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Representation theory and homological algebra

Fall term — 2020-2021

Final exam

8pt **1.** Let A be a ring.

(*a*) Given a left A-module M, prove that the following are equivalent.

(P.1) M is **projective**;

(P.2) $\operatorname{Ext}^n_A(M,N)=0$ for all left A-modules N and all $n\in\mathbb{N}$;

(P.3) $\operatorname{Ext}^1_A(M,N)=0$ for all left A-modules N.

(*b*) Given a left A-module N, prove that the following are equivalent

(I.1) N is **injective**;

(I.2) $\operatorname{Ext}^n_A(M,N)=0$ for all left A-modules M and all $n\in\mathbb{N}$;

(I.3) $\operatorname{Ext}^1_A(M,N)=0$ for all left A-modules M.

(*c*) Recall that a left A-module N is **divisible** if given $a\in A$ nonzero, the map $N\to N$ given by left multiplication by a is surjective. Prove that a quotient of a divisible module is divisible.

(*d*) Assume that A is a PID and recall that in this case a module is injective if and only if it is divisible. Prove that a submodule of a projective module is projective.

Solution.

(*a*) We will first prove that (P.1) implies (P.2). Indeed, if M is projective, then a projective resolution $P_\bullet$ of M is given by $P_0=M$, $P_n=0$ and $d_n:P_n\to P_{n-1}$ given by the zero map for all $n\in\mathbb{N}$, together with the augmentation $\epsilon:P_0\to M$ given by the identity of M. Then $H^n(\operatorname{Hom}_A(P_\bullet,N))=0$ for all $n\in\mathbb{N}$, as was to be shown.

The fact that (P.2) implies (P.3) is trivial, so it remains to prove that (P.3) implies (P.1). We recall that M is projective if an only if given any short exact sequence

$$0\longrightarrow N'\xrightarrow{f} N\xrightarrow{g} N''\longrightarrow 0$$

of left A-modules, then

$$0 \longrightarrow \mathrm{Hom}_A(M,N') \xrightarrow{f_*} \mathrm{Hom}_A(M,N) \xrightarrow{g_*} \mathrm{Hom}_A(M,N'') \longrightarrow 0$$

is also exact. Since $\{\mathrm{Ext}_A^\bullet(M,-)\}_{\bullet\in\mathbb{N}_0}$ for a δ-functor, we have the long exact sequence

$$0 \to \mathrm{Hom}_A(M,N') \overset{f_*}{\to} \mathrm{Hom}_A(M,N) \overset{g_*}{\to} \mathrm{Hom}_A(M,N'') \overset{\delta^0}{\to} \mathrm{Ext}_A^1(M,N') \to \dots$$

the result follows from the fact that $\mathrm{Ext}_A^1(M,N')$ vanishes.

(b) We will first prove that (I.1) implies (I.2). Indeed, if N is injective, then a injective resolution $I^\bullet$ of N is given by $I^0 = N$, $I^n = 0$ and $d^n : I^n \to I^{n+1}$ given by the zero map for all $n \in \mathbb{N}_0$, together with the coaugmentation $\eta : N \to I^0$ given by the identity of N. Then $H^n(\mathrm{Hom}_A(M,I^\bullet)) = 0$ for all $n \in \mathbb{N}$, as was to be shown.

The fact that (I.2) implies (I.3) is trivial, so it remains to prove that (I.3) implies (I.1). We recall that N is injective if an only if given any short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

of left A-modules, then

$$0 \longrightarrow \mathrm{Hom}_A(M'',N) \xrightarrow{f_*} \mathrm{Hom}_A(M,N) \xrightarrow{g_*} \mathrm{Hom}_A(M',N) \longrightarrow 0$$

is also exact. Since $\{\mathrm{Ext}_A^\bullet(-,N)\}_{\bullet\in\mathbb{N}_0}$ for a δ-functor, we have the long exact sequence

$$0 \to \mathrm{Hom}_A(M'',N) \overset{g^*}{\to} \mathrm{Hom}_A(M,N) \overset{f^*}{\to} \mathrm{Hom}_A(M',N) \overset{\delta^0}{\to} \mathrm{Ext}_A^1(M'',N) \to \dots$$

the result follows from the fact that $\mathrm{Ext}_A^1(M'',N)$ vanishes.

(c) Let N/N' be a quotient of a divisible A-module N by a submodule $N' \subseteq N$. Let $a \in A$ be nonzero and $\bar{n} \in N/N'$ be the class of $n \in N$. Since N is divisible, we see that there exists $m \in N$ such that $n = am$, which in turn implies that $\bar{n} = a\bar{m}$ in N/N', so N/N' is divisible.

(d) Let M be a projective module and $M' \subseteq M$ be a submodule. We have thus the short exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \tag{1}$$

where $M'' = M/M'$. It suffices to prove that $\mathrm{Ext}^1_A(M',N)$ for all A-modules N. Since the quotient of a divisible module is divisible, we see that any module N has an injective resolution of the form

$$0 \to N \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \to 0$$

Indeed, if $\eta : N \to I^0$ is any injection with I^0 injective, *i.e.* divisible, then $I^1 = I^0/\mathrm{Img}(\eta)$ is divisible, so injective. In particular, $\mathrm{Ext}^2_A(X,N) = H^2(\mathrm{Hom}_A(X,I^\bullet)) = 0$ for any left A-module X. If we apply the δ-functor $\{\mathrm{Ext}^\bullet_A(-,N)\}_{\bullet\in\mathbb{N}_0}$ to (1), we obtain the long exact sequence

$$0 \to \mathrm{Hom}_A(M'',N) \xrightarrow{g^*} \mathrm{Hom}_A(M,N) \xrightarrow{f^*} \mathrm{Hom}_A(M',N) \to \mathrm{Ext}^1_A(M'',N) \longrightarrow \mathrm{Ext}^1_A(M,N) \longrightarrow \mathrm{Ext}^1_A(M',N) \longrightarrow 0$$

where we have used that $\mathrm{Ext}^2_A(X,N) = 0$ for all A-modules X. Since $\mathrm{Ext}^1_A(M,N) = 0$, because M is projective, we conclude that $\mathrm{Ext}^1_A(M',N) = 0$. By the first item of this exercise we conclude that M' is projective, as was to be shown.

15pt **2.** *Koszul complex.* Let A be a k-algebra, where k is a commutative ring with unit. We denote by ${}_A\mathrm{DGMod}_A$ the category of complexes of A-bimodules, and recall that A with the usual left and right products by elements of A is the so-called **regular** bimodule.

(*a*) Given $x \in \mathscr{Z}(A)$, let $K(x) \in {}_A\mathrm{DGMod}_A$ be the complex of A-bimodules defined $K(x)_i = A$ for $i \in \{0,1\}$, $K(x)_i = 0$ for $i \in \mathbb{Z} \setminus \{0,1\}$ and with differential ∂^x such that $\partial^x_1 : K(x)_1 \to K(x)_0$ is the multiplication by x. Verify that this indeed defines a complex of A-bimodules. From now on, we will denote by e_x the unit 1_A of $K(x)_1 = A$, so $\partial^x(e_x) = x$.

(*b*) Given a finite sequence $\bar{x} = (x_1,\dots,x_n) \in \mathscr{Z}(A)^n$ of central elements, define the graded A-bimodule $K(\bar{x})$ by

$$K(\bar{x})_i = \bigoplus_{\substack{(i_1,\dots,i_n)\in\mathbb{Z}^n \\ i_1+\dots+i_n=i}} K(x_1)_{i_1} \otimes_A \cdots \otimes_A K(x_n)_{i_n}.$$

Prove that $K(\bar{x})_i = 0$ if $i \in \mathbb{Z} \setminus [\![0,n]\!]$.

(*c*) Given $I \subseteq [\![1,n]\!]$ with $\#(I) \in [\![0,n]\!]$, denote by $e_I \in K(\bar{x})_{\#(I)}$ the element

$$e_I = E_1 \otimes_A \cdots \otimes_A E_n, \tag{2}$$

where $E_i = e_{x_i} \in K(x_i)_1$ if $i \in I$ and $E_i = 1_A \in K(x_i)_0$ if $i \notin I$. Prove that the elements (2) induce isomorphisms

$$K(\bar{x})_i \cong \bigoplus_{\substack{I \subseteq [\![1,n]\!] \\ \#(I) = i}} A,$$

of A-bimodules for $i \in [\![0,n]\!]$, where A denotes the regular A-bimodule.

(*d*) Define the morphism of graded A-bimodules $d : K(\bar{x}) \to K(\bar{x})[-1]$ given by

$$d(\alpha_1 \otimes_A \cdots \otimes_A \alpha_n) = \sum_{j=1}^{n} (-1)^{\epsilon_j} \alpha_1 \otimes_A \cdots \otimes_A \alpha_{j-1} \otimes_A \partial^{x_j}(\alpha_j) \otimes_A \alpha_{j+1} \otimes_A \cdots \otimes_A \alpha_n,$$

where $\epsilon_j = |\alpha_1| + \cdots + |\alpha_{j-1}|$. Verify that this indeed defines a complex of A-bimodules on $K(\bar{x})$.

(*e*) Prove that $K(x_1, x_2)$ is isomorphic to the complex

$$0 \longrightarrow A \xrightarrow{d_2} A^2 \xrightarrow{d_1} A \longrightarrow 0$$

where $d_2(a) = (-x_2 a, x_1 a)$ and $d_1(a,b) = a x_1 + b x_2$, for $a, b \in A$.

(*f*) If $I = \{i_1 < \cdots < i_\ell\}$ with $\ell \in [\![1,n]\!]$, prove that

$$d(e_I) = \sum_{j=1}^{\ell} (-1)^{j-1} x_{i_j} e_{I \setminus \{i_j\}}.$$

(*g*) Given left A-modules M, N and a morphism $f : M \to N$ of left A-modules, define the k-module $H_n(\bar{x}, M) = H_n(K(\bar{x}) \otimes_A M, d \otimes_A \mathrm{id}_M)$ and the k-linear map $H_n(\bar{x}, f) : H_n(\bar{x}, M) \to H_n(\bar{x}, N)$ as $H_n(\mathrm{id}_{K(\bar{x})} \otimes_A f)$. Prove that this defines an additive functor $H_n(\bar{x}, -) : {}_A\mathsf{Mod} \to {}_k\mathsf{Mod}$ for all $n \in \mathbb{N}_0$. Moreover, prove that $H_0(\bar{x}, M) = M/(x_1, \dots, x_n).M$

(*h*) Prove that the family $\{H_n(\bar{x}, -)\}_{n \in \mathbb{N}_0}$ forms a homological δ-functor.

[bonus] (*i*) Let (C, d_C) be a complex of left A-modules and $x \in \mathscr{Z}(A)$. Prove that there is a short exact sequence

$$0 \to H_0\big(x, H_n(C, d_C)\big) \to H_n\big(\mathsf{Tot}^{\oplus}(K(x) \otimes_A C)\big) \to H_1\big(x, H_{n-1}(C, d_C)\big) \to 0$$

of k-modules for all $n \in \mathbb{Z}$.

Hint : Tensor the complex $0 \to A \to K(x) \to A[-1] \to 0$ with C over A.

[bonus] (*j*) We say that $\bar{x} = (x_1, \dots, x_n) \in \mathscr{Z}(A)^n$ is a **regular sequence** on a left A-module M if left multiplication by x_i on $M/(x_{i+1}, \dots, x_n).M$ is injective for all $i \in [\![1, n]\!]$. Prove that $H_i(\bar{x}, M) = 0$ for all $i \in \mathbb{N}$ if $\bar{x}$ is a regular sequence on M.

[bonus] (*k*) Let $\bar{x} = (x_1, \dots, x_n) \in \mathscr{Z}(A)^n$ be a regular sequence on A, and $I \subseteq A$ be the right ideal generated by $\{x_1, \dots, x_n\}$. Construct a free resolution of A/I in the category right A-modules and prove that

$$H_i(\bar{x}, M) \cong \mathrm{Tor}_i^A(A/I, M),$$

for all left A-modules M and all $i \in \mathbb{N}_0$.

[bonus] (*l*) Let $A = k[X_1, \dots, X_n]$ be the polynomial ring. Prove that $\bar{x} = (X_1, \dots, X_n)$ is a regular sequence and compute $\mathrm{Tor}_i^A(k, k)$ for $i \in \mathbb{N}_0$.

Solution.

(*a*) It is clear that Since $\partial_i^x = 0$ for all $i \in \mathbb{Z} \setminus \{1\}$, we see that $\partial_i^x \circ \partial_{i+1}^x = 0$, for all $i \in \mathbb{Z}$. Moreover, it is clear that $\partial_1^x : A \to A$ is a morphism of A-bimodules, since x is central.

(*b*) Since $K(x_i)_j$ vanishes unless $j \in \{0, 1\}$, we see that

$$K(\bar{x})_i = \bigoplus_{\substack{(i_1, \dots, i_n) \in \{0,1\}^n \\ i_1 + \dots + i_n = i}} K(x_1)_{i_1} \otimes_A \dots \otimes_A K(x_n)_{i_n}.$$

In particular, if $i_1, \dots, i_n \in \{0, 1\}$, then $i = i_1 + \dots + i_n \in [\![0, n]\!]$, which implies that $K(\bar{x})_i = 0$ if $i \notin [\![0, n]\!]$.

(*c*) We note first that, given $i \in [\![0, n]\!]$, there is a bijection

$$\psi : \{(i_1, \dots, i_n) \in \{0, 1\}^n : i_1 + \dots + i_n = i\} \to \{I \subseteq [\![1, n]\!] : \#(I) = i\}$$

sending $(i_1, \dots, i_n)$ to $I = \{j \in [\![1, n]\!] : i_j = 1\}$. Given $(i_1, \dots, i_n) \in \{0, 1\}^n$, define the linear map

$$f_{(i_1, \dots, i_n)} : A \to K(x_1)_{i_1} \otimes_A \dots \otimes_A K(x_n)_{i_n}$$

sending $a \in A$ to ae_I, where $I = \psi(i_1, \dots, i_n) = \{j \in [\![1, n]\!] : i_j = 1\}$. It is clear that $ae_I = e_I a$, so $f_{(i_1, \dots, i_n)}$ is a morphism of A-bimodules. Moreover, since $K(x_j)_{i_j} = A$, we see that

$$K(x_1)_{i_1} \otimes_A \dots \otimes_A K(x_n)_{i_n} = A \otimes_A \dots \otimes_A A \cong A,$$

where the last isomorphism sends $a_1 \otimes_A \dots \otimes_A a_n$ to $a_1 \dots a_n$, for $a_1, \dots, a_n \in A$. Composing $f_{(i_1, \dots, i_n)}$ for $(i_1, \dots, i_n) \in \{0, 1\}^n$ with the last isomorphism we get the dientity of A, which implies that $f_{(i_1,\dots,i_n)}$ is an isomorphism. Finally, from these maps we obtain the isomorphism of A-bimodules

$$\bigoplus_{\substack{(i_1,\dots,i_n)\in\{0,1\}^n\\ i_1+\dots+i_n=i}} f_{(i_1,\dots,i_n)} : \bigoplus_{\substack{(i_1,\dots,i_n)\in\{0,1\}^n\\ i_1+\dots+i_n=i}} A \to \bigoplus_{\substack{(i_1,\dots,i_n)\in\{0,1\}^n\\ i_1+\dots+i_n=i}} K(x_1)_{i_1}\otimes_A\cdots\otimes_A K(x_n)_{i_n} = K(\bar{x}).$$

(*d*) Since each differential ∂^{x_i} is a morphism of A-modules, we see that d is a morphism of A-modules as well. It only remains to prove that $d\circ d = 0$.

$$d\big(d(\alpha_1\otimes_A\cdots\otimes_A\alpha_n)\big) = \sum_{j=1}^{n}(-1)^{\epsilon_j} d\big(\alpha_1\otimes_A\cdots\otimes_A\alpha_{j-1}\otimes_A\partial^{x_j}(\alpha_j)\otimes_A\alpha_{j+1}\otimes_A\cdots\otimes_A\alpha_n\big)$$

$$= \sum_{j=1}^{n}(-1)^{\epsilon_j}\sum_{i=1}^{j-1}(-1)^{\epsilon_i}\alpha_1\otimes_A\cdots\otimes_A\partial^{x_i}(\alpha_i)\otimes_A\cdots\otimes_A\partial^{x_j}(\alpha_j)\otimes_A\cdots\otimes_A\alpha_n$$

$$+\sum_{j=1}^{n}\alpha_1\otimes_A\cdots\otimes_A\alpha_{j-1}\otimes_A\partial^{x_j}\big(\partial^{x_j}(\alpha_j)\big)\otimes_A\alpha_{j+1}\otimes_A\cdots\otimes_A\alpha_n$$

$$+\sum_{j=1}^{n}(-1)^{\epsilon_j}\sum_{i=j+1}^{n}(-1)^{\epsilon_i+1}\alpha_1\otimes_A\cdots\otimes_A\partial^{x_j}(\alpha_j)\otimes_A\cdots\otimes_A\partial^{x_i}(\alpha_i)\otimes_A\cdots\otimes_A\alpha_n$$

$$=\sum_{1\le i<j\le n}(-1)^{\epsilon_j+\epsilon_i}\alpha_1\otimes_A\cdots\otimes_A\partial^{x_i}(\alpha_i)\otimes_A\cdots\otimes_A\partial^{x_j}(\alpha_j)\otimes_A\cdots\otimes_A\alpha_n$$

$$+\sum_{1\le j<i\le n}(-1)^{\epsilon_j+\epsilon_i+1}\alpha_1\otimes_A\cdots\otimes_A\partial^{x_j}(\alpha_j)\otimes_A\cdots\otimes_A\partial^{x_i}(\alpha_i)\otimes_A\cdots\otimes_A\alpha_n = 0,$$

where we have used that $\partial^{x_j}\circ\partial^{x_j}=0$.

(*e*) This follows immediately from the definitions. Indeed, note that $K(x_1,x_2)_2 = \{ae_{x_1}\otimes_A e_{x_2} : a\in A\}$, $K(x_1,x_2)_1 = \{ae_{x_1}\otimes_A 1_A : a\in A\}\oplus\{b1_A\otimes_A e_{x_2} : b\in A\}$ and $K(x_1,x_2)_0 = \{a1_A\otimes_A 1_A : a\in A\}$. Moreover, the differential sends $ae_{x_1}\otimes_A e_{x_2}$ to $ax_1 1_A\otimes_A e_{x_2} - ax_2 e_{x_1}\otimes_A 1_A$, and $ae_{x_1}\otimes_A 1_A + b1_A\otimes_A e_{x_2}$ to $(ax_1+bx_2)1_A\otimes_A 1_A$.

(*f*) Since $\partial_0^{x_h}=0$ for all $h\in[\![1,n]\!]$ and

$$e_I = E_1\otimes_A\cdots\otimes_A E_n,$$

where $E_{i_j} = e_{x_{i_j}}\in K(x_{i_j})_1$ for $j\in[\![1,\ell]\!]$ and $E_i = 1_A\in K(x_i)_0$ if $i\notin I$, we see that

$$d(e_I) = \sum_{j=1}^{\ell}(-1)^{j-1}E_1\otimes_A\cdots\otimes_A\partial^{x_{i_j}}(E_{i_j})\otimes_A\cdots\otimes_A E_n = \sum_{j=1}^{\ell}(-1)^{j-1}x_{i_j}e_{I\setminus\{i_j\}}.$$

(*g*) It is clear that $H_n(\bar{x},\mathrm{id}_M)$ is the identity of $H_n(\bar{x},M)$, since $\mathrm{id}_{K(\bar{x})}\otimes_A\mathrm{id}_M$ is the identity of $K(\bar{x})\otimes_A M$ and taking n-homology is a functor. Analogously, given morphisms of left A-modules $f : M'\to M$ and $g : M\to M''$, we see that

$$\mathrm{id}_{K(\bar{x})}\otimes_A(g\circ f) = (\mathrm{id}_{K(\bar{x})}\otimes_A g)\circ(\mathrm{id}_{K(\bar{x})}\otimes_A f)$$

which implies that $H_n(\bar{x}, g \circ f) = H_n(\bar{x}, g) \circ H_n(\bar{x}, f)$, since taking n-homology is a functor. Furthermore, given morphisms of left A-modules $f, g : M' \to M$, we see that

$$\mathrm{id}_{K(\bar{x})} \otimes_A (f+g) = (\mathrm{id}_{K(\bar{x})} \otimes_A f) + (\mathrm{id}_{K(\bar{x})} \otimes_A g)$$

which implies that $H_n(\bar{x}, f+g) = H_n(\bar{x}, f) + H_n(\bar{x}, g)$, since taking n-homology is an additive functor.

On the other hand, we see that, by definition, $H_0(\bar{x}, M)$ is given as the quotient of M by the image of $d_1 \otimes_A \mathrm{id}_M$. Moreover, the map $d_1 \otimes_A \mathrm{id}_M$ can be equivalently described as the map $M^n \to M$ sending $(m_1, \dots, m_n) \in M^n$ to $x_1 m_1 + \dots + x_n m_n$, so its image is $(x_1, \dots, x_n).M$. As a consequence, $H_0(\bar{x}, M) = M/(x_1, \dots, x_n).M$.

(h) Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & N' & \xrightarrow{h} & N & \xrightarrow{k} & N'' & \longrightarrow & 0
\end{array}$$

be a commutative diagram with exact rows. By tensoring with the complex $K(\bar{x})$ on the left over A we get the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K(\bar{x}) \otimes_A M' & \xrightarrow{\mathrm{id}_{K(\bar{x})} \otimes_A f} & K(\bar{x}) \otimes_A M & \xrightarrow{\mathrm{id}_{K(\bar{x})} \otimes_A g} & K(\bar{x}) \otimes_A M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{K(\bar{x})} \otimes_A \alpha} & & \downarrow{\scriptstyle \mathrm{id}_{K(\bar{x})} \otimes_A \beta} & & \downarrow{\scriptstyle \mathrm{id}_{K(\bar{x})} \otimes_A \gamma} & & \\
0 & \longrightarrow & K(\bar{x}) \otimes_A N' & \xrightarrow{\mathrm{id}_{K(\bar{x})} \otimes_A h} & K(\bar{x}) \otimes_A N & \xrightarrow{\mathrm{id}_{K(\bar{x})} \otimes_A k} & K(\bar{x}) \otimes_A N'' & \longrightarrow & 0
\end{array} \qquad (3)$$

of complexes of k-modules. Moreover, it has exact rows, since the functor $K(\bar{x})_i \otimes_A (-)$ is exact, because $K(\bar{x})_i$ is a free right A-module and in particular flat. By applying Thm. 1.42 to (3), we obtain the commutative diagram

$$\begin{array}{ccccccccccc}
\dots \to & H_{n+1}(\bar{x}, M'') & \xrightarrow{\delta_{n+1}} & H_n(\bar{x}, M') & \xrightarrow{H_n(\bar{x}, f)} & H_n(\bar{x}, M) & \xrightarrow{H_n(\bar{x}, g)} & H_n(\bar{x}, M'') & \xrightarrow{\delta_n} & H_{n-1}(\bar{x}, M') & \to \dots \\
 & \downarrow{\scriptstyle H_{n+1}(\bar{x}, \gamma)} & & \downarrow{\scriptstyle H_n(\bar{x}, \alpha)} & & \downarrow{\scriptstyle H_n(\bar{x}, \beta)} & & \downarrow{\scriptstyle H_n(\bar{x}, \gamma)} & & \downarrow{\scriptstyle H_{n-1}(\bar{x}, \alpha)} & \\
\dots \to & H_{n+1}(\bar{x}, N'') & \xrightarrow{\delta_{n+1}} & H_n(\bar{x}, N') & \xrightarrow{H_n(\bar{x}, h)} & H_n(\bar{x}, N) & \xrightarrow{H_n(\bar{x}, k)} & H_n(\bar{x}, N'') & \xrightarrow{\delta_n} & H_{n-1}(\bar{x}, N') & \to \dots
\end{array}$$

of k-modules with exact rows. This proves that the family $\{H_n(\bar{x}, -)\}_{n \in \mathbb{N}_0}$ forms a homological δ-functor, as was to be shown.

(i) Consider the shor exact sequence

$$0 \longrightarrow A \longrightarrow K(x) \longrightarrow A[-1] \longrightarrow 0$$

of complexes of A-bimodules, where the first nonzero map is the inclusion of A, who is concentrated in homological degree zero, into $K(x)$, and the second map is the canonical projection onto $A[-1]$, which is A concentrated in homological degree 1. By applying $C \otimes_A (-)$ we obtain a short exact sequence

$$0 \longrightarrow C \otimes_A A \longrightarrow C \otimes_A K(x) \longrightarrow C \otimes_A A[-1] \longrightarrow 0$$

of double complexes of k-modules. If we apply the exact functor $\mathsf{Tot}^{\oplus}$ and we use that that $\mathsf{Tot}^{\oplus}(C \otimes_A A) \cong C$ and $\mathsf{Tot}^{\oplus}(C \otimes_A A[-1]) \cong C[-1]$, we get the short exact sequence of complexes

$$0 \longrightarrow C \longrightarrow \mathsf{Tot}^{\oplus}\big(C \otimes_A K(x)\big) \longrightarrow C[-1] \longrightarrow 0 \tag{4}$$

of complexes of k-modules. Applying Thm. 1.42 to the previous short exact sequence of complexes we obtain a long exact sequence

$$\begin{array}{l} \dots \longrightarrow H_n(C) \longrightarrow H_n\Big(\mathsf{Tot}^{\oplus}\big(C \otimes_A K(x)\big)\Big) \longrightarrow H_n(C[-1]) \xrightarrow{\partial_n} \\ \hookrightarrow H_{n-1}(C) \longrightarrow H_{n-1}\Big(\mathsf{Tot}^{\oplus}\big(C \otimes_A K(x)\big)\Big) \longrightarrow H_{n-1}(C[-1]) \longrightarrow \dots \end{array} \tag{5}$$

of k-modules. Note that $H_m(C[-1]) = H_{m-1}(C)$ for all $m \in \mathbb{Z}$. Moreover, a diagram chasing of (4) shows that $\partial_n : H_{n-1}(C) \to H_{n-1}(C)$ is precisely the map given by multiplication by x. Hence, (5) gives us the short exact sequence

$$0 \to H_0\big(x, H_n(C, d_C)\big) \to H_n\Big(\mathsf{Tot}^{\oplus}(K(x) \otimes_A C)\Big) \to H_1\big(x, H_{n-1}(C, d_C)\big) \to 0$$

of k-modules for all $n \in \mathbb{Z}$.

(*j*) We prove the result by induction on $n \in \mathbb{N}$. If $n = 1$, then $H_i(x, M)$ is n-homology of the complex

$$0 \to M \to M \to 0$$

concentrated in homological degrees 0 and 1, where the nonzero map is given by multiplication by x. In particular, $H_i(x, M)$ vanishes for all integers $i \geq 2$. Moreover, since the nonzero map above is injective by assumption, we see that $H_1(x, M) = 0$.

Assume now that the result holds for all integers in $[\![1, n-1]\!]$. We will prove it holds for n. We first note the isomorphism

$$K(\bar{x}) \otimes_A M \cong \mathsf{Tot}^{\oplus}\Big(K(x_1) \otimes \big(K(\bar{x}') \otimes_A M\big)\Big)$$

of complexes of k-modules, where $\bar{x'} = (x_2, \dots, x_n)$. Using the previous item with $C = K(\bar{x}') \otimes_A M$ we get the short exact sequence

$$0 \to H_0\big(x_1, H_i(\bar{x'}, M)\big) \to H_i(\bar{x}, M) \to H_1\big(x_1, H_{i-1}(\bar{x'}, M)\big) \to 0$$

of k-modules for all $i \in \mathbb{N}$. If $i \geq 2$, then $H_i(\bar{x}, M) = 0$, since $H_i(\bar{x'}, M) = H_{i-1}(\bar{x'}, M) = 0$ In particular, by inductive hypothesis. If $i = 1$, then $H_1(\bar{x'}, M) = 0$. Moreover, $H_0(\bar{x'}, M) = M/(x_2, \dots, x_n).M$ and by hypothesis the map given by multiplication by x_1 on $H_1(\bar{x'}, M) = M/(x_2, \dots, x_n).M$ is injective, which tells us that $H_1\big(x_1, H_0(\bar{x'}, M)\big)$ also vanishes, by the proof for the case with $n = 1$. As a consequence, $H_1(\bar{x}, M) = 0$, as we wanted to show.

(k) Since $\bar{x}$ is a regular sequence on A, we conclude that the complex $K(\bar{x})$ is acyclic in positive homological degrees. Moreover, we now that $H_0(\bar{x}, A) = A/I$, where I is the right ideal generated by $\{x_1, \dots, x_n\}$. this implies that $(K(\bar{x}), d)$ together with the canonical projection $K(\bar{x})_0 \to A/I$ is a resolution of A/I in the category of right A-modules. Since each $K(\bar{x})_i$ is free for all $i \in \mathbb{N}_0$, we conclude that it gives a projective resolution. In particular, $\mathrm{Tor}_i^A(A/I, M) \cong H_i(K(\bar{x}) \otimes_A M) = H_i(\bar{x}, M)$, for all $i \in \mathbb{N}_0$, as was to be shown.

(l) It is easy to see that $k[X_1, \dots, X_n]/(X_{i+1}, \dots, X_n) \cong k[X_1, \dots, X_i]$ and the multiplication by X_i is injective since $X_i \neq 0$ and $k[X_1, \dots, X_i]$. In consequence, $(X_1, \dots, X_n)$ is a regular sequence on $A = k[X_1, \dots, X_n]$. Let I be the ideal generated by $\{X_1, \dots, X_n\}$. On the other hand, since $A/I \cong k$, the previous item tells us that $\mathrm{Tor}_i^A(k, k) \cong H_i(\bar{x}, k)$, for all $i \in \mathbb{N}_0$. We will thus compute the homology of the complex $K(\bar{x}) \otimes_A k$. However, in this case $d_i \otimes_A \mathrm{id}_k$ vanishes, which tells us that the i-th homology of $K(\bar{x}) \otimes_A k$ is precisely $K(\bar{x})_i \otimes_A k$ for all $i \in \mathbb{N}_0$. By the third item we conclude that

$$H_i(\bar{x}, k) \cong \bigoplus_{\substack{I \subseteq [\![1,n]\!] \\ \#(I) = i}} k$$

for $i \in [\![0, n]\!]$ and $H_i(\bar{x}, k)$ vanishes if $i \in \mathbb{Z} \setminus [\![0, n]\!]$.

6pt **3.** Let G be a finite group of $n = \#(G)$ elements and let $N(G) = \sum_{g \in G} g$ be its norm.

(a) Let $B_\bullet = (\mathrm{Bar}(\mathbb{Z}G)_\bullet, d_\bullet)_{\bullet \in \mathbb{N}_0}$ be the Bar resolution of the trivial left $\mathbb{Z}G$-module $\mathbb{Z}$. Denote by $F : B_\bullet \to B_\bullet$ be the morphism of complexes of $\mathbb{Z}G$-modules given by multiplication by $n - N(G)$ on B_0 and by n on B_i if $i \neq 0$. Prove that F is null-homotopic.
Hint : For $i \in \mathbb{Z}$, consider the $\mathbb{Z}G$-linear maps $h_i : B_i \to B_{i+1}$ sending $[g_1|\dots|g_i]$ to $(-1)^{i+1} \sum_{g \in G} [g_1|\dots|g_i|g]$ if $i \in \mathbb{N}_0$, and zero otherwise.

(b) Let M be a left $\mathbb{Z}G$-module. Prove that the action by n on $H^i(G, M)$ vanishes for all $i \in \mathbb{N}$, *i.e.* $H^i(G, M)$ is a $(\mathbb{Z}/n.\mathbb{Z})$-module.

Solution.

(a) We will prove that $F_i = h_{i-1} \circ d_i + d_{i+1} \circ h_i$ for all $i \in \mathbb{N}_0$, where $d_0 = 0$. Since the morphisms d_i and h_i are $\mathbb{Z}G$-linear, it suffices to prove the previous identity for an element of the form $[g_1|\dots|g_i]$, with $g_1, \dots, g_i \in G$ and $i \in \mathbb{N}_0$. For $i = 0$, we see that

$$(d_1 \circ h_0)([]) = -\sum_{g\in G} d_1\big([g]\big) = -\sum_{g\in G}\big(g[] - []\big) = \big(n - N(G)\big)[] = F_0([]),$$

which tells us that $F_0 = d_1 \circ h_0$, as we wanted to show.

Assume now that $i \in \mathbb{N}$. Then, we have that

$$\begin{aligned}
(d_{i+1} \circ h_i)\big([g_1|\dots|g_i]\big) &= (-1)^{i+1}\sum_{g\in G} d_{i+1}\big([g_1|\dots|g_i|g]\big) \\
= (-1)^{i+1}\sum_{g\in G}\Bigg(& g_1[g_2|\dots|g_i|g] + \sum_{j=1}^{i-1}(-1)^j[g_1|\dots|g_{j-1}|g_jg_{j+1}|g_{j+2}|\dots|g_i|g] \\
&+ (-1)^i[g_1|\dots|g_{i-1}|g_ig] - (-1)^i[g_1|\dots|g_i]\Bigg),
\end{aligned}$$

as well as

$$\begin{aligned}
(h_{i-1} \circ d_i)\big([g_1|\dots|g_i]\big) &= g_1h_{i-1}\big([g_2|\dots|g_i]\big) \\
&+ h_{i-1}\Bigg(\sum_{j=1}^{i-1}(-1)^j[g_1|\dots|g_{j-1}|g_jg_{j+1}|g_{j+2}|\dots|g_i] + (-1)^i[g_1|\dots|g_{i-1}]\Bigg) \\
= (-1)^i\sum_{g\in G}\Bigg(& g_1[g_2|\dots|g_i|g] + \sum_{j=1}^{i-1}(-1)^j[g_1|\dots|g_{j-1}|g_jg_{j+1}|g_{j+2}|\dots|g_i|g] \\
&+ (-1)^i[g_1|\dots|g_{i-1}|g]\Bigg).
\end{aligned}$$

Hence,

$$\begin{aligned}
(h_{i-1} \circ d_i + d_{i+1} \circ h_i)\big([g_1|\dots|g_i]\big) &= \sum_{g\in G}[g_1|\dots|g_i] = n[g_1|\dots|g_i] \\
&= F_i\big([g_1|\dots|g_i]\big),
\end{aligned}$$

where we have used that

$$\sum_{g\in G}[g_1|\dots|g_{i-1}|g_ig] = \sum_{g\in G}[g_1|\dots|g_{i-1}|g].$$

In consequence, $F_i = h_{i-1} \circ d_i + d_{i+1} \circ h_i$ for all $i \in \mathbb{N}$, as was to be shown. This shows that F is null-homotopic.

(b) Recall that $H^i(G, M)$ can be computed as the i-th cohomology group of the cochain complex $C^\bullet = \mathrm{Hom}_{\mathbb{Z}G}(B_\bullet, M)$. We shall denote by $d^i : C^i \to C^{i+1}$ the i-th component of the differential of $C^\bullet$, and we recall that $d^i(f) = f \circ d_{i+1}$, for $f \in \mathrm{Hom}_{\mathbb{Z}G}(B_i, M)$. Define $\hat{F}^i : C^i \to C^i$ for $i \in \mathbb{N}_0$ as $\hat{F}^i(f) = f \circ F_i$, for $f \in \mathrm{Hom}_{\mathbb{Z}G}(B_i, M)$. Note that $\hat{F}^i$ coincides with the map $C^i \to C^i$ given by multiplication by n for $i \in \mathbb{N}$, which induces in turn the map $H^i(C) \to H^i(C)$ given by multiplication by n for $i \in \mathbb{N}$. From the previous homotopy $(h_i)_{i \in \mathbb{N}_0}$ define also $\hat{h}^i : C^i \to C^{i-1}$ as $\hat{h}_i(f) = f \circ h_{i-1}$, for $f \in \mathrm{Hom}_{\mathbb{Z}G}(B_i, M)$. Then, $F_i = h_{i-1} \circ d_i + d_{i+1} \circ h_i$ implies that $\hat{F}^i = \hat{h}^{i+1} \circ d^i + d^{i-1} \circ \hat{h}^i$ for all $i \in \mathbb{N}_0$. In particular, the map $H^i(C) \to H^i(C)$ given by multiplication by n coincides with the zero map, and the result follows.