

# Representation theory

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# Chapter I

## Generalities on Modules

All rings are associative and unital.

### 1 Modules and algebras

#### 1.1 Module over a ring

**Definition 1.1** Let  $R$  a ring. A **left  $R$ -module**  $M$  is an abelian group together with a map  $R \times M \rightarrow M$  sending  $(a, m)$  to  $a.m$  satisfying for any  $a, a' \in A, m \in M$

- $(a + a').m = a.m + a'.m$ ;
- $a.(m + m') = a.m + a.m'$ ;
- $(aa').m = a.(a'.m)$ ;
- $1_A.m = m$ .

**Example 1.2**  $R = k$  field then  $k$ -vector space.  $R = \mathbb{Z}$  then abelian group.

$R$  is an  $R$ -module over itself.

If  $M$  is an abelian group, then  $M$  is a left  $\text{End}_{\mathbb{Z}}(M)$ -module.

Representation point of view.

**Proposition 1.3**

Let  $M$  be an abelian group. Then  $M$  is a left  $R$ -module if and only if there exists a ring homomorphism  $\rho : R \rightarrow \text{End}(M)$ .

Difference between left and right modules.

If  $R$  is commutative, then a left  $R$ -module is a right  $R$ -module.

A left  $R$ -module is a right  $R^{\text{op}}$ -module.

**Definition 1.4** Let  $R$  and  $R'$  be rings. A  **$R$ - $R'$ -bimodule**  $M$  is an abelian group  $M$  which is a left  $R$ -module, a right  $R'$ -module, and such that  $(a.m).b = a.(m.b)$  for any  $a \in R$ ,  $m \in M$  and  $b \in R'$ .

If  $R$  is commutative, then a  $R$ -module is automatically a  $R$ -bimodule.

**Definition 1.5** Let  $M$  and  $N$  be  $R$ -modules. A **morphism** of  $R$ -modules (or a  $R$ -linear application) is a morphism of abelian groups such that  $f : M \rightarrow N$  such that  $f(xm) = xf(m)$

## 1.2 Algebras

In all what follows,  $k$  will be a commutative ring.

**Definition 1.6** A  **$k$ -algebra** is a unital ring with a structure of  $k$ -module such that  $\lambda(ab) = (\lambda a)b = a(\lambda b)$  for  $\lambda \in k$  and  $a, b \in A$ .

**Example 1.7** Typical examples in this course:  $k = \mathbb{Z}$  or  $k$  is a field. The algebra  $A$  could be  $\mathcal{M}_n(k)$ ,  $k[X]$ ,  $k[X, Y]$ ,  $\mathcal{T}_n^+(k) \subset \mathcal{M}_n(k)$  upper triangular matrices.  $A = kG$  where  $G$  is a group.

A **morphism of  $k$ -algebras** is a map  $f : A \rightarrow B$  which is a ring morphism and a  $k$ -module morphism.

Notions of subalgebras and ideals (left, right and two-sided).

**Remark 1.8** A ring is always a  $\mathbb{Z}$ -algebra.

The map  $k \rightarrow A$  sending  $\lambda$  to  $\lambda 1_A$  is a  $k$ -algebra morphism whose image is in the center of  $A$ .

## 1.3 Modules over algebras

**Definition 1.9** A **left  $A$ -module**  $M$  is a left  $A$ -module thinking of  $A$  as a ring.

Because of the map  $k \rightarrow A$ , a  $A$ -module is automatically a  $k$ -module. And we have  $(\lambda a)m = \lambda(am) = a(\lambda m)$ . (Scalars commute with everything).

So in other words, any ring map  $A \rightarrow \text{End}(M)$  factors through a  $k$ -algebra map  $A \rightarrow \text{End}_k(M)$ .

**Proposition 1.10**

Let  $G$  be a group, and  $k$  be a field. Let  $M$  be a  $k$ -vector space. It has a structure of  $kG$ -modules if and only if there exists a group morphism  $\rho : G \rightarrow \text{Aut}(M)$ .  $(\rho, M)$  is called a representation of the group  $G$ .

## 1.4 Submodules, quotients and direct sums

### Submodules

**Definition 1.11** Let  $M$  be a left  $A$ -module. A **submodule**  $N \subset M$  is a subgroup which is stable under  $A$ -multiplication.

For example, the submodules of  $A$  seen as a left  $A$ -module are the left ideals of  $A$ .

### Quotient

**Proposition 1.12**

Let  $M$  be a  $A$ -module, and  $N \subset M$  a submodule, then  **$M/N$**  has a natural structure of  $A$ -module, and the projection  $M \rightarrow M/N$  is  $A$ -linear.

If  $N, N'$  are submodules of  $M$ , so are  $N + N'$  and  $N \cap N'$ .

### Submodules and morphisms

**Proposition 1.13**

Let  $f : M \rightarrow N$  be a morphism of  $A$ -modules, then  **$\text{Ker } f$**  and  **$\text{Im } f$**  are  $A$ -modules.

If  $M' \subset M$  is a submodule. Then there exists a unique  $\bar{f} : M/M' \rightarrow N$  such that  $\bar{f} \circ p = f$  if and only if  $M' \subset \text{Ker } f$ .

In particular  $f$  induces an isomorphism  $M/\text{Ker } f \simeq \text{Im } f$ .

**Definition 1.14** For  $f : M \rightarrow N$  a morphism, we define  $\text{Coker } f := N/\text{Im } f$  the **cokernel** of  $f$ . It is a  $A$ -module.

## Direct sum

### Proposition 1.15

Let  $M$  and  $N$  be  $A$ -modules. Then  $M \times N$  has naturally a structure of  $A$ -module.

We denote it as  **$M \oplus N$**  (external direct sum).

Note that if  $M_1$  and  $M_2$  are submodules of  $M$ , such that  $M_1 \cap M_2 = \{0\}$ , then  $M_1 + M_2 \simeq M_1 \oplus M_2$ . (so internal direct sums coincide with external ones).

If  $M$  and  $N$  are modules,  $M$  is naturally isomorphic to a submodule of  $M \oplus N$  and its quotient is isomorphic to  $N$ . However, if  $N \subset M$  is a submodule,  $M$  is not isomorphic in general to  $N \oplus M/N$ .

### Proposition 1.16

Let  $X$  be a  $A$ -module.

If there exist  $p_1, p_2 \in \text{End}_A(X)$  such that

$$p_1 \circ p_2 = p_2 \circ p_1 = 0 \quad p_i^2 = p_i \quad \text{and} \quad p_1 + p_2 = \text{Id}_X,$$

then  $X$  is isomorphic to  $\text{Im } p_1 \oplus \text{Im } p_2$ .

**Example 1.17** Assume  $1_A = e_1 + e_2$  with  $e_i^2 = e_i$  (idempotent),  $e_1 e_2 = e_2 e_1 = 0$  (orthogonal), then  $A \simeq Ae_1 \oplus Ae_2$  as a left  $A$ -module.

For example if  $A = \mathcal{M}_n(k)$ , then one can prove that  $A \simeq (k^n)^n$  as a left  $A$ -module.

## 2 Tensor products and Hom

### 2.1 Homomorphism module

Let  $M$  and  $N$  be  $A$ -modules. Then  **$\text{Hom}_A(M, N)$**  has a structure of  $\text{End}(N)$ - $\text{End}(M)$ -bimodule (in particular it is a  $k$ -bimodule) given by right and left



composition.

As a consequence, if  $M$  is a  $A$ - $B$ -bimodule and  $N$  a  $A$ - $C$ -bimodule. Then  $\text{Hom}_A(M, N)$  has a structure of  $B$ - $C$ -bimodule, given by

$$b.f.c(m) := f(mb)c, \text{ for } m \in M, b \in B, c \in C \text{ and } f \in \text{Hom}(M, N).$$

**Example 2.1** If  $M$  is a left  $A$ -module, then  $\text{Hom}_k(M, k) = M^*$  and  $M^\vee := \text{Hom}_A(M, A)$  are right  $A$ -modules.  $A^* = \text{Hom}_k(A, k)$  is a left  $A$ -module (in fact it is a  $A$ -bimodule.)

If  $A = kG$ , since  $g \mapsto g^{-1}$  is an isomorphism  $kG \rightarrow kG^{\text{op}}$ , then if  $V$  is a  $G$ -representation, then  $V^*$  is naturally a  $kG^{\text{op}}$ -module, hence a  $kG$ -module.

If  $B$  is a subalgebra of  $A$ , then  $A$  is naturally a  $B$ -module. Then for a  $A$ -module  $M$ , we have

$${}_B M \simeq \text{Hom}_A(A_B, M) \text{ as } B\text{-modules}$$

**Proposition 2.2**

1. For each  $A$ -module  $M$ , there is an isomorphism of  $A$ -module  $\text{Hom}_A(A, M) \simeq M$ .
2. There is an algebra isomorphism  $\text{End}_A(A) \simeq A^{\text{op}}$ .

$$\text{Hom}(M \oplus M', N \oplus N') \simeq \begin{bmatrix} \text{Hom}(M, N) & \text{Hom}(M', N) \\ \text{Hom}(M, N') & \text{Hom}(M', N') \end{bmatrix} \text{ as } k\text{-module.}$$

**Proposition 2.3**

Let  $M$  and  $N$  be modules. Then we have an isomorphism  $\text{End}(M \oplus N) \simeq \begin{bmatrix} \text{End}(M) & \text{Hom}(N, M) \\ \text{Hom}(M, N) & \text{End}(N) \end{bmatrix}$  as a  $k$ -algebra.

## 2.2 Tensor product

Let  $M$  be a right  $A$ -module and  $N$  be a left  $A$ -module. We define the space  $M \otimes_A N$  as the  $k$ -free module generated by the  $m \otimes n$  mod out by the submodule generated by

- $(m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n$

- $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$
- $(ma) \otimes n = m \otimes (an)$ ,  $a \in A$ ,  $m \in M$ ,  $n \in N$ ;

If  $M$  is a  $B$ - $A$ -bimodule, and if  $N$  is a  $A$ - $C$ -bimodule, then  $M \otimes_A N$  is a  $B$ - $C$ -bimodule.

**Proposition 2.4**

1. there is a canonical isomorphism  $A \otimes_A X \simeq X$ .
2. there is a unique isomorphism  $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$  sending  $(x \otimes y) \otimes z$  to  $x \otimes (y \otimes z)$
3. If  $A$  is commutative then there is a unique isomorphism  $X \otimes Y \simeq Y \otimes X$  sending  $x \otimes y$  to  $y \otimes x$ .
4. There is a canonical isomorphism

$$(M \oplus M') \otimes_A N \simeq (M \otimes_A N) \oplus (M' \otimes_A N)$$

5. If  $f : X_1 \rightarrow X_2$  and  $g : Y_1 \rightarrow Y_2$  are module morphisms, then there exists a unique module morphism  $f \otimes g : X_1 \otimes Y_1 \rightarrow X_2 \otimes Y_2$  sending  $x \otimes y$  on  $f(x) \otimes g(y)$ .

**Extension of scalars**

If  $A \rightarrow B$  is a morphism of algebras, it makes  $B$  a  $A$ -module, hence we can define  $B \otimes_A M$  for any  $A$ -module  $M$ .

**Example 2.5**  $B \otimes_A A[X] \simeq B[X]$ .

If  $G$  is a group and  $H$  is a subgroup. We have an injection  $kH \rightarrow kG$ . So for any  $kH$ -module  $M$ , there is a  $kG$ -module defined by  $\text{Ind}_G^H(M) := kG \otimes_{kH} M$ .

**Tensor product of algebras**

**Theorem 2.6**

$A \otimes_K B$  is an algebra.

The data of a  $A$ - $B$ -bimodule is the same of a  $A \otimes B^{\text{op}}$ -module. And same for the morphisms.

## 2.3 Adjunction formula

### Theorem 2.7

Let  $A$  and  $B$  be algebras, let  $X$  be a  $A$ -module, let  $Y$  be a  $B$ - $A$ -bimodule and let  $Z$  be a  $B$ -module. Then there is a canonical isomorphism

$$\mathrm{Hom}_A({}_A X, \mathrm{Hom}_B({}_B Y_{A,B} Z)) \simeq \mathrm{Hom}_B({}_B Y \otimes_A X, {}_B Z).$$

**Example 2.8** Let  $H$  be a subgroup of  $G$ . Let  $M$  be a representation of  $H$  and  $N$  be a representation of  $G$ . Then we have

$$\mathrm{Hom}_{kG}(\mathrm{Ind}_G^H(M), N) \simeq \mathrm{Hom}_{kH}(M, \mathrm{Res}_H^G(N)).$$

## 3 Finite and infinite modules

### 3.1 Product and sums

Let  $I$  be a set and  $(M_i)_{i \in I}$  be a collection of  $A$ -modules. Then

$$\prod_{i \in I} M_i := \{(m_i, i \in I), m_i \in M_i\}$$

is naturally a  $A$ -module.

We define  $\bigoplus_{i \in I} M_i$  the subset of  $\prod_i M_i$  consisting of finitely supported  $I$ -uples. It is a  $A$ -submodule.

### Proposition 3.1

For any sets  $I$  and  $J$ , and modules  $(M_i)$ ,  $(N_j)$ , there is an isomorphism

$$\mathrm{Hom}_A\left(\bigoplus_i M_i, \prod_j N_j\right) \simeq \prod_{(i,j)} \mathrm{Hom}(M_i, N_j).$$

### 3.2 Free modules

If  $I$  a set, define  $A^I := \{f : I \rightarrow A\}$  and  $A^{(I)} := \{f : I \rightarrow A \text{ with finite support}\}$ .

**Definition 3.2** A  $A$ -module is called **free** if it admits a basis, that is a family of elements  $(x_i)_{i \in I}$  that is linearly independent (every finite linear combination...) that generates it.

**Theorem 3.3**

Any free  $A$ -module is of the form  $A^{(I)}$ .

**Theorem 3.4**

Every  $A$ -module is a quotient of a free  $A$ -module.

As a corollary, any  $A$ -module  $M$  is the cokernel of a  $A$ -module morphism between free modules.

### 3.3 Finite modules

**Definition 3.5** A **finitely generated**  $A$ -module (or **module of finite type**)  $M$  is a module of the form  $\langle X \rangle$  for  $X$  a finite subset of  $M$ .

A module  $M$  is of finite type if and only if there exists a map  $A^n \rightarrow M$ . However, in general it could happen that the kernel of this map is not finitely generated. If it is,  $M$  is called **finitely presented** and there exists a map  $A^m \rightarrow A^n$  such that  $M$  is isomorphic to the cokernel.

**Particular cases:**

If  $A$  is a finite dimensional  $k$ -algebra, then

module of finite type = module of finite dimension = module of finite presentation

In this case, it is clearly closed under kernel and cokernel.

The same is true if  $A = \mathbb{Z}$ . Any subgroup of a finitely generated abelian group is finitely generated.

Moreover any subgroup of a finitely generated free abelian group is a free abelian group.

We know well the structure of finitely generated abelian groups (built from  $\mathbb{Z}$  and  $\mathbb{Z}_{p^\alpha}$ ) but non finitely generated abelian groups are much more complicated:  $\mathbb{R}$ ,  $\mathbb{Q}$ , ...

# Chapter II

## Categories of modules

### 1 Linear categories and functors

#### 1.1 Definition

**Definition 1.1** A  $k$ -linear category  $\mathcal{C}$  is a collection of objects (also denoted by  $\mathcal{C}$ ) and for each  $X, Y$  a  $k$ -module  $\text{Hom}_{\mathcal{C}}(X, Y)$  together with a  $k$ -bilinear map

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) \times \text{Hom}_{\mathcal{C}}(Y, Z) &\rightarrow \text{Hom}_{\mathcal{C}}(X, Z) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

satisfying  $h \circ (g \circ f) = (h \circ g) \circ f$  and with the following properties

- for each  $X \in \mathcal{C}$ , there is  $1_X \in \text{End}_{\mathcal{C}}(X)$  such that  $f \circ 1_X = f$  and  $1_X \circ g = g$  for any  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$  and  $g \in \text{Hom}_{\mathcal{C}}(Y, X)$ ;
- there is an object  $0 \in \mathcal{C}$  such that  $\text{Hom}_{\mathcal{C}}(X, 0) = \text{Hom}_{\mathcal{C}}(0, X) = 0$  for all  $X$ ;
- for each  $X, Y$  in  $\mathcal{C}$  there is an object  $X \oplus Y$  such that

$$\text{Hom}(X \oplus Y, Z) \simeq \text{Hom}_{\mathcal{C}}(X, Z) \oplus \text{Hom}_{\mathcal{C}}(Y, Z) \text{ and}$$

$$\text{Hom}(Z, X \oplus Y) \simeq \text{Hom}_{\mathcal{C}}(Z, X) \oplus \text{Hom}_{\mathcal{C}}(Z, Y) \text{ (as } k\text{-modules).}$$

The category of  $A$ -modules  $\text{Mod } A$  is such a category. The category of finitely generated  $A$ -modules  $\text{mod } A$  is also such a category.

Note that in  $\text{Mod } A$ , the isomorphisms above are given by

$$\begin{aligned} \text{Hom}_A(X \oplus Y, Z) &\simeq \text{Hom}_A(X, Z) \oplus \text{Hom}_A(Y, Z) \\ f &\mapsto (f \circ i_X, f \circ i_Y) \\ f \circ p_X + g \circ p_Y &\mapsto (f, g) \end{aligned}$$

## 1.2 Linear functors

**Definition 1.2** A  $k$ -linear covariant (resp. contravariant) functor  $F$  between two  $k$ -linear categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  is the data of an object  $FX \in \mathcal{C}_2$  for each object  $X \in \mathcal{C}_1$ , and a  $k$ -linear map

$$F_{X,Y} : \text{Hom}_{\mathcal{C}_1}(X, Y) \rightarrow \text{Hom}_{\mathcal{C}_2}(FX, FY)$$

such that

- $F(f \circ g) = F(f) \circ F(g)$  (resp.  $F(f \circ g) = F(g) \circ F(f)$ );
- $F(1_X) = 1_{FX}$  for each  $X \in \mathcal{C}_1$ ;
- $F(0) = 0$ ;
- $F(X \oplus Y) \simeq FX \oplus FY$  and these maps are compatible with the isomorphisms for the Hom, i.e. the following commutes

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}_1}(X \oplus Y, Z) & \xrightarrow{\sim} & \text{Hom}_{\mathcal{C}_2}(F(X \oplus Y), Z) \\ \downarrow \sim & & \downarrow \sim \\ & & \text{Hom}_{\mathcal{C}_2}(FX \oplus FY, Z) \\ & & \downarrow \sim \\ \text{Hom}_{\mathcal{C}_1}(X, Z) \oplus \text{Hom}_{\mathcal{C}_1}(Y, Z) & \xrightarrow{\sim} & \text{Hom}_{\mathcal{C}_2}(FX, FZ) \oplus \text{Hom}_{\mathcal{C}_2}(FY, FZ) \end{array}$$

A composition of linear functors is clearly a linear functor.

**Definition 1.3** Let  $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  be a  $k$ -linear functor. If for any  $Y \in \mathcal{C}_2$  there exists  $X \in \mathcal{C}_1$  such that  $FX$  is isomorphic to  $Y$ , we say that  $F$  is **dense**. If for any  $X, Y$ , the map  $F_{X,Y}$  is an isomorphism, we say that  $F$  is **fully faithful**. A functor which is dense and fully faithful is called an **equivalence**, and the categories  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are said to be equivalent  $k$ -linear categories.

A **natural transformation**  $\eta : F \rightarrow G$  between two functors  $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  and  $G : \mathcal{C}_1 \rightarrow \mathcal{C}_2$  assigns  $\eta_X \in \text{Hom}_{\mathcal{C}_2}(FX, GX)$  for each  $X \in \mathcal{C}_1$  such that  $\eta_Y \circ F_{X,Y}(f) = G_{X,Y}(f) \circ \eta_X$  for any  $f \in \text{Hom}_{\mathcal{C}_1}(X, Y)$ .

If moreover each  $\eta_X$  is invertible, we say that there is a **functorial isomorphism** between  $F$  and  $G$ .

For example if  $B \rightarrow A$  is a morphism of algebras, then  ${}_A M \mapsto_B M$  from  $\text{Mod } A \rightarrow \text{Mod } B$  is a functor. For example,  $\text{Mod } A \rightarrow \text{Mod } k$ , or  $\text{Mod } A \rightarrow \text{Mod } \mathbb{Z}$  are functors called **forgetful functors**.

## 1.3 Functors Hom and $\otimes$

Hom and  $\otimes$  are the main examples of functors in representation theory.

### Theorem 1.4

Let  ${}_A M_B$  be a  $A$ - $B$ -bimodule, then

- $\text{Hom}_A(M, -)$  is a covariant functor  $\text{Mod } A \rightarrow \text{Mod } B$ ;
- $\text{Hom}_A(-, M)$  is a contravariant functor  $\text{Mod } A \rightarrow \text{Mod } B$ ;
- $- \otimes_A M$  is a covariant functor  $\text{Mod } A^{\text{op}} \rightarrow \text{Mod } B^{\text{op}}$
- $M \otimes_B -$  is a covariant functor  $\text{Mod } B \rightarrow \text{Mod } A$ .

### Proposition 1.5

All the isomorphisms described in the previous chapter subsections 2.1, 2.2 and 2.3 are functorial isomorphisms.

For example the functor  $\text{Hom}_A(A, -)$  is isomorphic to  $\text{Id} : \text{Mod } A \rightarrow \text{Mod } A$ .

The contravariant functors (from  $\text{Mod } A$  to  $\text{Mod } k$ )  $\text{Hom}_A(-, \text{Hom}_B({}_B Y_A, {}_B Z))$  and  $\text{Hom}_B({}_B Y \otimes_A -, {}_B Z)$  are isomorphic.

## 2 Short exact sequences

### 2.1 Abelian category

**Definition 2.1** Let  $X, Y$  and  $Z$  be  $A$ -modules. A sequence  $X \xrightarrow{f} Y \xrightarrow{g} Z$  of morphisms is called **exact** if  $\text{Ker } g = \text{Im } f$ .

A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0;$$

so equivalently,  $f$  is injective,  $\text{Ker } g = \text{Im } f$  and  $g$  is surjective.

For example if  $N \subset M$  is a submodule, there is a natural short exact sequence of the form

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{p} M/N \longrightarrow 0.$$

Here is the fundamental property of the module category:

**Proposition 2.2**

Let  $f : M \rightarrow N$ , then there exist two short exact sequences:

$$0 \longrightarrow K \longrightarrow M \xrightarrow{p} I \longrightarrow 0 \quad 0 \xrightarrow{i} I \longrightarrow N \longrightarrow C \longrightarrow 0$$

such that  $i \circ p = f$ .

This is the property that makes the category  $\text{Mod } A$  an abelian category.

NB: if  $k$  is a field, and  $A$  is a finite dimensional  $k$ -algebra, then  $\text{mod } A$  is also an abelian category. Indeed, if  $M$  and  $N$  are finitely generated so are  $K$ ,  $I$  and  $C$ . The same is true for finitely generated abelian groups.

## 2.2 Monomorphisms and epimorphisms

**Definition 2.3** A morphism  $f : X \rightarrow Y$  is called a monomorphism if  $f \circ g = f \circ h \Rightarrow g = h$  (or equivalently  $f \circ g = 0 \Rightarrow g = 0$ ).

A morphism  $f : X \rightarrow Y$  is called an epimorphism if  $g \circ f = h \circ f \Rightarrow g = h$  (or equivalently  $g \circ f = 0 \Rightarrow g = 0$ ).

**Proposition 2.4**

1. A  $A$ -linear map  $f : X \rightarrow Y$  is a monomorphism if and only if  $f$  is injective.
2. A  $A$ -linear map  $f : X \rightarrow Y$  is an epimorphism if and only if  $f$  is surjective.

**Proposition 2.5**

Let  $f : X \rightarrow Y$  be morphism in  $\text{Mod } A$ . Then



1. for any morphism  $g : U \rightarrow X$  such that  $f \circ g = 0$  there exists a unique morphism  $h : U \rightarrow \text{Ker} f$  such that  $g = i \circ h$  where  $i : \text{Ker} f \rightarrow X$ .
2. for any morphism  $g : Y \rightarrow Z$  such that  $g \circ f = 0$  there exists a unique morphism  $h : \text{Coker} f \rightarrow Z$  such that  $g = h \circ p$  where  $p : Y \rightarrow \text{Coker} f$ .

### Corollary 2.6

Let  $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi \downarrow & & \downarrow \psi \\ X' & \xrightarrow{f'} & Y' \end{array}$  be a commutative square (that is  $\psi \circ f = f' \circ \varphi$ ), then

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi \downarrow & & \downarrow \psi \\ X' & \xrightarrow{f'} & Y' \end{array}$$

it can be completed into a commutative diagram

$$\begin{array}{ccccccc} \text{Ker} f & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & \text{Coker} f \\ \downarrow & & \downarrow \varphi & & \downarrow \psi & & \downarrow \\ \text{Ker} f' & \longrightarrow & X' & \xrightarrow{f'} & Y' & \longrightarrow & \text{Coker} f' \end{array}$$

## 2.3 Split short exact sequences

**Definition 2.7** A short exact sequence  $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$  is a **split short exact sequence** if there exists an isomorphism  $h : Y \rightarrow X \oplus Z$  such that there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \longrightarrow 0 \\ & & \parallel & & \downarrow h & & \parallel \\ 0 & \longrightarrow & X & \xrightarrow{i} & X \oplus Z & \xrightarrow{p} & Z \longrightarrow 0 \end{array}$$

A morphism  $f : X \rightarrow Y$  is said to be a **section** if there exists  $f' : Y \rightarrow X$  with  $f' \circ f = 1_X$ .

A morphism  $g : Y \rightarrow Z$  is said to be a **retraction** if there exists  $g' : Z \rightarrow Y$  such that  $g \circ g' = 1_Z$ .

**Proposition 2.8**

A short exact sequence  $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$  if and only if one of the following occurs:

- $f$  is a section;
- $g$  is a retraction.

**Example 2.9** In the category  $\text{Mod } k$ , every short exact sequence splits.

As we will see later, it is also the case in  $\text{Mod } CG$  where  $G$  is a finite group.

It is not the case in  $\text{Mod } \mathbb{Z}$ , for instance  $0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$  does not split.

## 2.4 Push forward and pull back

### Pull back

**Definition 2.10** Let  $g_1 : Y_1 \rightarrow Y$  and  $g_2 : Y_2 \rightarrow Y$  be morphisms. Then a **pull back** of  $g_1$  and  $g_2$  is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y_1 \\ \downarrow f_2 & & \downarrow g_1 \\ Y_2 & \xrightarrow{g_2} & Y \end{array}$$

such that for any commutative square

$$\begin{array}{ccccc} Z & & & & \\ & \searrow h_1 & & & \\ & & X & \longrightarrow & Y_1 \\ & \searrow h_2 & \downarrow & & \downarrow g_1 \\ & & Y_2 & \xrightarrow{g_2} & Y \end{array}$$

there exists a unique  $h : Z \rightarrow X$  such that  $h_1 = f_1 \circ h$  and  $h_2 = f_2 \circ h$ .

**Definition 2.11** Let  $f_1 : X \rightarrow X_1$  and  $f_2 : X \rightarrow X_2$  be morphisms. Then a **push forward** of  $f_1$  and  $f_2$  is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f_1} & X_1 \\ \downarrow f_2 & & \downarrow g_1 \\ X_2 & \xrightarrow{g_2} & Y \end{array}$$

such that for any commutative square

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y_1 \\ \downarrow f_2 & & \downarrow h_1 \\ Y_2 & \xrightarrow{\quad} & Y \end{array} \quad ;$$

$\begin{array}{ccc} & & Z \\ & \nearrow h_2 & \\ & & \end{array}$

there exists a unique  $h : Y \rightarrow Z$  such that  $h_1 = h \circ g_1$  and  $h_2 = h \circ g_2$ .

**Example 2.12** Let  $f : X \rightarrow Y$  be a morphism. The commutative square  $\text{Ker } f \rightarrow 0$  is a pull back.

$$\begin{array}{ccc} \text{Ker } f & \longrightarrow & 0 \\ \downarrow i & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

The commutative square  $X_1 \times X_2 \xrightarrow{p_1} X_1$  is a pull back.

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{p_1} & X_1 \\ \downarrow p_2 & & \downarrow \\ X_2 & \longrightarrow & 0 \end{array}$$

**Proposition 2.13**

There exist pull backs and push outs in the category  $\text{Mod } A$ .

*Proof :* The pull back of  $(g_1 : Y_1 \rightarrow Y, g_2 : Y_2 \rightarrow Y)$  is given by

$$X := \text{Ker}(g_1 - g_2 : Y_1 \oplus Y_2 \rightarrow Y).$$

The push-out of  $(f_1 : X \rightarrow X_1, f_2 : X \rightarrow X_2)$  is given by

$$Y = \text{Coker}((f_1, -f_2) : X \rightarrow X_1 \oplus X_2).$$

□

**Theorem 2.14**

Let  $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} 0$  be a short exact sequence.

1. For any  $z : Z' \rightarrow Z$ , there exists a commutative diagram where the horizontal lines are exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow z & & \\ 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \end{array}$$

2. For any  $x : X \rightarrow X'$ , there exists a commutative diagram where horizontal lines are exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow x & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

### 3 Short exact sequences and functors

#### 3.1 Exact functors

- Definition 3.1**
1. A covariant functor  $F$  is **left exact** if for any  $0 \rightarrow X \rightarrow Y \rightarrow Z$  the sequence  $0 \rightarrow FX \rightarrow FY \rightarrow FZ$  is exact
  2. A covariant functor is called **right exact** if for any exact sequence  $X \rightarrow Y \rightarrow Z \rightarrow 0$ , the sequence  $FX \rightarrow FY \rightarrow FZ \rightarrow 0$  is exact.
  3. A contravariant functor  $F$  is **left exact** if for any  $X \rightarrow Y \rightarrow Z \rightarrow 0$ , the sequence  $0 \rightarrow FZ \rightarrow FY \rightarrow FX$  is exact.
  4. A contravariant functor  $F$  is **right exact** if for any  $0 \rightarrow X \rightarrow Y \rightarrow Z$ , the sequence  $FZ \rightarrow FY \rightarrow FX \rightarrow 0$  is exact.
  5. A functor is called **exact** if it is both left and right exact. So it sends any short exact sequence to a short exact sequence.

**Theorem 3.2**

Let  $M$  be a  $A$ - $B$ -bimodule. We have the following:

1. the functors  $\text{Hom}_A(M, -)$  and  $\text{Hom}_A(-, M)$  are left exact;
2. the functors  $M \otimes_B -$  and  $- \otimes_A M$  are right exact.

*Proof:* Here we need to show a statement a bit more precise. We will show that a sequence  $0 \rightarrow X \rightarrow Y \rightarrow Z$  is exact if and only if for all  $M \in \text{Mod } A$  the sequence  $0 \rightarrow \text{Hom}(M, X) \rightarrow \text{Hom}(M, Y) \rightarrow \text{Hom}(M, Z)$  is exact, and the similar statement for the other functors.  $\square$

### 3.2 Projective, injective and flat modules

**Definition 3.3** An  $A$ -module  $P$  is said to be **projective** if the functor  $\text{Hom}_A(P, -)$  is exact.

An  $A$ -module  $I$  is said to be **injective** if  $\text{Hom}_A(-, I)$  is exact.

An  $A$ -module  $F$  is said to be **flat** if  $F \otimes_A -$  is exact.

The following is clear from the definition.

**Proposition 3.4**

1. A  $A$ -module  $P$  is projective if and only if for any epimorphism  $f : X \rightarrow Y$  and morphism  $u : P \rightarrow Y$ , there exists a morphism  $v : P \rightarrow X$  such that  $f \circ v = u$ .
2. A  $A$ -module  $I$  is injective if and only if for any monomorphism  $f : X \rightarrow Y$ , and any morphism  $u : X \rightarrow I$ , there exists a morphism  $v : Y \rightarrow I$  such that  $v \circ f = u$ .

**Lemma 3.5**

1. Let  $(M_i)_{i \in E}$  be a family of  $A$ -modules. Then  $\bigoplus_{i \in E} M_i$  is projective if and only if  $M_i$  is projective for any  $i \in E$ .

2. Let  $(M_i)_{i \in E}$  be a family of  $A$ -modules. Then  $\prod_{i \in E} M_i$  is injective if and only if  $M_i$  is injective for any  $i \in E$ .
3. Let  $(M_i)_{i \in E}$  be a family of  $A$ -modules. Then  $\bigoplus_{i \in E} M_i$  is flat if and only if  $M_i$  is flat for any  $i \in E$ .

### 3.3 Existence of projective and flat modules

**Theorem 3.6**

A module  $M$  is projective if and only if it is a direct summand of a free module.

*Proof* : The proof here comes from the fact that  $\text{Hom}_A(A, M) \simeq M$ , it is then clear that  $A$  is projective. Then by the previous lemma we clearly have that any free module is projective and so is any direct summand of a free module.

Now given a projective module  $P$ , we can take a free cover  $F \rightarrow P \rightarrow 0$  of  $P$ . Then since  $P$  is projective, the map  $F \rightarrow P$  is a retraction therefore  $P$  is isomorphic to a direct summand of  $F$ .

□

**Theorem 3.7**

Free  $\Rightarrow$  projective  $\Rightarrow$  flat.

*Proof* : This is an easy consequence of the previous lemma, and of the fact that  $A$  is flat. □

We will see later that for certain nice rings (Noetherian) finitely generated projective modules coincide with finitely generated flat modules.

## 3.4 Existence of injective modules

Case where  $k$  is a field

**Lemma 3.8**

If  $k$  is a field, then  $k$  is injective in  $\text{Mod } k$ .

*Proof* : This comes from the fact that all short exact sequences splits in  $\text{Mod } k$ .  $\square$

As a consequence, and using the natural embedding  $M \rightarrow M^{**}$  we obtain the following.

**Theorem 3.9**

Let  $k$  be a field and  $A$  be a  $k$ -algebra. Then we have

$M \in \text{Mod } A$  is projective  $\Rightarrow M^* = \text{Hom}_k(M, k) \in \text{Mod } A^{\text{op}}$  is injective.

As an immediate corollary, we obtain that  $A^*$  is naturally a left  $A$ -module injective.

Note that in the case of where  $A$  is finite dimensional, the  $k$ -duality induces a bijection between projective and injective objects in  $\text{mod } A$  (finite dimensional  $A$ -modules).

### Case of abelian groups

The general case is much more complicated. Already for  $A = \mathbb{Z}$  it is difficult to exhibit injective  $\mathbb{Z}$ -modules. For example, using the embedding  $\mathbb{Z} \rightarrow \mathbb{Q}$ , one can see that  $\mathbb{Z}$  is not an injective object.

However, the aim here is to prove that  $\mathbb{Q}$  is injective. To prove this, we will use the following criterion.

**Theorem 3.10 (Baer's criterion)**

Let  $A$  be a  $k$ -algebra. Then a  $A$ -module  $M$  is injective if and only if for any submodule  $J \subset A$ , the map  $\text{Hom}_A(A, M) \rightarrow \text{Hom}_A(J, M)$  is surjective.

Necessity is clear. The converse direction is more involved and uses Zorn lemma, we refer to Assem (Theorem 3.4 in Chapter IV) for a complete proof.

But this lemma implies easily the following:

**Proposition 3.11**

$\mathbb{Q}$  and  $\mathbb{Q}/\mathbb{Z}$  are injective as  $\mathbb{Z}$ -modules.

**General case**

This leads us to introduce an other notion of dual.

**Definition 3.12** Let  $M \in \text{Mod } A$ , we define  $M^\wedge := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \text{Mod } A^{\text{op}}$  the **Pontryagin dual** of  $M$ .

We have then the same kind of properties that for the  $k$ -dual.

**Lemma 3.13**

A map  $X \rightarrow Y$  in  $\text{Mod } A$  is injective if and only if the corresponding map  $Y^\wedge \rightarrow X^\wedge$  in  $\text{Mod } A^{\text{op}}$  is surjective.

**Theorem 3.14**

A right  $A$ -module  $X$  is flat if and only if the  $A$ -module  $X^\wedge$  is injective.

As a corollary, we then obtain that  $A^\wedge$  is an injective left  $A$ -module. It is unfortunately in general not finitely generated.



# Chapter III

## Decomposition theorems

### 1 Noetherian and Artinian

#### 1.1 Noetherian and Artinian modules

**Definition 1.1** 1. A  $A$ -module  $M$  is said to be **Artinian** if for any decreasing sequence  $M_0 \supseteq M_1 \supseteq \cdots$  of submodules there exists  $n$  such that  $M_j = M_n \forall j \geq n$ .

2. A  $A$ -module  $M$  is said to be **Noetherian** if for any increasing sequence  $M_0 \subseteq M_1 \subseteq \cdots$  of submodules there exists  $n$  such that  $M_j = M_n \forall j \geq n$ .

**Example 1.2** If  $k$  is a field, then any finite dimensional  $A$ -module is both Artinian and Noetherian.

$\mathbb{Z}$  or more generally any principal ring is Noetherian. But  $\mathbb{Z}$  is not Artinian. The ring  $\mathbb{Z}/n\mathbb{Z}$  is Artinian and Noetherian.

**Proposition 1.3**

Let  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$  be a short exact sequence of  $A$ -modules. Then we have

1.  $X$  and  $Z$  are Artinian if and only if so is  $Y$ .
2.  $X$  and  $Z$  are Noetherian if and only if so is  $Y$ .

*Proof:* If  $Y$  is Artinian, then so is  $X$  since it is a submodule of  $Y$ . If  $Z_0 \supseteq Z_1 \supseteq \cdots$  is a decreasing chain of submodules of  $Z$ , then  $p^{-1}(Z_0) \supseteq p^{-1}(Z_1) \supseteq$

$\dots$  is a decreasing chain of submodules of  $Y$ . So  $p^{-1}(Z_\ell) = p^{-1}(Z_{\ell+1})$  which implies  $Z_\ell = Z_{\ell+1}$ .

Conversely, let  $Y_0 \supseteq Y_1 \supseteq \dots$  be a decreasing chain of submodules of  $Y$ . Then we obtain the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X \cap Y_{\ell+1} & \longrightarrow & Y_{\ell+1} & \longrightarrow & Z \cap Y_{\ell+1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X \cap Y_\ell & \longrightarrow & Y_\ell & \longrightarrow & Z \cap Y_\ell \longrightarrow 0
 \end{array}$$

If both left and right maps are equalities, then so is the middle one.  $\square$

**Theorem 1.4**

Let  $M$  be a  $A$ -module. Then  $M$  is Noetherian if and only if every submodule of  $M$  is finitely generated.

*Proof:* Let  $N$  be a submodule of  $M$ , and consider the set  $\mathcal{E}$  of submodules of  $N$  which are finitely generated. It is non empty since  $\{0\}$  is finitely generated. Since  $M$  is Noetherian, any increasing chain has an upper bound. So by Zorn's lemma, it has a maximal element  $L$ . If  $L \neq N$ , then there exists  $x \in N \setminus L$ , and  $\langle L, x \rangle$  is a finitely generated submodule of  $N$ , which contradicts maximality. So  $N$  is maximal, and so  $N$  is finitely generated.

Conversely, let  $M_0 \subseteq M_1 \dots$  be an increasing chain of submodules in  $M$ . Then the union of the  $M_i$  is a submodule of  $M$ . It has a finite set of generators, so there exists  $n$  such that every generator is in  $M_n$ , and so the union of the  $M_i$  equals  $M_n$ .  $\square$

**Corollary 1.5**

If  $A$  is a Noetherian left module, then any finitely generated  $A$ -module is finitely presented.

## 1.2 Noetherian and Artinian algebras

- Definition 1.6**
1. An algebra  $A$  is called **left Artinian** if the module  ${}_A A$  is Artinian.
  2. An algebra  $A$  is called **left Noetherian** if the module  ${}_A A$  is Noetherian.

### Theorem 1.7

Let  $A$  be a  $k$ -algebra.

1. If  $A$  is left Artinian, then any  $A$ -module of finite type is Artinian.
2. If  $A$  is left Noetherian, then any  $A$ -module of finite type is Noetherian.

*Proof:* It follows directly from Proposition 1.3. □

### Corollary 1.8

If  $A$  is Artinian or Noetherian, then the category  $\text{mod } A$  of finitely generated  $A$ -modules is an Abelian category.

## 2 Indecomposable modules and algebras

### 2.1 Idempotents

**Definition 2.1** An algebra  $A$  is said to be **connected** if it is not isomorphic to the product of two non trivial algebras.

A  $A$ -module  $M$  is said to be **indecomposable** if it is not isomorphic to the direct sum of two proper submodules.

The key notion here is the notion of **idempotents**. Indeed, if  $A = \prod_i A_i$ , then denoting by  $e_i := (0, \dots, 1_{A_i}, 0, \dots)$  we have the following relations

$$e_i^2 = e_i, \quad e_i e_j = 0 \text{ for } i \neq j, \quad 1_A = \sum_i e_i \text{ and } e_i \in Z(A).$$

Moreover  $A_i \simeq e_i A e_i$ .

Similarly, let  $M = \bigoplus_i M_i$  be decomposable. Denote by  $p_j$  and  $i_j$  the projections and injections, and set  $e_j := p_j \circ i_j \in \text{End}_A(M)$ . Then we have

$$e_i^2 = e_i, \quad e_i e_j = 0 \text{ for } i \neq j, \quad \text{Id}_M = \sum_i e_i$$

So roughly speaking, an algebra will be connected if it has very few idempotent, and a module  $M$  will be indecomposable if its endomorphism algebra has also very few idempotents.

## 2.2 Algebras and bloc decomposition

The first result show that the conditions above on the idempotents is sufficient to decompose an algebra.

### **Theorem 2.2**

Let  $A$  be a  $k$  algebra. Assume that  $1_A = \sum_{i=1}^s e_i$  with the properties

$$e_i^2 = e_i, \quad e_i e_j = 0 \text{ for } i \neq j, \quad \text{and } e_i \in Z(A),$$

then  $A$  is isomorphic to  $\prod_{i=1}^s A_i$  with  $A_i = A e_i$ .

*Proof:* First note that  $A_i = A e_i = e_i A e_i$  is an algebra, and a tw-sided ideal of  $A$ .

The isomorphisms  $A \rightarrow \prod_{i=1}^s A_i$  and  $\prod_{i=1}^s A_i \rightarrow A$  are given by

$$a \mapsto (a e_1, \dots, a e_s) \quad \text{and} \quad (b_1, \dots, b_s) \mapsto \sum_i b_i.$$

One easily checks that these are isomorphisms of algebras and inverse one of each other.  $\square$

### **Theorem 2.3**

Let  $A$  be a  $k$ -algebra which is Noetherian or Artinian. Then  $A$  is isomorphic to a finite product of connected algebras which are uniquely determined.

*Proof*: The existence comes from Noetherianity or Artinianity.

Unicity is quite easy, using the fact that the product of two two-sided ideal is a two-sided ideal. So if we have

$$A_1 \times \cdots \times A_s = B_1 \times \cdots \times B_t,$$

then we have  $A_i = AA_i = \prod_j (B_j A_i)$  and since  $A_i$  is connected we obtain  $A_i = B_j A_i$ . Using then  $B_j = B_j A$ , we obtain  $A_i = B_j$ . Finally we use the fact that

$$\prod_{\ell \neq i} A_\ell = A/A_i = A/B_j = \prod_{k \neq j} B_k$$

and conclude by induction. □

### **Theorem 2.4**

Let  $A = A_1 \times A_2$  be the direct product of two  $k$  algebras. Then there is an equivalence of categories

$$\text{Mod } A \simeq \text{Mod } A_1 \times \text{Mod } A_2.$$

*Proof*: The functor is given by  $M \mapsto (e_1 M, e_2 M)$  where the  $e_i$  are the idempotents defined above. One then needs to show that for  $M, N \in \text{Mod } A$ , then

$$\text{Hom}_A(e_1 M, e_2 N) = 0 \quad \text{and} \quad \text{Hom}_A(e_1 M, e_1 N) = \text{Hom}_{A_1}(e_1 M, e_1 N).$$

□

## **2.3 Indecomposable modules and local rings**

**Definition 2.5** A  $k$ -algebra is said to be **local** if it has a unique maximal left ideal.

The link between these two notions is given by the Theorem below.

**Theorem 2.6**

A module  $M$  which is Artinian and Noetherian is indecomposable if and only if  $\text{End}_A(M)$  is local.

In order to prove this result, we need two lemmas.

**Lemma 2.7**

Let  $A$  be a  $k$ -algebra. Then  $A$  is local if and only if for any  $x \in A$ ,  $x$  or  $1 - x$  is invertible.

*Proof:* One direction is clear, since if both  $x$  and  $1 - x$  are non invertible, then they are both in the maximal ideal  $J$ , which implies  $J = A$ .

For the other direction, we prove that the set  $J$  of non left invertible elements of  $A$  is an ideal. It satisfies clearly  $AJ \subset J$ . Now let  $x$  and  $y$  be in  $J$  such that  $x - y$  has a left inverse  $a$ . Since  $ax$  is in  $J$ , then  $1 - ax = ay$  is invertible which is a contradiction. Finally  $J$  clearly contains all proper ideals of  $A$ , so  $A$  is local.  $\square$

**Lemma 2.8 (Fitting's lemma)**

Let  $f \in \text{End}_A(M)$  where  $M$  is Noetherian and Artinian, then there exists  $n \geq 1$  such that

$$M = \text{Ker } f^n \oplus \text{Im } f^n$$

*Proof:* By Artinianity and Noetherianity, there exists  $n$  such that  $\text{Ker } f^{n+1} = \text{Ker } f^n$  and  $\text{Im } f^{n+1} = \text{Im } f^n$ . One easily checks that  $\text{Ker } f^n \cap \text{Im } f^n = \{0\}$ . And if  $y \in M$ , taking  $x$  such that  $f^n(x) = f^{2n}(y)$  we obtain

$$y = (x - f^n(y)) + f^n(y) \in \text{Ker } f^n \oplus \text{Im } f^n.$$

$\square$

## 2.4 Krull-Schmidt decomposition

### Theorem 2.9 (Azumaya-Krull-Remak-Schmidt)

Let  $A$  be a  $k$ -algebra and let  $M \in \text{Mod } A$ . If  $M$  is Noetherian or Artinian, then  $M$  is isomorphic to a finite direct sum of indecomposable modules.

If  $M$  is both Artinian and Noetherian, then the decomposition is essentially unique.

*Proof* : The proof of existence is the same as for the bloc decomposition of algebras.

Assume that  $\bigoplus_{i=1}^m M_i = \bigoplus_{j=1}^n N_j$ . We proceed on induction on  $m$ . Using that  $\text{End}(M_1)$  is local, we obtain a  $j$  such that  $p_{M_1} \circ i_{N_j} \circ p_{N_j} \circ i_{M_1}$  is invertible. Then using the fact that  $\text{End}(N_j)$  is local, we prove that both  $p_{M_1} \circ i_{N_j}$  and  $p_{N_j} \circ i_{M_1}$  are invertible. So  $N_j$  is isomorphic to  $M_1$ . Finally we need to construct an isomorphism  $\varphi : M \rightarrow M$  such that there exists a commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{i_1} & M \\ p_{N_j} \circ i_{M_1} \downarrow & & \downarrow \varphi \\ N_j & \xrightarrow{i_j} & M \end{array}$$

And we apply the induction hypothesis.

□

### Corollary 2.10

If  $k$  is a field and if  $A$  is a  $k$ -algebra, then any finite dimensional  $A$ -module can be decomposed into a unique finite direct sum of indecomposable modules.

## 3 Simple and Semi-simple

### 3.1 Simple modules

**Definition 3.1** A  $A$ -module  $S \neq 0$  is called **simple** if it has no non zero proper submodule.

**Example 3.2** If  $k$  is a field, then any 1-dimensional  $A$ -module is simple. In  $\text{mod } k$ , there is only one simple module up to isomorphism  $k$ .

The  $\mathcal{M}_n(k)$ -module  $k^n$  is simple.

Any division  $k$ -algebra  $D$  over  $k$  is a simple  $D$ -module. So there might be infinite dimensional simple module (e.g.  $D = \mathbb{C}(X)$ ).

$\mathbb{Z}_n$  is simple if and only if  $n$  is prime.

For  $G = \mathbb{D}_4$  the dihedral group, consider the 2-dimensional representation given by

$$\begin{aligned} \rho: G &\longrightarrow \text{GL}_2(\mathbb{C}) \\ r &\longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ s &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

is simple.

We have the following characterization for simple modules.

#### **Lemma 3.3**

For a  $A$ -module  $S$  the following are equivalent:

1.  $S$  is simple;
2.  $\forall x \neq 0 \in S, Ax = S$ ;
3.  $\forall x \neq 0 \in S, S \simeq A/\text{Ann}(x)$ ;

#### **Lemma 3.4 (Schur's lemma)**

Let  $S$  be a simple  $A$ -module.

1. then the  $k$ -algebra  $\text{End}_A(S)$  is a division algebra.
2. if moreover  $k = \bar{k}$  is an algebraically closed field and  $S$  is finite dimensional, then  $\text{End}_A(S) \simeq k$ .



**Proposition 3.5**

Let  $k = \bar{k}$  be an algebraically closed field, and  $A$  be a commutative  $k$ -algebra. Then any finite dimensional simple  $A$ -module is 1-dimensional.

### 3.2 Composition series

**Definition 3.6** Let  $M$  be a non zero  $A$ -module. A sequence of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

is called a **composition series** for  $M$  if  $M_{i+1}/M_i$  is simple for any  $i$ . These quotients are called **composition factors**.

**Example 3.7**  $\mathbb{Z}_{p^m}$  has a unique composition series:

$$p^m \mathbb{Z}_{p^m} \subset p^{m-1} \mathbb{Z}_{p^m} \subset \cdots \subset \mathbb{Z}_{p^m}.$$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$  has three composition series.

Let  $M_\ell = k^\ell$  be the  $\mathcal{T}_n(k)$ -module given as in the exercises. Then  $M_n$  has a unique composition series

$$M_0 \subset M_1 \subset \cdots \subset M_n.$$

**Proposition 3.8**

A  $A$ -module has a composition series if and only if it is Artinian and Noetherian.

*Proof* : We use Artinianity to prove that any module has a simple submodule. We then construct a composition series inductively, that stops by Noetherianity.

The converse can be easily shown by induction on the length of the composition series using Proposition 1.3.  $\square$

**Theorem 3.9 (Jordan Hölder)**

If  $A$ -module  $M$  admits two composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M, \quad 0 = N_0 \subset M_1 \subset \cdots \subset N_n = M,$$

then  $m = n$ , and there exists a permutation  $\sigma \in \mathfrak{S}_m$  such that  $M_{\sigma(i)+1}/M_{\sigma(i)} \simeq N_{i+1}/N_i$ .

*Proof:* The proof is done by induction on  $m$ . The idea is to extract from

$$0 \subseteq M_{m-1} \cap N_1 \subseteq \cdots \subseteq M_{m-1} \subseteq \cdots \subseteq M_{m-1} + N_{n-1} \subseteq M$$

a composition series of  $M$  of the form

$$0 = M'_0 \subset M'_1 \subset \cdots \subset M'_{n-2} \subseteq M_{m-1} \subset M,$$

such that each quotient is of the form  $N_{i+1}/N_i$  for some  $i$ , and then apply induction.  $\square$

### 3.3 Semi-simple algebras

**Definition 3.10** A  $A$ -module is called **semi-simple** if it is the direct sum (possibly infinite) of simple modules.

A  $k$ -algebra is called **semi-simple** if it is semi-simple as a left module over itself.

**Theorem 3.11 (Maschke's theorem)**

Let  $G$  be a finite group and  $k$  be a field such that  $|G|$  is invertible in  $k$ . Then any finite dimensional  $kG$ -module is semi-simple. In particular  $kG$  is semi-simple.

*Proof:* If  $W \subset V$  is a submodule, the idea is first to construct a  $k$ -linear retraction  $V \rightarrow W$ , and to modify it, using the fact that  $|G|$  is invertible to obtain a  $kG$ -linear retraction.  $\square$

**Theorem 3.12 (Artin Wedderburn)**

A  $k$ -algebra  $A$  is semi-simple if and only if it is isomorphic to

$$\mathcal{M}_{n_1}(D_1) \times \dots \times \mathcal{M}_{n_s}(D_s)$$

where  $D_s$  are division  $k$ -algebras.

*Proof:* It is not hard to see that  $\mathcal{M}_n(D)$  is semi-simple since  $D^n$  is a simple  $\mathcal{M}_n(D)$ -module.

Conversely, decomposing  $A$  into a sum of simple, we first show that the sum is finite, since  $1_A$  is a finite sum of elements in this decomposition. Then we conclude using Schur's lemma 3.4, the fact that  $\text{End}_A(A) = A^{\text{op}}$ , and that

$$\text{End}_A(S^n) \simeq \mathcal{M}_n(\text{End}(S)) \text{ and } \text{Hom}_A(S, S') = 0 \text{ if } S \neq S'$$

□

**Corollary 3.13**

Let  $k = \bar{k}$  be an algebraically closed field. A finite dimensional  $k$ -algebra  $A$  is semi-simple if and only if it is isomorphic to

$$\mathcal{M}_{n_1}(k) \times \dots \times \mathcal{M}_{n_s}(k).$$

Moreover, there exist exactly  $s$  isomorphism classes of simple modules, of dimension  $n_1, \dots, n_s$  respectively.

### 3.4 The module category for a semi-simple algebra

**Proposition 3.14**

A module  $S$  is semi-simple if and only if every submodule of  $S$  is a direct summand of  $S$ .

**Theorem 3.15**

For an algebra  $A$ , the following are equivalent:

1.  $A$  is semi-simple;
2. every  $A$ -module is semi-simple;
3. every short exact sequence in  $\text{Mod } A$  splits;
4. every  $A$ -module is projective;
5. every  $A$ -module is injective.