HOMOLOGICAL ALGEBRA
Fall — 2020–2021

Exercise sheet 1: More on complexes

Do exercises 1.26, 1.28, 1.30, 1.31, 1.32, 1.34, 1.38, 1.39, 1.42, 1.43, 1.45 and 1.46 of the lectures. We also propose the following complementary exercises.

Complexes coming from differential geometry (from the point of view of an algebraist)

1. Given a commutative $\mathbb{R}$-algebra A and an A-module N, prove that

$$\mathrm{Der}_{\mathbb{R}}(A,N) = \left\{ d : A \to M : \begin{array}{c} d \text{ is } \mathbb{R}\text{-linear and} \\ d(fg) = d(f)g + f\,d(g) \text{ for all } f,g \in A \end{array} \right\}$$

is an A-module via $(f\,d)(g) = f\,d(g)$, for all $f,g \in A$ and $d \in \mathrm{Der}_{\mathbb{R}}(A,N)$. Moreover, prove that if $d,d' \in \mathrm{Der}_{\mathbb{R}}(A) = \mathrm{Der}_{\mathbb{R}}(A,A)$, then $[d,d'] = d \circ d' - d' \circ d \in \mathrm{Der}_{\mathbb{R}}(A)$.

2. Given a commutative $\mathbb{R}$-algebra A and a projective A-module of finite type P, define

$$T_A P = \bigoplus_{n \in \mathbb{N}_0} P^{\otimes_A n},$$

where $P^{\otimes_A 0} = A$.

(*a*) Prove that $T_A P$ is an A-algebra for the product induced by

$$(p_1 \otimes \cdots \otimes p_n) \cdot (p_{n+1} \otimes \cdots \otimes p_{n+m}) = p_1 \otimes \cdots \otimes p_{n+m},$$

for all $p_1, \dots, p_{n+m} \in P$, $n,m \in \mathbb{N}$, called the **tensor algebra**.

(*b*) Define the **exterior algebra** given as the quotient algebra

$$\Lambda_A P = T_A P / \langle \{ p \otimes p : p \in P \} \rangle,$$

whose product is typically denoted by $\wedge$. Prove that the $\mathbb{N}_0$-grading of $T_A P$ induces an $\mathbb{N}_0$-grading on $\Lambda_A P$, whose n-th homogeneous component is denoted by $\Lambda_A^n P$, and that there exists $r \in \mathbb{N}$ such that $\Lambda_A^n P = 0$ for all $n > r$.

3. Let M be a smooth manifold of dimension d and let $A = C^{\infty}(M,\mathbb{R})$ be the $\mathbb{R}$-algebra of smooth functions on M. Then $\mathrm{Der}_{\mathbb{R}}(A)$ is called the **module of vector fields** on M, and it is denoted by $\mathfrak{X}(M)$. By the Serre-Swan theorem[1], $\mathfrak{X}(M)$ is a projective A-module of finite type, which implies that $\Omega^1(M) = \mathrm{Hom}_A(\mathfrak{X}(M), A)$ is also a projective A-module of finite type. Define the $\mathbb{R}$-linear map

$$d : A \to \Omega^1(M) = \mathrm{Hom}_A\big(\mathfrak{X}(M), A\big) \tag{1}$$

given by $d(f)(X) = X(f)$, for all $f \in A$ and $X \in \mathfrak{X}(M)$. Prove that d is well defined and that $d \in \mathrm{Der}_{\mathbb{R}}(A, \Omega^1(M))$.

1. See Thm. 11.32 in Nestruev, J. *Smooth manifolds and observables*, Graduate Texts in Mathematics, 220. Springer-Verlag, New York, 2003. xiv+222 pp.

4. Assume the hypotheses as in the previous item. Let $\Omega(M) = \Lambda_A \Omega^1(M)$, and $\Omega^p(M) = \Lambda_A^p \Omega^1(M)$ for $p \in \mathbb{N}_0$. It is a $\mathbb{N}_0$-graded A-module by Exercise **2**.

(a) Prove that the morphism of A-modules

$$\phi : \Lambda_A^p \Omega^1(M) \to \mathsf{Hom}_A\left(\Lambda_A^p \mathfrak{X}(M), A\right) \tag{2}$$

given by

$$\phi(\omega_1 \wedge \cdots \wedge \omega_p)(X_1 \wedge \cdots \wedge X_p) = \det\left(\omega_i(X_j)\right)_{i,j \in [\![1,p]\!]}$$

for all $\omega_1, \dots, \omega_p \in \Omega^1(M)$ and $X_1, \dots, X_p \in \mathfrak{X}(M)$, is an isomorphism.

(b) For $p \in \mathbb{N}_0$, define the $\mathbb{R}$-linear map

$$\mathbf{d} : \Omega^p(M) \to \Omega^{p+1}(M) \tag{3}$$

given by

$$\begin{aligned}
&\phi\big(\mathbf{d}(\omega_1 \wedge \cdots \wedge \omega_p)\big)(X_1 \wedge \cdots \wedge X_{p+1}) \\
&= \sum_{i=1}^{p} (-1)^{i+1} d\Big(\phi(\omega_1 \wedge \cdots \wedge \omega_p)(X_1 \wedge \cdots \wedge \hat{X}_i \wedge \dots X_{p+1})\Big)(X_i) \\
&+ \sum_{i<j}^{p} (-1)^{i+j} \phi(\omega_1 \wedge \cdots \wedge \omega_p)([X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_{p+1}),
\end{aligned}$$

for all $\omega_1, \dots, \omega_p \in \Omega^1(M)$ and $X_1, \dots, X_{p+1} \in \mathfrak{X}(M)$, where the hat indicates that the corresponding element is omitted. We also denote by $\mathbf{d}$ the unique $\mathbb{R}$-linear map $\mathbf{d} : \Omega(M) \to \Omega(M)$ whose restriction to $\Omega^p(M)$ is the map (3). Prove that

(i) $\mathbf{d}(\omega \wedge \omega') = \mathbf{d}(\omega) \wedge \omega' + (-1)^p \omega \wedge \mathbf{d}(\omega')$ for all elements $\omega \in \Omega^p(M)$ and $\omega' \in \Omega^{p'}(M)$,

(ii) $\mathbf{d}|_{\Omega^0(M)}$ coincides with d defined in (1).

(c) Prove that $\mathbf{d} \circ \mathbf{d} = 0$.

The complex

$$0 \longrightarrow C^\infty(M, \mathbb{R}) \xrightarrow{\mathbf{d}} \Omega^1(M) \xrightarrow{\mathbf{d}} \Omega^2(M) \xrightarrow{\mathbf{d}} \dots \tag{4}$$

is called the **de Rham complex** of M and its cohomology $H_{\mathsf{dR}}(M, \mathbb{R})$ is called the **de Rham cohomology** of M.

5. Assume the hypotheses and terminology of the two previous exercises and let $M = \mathbb{R}^3$.

(a) Prove the isomorphism $\mathfrak{X}(M) \simeq A^3$ of A-modules, which implies the isomorphism $\Omega^1(M) \simeq A^3$. Deduce the isomorphisms $\Omega^2(M) \simeq A^3$, $\Omega^3(M) \simeq A$ and $\Omega^p(M) = 0$ for all integers $p > 3$.

(b) Prove that the de Rham complex of M is isomorphic to the complex

$$0 \longrightarrow A \xrightarrow{\mathsf{grad}} A^3 \xrightarrow{\mathsf{rot}} A^3 \xrightarrow{\mathsf{div}} A \longrightarrow 0 \tag{5}$$

where grad denotes the gradient, rot is the rotor and div is the divergence. Compute its cohomology.

Complexes coming from algebraic topology

6. Let Δ be the category whose objects are given by all the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$, for all $n \in \mathbb{N}_0$ and whose morphisms are the nondecreasing monotone functions. Given $n \in \mathbb{N}$ and $i \in [\![0,n]\!]$, let $\varepsilon_{i,n} : [n-1] \to [n]$ be the unique injective map in Δ such that $i \notin \mathsf{Img}(\varepsilon_{i,n})$, and given $n \in \mathbb{N}_0$ and $i \in [\![0,n]\!]$, let $\eta_{i,n} : [n+1] \to [n]$ be the unique surjective map in Δ such that $\#(\eta_{i,n}^{-1}(i)) = 2$. More explicitly,

$$\varepsilon_{i,n}(j) = \begin{cases} j, & \text{if } 0 \le j < i, \\ j+1, & \text{if } i \le j \le n-1, \end{cases} \quad \text{and } \eta_{i,n}(j) = \begin{cases} j, & \text{if } 0 \le j \le i, \\ j-1, & \text{if } i < j \le n+1. \end{cases}$$

(*a*) Prove that

$$\begin{aligned} \varepsilon_{j,n+1} \circ \varepsilon_{i,n} &= \varepsilon_{i,n+1} \circ \varepsilon_{j-1,n}, && \text{if } 0 \le i < j \le n+1, \\ \eta_{j,n-1} \circ \eta_{i,n} &= \eta_{i,n-1} \circ \eta_{j+1,n}, && \text{if } 0 \le i \le j \le n-1, \\ \eta_{j,n} \circ \varepsilon_{i,n+1} &= \begin{cases} \varepsilon_{i,n} \circ \eta_{j-1,n-1}, & \text{if } 0 \le i < j \le n, \\ \mathsf{id}_{[n]}, & \text{if } i = j \text{ or } i = j+1, \\ \varepsilon_{i-1,n} \circ \eta_{j,n-1}, & \text{if } 1 \le j+1 < i \le n+1. \end{cases} \end{aligned}$$

(*b*) Prove that any morphism $f : [n] \to [m]$ has a unique decomposition $f = \varepsilon \circ \eta$, where ε is an injective map in Δ, η is a surjective map in Δ, and they are uniquely given as compositions of the form

$$\varepsilon = \varepsilon_{i_1,m} \circ \cdots \circ \varepsilon_{i_s,n-r+1}$$

with $0 \le i_s < \cdots < i_1 \le m$ and

$$\eta = \eta_{j_1,n-r} \circ \cdots \circ \eta_{j_r,n-1}$$

with $0 \le j_1 < \cdots < j_r < n$.
Hint : let $\{i_s < \cdots < i_1\}$ be the complement of the image of f in $[m]$ and $\{j_1 < \cdots < j_r\}$ the subset of $[n]$ given by all elements $j \in [\![0, n-1]\!]$ satisfying that $f(j) = f(j+1)$.

Definition 1 *A* ***simplicial object*** *in a category $\mathscr{C}$ is a functor $S : \Delta^{\mathsf{op}} \to \mathscr{C}$, whereas a* ***cosimplicial object*** *is a functor $S : \Delta \to \mathscr{C}$. A (resp., co)simplicial object in the category of sets is called a (resp., co)simplicial set, a (resp., co)simplicial object in the category of topological spaces is a (resp., co)simplicial topological space, a (resp., co)simplicial object in the category of modules over a ring is a (resp., co)simplicial module, etc. We denote by $\mathsf{Fun}(\Delta^{\mathsf{op}}, \mathscr{C})$ (resp., $\mathsf{Fun}(\Delta, \mathscr{C})$) the category of (resp., co)simplicial objects in $\mathscr{C}$.*

7. Prove that a collection $\{M_n\}_{n\in\mathbb{N}_0}$ of objects in a category $\mathscr{C}$ together with morphisms $\partial_{i,n} : M_n \to M_{n-1}$ for $n \in \mathbb{N}$ and $i \in [\![0,n]\!]$, and morphisms $\sigma_{i,n} : M_n \to M_{n+1}$ for $n \in \mathbb{N}_0$ and $i \in [\![0,n]\!]$, such that

$$\begin{aligned} \partial_{i,n} \circ \partial_{j,n+1} &= \partial_{j-1,n} \circ \partial_{i,n+1}, && \text{if } 0 \le i < j \le n+1, \\ \sigma_{i,n} \circ \sigma_{j,n-1} &= \sigma_{j+1,n} \circ \sigma_{i,n-1}, && \text{if } 0 \le i \le j \le n-1, \\ \partial_{i,n+1} \circ \sigma_{j,n} &= \begin{cases} \sigma_{j-1,n-1} \circ \partial_{i,n}, & \text{if } 0 \le i < j \le n, \\ \mathsf{id}_{M_n}, & \text{if } i = j \text{ or } i = j+1, \\ \sigma_{j,n-1} \circ \partial_{i-1,n}, & \text{if } 1 \le j+1 < i \le n+1. \end{cases} \end{aligned} \tag{6}$$

uniquely determines a functor $M : \Delta^{\mathrm{op}} \to \mathscr{C}$ such that $M([n]) = M_n$ for $n \in \mathbb{N}_0$, $M(\varepsilon_{i,n}) = \partial_{i,n}$ for $n \in \mathbb{N}$ and $i \in [\![0,n]\!]$, and $M(\eta_{i,n}) = \sigma_{i,n}$ for $n \in \mathbb{N}_0$ and $i \in [\![0,n]\!]$.

Definition 2 *A **semisimplicial object** in a category $\mathscr{C}$ is a collection $\{M_n\}_{n\in\mathbb{N}_0}$ of objects in $\mathscr{C}$ together with morphisms $\partial_{i,n} : M_n \to M_{n-1}$ for $n \in \mathbb{N}$ and $i \in [\![0,n]\!]$, satisfying the first identity of* (6)*. The reader can formulate the analogous definition of **semicosimplicial object**.*

8. Let $\{M_n\}_{n\in\mathbb{N}_0}$ be a semisimplicial object in ${}_A\mathsf{Mod}$, for some ring A, for morphisms $\partial_{i,n} : M_n \to M_{n-1}$ for $n \in \mathbb{N}$ and $i \in [\![0,n]\!]$. Define $d_n : M_n \to M_{n-1}$ as $\sum_{i=0}^{n}(-1)^i \partial_{i,n}$. Prove that $d_{n-1} \circ d_n = 0$ for all $n \in \mathbb{N}$. This complex (M,d) is called the **unnormalized chain complex** associated with the semisimplicial object and its homology is called the **simplicial homology** of the semisimplicial object.

9. Let

$$|-| : \Delta \to \mathsf{Top}$$

be the assignement sending the standard simplex

$$|[n]| = \left\{(t_0,\dots,t_n) \in \mathbb{R}^{n+1}_{\geq 0} : \sum_{i=0}^{n} t_i = 1\right\}$$

to every $[n]$, for $n \in \mathbb{N}_0$, and, given a map $f : [n] \to [m]$ in Δ, let $|f| : |[n]| \to |[m]|$ be the map sending $(t_0,\dots,t_n)$ to the $(m+1)$-tuple $(t'_0,\dots,t'_m)$ such that

$$t'_j = \sum_{i\in f^{-1}(\{j\})} t_i,$$

for all $j \in [\![0,m]\!]$. Prove that this defines a functor and thus a cosimplicial topological space.

10. A **combinatorial simplicial complex** is a totally ordered finite set $(V,\preceq)$ together with a collection K of nonempty subsets of V satisfying that, if $\sigma \subseteq \tau$ are nonempty subsets of V and $\tau \in K$, then $\sigma \in K$. Given $n \in \mathbb{N}_0$ define

$$SS_n(K) = \left\{(v_0,\dots,v_n) \in K^{n+1} : v_0 \preceq \cdots \preceq v_n\right\}.$$

Given $f : [n] \to [m]$ in Δ, define $SS_m(K) \to SS_n(K)$ as the unique map sending $(v_0,\dots,v_m)$ to $(v_{f(0)},\dots,v_{f(n)})$. Prove that this defines a functor

$$SS(K) : \Delta^{\mathrm{op}} \to \mathsf{Set}$$

and thus a simplicial set.

11. Given a topological space and $n \in \mathbb{N}_0$, define $S_n(X)$ to be the simplicial set $\mathsf{Hom}_{\mathsf{Top}}(|[n]|, X)$ and given a continuous map $f : X \to Y$, let $S_n(f) : S_n(X) \to S_n(Y)$ send $\sigma \in \mathsf{Hom}_{\mathsf{Top}}(|[n]|, X)$ to $f \circ \sigma$. Prove that this defines a functor

$$S : \mathsf{Top} \to \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set}).$$

Moreover, if A is a ring, the previous functor induces a functor

$$S(-,A) : \mathsf{Top} \to \mathsf{Fun}(\Delta^{\mathrm{op}}, {}_A\mathsf{Mod})$$

where $S_n(X,A)$ is the free A-module generated by $S_n(X)$ and $S_n(f,A)$ is the unique morphism of A-modules extending $S_n(f)$. The unnormalized complex associated with the simplicial module $S(X,A)$ is called the **complex of singular chains** of X, and its homology is called the **singular homology** of X.