begin{aligned} d_3(cbab^2 \otimes 1) &= cbab \otimes b + ca^2 \otimes ba - b^2ab \otimes c, \\ d_4(cbab^2ab \otimes 1) &= cbab^2 \otimes ab - cbabab \otimes a + b^2ab^2 \otimes ac + bab^2ab \otimes c \end{aligned}$$

and for $n \geqslant 5$, we have

$$\begin{aligned} d_n(cbab^2(ab)^{n-3} \otimes 1) = {} & cbab^2(ab)^{n-4} \otimes ab + (-1)^n b^2ab^2(ab)^{n-4} \otimes ac \\ & + (-1)^n bab^2ab^2(ab)^{n-5} \otimes ac. \end{aligned}$$

Proof. For the case $n \geqslant 5$, we use induction on n. The case $n = 5$ is satisfied. Suppose $n \geqslant 6$ and the formula is true for $n-1$. Then

$$\begin{aligned}
& d_n(cbab^2(ab)^{n-3} \otimes 1) \\
&= cbab^2(ab)^{n-4} \otimes ab - i_{n-1}d_{n-1}(cbab^2(ab)^{n-4} \otimes ab) \\
&= cbab^2(ab)^{n-4} \otimes ab - i_{n-1}[(cbab^2(ab)^{n-5} \otimes ab + (-1)^{n-1}b^2ab^2(ab)^{n-5} \otimes ac \\
&\quad + (-1)^{n-1}bab^2ab^2(ab)^{n-6} \otimes ac)ab] \\
&= cbab^2(ab)^{n-4} \otimes ab + (-1)^n i_{n-1}(b^2ab^2(ab)^{n-5} \otimes abac + bab^2ab^2(ab)^{n-6} \otimes abac) \\
&= cbab^2(ab)^{n-4} \otimes ab + (-1)^n b^2ab^2(ab)^{n-4} \otimes ac \\
&\quad + (-1)^n i_{n-1}(bab^2ab^2(ab)^{n-6} \otimes abac - (-1)^{n-2}babab(ab)^{n-4} \otimes ac) \\
&= cbab^2(ab)^{n-4} \otimes ab + (-1)^n b^2ab^2(ab)^{n-4} \otimes ac + (-1)^n bab^2ab^2(ab)^{n-5} \otimes ac.
\end{aligned}$$

□

Proposition 11.

$$d_3(cb^2ab \otimes 1) = cb^2 \otimes ab - cbab \otimes a - bab^2 \otimes c$$

and for $n \geqslant 4$, we have

$$d_n(cb^2(ab)^{n-2} \otimes 1) = cb^2(ab)^{n-3} \otimes ab + (-1)^n bab^3(ab)^{n-4} \otimes ac.$$

Proof. Induction on n. The case $n = 4$ is satisfied. Suppose $n \geqslant 5$ and the formula is true for $n-1$. Then

$$\begin{aligned}
& d_n(cb^2(ab)^{n-2} \otimes 1) \\
&= cb^2(ab)^{n-3} \otimes ab - i_{n-1}d_{n-1}(cb^2(ab)^{n-3} \otimes ab) \\
&= cb^2(ab)^{n-3} \otimes ab - i_{n-1}[(cb^2(ab)^{n-4} \otimes ab + (-1)^{n-1}bab^3(ab)^{n-5} \otimes ac)ab] \\
&= cb^2(ab)^{n-3} \otimes ab - i_{n-1}((-1)^{n-1}bab^3(ab)^{n-5} \otimes abac) \\
&= cb^2(ab)^{n-3} \otimes ab + (-1)^n bab^3(ab)^{n-4} \otimes ac.
\end{aligned}$$

□

Proposition 8 gives the expression of $d_n(x \otimes 1)$, where the chain x is generated by three letters, there is only one c in x and b^2 is not a subword of x. Proposition 9, 10 and 11 are the case when x is generated by three letters, there is only one c in x and there is only one b^2 in x.

Based on Proposition 9-11, we can prove the following Proposition 12-16 by induction. The proof is very similar to Proposition 9-11.

Proposition 12. For $j_1 + j_2 = n-3$ and $j_1 \geqslant 2$, $j_2 \geqslant 0$, we have

$$d_n(cb(ab)^{j_1}b(ab)^{j_2}b \otimes 1) = \begin{cases} cb(ab)^{n-3}b \otimes b + (-1)^{n-1}bab^2(ab)^{n-3} \otimes 1 & j_2 = 0 \\ cb(ab)^{n-4}bab \otimes b + cb(ab)^{n-2} \otimes 1 & j_2 = 1 \\ cb(ab)^{j_1}b(ab)^{j_2} \otimes b & j_2 \geqslant 2 \end{cases}$$

Proposition 13.

$$d_n(cb(ab)^{n-4}b^2ab \otimes 1) = cb(ab)^{n-4}b^2 \otimes ab - cb(ab)^{n-4}bab \otimes a + (-1)^{n-1}bab^2(ab)^{n-3} \otimes 1$$

for $n \geqslant 6$.

$$d_n(cb(ab)^{n-5}bab^2ab \otimes 1) = cb(ab)^{n-5}bab^2 \otimes ab - cb(ab)^{n-5}babab \otimes a - cb(ab)^{n-2} \otimes 1$$

for $n \geqslant 7$.

$$d_n(cb(ab)^{j_1}b(ab)^{j_2}bab \otimes 1) = cb(ab)^{j_1}b(ab)^{j_2}b \otimes ab - cb(ab)^{j_1}b(ab)^{j_2+1} \otimes a$$

for $j_1 + j_2 = n - 4$ and $j_1 \geqslant 2$, $j_2 \geqslant 2$.

Proposition 14. Let $j_1 + j_2 + j_3 = n - 3$ and $j_1 \geqslant 2$, $j_2 \geqslant 0$, $j_3 \geqslant 2$. When $j_2 = 0$, we have

$$d_n(cb(ab)^{j_1}b^2(ab)^{j_3} \otimes 1) = cb(ab)^{j_1}b^2(ab)^{j_3-1} \otimes ab + (-1)^{n-1}bab^2(ab)^{n-3} \otimes 1.$$

When $j_2 = 1$, we have

$$d_n(cb(ab)^{j_1}bab^2(ab)^{j_3} \otimes 1) = cb(ab)^{j_1}bab^2(ab)^{j_3-1} \otimes ab + (-1)^{j_3}cb(ab)^{j_2} \otimes 1.$$

When $j_2 \geqslant 2$, we have

$$d_n(cb(ab)^{j_1}b(ab)^{j_2}b(ab)^{j_3} \otimes 1) = cb(ab)^{j_1}b(ab)^{j_2}b(ab)^{j_3-1} \otimes ab.$$

Let $x_1, \cdots, x_r$ be n-chains, $u_1, \cdots, u_r \in M$ and $l \geqslant 1$. We formally write

$$c^l(\sum_{i=1}^{r} x_i \otimes u_i) = \sum_{i=1}^{r} c^l x_i \otimes u_i.$$

Let $x = c^l b^{j_1}ab^{j_2} \cdots ab^{j_s}$ be an n-chain. We denote $c^l d_{n-l}(b^{j_1}ab^{j_2} \cdots ab^{j_s} \otimes 1)$ by $\star$. Then Proposition 12, 13 and 14 can be rewritten as

$$d_n(cb(ab)^{j_1}b(ab)^{j_2}b(ab)^{j_3} \otimes 1) = \begin{cases} \star + (-1)^{n-1}bab^2(ab)^{n-3} \otimes 1 & j_2 = 0 \\ \star & j_2 \geqslant 1 \end{cases}$$

for $j_1 + j_2 + j_3 = n - 3$ and $j_1 \geqslant 2$, $j_2 \geqslant 0$, $j_3 \geqslant 0$.

Proposition 15. Let $x = cb(ab)^{j_1}b(ab)^{j_2}b(ab)^{j_3}$ and $j_1 + j_2 + j_3 = n - 3$. Then

$$d_n(x \otimes 1) = \star + (-1)^{n-1}b^2ab^2(ab)^{n-4} \otimes a + (-1)^{n-1}bab^2ab^2(ab)^{n-5} \otimes a$$

for $j_1 = 1$, $j_2 \geqslant 1$ and $j_3 = 0$.

$$d_n(x \otimes 1) = \star + (-1)^{n-1}bab^3(ab)^{n-4} \otimes a$$

for $j_1 = 0$, $j_2 \geqslant 1$ and $j_3 = 0$.

$$d_n(x \otimes 1) = \star$$

for $0 \leqslant j_1 \leqslant 1$, $j_2 \geqslant 2$ and $j_3 \geqslant 1$.

Proposition 16.

$$\begin{aligned}
d_6(cbab^2ab^2ab\otimes 1) &= \star + b^2ab^2abab\otimes c + bab^2ab^2ab\otimes c,\\
d_4(cbab^3\otimes 1) &= \star + b^3ab\otimes c,\\
d_5(cbab^3ab\otimes 1) &= \star - b^3ab^2\otimes ac - b^2ab^2ab\otimes c - bab^2ab^2\otimes c - bab^2abab\otimes 1,\\
d_5(cb^2ab^2ab\otimes 1) &= \star - bab^3ab\otimes c,\\
d_4(cb^3ab\otimes 1) &= \star + bab^3\otimes c.
\end{aligned}$$

$$\begin{aligned}
d_n(cbab^2ab^2(ab)^{n-5}\otimes 1) = \star &+ (-1)^n b^2ab^2abab^2(ab)^{n-7}\otimes ac\\
&+ (-1)^n bab^2ab^2ab^2(ab)^{n-7}\otimes ac
\end{aligned}$$

for $n \geqslant 7$.

$$\begin{aligned}
d_n(cbab^3(ab)^{n-4}\otimes 1) = \star &+ (-1)^n b^3ab^2(ab)^{n-5}\otimes ac + (-1)^n b^2ab^2ab^2(ab)^{n-6}\otimes ac\\
&+ (-1)^n bab^2ab^3(ab)^{n-6}\otimes ac + (-1)^n bab^2(ab)^{n-3}\otimes 1
\end{aligned}$$

for $n \geqslant 6$.

$$d_n(cb^2ab^2(ab)^{n-4}\otimes 1) = \star + (-1)^n bab^3ab^2(ab)^{n-6}\otimes ac$$

for $n \geqslant 6$.

$$d_n(cb^3(ab)^{n-3}\otimes 1) = \star + (-1)^n bab^4(ab)^{n-5}\otimes ac$$

for $n \geqslant 5$.

Proposition 12-16 give the expression of $d_n(x\otimes 1)$, where the chain x is generated by three letters, there is only one c in x and there is only two b^2 in x.

Proposition 17.

$$d_n(cb^n\otimes 1) = cb^{n-1}\otimes b + (-1)^{n-1}bab^{n-1}\otimes 1 + a^n\otimes c$$

for $n \geqslant 2$.

$$d_n(cb^{n-1}ab\otimes 1) = cb^{n-1}\otimes ab - cb^{n-2}ab\otimes a + (-1)^n bab^{n-1}\otimes c$$

for $n \geqslant 3$.

$$d_n(cb^{j_1}(ab)^{j_2}\otimes 1) = cb^{j_1}(ab)^{j_2-1}\otimes ab + (-1)^n bab^{j_1+1}(ab)^{j_2-2}\otimes ac$$

for $j_1 + j_2 = n$ and $j_1 \geqslant 1$, $j_2 \geqslant 2$.

Proof. The first and second formulas can be proved by induction on n. We now prove the last one by induction on j_2. Fix $j_1 \geqslant 1$ and let $n = j_1 + j_2$. For the base case $j_2 = 2$, we have

$$d_n(cb^{j_1}abab\otimes 1)$$

$$
\begin{aligned}
&= cb^{j_1}ab \otimes ab - i_{n-1}d_{n-1}(cb^{j_1}ab \otimes ab) \\
&= cb^{j_1}ab \otimes ab - i_{n-1}[(cb^{j_1} \otimes ab - cb^{j_1-1}ab \otimes a + (-1)^{n-1}bab^{j_1} \otimes c)ab] \\
&= cb^{j_1}ab \otimes ab - i_{n-1}((-1)^{n-1}bab^{j_1} \otimes bac) \\
&= cb^{j_1}ab \otimes ab + (-1)^{n}bab^{j_1+1} \otimes ac.
\end{aligned}
$$

Suppose $j_2 \geqslant 3$ and the formula is true for $j_2 - 1$. Then

$$
\begin{aligned}
&d_n(cb^{j_1}(ab)^{j_2} \otimes 1) \\
&= cb^{j_1}(ab)^{j_2-1} \otimes ab - i_{n-1}d_{n-1}(cb^{j_1}(ab)^{j_2-1} \otimes ab) \\
&= cb^{j_1}(ab)^{j_2-1} \otimes ab - i_{n-1}[(cb^{j_1}(ab)^{j_2-2} \otimes ab + (-1)^{n-1}bab^{j_1+1}(ab)^{j_2-3} \otimes ac)ab] \\
&= cb^{j_1}(ab)^{j_2-1} \otimes ab - i_{n-1}((-1)^{n-1}bab^{j_1+1}(ab)^{j_2-3} \otimes abac) \\
&= cb^{j_1}(ab)^{j_2-1} \otimes ab + (-1)^{n}bab^{j_1+1}(ab)^{j_2-2} \otimes ac.
\end{aligned}
$$

□

Finally we could claim that

$$
\begin{aligned}
&d_n(c^n a \otimes 1) = c^n \otimes a + c^{n-1}b \otimes c + c^{n-1}a \otimes b \\
&\quad + \sum_{2 \leqslant i \leqslant n,\, i\, even} (c^{n-i}b^i \otimes a - c^{n-1-i}b^i ab \otimes 1 + c^{n-1-i}a^{i+1} \otimes b), \\
&n \geqslant 1; \\
&d_n(c^{n+1-k}a^k \otimes 1) = c^{n+1-k}a^{k-1} \otimes a + c^{n-k}b^k \otimes c + c^{n-k}b^{k-1}ab \otimes 1, \\
&2 \leqslant k \leqslant n,\ n \geqslant 2; \\
&d_n(c^n b \otimes 1) = c^n \otimes b + c^{n-1}b \otimes a + c^{n-1}a \otimes c \\
&\quad + \sum_{2 \leqslant i \leqslant n,\, i\, even} (c^{n-i}a^i \otimes b + c^{n-1-i}b^{i+1} \otimes a - c^{n-1-i}b^{i-1}ab^2 \otimes 1), \\
&n \geqslant 1; \\
&d_n(c^{n+1-k}b^k \otimes 1) = c^{n+1-k}b^{k-1} \otimes b + (-1)^{k-1}c^{n-k}bab^{k-1} \otimes 1 + c^{n-k}a^k \otimes c \\
&\quad + (-1)^{k-1} \sum_{\substack{i=k-1,\, k+1,\, k+3,\, \cdots, \\ i \leqslant n-3}} (c^{n-3-i}b^3ab^i \otimes 1 - c^{n-3-i}bab^{i+2} \otimes 1), \\
&2 \leqslant k \leqslant n,\ n \geqslant 2; \\
&d_n(c^{n-1}bab \otimes 1) = c^{n-1}b \otimes ab - c^{n-1}a \otimes ba + c^{n-2}bab \otimes c \\
&\quad + \sum_{2 \leqslant i \leqslant n-1,\, i\, even} (c^{n-1-i}b^i ab \otimes a - c^{n-1-i}a^{i+1} \otimes ba + c^{n-2-i}b^{i+1}ab \otimes c \\
&\quad - c^{n-2-i}b^{i-1}ab^3 \otimes c), \\
&n \geqslant 2.
\end{aligned}
$$

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