Finitely generated modules

Exercise 1. Let M be a A-module. Assume that N is a submodule of M which is finitely generated and such that M/N is finitely generated. Show that M is finitely generated.

Exercise 2. The aim here is to show that any subgroup of a finitely generated subgroup is finitely generated without using the classification of finitely generated subgroups.

1. Show by induction on n that any subgroup of $\mathbb{Z}^n$ is finitely generated.
2. Let G be a finitely generated abelian group. Deduce that any subgroup of G is finitely generated.

Tensor product and induction

Exercise 3. Let H be a subgroup of a finite group G, and W a representation of H. Denote by $\{x_1, \ldots, x_\ell\}$ some representants of the classes G/H. We define $V := \mathrm{Ind}_H^G(W) = kG \otimes_{kH} W$, and $\rho_V : G \to \mathrm{GL}(V)$ the corresponding morphism.

1. Let the subspace $W_i := \mathrm{vect}(x_i \otimes w, w \in W)$ of V. Show that $V = \bigoplus_i W_i$. Deduce the dimension of V.
2. Let $g \in G$, and $1 \leq i \leq \ell$. Show that there exists $1 \leq j \leq \ell$ such that $\rho_V(g)(W_i) \subset W_j$.
3. Deduce how to construct the morphism ρ_V.
4. Let $G := \mathfrak{S}_3$, $H := \mathfrak{A}_3$ and W be the one-dimensional representation $\mathfrak{A}_3 \to \mathbb{C}^*$ sending the 3-cycle (123) to the primitive third root of unity j. Describe $V := \mathrm{Ind}_H^G(W) = kG \otimes_{kH} W$.

Categories and functors

Exercise 4. Let Q be a quiver, and kQ its path algebra. A representation V of Q is the data $V = \{(V_i, i \in Q_0), (v_\alpha, \alpha \in Q_1\}$, where V_i is a finite dimensional k-vector space, and $v_\alpha \in \mathrm{Hom}_k(V_{s(\alpha)}, V_{t(\alpha)})$ is a k-linear map.

For $V = (V_i, v_\alpha)$ and $W = (W_i, w_\alpha)$ two representations of Q, we define a morphism $\varphi : V \to W$ to be a collection of k-linear maps $\varphi_i : V_i \to W_i$ for any $i \in Q_0$ such that for any $a \in Q_1$, $\varphi_{t(a)} \circ v_a = w_a \circ \varphi_{s(a)}$.

1. Show that $\mathrm{Rep}_k(Q)$ is a k-linear category.
2. Let M be a finite dimensional kQ-module, and $\rho : kQ \to \mathrm{End}_k(M)$ the corresponding morphism. Denote by $M_i := e_i M$. Show that $M \simeq \oplus_{i \in Q_0} M_i$ as a k-vector space.

3. Let $\alpha : i \to j$ be an arrow of Q. Show that $\rho(\alpha)$ restricts to a k-linear map $M_i \to M_j$.

4. Show that $F : \operatorname{mod} kQ \to \operatorname{Rep}_k(Q)$ sending M to $(M_i, \rho(\alpha)_{|M_{s(\alpha)}})$ is an equivalence of categories.

5. Describe the inverse functor $G : \operatorname{Rep}_k(Q) \to \operatorname{mod} kQ$.

Now let Q be the quiver $Q : 1 \to 2$.

6. Let M be the module k^2 with left multiplication action of kQ using the isomorphism $kQ \simeq \mathcal{T}_2(k)$. Describe the corresponding representation.

7. Find two representations S and S' and a short exact sequence $0 \to S \to M \to S' \to 0$ which does not split.

8. Describe the representations associated to the modules kQ, kQe_1 and kQe_2.

9. Describe the representation associated to the module $(kQ)^*$.

10. Let V be a representation of Q. Show that V is isomorphic to a direct sum of copies of S, S' and M.

Exercise 5. Let $\varphi : A \to B$ be a morphism of k-algebras.

1. Show that the two functors $F : \operatorname{Hom}_B({}_BB_A, -) : \operatorname{Mod} B \to \operatorname{Mod} A$ and ${}_AB \otimes_B - : \operatorname{Mod} B \to \operatorname{Mod} A$ are isomorphic.

2. Show that the two functors $G : \operatorname{Hom}_A({}_AB_B, -) : \operatorname{Mod} A \to \operatorname{Mod} B$ and ${}_BB \otimes_A - : \operatorname{Mod} A \to \operatorname{Mod} B$ are isomorphic.

3. Show that the composition $F \circ G : \operatorname{Mod} A \to \operatorname{Mod} A$ is isomorphic to $\operatorname{Id}_{\operatorname{Mod} A}$.

Exercise 6. Let $F : \mathcal{C} \to \mathcal{D}$ be a k-linear (covariant) functor between abelian categories. Show that F sends a split short exact sequence on a split short exact sequence.

Exact sequences

Exercise 7 (Snake lemma)**.** We consider the following diagram in $\operatorname{Mod} A$, where the lines are exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \longrightarrow & 0 \end{array}$$

Show that it induces an exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Ker} h \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker} g \longrightarrow \operatorname{Coker} h \longrightarrow 0 .$$

Exercise 8. In an abelian category, we consider the following diagram where lines are exact.

$$\begin{array}{ccccccccc} X_1 & \xrightarrow{u_1} & X_2 & \xrightarrow{u_2} & X_3 & \xrightarrow{u_3} & X_4 & \xrightarrow{u_4} & X_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ Y_1 & \xrightarrow{v_1} & Y_2 & \xrightarrow{v_2} & Y_3 & \xrightarrow{v_3} & Y_4 & \xrightarrow{v_4} & Y_5 \end{array}$$

1. Show that if f_5 is a monomorphism, if f_2 and f_4 are epimorphism, then f_3 is an epimorphism.

2. Show that if f_1 is an epimorphism, if f_2 and f_4 are monomorphism, then f_3 is a monomorphism.

Exercise 9. Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be a short exact sequence. Show that it splits if and only if for any M the sequence

$$0 \longrightarrow \mathrm{Hom}(M, X) \longrightarrow \mathrm{Hom}(M, Y) \longrightarrow \mathrm{Hom}(M, Z) \longrightarrow 0$$

is exact.

Exercise 10. We consider the following diagram in an abelian category, where the lines are exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \longrightarrow & 0 \end{array}$$

1. Show that if f is a retraction and g a section, then h is a section.

2. State and show the dual statement.

Projective-injective-flat

Exercise 11. Let A be a commutative k-algebra. Show that if P and P' are projective A-module, then so is $P \otimes_A P'$.

Exercise 12. Let I be an injective A-module, and $A \to B$ an algebra morphism. Show that $\mathrm{Hom}_A(B, I)$ is an injective B-module.

Exercise 13.

1. Show that $X \in \mathrm{Mod}\, A^{\mathrm{op}}$ is flat if and only if for any $J \subset A$, the natural morphism $X \otimes_A J \to XJ$ is an isomorphism.

2. Deduce that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module.