Basics on homotopy theory

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§1. Conventions

We will denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of (strictly) positive integers, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ the set of nonnegative integers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{R}$ the set of all real numbers. Given two integers $m \leq n$, we define $[\![m, n]\!] = \{x \in \mathbb{Z} : m \leq x \leq n\}$. Given two real numbers $a \leq b$, $[a, b] = \{t \in \mathbb{R} : a \leq t \leq b\}$ is the usual closed interval, whereas if $a < b$, $]a, b[= \{t \in \mathbb{R} : a < t < b\}$ is the usual open interval. The half-open intervals $[a, b[$ and $]a, b]$ are defined analogously. More generally, for $n \in \mathbb{N}$, given $x_0 \in \mathbb{R}^n$ and $r > 0$, $B_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$ is the open ball and $\bar{B}_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$ is the closed ball (for the Euclidean norm). Given $x, y \in \mathbb{R}^n$, we denote by $[x, y] = \{tx + (1 - t)y : t \in [0, 1]\}$ the segment they form. Recall also that we set $\mathbb{R}^0 = \{*\}$.

If I is a set, $\#(I)$ denotes its cardinal. For a map $f : X \to Y$, the image of f will be denoted by $\mathrm{Img}(f)$. Moreover, given a set X, $\mathrm{id}_X : X \to X$ will usually denote the identity on X. If A is a ring, $A^\times$ denotes the group of invertible elements of A.

§2. Basic topological definitions I

2.1. We assume that the reader is familiar with the basic notions of topology, such as topological space, continuous maps, etc. We refer the reader to [Kel1975] or [Mun2000] for the basics on set topology. We will review however some of the basic notions that we will need in the sequel, in particular those for which we use a slightly different definition. We will embark afterwards in our study of homotopy theory, in particular of the fundamental group, making use of covering spaces. Our exposition will be as self-contained as possible, but some of the results are left as exercises, many of which will be discussed in class. Our exposition is greatly influenced by the nice presentation in [Wed2016]. For further details, see [GH1981] and [Mun1984].

2.2. We will sometimes denote a topological space $(X, \mathcal{T})$ simply by X if the topology is clear. Topological spaces are not necessarily Hausdorff, unless it is explicitly stated. Moreover, to avoid confusion we remark that a neighborhood of a point x in a topological space is a subset $Y \subseteq X$ such that there is an open set $U \subseteq X$ satisfying that $x \in U \subseteq Y$. In particular, a neighborhood of a point is not necessarily open. We also recall that a map (regardless of continuity) $f : X \to Y$ between topological spaces is called **closed** (resp., **open**) if $f(A) \subseteq Y$ is closed (resp., open) for every closed (resp., open) subset $A \subseteq X$.

2.3. Exercise. *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. Let $\{Y_i\}_{i \in I}$ be a family of subsets of Y such that one of the following two conditions is satisfied:*

(i) *$\cup_{i \in I} Y_i^\circ = Y$, where Y_i° denotes the interior of Y_i in Y;*

(ii) *$\cup_{i \in I} Y_i = Y$, Y_i is closed for all $i \in I$, and for all $y \in Y$ there exist an open set $V \subseteq Y$ and a finite subset $J \subseteq I$ such that $y \in V$ and $V \cap Y_i = \emptyset$ for $i \in J$.*

*We recall that a covering of a topological space satisfying the last condition of (ii) is said to be **locally finite**. Prove that f is closed (resp., open) if and only if the map $f|_{f^{-1}(Y_i)} : f^{-1}(Y_i) \to Y_i$ is closed (resp., open) for all $i \in I$.*

2.4. Recall that a subset $A \subseteq X$ of a topological space is called **relatively Hausdorff** if given $x, y \in A$, there exist open sets $U, V \subseteq X$ such that $U \cap V = \emptyset$, $x \in U$ and $y \in V$. Recall that a continuous map $f : X \to Y$ between topological spaces is called **separated** if the induced map

$$\Delta_f : X \to X \times_Y X = \{(x, x') \in X \times X : f(x) = f(x')\}$$

sending $x \in X$ to (x, x) is closed, where $X \times_Y X$ has the subspace topology of $X \times X$. Using the continuous map $X \times_Y X \to X$ given by $(x, x') \mapsto x$, we see that f is separated if and only if $\mathsf{Img}(\Delta_f) \subseteq X \times_Y X$ is a closed subset. We will also denote this image by Δ_f if there is no possible confusion.

2.5. Exercise. *Let $f : X \to Y$ be a continuous map of topological spaces. Prove that the following condition are equivalent:*

(i) f is separated;

(ii) $f^{-1}(A) \subseteq X$ is relatively Hausdorff for every relatively Hausdorff $A \subseteq Y$;

(iii) $f^{-1}(\{y\}) \subseteq X$ is relatively Hausdorff for every $y \in Y$.

2.6. Using the previous exercise we see easily that, given $f : X \to Y$ and $g : Y \to Z$ two continuous maps of topological spaces, then $g \circ f$ is separated if f and g are so, and f is separated if $g \circ f$ is so.

2.7. Recall that a continuous map $f : X \to Y$ of topological spaces is called an **embedding** if the induced map $f : X \to \mathsf{Img}(f)$ is a homeomorphism, where the image of f is endowed with the subspace topology.

2.8. Exercise. *Let $f : X \to Y$ be a continuous map of topological spaces. Assume that there exists a continuous map $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$. Prove that f is an embedding.*

2.9. Recall that a topological space X **compact** if given any open covering $\{U_i\}_{i \in I}$ of X, *i.e.* $\cup_{i \in I} U_i = X$, there exists a finite subset $I_0 \subseteq I$ such that $\cup_{i \in I_0} U_i = X$.

2.10. Exercise. *Let X and Y be topological spaces, and let $A \subseteq X$ and $B \subseteq Y$ be subspaces.*

(i) Prove that if A and B are compact subspaces and $W \supseteq A \times B$ is an open subset of $X \times Y$, there exist open sets $U \subseteq X$ and $V \subseteq Y$ such that $A \subseteq U$, $B \subseteq V$ and $U \times V \subseteq W$.

(ii) Prove that X is compact if and only if the projection map $X \times Y \to Y$ is closed for all topological spaces Y. Hint. For the converse implication, suppose that X is not compact, pick a covering $\{U_i\}_{i \in I}$ of X such that $\cup_{i \in I'} U_i \neq X$ for every finite subset $I' \subseteq I$, consider the space $\mathcal{I} = \{I' \subseteq I : I' \neq \emptyset \text{ is finite}\} \cup \{I\}$ endowed with the topology generated by the family $\{U_J : J \subseteq I \text{ such that } J \text{ is finite}\}$ where

$$U_J = \{I' \in \mathcal{I} : I' \supseteq J\},$$

as well as the set $A \subseteq X \times \mathcal{I}$ given by

$$A = \{(x, I') \in X \times \mathcal{I} : x \notin \bigcup_{i \in I'} U_i\}.$$

2.11. Recall that a continuous map $f : X \to Y$ between topological spaces is called **proper** if given any continuous map $g : Z \to Y$ of topological spaces, the projection $X \times_Y Z \to Z$ on the second factor is closed, where $X \times_Y Z = \{(x, z) \in X \times Z : f(x) = g(z)\}$ is endowed with the subspace topology of $X \times Z$.

2.12. Exercise. *Let $f : X \to Y$ be a continuous map of topological spaces. Prove that the following condition are equivalent:*

(i) f is proper;

(ii) f is closed and $f^{-1}(\{y\}) \subseteq X$ is a compact subspace for every $y \in Y$;

(iii) f is closed and $f^{-1}(A) \subseteq X$ is a compact subspace for every compact subspace $A \subseteq Y$;

(iv) the map $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is closed, for every topological space Z.

Hint. Prove (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv). For the third implication, consider $A \subseteq X \times_Y Z$ closed and $z_0 \in Z \setminus \pi_2(A)$, where $\pi_2 : X \times_Y Z \to Z$ is the canonical projection. After considering the simple case $z_0 \notin \mathsf{Img}(\pi_2) = g^{-1}(f(X))$, suppose the opposite and construct open subsets $U \subseteq X$ and $V \subseteq Y$ such that $(U \times V) \cap A = \emptyset$ by means of Exercise 2.10, (i). Take finally the open neighborhood $V \cap g^{-1}(Y \setminus f(X \setminus U)) \subseteq Z \setminus \pi_2(A)$ of z_0.

2.13. Using the previous exercise we see easily that, given $f : X \to Y$ and $g : Y \to Z$ two continuous maps of topological spaces, then $g \circ f$ is proper if f and g are so.

2.14. Exercise. *Prove that any closed embedding is proper.*

2.15. Exercise. *Let $f : X \to Y$ and $g : Y \to Z$ be two continuous maps of topological spaces.*

(i) Define $\Gamma_f : X \to X \times_Z Y$ as the continuous map sending $x \in X$ to $(x, f(x))$, where

$$X \times_Z Y = \{(x,y) \in X \times Y : (g \circ f)(x) = g(y)\}$$

is endowed with the subspace topology. Use the canonical projection $X \times_Z Y \to X$ to prove that Γ_f is an embedding.

(ii) Show that $\mathsf{Img}(\Gamma_f) = (f \times \mathrm{id}_Y)^{-1}(\mathsf{Img}(\Delta_g))$. Deduce that Γ_f is a closed map if g is separated.

(iii) Show that f is the composition of Γ_f and the canonical projection $X \times_Z Y \to Y$. Deduce that f is proper if $g \circ f$ is so and g is separated.

(iv) Prove that g is proper if $g \circ f$ is so and f is surjective.

§3. Basic topological definitions II: Connectedness

3.1. We recall that a topological space X is said to be **connected** if $X \neq \emptyset$ and for every pair $U, V \subseteq X$ of open sets of X such that $U \cap V = \emptyset$, $X = U \cup V$, either $U = \emptyset$ or $V = \emptyset$. [1]

3.2. Exercise. *Prove that, for every pair of real numbers $a \leq b$, the interval $[a,b] \subseteq \mathbb{R}$ is connected, for the subspace topology of the usual topology of $\mathbb{R}$. Unless otherwise stated, the topology of $\mathbb{R}^n$, for $n \in \mathbb{N}$ will always be the usual one (i.e., the one induced by any norm), and the topology of any of the intervals of $\mathbb{R}$ or balls of $\mathbb{R}^n$ will always be the subspace topology.*

3.3. Exercise. *(i) Prove that, given a topological space X and a family $\{Y_i\}_{i \in I}$ of connected subspaces of X (i.e. $Y_i \subseteq X$ is a connected topological space for the subspace topology) such that $\cap_{i \in I} Y_i \neq \emptyset$, then $\cup_{i \in I} Y_i$ is a connected subspace of X.*

(ii) Prove that, if $f : X \to Y$ is a surjective continuous map between topological spaces X and Y, and X is connected, then Y is connected as well.

3.4. Given a topological space X and a point $x \in X$, we define the **(connected) component** of x as the union of all the connected subspaces of X containing x. This is a connected subspace of X, by item (i) of the previous exercise, that we will typically denote by C_x.

3.5. Exercise. *(i) Let $A \subseteq X$ be a connected subspace of a topological space X. Prove that $\bar{A} \subseteq X$ is also connected.*

(ii) Prove that C_x is closed for a topological space X and a point $x \in X$.

(iii) Consider the topological space $\mathbb{Q}$, with the subspace topology of $\mathbb{R}$. Prove that the connected component of any $x \in \mathbb{Q}$ is $\{x\}$, which is not open in $\mathbb{Q}$.

[1] This is the first deviation we take: our definition of connected space explicitly excludes the case when it is empty, in contrast to some classical textbooks, such as [Kel1975]. Since the case of an empty topological space is somehow pathological, for it forces us to explicitly exclude it from many equivalent conditions to that of being connected as well as many properties satisfied by them, we have to decided to exclude it from the definition.

3.6. Exercise. *A nonempty topological space X is called* ***irreducible*** *(or* ***hyperconnected****) if given any pair of nonempty open subsets $U, V \subseteq X$, $U \cap V \neq \emptyset$.*

(i) Prove that every irreducible space is connected.

(ii) Prove that a nonempty topological space is irreducible if and only if every nonempty subset is nowhere dense (i.e. the interior of its closure is empty), if and only if every nonempty open subset is dense.

3.7. A topological space X is said to be **locally connected** if for every $x \in X$ there is a fundamental system of connected neighborhoods of x, *i.e.* for every $x \in X$ and every open set $V \subseteq X$ including x there exists a connected subspace W of X and an open set U such that $x \in U \subseteq W \subseteq V$.

3.8. Exercise. *Let*

$$X = \{(0,1)\} \cup \Big([0,1] \times \{0\}\Big) \cup \bigcup_{n \in \mathbb{N}_0} \Big(\{1/2^n\} \times [0,1]\Big) \subseteq \mathbb{R}^2.$$

We give X the subspace topology of $\mathbb{R}^2$. Prove that X is connected but it is not locally connected. Give an example of a locally connected topological space that is not connected.

§4. Basic topological definitions III: Path connectedness

4.1. Given a topological space X, a **(continuous) path** is a continuous map $\gamma : [0,1] \to X$. The point $\gamma(0)$ is called the **initial** (or **starting**) **point** of γ and $\gamma(1)$ is called the **end point** of γ. Moreover, we say that γ is a **loop** (based at $\gamma(0)$) if $\gamma(0) = \gamma(1)$.

4.2. Let $\gamma, \eta : [0,1] \to X$ be two paths in X such that $\gamma(1) = \eta(0)$. These paths are called **composable**. Define $\gamma^- : [0,1] \to X$ by $\gamma^-(t) = \gamma(1-t)$, for $t \in [0,1]$, and $\gamma \cdot \eta : [0,1] \to X$ by

$$(\gamma \cdot \eta)(t) = \begin{cases} \gamma(2t), & \text{if } t \in [0,1/2], \\ \eta(2t-1), & \text{if } t \in [1/2,1]. \end{cases}$$

It is clear that γ^- and $\gamma \cdot \eta$ are paths in X. They are called the **inverse path** of γ and the **concatenation (or product)** of γ and η, respectively.

4.3. Remark. *Note that the order of the paths in the definition of the concatenation does not follow the order used for the composition of functions. This is however the most common convention in the literature, so we decided to stick to it.*

4.4. Define the following relation on a topological space X: given $x, y \in X$ we say that $x \sim y$ if there is a path in X with starting point x and end point y. It is easy to see that this is an equivalence relation, so it defines the set $\pi_0(X)$ formed by all equivalence classes. An element of $\pi_0(X)$ (*i.e.* an equivalence class of X under the relation $\sim$) is called a **path connected component** of X. We will say that X is **path connected** if $\pi_0(X)$ has cardinal 1. Using Exercise 3.2, one sees that any path connected space is connected.

4.5. Remark. *Note that a path connected space is a fortiori nonempty.*

4.6. Exercise. *Let*

$$A = \Big\{(x, \sin(1/x)) : x \in \,]0,\pi]\Big\} \subseteq \mathbb{R}^2$$

be a subset of $\mathbb{R}^2$, provided with the usual topology. Prove that A is path connected (so a fortiori connected) but $\bar{A}$ is not path connected. Note however that $\bar{A}$ is connected by Exercise 3.5, (i).

4.7. A topological space X is said to be **locally path connected** if for every $x \in X$ there is a fundamental system of path connected neighborhoods of x.

4.8. Exercise. *Let*

$$X = \Big(\{0\} \times [0,1]\Big) \cup \Big([0,1] \times \{0\}\Big) \cup \bigcup_{n \in \mathbb{N}_0} \Big(\{1/2^n\} \times [0,1]\Big) \subseteq \mathbb{R}^2.$$

We give X the subspace topology of $\mathbb{R}^2$. Prove that X is path connected but it is not locally path connected. Give an example of a locally path connected space that is not path connected.

4.9. Proposition. *Let X be a topological space such that every point $x \in X$ has a path connected neighborhood. Then, a subspace $Y \subseteq X$ is a connected component of X if and only if it is a path connected component of X. Moreover, all the (path) connected components of X are open (and closed).*

Proof. Since connected and path connected components are nonempty, we assume without loss of generality that $X \neq \emptyset$. Let $x \in X$ be any element, and let C_x (resp., PC_x) denote the (resp., path) connected component of x. We first show that PC_x is open. Let $V_x \subseteq X$ be a path connected neighborhood of x. Since V_x is path connected, we see that $PC_x \cup V_x$ is also path connected, so by definition of path connected component we obtain that $PC_x \cup V_x = PC_x$, which in turn implies that the latter is open.

Since any path connected set is connected, we see that $PC_y \subseteq C_x$ for all $y \in C_x$. Consider the canonical projection $p : C_x \to \pi_0(X)$ sending y to PC_y. By the axiom of choice there is an injective map $s : \mathsf{Img}(p) \to C_x$ such that $p \circ s = \mathrm{id}_{\mathsf{Img}(p)}$. Let $A_x = \mathsf{Img}(s)$. Then, $C_x = \sqcup_{y \in A_x} PC_y$. Since C_x is component, A_x has cardinal one. Indeed, if $\#(A) \geq 2$, $C_x = PC_y \sqcup (\cup_{y' \in A \setminus \{y\}} PC_{y'})$ gives a contradiction. Hence, C_x is path connected, and the inclusion $PC_x \subseteq C_x$ tells us that $PC_x = C_x$, as was to be shown. Since the connected components are closed (by Exercise 3.5, (ii)), the (path) connected components are also closed. □

4.10. Corollary. *Let X be a locally path connected topological space. Then, the path connected components coincide with the connected components, which are open and closed. Moreover, every point has a fundamental system of open connected neighborhoods.*

Proof. The first part is a direct consequence of the previous result. For the last part, it suffices to prove that given any open neighborhood W of a point $x \in X$, there is an open connected set V such that $x \in V \subseteq W$. Then, the subspace W of X satisfies the hypotheses of the previous proposition, which in turn implies that $W = \sqcup_{i \in I} P_i$ is the disjoint union of its path connected components P_i, which are path connected open subsets of W, so *a fortiori* path connected open subsets of X. Since $x \in W = \sqcup_{i \in I} P_i$, there is $i_0 \in I$ such that $x \in P_{i_0} \subseteq W$. Taking $V = P_{i_0}$ This proves the corollary. □

§5. Algebraic interlude I: Basics on category theory

We present the following definition, which provides a very useful general language to deal with the different situations we will encounter. For a more detailed and comprehensive exposition, see [Mac1971]. For a discussion on how to safely frame category theory within set theory (or first order logic) see [Mur2006] and the references therein.

5.1. Definition. *A **category** $\mathscr{C}$ is a tuple $(\mathscr{C}_0, \mathscr{C}_1, s_{\mathscr{C}}, t_{\mathscr{C}}, \circ_{\mathscr{C}}, i_{\mathscr{C}})$ where*

*(i) $\mathscr{C}_0$ is a collection of elements, called the **objects** of $\mathscr{C}$ and typically denoted by $X, Y, Z, \dots$;*

*(ii) $\mathscr{C}_1$ is a collection of elements, called the **morphisms** of $\mathscr{C}$ and typically denoted by $f, g, h, \dots$;*

(iii) $s_{\mathscr{C}}, t_{\mathscr{C}} : \mathscr{C}_1 \to \mathscr{C}_0$ are two maps, called the ***source map*** and the ***target map***, and whose inverse image

$$s_{\mathscr{C}}^{-1}(\{X\}) \cap t_{\mathscr{C}}^{-1}(\{Y\})$$

is denoted by $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ *and is called the* ***space of morphisms*** *from* X *to* Y*;*

(iv) $\circ_{\mathscr{C}} : \mathscr{C}_1 \times_{s,t} \mathscr{C}_1 \to \mathscr{C}_1$ *is a map called the* ***composition****, sending a pair* (f, g) *to* $f \circ_{\mathscr{C}} g$*, where*

$$\mathscr{C}_1 \times_{s,t} \mathscr{C}_1 = \big\{(f, g) \in \mathscr{C}_1 \times \mathscr{C}_1 : s_{\mathscr{C}}(f) = t_{\mathscr{C}}(g)\big\};$$

(v) $i_{\mathscr{C}} : \mathscr{C}_0 \to \mathscr{C}_1$ *is a map, called the* ***identity****, sending every element* X *to a distinguished morphism* $i_{\mathscr{C}}(X) = \mathrm{id}_X$*, called the* ***identity morphism****;*

satisfying that

(CAT.1) *given* $f, g, h \in \mathscr{C}_1$ *such that* $s_{\mathscr{C}}(f) = t_{\mathscr{C}}(g)$ *and* $s_{\mathscr{C}}(g) = t_{\mathscr{C}}(h)$*, then*

$$(f \circ_{\mathscr{C}} g) \circ_{\mathscr{C}} h = f \circ_{\mathscr{C}} (g \circ_{\mathscr{C}} h).$$

(CAT.2) *For every* $X \in \mathscr{C}_0$ *and every* $f, g \in \mathscr{C}_1$ *such that* $s_{\mathscr{C}}(f) = X = t_{\mathscr{C}}(g)$*, we have*

$$f \circ_{\mathscr{C}} \mathrm{id}_X = f \text{ and } \mathrm{id}_X \circ_{\mathscr{C}} g = g.$$

5.2. Remark. *More intuitively, a category can be simply defined as a collection of objects* $\mathscr{C}_0$ *and for every pair of objects* $(X, Y) \in \mathscr{C}_0^2$ *a set* $\mathsf{Hom}_{\mathscr{C}}(X, Y)$*, called the space of morphisms from* X *to* Y*, satisfying that* $\mathsf{Hom}_{\mathscr{C}}(X, Y) = \mathsf{Hom}_{\mathscr{C}}(X', Y')$ *if and only if* $X = X'$ *and* $Y = Y'$*, as well as maps*

$$\circ_{X,Y,Z} = \circ_{\mathscr{C}}|_{\mathsf{Hom}_{\mathscr{C}}(Y,Z) \times \mathsf{Hom}_{\mathscr{C}}(X,Y)} : \mathsf{Hom}_{\mathscr{C}}(Y, Z) \times \mathsf{Hom}_{\mathscr{C}}(X, Y) \to \mathsf{Hom}_{\mathscr{C}}(X, Z)$$

for every triple of objects $(X, Y, Z) \in \mathscr{C}_0^3$*, and a distinguished morphism* $\mathrm{id}_X \in \mathsf{Hom}_{\mathscr{C}}(X, X)$ *for every object* $X \in \mathscr{C}_0$ *satisfying the axioms (CAT.1) and (CAT.2). If the category is clear, we will denote its composition* $\circ_{\mathscr{C}}$ *simply by* $\circ$*.*

5.3. Exercise. (i) *Let* $\mathscr{C}_0$ *be the collection of all sets,* $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ *the set of all maps from* X *to* Y*,* $\circ_{\mathscr{C}}$ *the usual composition of maps and* id_X *is the identity map from* X *to itself. Prove that this forms a category, denoted by* Set*.*

(ii) *Let* $\mathscr{C}_0$ *be the collection of all groups,* $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ *the set of all morphisms of groups from* X *to* Y*,* $\circ_{\mathscr{C}}$ *the usual composition of morphisms of groups and* id_X *is the identity map from* X *to itself. Prove that this forms a category, denoted by* Grp*.*

(iii) *Let* $\mathscr{C}_0$ *be the collection of all topological spaces,* $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ *the set of all continuous maps from* X *to* Y*,* $\circ_{\mathscr{C}}$ *the usual composition of maps and* id_X *is the identity map from* X *to itself. Prove that this forms a category, denoted by* Top*.*

(iv) *Let* G *be a group and* $\mathscr{C}_0$ *be the collection of all sets provided with a right action of* G*,* $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ *the set of all* G*-linear (or* G*-equivariant) maps from* X *to* Y*,* $\circ_{\mathscr{C}}$ *the usual composition of maps and* id_X *is the identity map from* X *to itself (see Section 8 for more details). Prove that this forms a category, denoted by* $\mathsf{Set}\text{-}G$*.*

5.4. Let $\mathscr{C}$ be a category. A morphism $f \in \mathsf{Hom}_{\mathscr{C}}(X, Y)$ is called an **isomorphism** if there exists a morphism $g \in \mathsf{Hom}_{\mathscr{C}}(Y, X)$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In this case, we also say that X and Y are **isomorphic** objects, and we will write it $X \cong Y$. The set of isomorphisms from X to Y is denoted by $\mathsf{Iso}_{\mathscr{C}}(X, Y)$, and we define $\mathsf{Aut}_{\mathscr{C}}(X, Y)$ as the group $\mathsf{Iso}_{\mathscr{C}}(X, X)$ under the composition $\circ_{\mathscr{C}}$, which is called the group of **automorphisms** of X.

5.5. Exercise. *Prove that there exists at most one morphism* $g \in \mathsf{Hom}_{\mathscr{C}}(Y, X)$ *satisfying the previous condition. This morphism, if it exists, is called the* ***inverse*** *of* f*.*

5.6. Exercise. *Prove that a morphism in* Set *or in* Grp *is an isomorphism if and only if it is bijective. Show that this does not hold in* Top.

5.7. Definition. *Given two categories $\mathscr{C}$ and $\mathscr{D}$, a* ***functor*** *F from $\mathscr{C}$ to $\mathscr{D}$, usually written as $F : \mathscr{C} \to \mathscr{D}$, is a pair of maps (F_0, F_1) of the form $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and $F_1 : \mathscr{C}_1 \to \mathscr{D}_1$ such that $t_{\mathscr{D}} \circ F_1 = F_0 \circ t_{\mathscr{C}}$, $s_{\mathscr{D}} \circ F_1 = F_0 \circ s_{\mathscr{C}}$, $F_1 \circ i_{\mathscr{C}} = i_{\mathscr{D}}$ and $F_1 \circ (\circ_{\mathscr{C}}) = (\circ_{\mathscr{D}}) \circ F$.*

5.8. Remark. *More intuitively, a functor $F : \mathscr{C} \to \mathscr{D}$ is given by a map $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ together with maps*

$$F_{X,Y} = F_1|_{\mathsf{Hom}_{\mathscr{C}}(X,Y)} : \mathsf{Hom}_{\mathscr{C}}(X,Y) \to \mathsf{Hom}_{\mathscr{D}}(F_0(X), F_0(Y))$$

for all pairs $(X,Y) \in \mathscr{C}_0^2$ satisfying that

$$F_{X,X}(\mathrm{id}_X) = \mathrm{id}_{F_0(X)} \text{ and } F_{X,Z}(f \circ g) = F_{Y,Z}(f) \circ F_{X,Y}(g),$$

for all $X, Y, Z \in \mathscr{C}_0$ and for all morphisms $f \in \mathsf{Hom}_{\mathscr{C}}(Y,Z)$ and $g \in \mathsf{Hom}_{\mathscr{C}}(X,Y)$. By abuse of notation, and if it is clear from the context, it is usual to omit the subscripts and to denote both F_0 and F_1 simply by F.

5.9. Exercise. *Let $\mathscr{C}$ be a category, and consider $F = (F_0, F_1) : \mathscr{C} \to \mathscr{C}$ given as follows. For every object $X \in \mathscr{C}_0$, set $F(X) = X$, and for every morphism $f : X \to Y$ in $\mathscr{C}_1$ between the objects X and Y, define $F_1(f) = f$. Prove that this defines a functor, called the* ***identity functor***.

5.10. Exercise. *Let $F = (F_0, F_1) :$* Top *$\to$* Set *be defined as follows. For every topological space X, set $F(X) = \pi_0(X)$, and for every continuous map $f : X \to Y$ between topological spaces X and Y, define $F_1(f) : \pi_0(X) \to \pi_0(Y)$ as the map sending PC_x to $PC_{f(x)}$. Prove that this defines a functor, called π_0.*

5.11. Exercise. *Let $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ be a functor between two categories $\mathscr{C}$ and $\mathscr{D}$, and let $f \in \mathscr{C}_1$ be an isomorphism. Prove that $F_1(f)$ is an isomorphism.*

5.12. A functor $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ is an **equivalence** if $F_{X,Y} : \mathsf{Hom}_{\mathscr{C}}(X,Y) \to \mathsf{Hom}_{\mathscr{D}}(F_0(X), F_0(Y))$ is bijective for all objects X and Y in $\mathscr{C}_0$, and for every object Z in $\mathscr{D}$, there is an object X in $\mathscr{C}_0$ such that $F_0(X) \cong Z$. If a functor satisfies the first part of the definition it is called **fully faithful**, whereas a functor satisfies the second part it is called **dense**.

5.13. Given functors $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ and $G = (G_0, G_1) : \mathscr{D} \to \mathscr{E}$, it is easy to see that $(G_0 \circ F_0, G_1 \circ F_1)$ is a functor from $\mathscr{C}$ to $\mathscr{E}$, called the **composition functor**, and it is denoted by $G \circ F$.

5.14. Given two functors $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ and $G = (G_0, G_1) : \mathscr{C} \to \mathscr{D}$, a **natural transformation** from F to G is a map $h : \mathscr{C}_0 \to \mathscr{D}_1$ such that $s_{\mathscr{D}} \circ h = F_0$, $t_{\mathscr{D}} \circ h = G_0$, and for every $f \in \mathscr{C}_1$, $G_1(f) \circ h(s_{\mathscr{C}}(f)) = h(t_{\mathscr{C}}(f)) \circ F_1(f)$. Equivalently, a natural transformation from F to G is a collection of morphisms $\{h_X = h(X) \in \mathsf{Hom}_{\mathscr{D}}(F_0(X), G_0(X)) : X \in \mathscr{C}_0\}$ such that $G_1(f) \circ h_X = h_Y \circ F_1(f)$, for every morphism $f \in \mathsf{Hom}_{\mathscr{C}}(X,Y)$, which can be represented graphically as the commutation of the following diagram

$$\begin{array}{ccc} F_0(X) & \xrightarrow{h_X} & G_0(X) \\ \downarrow{\scriptstyle F_1(f)} & & \downarrow{\scriptstyle G_1(f)} \\ F_0(Y) & \xrightarrow{h_Y} & G_0(Y) \end{array}$$

A **natural isomorphism** is a natural transformation h from F to G such that $h(X)$ is an isomorphism in $\mathscr{D}$, for all $X \in \mathscr{C}_0$. We say in this case that F and G are **naturally isomorphic**.

5.15. Exercise. *Let $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ be a functor between two categories $\mathscr{C}$ and $\mathscr{D}$. Prove that F is an equivalence if and only if there exists a functor $G = (G_0, G_1) : \mathscr{D} \to \mathscr{C}$ such that $F \circ G$ is naturally isomorphic to the identity functor of $\mathscr{D}$ and $G \circ F$ is naturally isomorphic to the identity functor of $\mathscr{C}$. The functor G is called a* ***quasi-inverse***.

5.16. Given a category $\mathscr{C}$, a subset $S \subseteq \mathscr{C}_0$ is called a **set of isomorphic classes of objects** of $\mathscr{C}$ if for every $Y \in \mathscr{C}_0$ there is $X \in S$ such that $X \cong Y$, and given $X, X' \in S$, $X \cong S'$ implies that $S = S'$.

5.17. Exercise. *Let $\mathscr{C}$ be a category with a set S of isomorphic classes of objects.*

(i) Assume that $S' \subseteq \mathscr{C}_0$ is another set of isomorphic classes of objects of $\mathscr{C}$. Prove that S and S' are in bijection.

(ii) and let $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ be an equivalence between two categories $\mathscr{C}$ and $\mathscr{D}$. Prove that the set $\{F(X) : X \in S\} \subseteq \mathscr{D}_0$ is a set of isomorphic classes of objects of $\mathscr{D}$.

§6. Homotopy I: The basics

6.1. Let $f, g : X \to Y$ be two continuous maps between topological spaces and let $A \subseteq X$ be a subset. We say that f is **homotopic to** g **relative to** A, a we will denote it by $f \simeq g \,(\mathsf{rel}\, A)$, if there is a continuous map $H : X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$, and $H(x,t) = f(x) = g(x)$, for all $x \in A$ and all $t \in [0,1]$. If $A = \emptyset$ we will simply say that **homotopic to** g, and we will denote it by $f \simeq g$.

6.2. Remark. *Note that $f \simeq g \,(\mathsf{rel}\, A)$ implies that $f(x) = g(x)$, for all $x \in A$.*

6.3. Lemma. *Let X, Y and Z be topological spaces, and $A \subseteq X$ be a subset.*

(i) Consider the relation on the set $\mathsf{Hom}_{\mathsf{Top}}(X,Y)$ of continuous maps from X to Y given by $f \simeq g \,(\mathsf{rel}\, A)$ defined before for $f, g : X \to Y$. This is an equivalence relation.

(ii) Given continuous maps $f, g : X \to Y$ and $f', g' : Y \to Z$ such that $f \simeq g \,(\mathsf{rel}\, A)$ and $f' \simeq g' \,(\mathsf{rel}\, f(A))$, then $f' \circ f \simeq g' \circ g \,(\mathsf{rel}\, A)$.

Proof. (i) It is clear that $f \simeq f \,(\mathsf{rel}\, A)$, for all $f \in \mathsf{Hom}_{\mathsf{Top}}(X,Y)$, using the homotopy $H(x,t) = f(x)$, for all $x \in X$ and $t \in [0,1]$. On the other hand, if $f \simeq g \,(\mathsf{rel}\, A)$ by means of the homotopy $H : X \times [0,1] \to Y$, then $G : X \times [0,1] \to Y$ given by $G(x,t) = H(x, 1-t)$, for all $x \in X$ and $t \in [0,1]$, is a homotopy from g to f relative to A, *i.e.* $g \simeq f \,(\mathsf{rel}\, A)$. Finally, if $f, g, h \in \mathsf{Hom}_{\mathsf{Top}}(X,Y)$ satisfy that $f \simeq g \,(\mathsf{rel}\, A)$ and $g \simeq h \,(\mathsf{rel}\, A)$ by means of homotopies $H : X \times [0,1] \to Y$ and $G : X \times [0,1] \to Y$, respectively, then the map $K : X \times [0,1] \to Y$ given by

$$K(x,t) = \begin{cases} H(x, 2t), & \text{if } t \in [0, 1/2], \\ G(x, 2t-1), & \text{if } t \in [1/2, 1], \end{cases}$$

is a homotopy from f to h relative to A.

(ii) Let $H : X \times [0,1] \to Y$ be a homotopy from f to g relative to A and $G : Y \times [0,1] \to Z$ be a homotopy from f' to g' relative to $f(A)$. Define the map $K : X \times [0,1] \to Z$ by $K(x,t) = G(H(x,t), t)$, for all $x \in X$ and $t \in [0,1]$. This gives a homotopy from $f' \circ f$ to $g' \circ g$ relative to A, as was to be shown. □

6.4. Exercise. *Let $U \subseteq \mathbb{R}^n$ for $n \in \mathbb{N}$, provided with the subspace topology, $A \subseteq U$ and $f, g : X \to U$ be two continuous functions defined on a topological space X such that $f(x) = g(x)$ for all $x \in A$ and that*

$$[f(x), g(x)] = \{tf(x) + (1-t)g(x) : t \in [0,1]\} \subseteq U$$

for all $x \in X$. Prove that $f \simeq g$ relative to A.

6.5. Exercise. *(i) Let $\mathscr{C}_0$ be the collection of all topological spaces, $\mathsf{Hom}_{\mathscr{C}}(X,Y)$ the set of all equivalence classes of continuous maps from X to Y under the equivalence relation given by homotopy, and id_X is the equivalence class of the identity map from X to itself. Prove that usual composition of maps induces a map*

$$\mathsf{Hom}_{\mathscr{C}}(Y,Z) \times \mathsf{Hom}_{\mathscr{C}}(X,Y) \to \mathsf{Hom}_{\mathscr{C}}(X,Z)$$

for all topological spaces X, Y, Z. Moreover, prove that the previous data forms a category, denoted by hTop, that is called the ***(naive) homotopy category****.*

(ii) Prove that the maps $(Q_0, Q_1) : \mathsf{Top} \to \mathsf{hTop}$ given by $Q_0(X) = X$ for all topological spaces X and such that

$$Q_{X,Y} : \mathsf{Hom}_{\mathsf{Top}}(X,Y) \to \mathsf{Hom}_{\mathsf{hTop}}(X,Y)$$

sends any continuous map $f : X \to Y$ to the unique equivalence class under the homotopy equivalence relation including f, form a functor, called the ***quotient*** *functor and denoted by Q.*

(iii) Prove that there exists a unique functor $\bar{\pi}_0 : \mathsf{hTop} \to \mathsf{Set}$ such that $\bar{\pi}_0 \circ Q = \pi_0$, where π_0 is the functor defined in Exercise 5.10. By abuse of notation, we will denote the functor $\bar{\pi}_0$ also by π_0.

6.6. Let $f : X \to Y$ be a continuous map between topological spaces. We say that f is **null homotopic** if there exists $y_0 \in Y$ such that f is homotopic to the constant map $c_{y_0} : X \to Y$ that sends every $x \in X$ to y_0. On the other hand, we say that f is a **homotopy equivalence** if there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_Y$. This is tantamount to say the image of f under the quotient functor $Q : \mathsf{Top} \to \mathsf{hTop}$ defined before is an isomorphism.

6.7. Exercise. *Let $n \in \mathbb{N}$ and $S^n \subseteq \mathbb{R}^{n+1}$ be the unit sphere. Prove that the inclusion $S^n \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}_{\mathbb{R}^{n+1}}\}$ is a homotopy equivalence that is not a homeomorphism.*

6.8. We say that a topological space X is **contractible** if X is contractible if the identity map id_X is null homotopic. Equivalently, X is contractible if it is isomorphic to the singleton space $\{*\}$ in the category hTop. A topological space X is said to be **locally contractible** if every point has a fundamental system of open contractible neighborhoods.

6.9. Exercise. *Recall that a set $A \subseteq \mathbb{R}^n$ is called* ***convex*** *(resp.,* ***star-shaped****) if for every pair of points $x, y \in A$, $tx + (1-t)y \in A$ for all $t \in [0,1]$ (resp., if there exists $x_0 \in A$ such that $tx_0 + (1-t)x \in A$ for all $t \in [0,1]$ and $x \in A$). We call x_0 a* ***star center*** *of the star-shaped set.*

(i) Prove that any convex set is star-shaped, and that any star-shaped set is contractible.

(ii) Prove that $X = S^1 \setminus \{1\} \subseteq \mathbb{R}^2$, with the subspace topology, is contractible but not star-shaped.

6.10. Exercise (Or why this definition of homotopy is (almost) useless in algebraic geometry)**.** *A topological space X is called* ***sober*** *if for every nonempty irreducible closed subset $Y \subseteq X$ there exists a unique $y \in X$, called the* ***generic point*** *of Y, such that $Y = \overline{\{y\}}$. Let X be a nonempty topological space having a point $x_0 \in X$ such that $\{x_0\}$ is dense in X. Prove that X is contractible. Deduce that an irreducible and sober topological space is contractible.*

Hint. Consider $H : X \times [0,1] \to X$ as $H(x,t) = x_0$ for $x \in X$ and $t \in \,]0,1]$ and $H(x,0) = x$ for $x \in X$.

§7. Homotopy II: The fundamental group

7.1. Given a point x in a topological space, we will denote by $\varepsilon_x : [0,1] \to X$ the constant path, *i.e.* $\varepsilon_x(t) = x$ for all $t \in [0,1]$. Moreover, if two paths γ and η in X satisfy that $\gamma \simeq \eta \,(\mathsf{rel}\{0,1\})$, we will say that γ is **path-homotopic** to η (or that there is a **homotopy as paths** from γ to η).

7.2. Lemma. *Let X be a nonempty topological space.*

(i) Let γ be a path and set $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$. Then

$$\varepsilon_{x_0} \cdot \gamma \simeq \gamma \simeq \gamma \cdot \varepsilon_{x_1} \,(\mathsf{rel}\{0,1\}).$$

(ii) Given paths $\gamma_1, \gamma_2, \eta_1, \eta_2$ in X such that $\gamma_2(0) = \gamma_1(1)$, $\eta_2(0) = \eta_1(1)$ and $\gamma_i \simeq \eta_i \,(\mathsf{rel}\{0,1\})$ for $i = 1,2$, we have

$$\gamma_1 \cdot \gamma_2 \simeq \eta_1 \cdot \eta_2 \,(\mathsf{rel}\{0,1\}).$$

(iii) Given any path γ in X with $x_0 = \gamma(0)$, $\gamma \cdot \gamma^- \simeq \varepsilon_{x_0} \,(\mathsf{rel}\{0,1\})$.

(iv) Given paths $\gamma_1, \gamma_2, \gamma_3$ in X such that $\gamma_{i+1}(0) = \gamma_i(1)$, for $i = 1,2$, we have

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \simeq \gamma_1 \cdot (\gamma_2 \cdot \gamma_3) \,(\mathsf{rel}\{0,1\}).$$

Proof. (i) Consider the maps $f_i : [0,1] \to [0,1]$ for $i = 1,2,3$ given by

$$f_1(t) = \begin{cases} 0, & \text{if } t \in [0,1/2], \\ 2t-1, & \text{if } t \in [1/2,1], \end{cases} \qquad f_3(t) = \begin{cases} 2t, & \text{if } t \in [0,1/2], \\ 1, & \text{if } t \in [1/2,1], \end{cases}$$

and $f_2(t) = t$, for all $t \in [0,1]$, which are clearly continuous. By Exercise 6.4, the three maps are homotopic relative to $\{0,1\}$, which in turn implies that the three paths $\gamma \circ f_i$ are homotopic relative to $\{0,1\}$, by Lemma 6.3, (ii). Since $\gamma \circ f_1 = \varepsilon_{x_0} \cdot \gamma$, $\gamma \circ f_2 = \gamma$ and $\gamma \circ f_3 = \gamma \cdot \varepsilon_{x_1}$, the claim follows.

(ii) Let $H_i : [0,1] \times [0,1] \to X$ be the homotopy from γ_i to η_i relative to $\{0,1\}$) for $i = 1,2$. Then, the map $H : [0,1] \times [0,1] \to X$ given by

$$H(s,t) = \begin{cases} H_1(2s,t), & \text{if } s \in [0,1/2] \text{ and } t \in [0,1], \\ H_2(2s-1,t), & \text{if } s \in [1/2,1] \text{ and } t \in [0,1], \end{cases}$$

gives a homotopy from $\gamma_1 \cdot \gamma_2$ to $\eta_1 \cdot \eta_2$ relative to $\{0,1\}$.

(iii) The map $H : [0,1] \times [0,1] \to [0,1]$ given by

$$H(s,t) = \begin{cases} \gamma\big(2s(1-t)\big), & \text{if } s \in [0,1/2] \text{ and } t \in [0,1], \\ \gamma\big(2(1-s)(1-t)\big), & \text{if } s \in [1/2,1] \text{ and } t \in [0,1], \end{cases}$$

which is clearly continuous, gives a homotopy from $\gamma \cdot \gamma^-$ to ε_{x_0} relative to $\{0,1\}$.

(iv) Let $\gamma : [0,1] \to X$ be the continuous path given by

$$\gamma(t) = \begin{cases} \gamma_1(3t), & \text{if } t \in [0,1/3], \\ \gamma_2(3t-1), & \text{if } t \in [1/3,2/3], \\ \gamma_3(3t-2), & \text{if } t \in [2/3,1], \end{cases}$$

and consider the maps $g_i : [0,1] \to [0,1]$ for $i = 1,2$ given by

$$g_1(t) = \begin{cases} 4t/3, & \text{if } t \in [0,1/2], \\ (2t+1)/3, & \text{if } t \in [1/2,1], \end{cases} \qquad g_2(t) = \begin{cases} 2t/3, & \text{if } t \in [0,1/2], \\ (4t-1)/3, & \text{if } t \in [1/2,1], \end{cases}$$

which are clearly continuous. It is a simple computation to verify that $\gamma \circ g_1 = (\gamma_1 \cdot \gamma_2) \cdot \gamma_3$ and $\gamma \circ g_2 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$. By Exercise 6.4, the maps g_1 and g_2 are homotopic relative to $\{0,1\}$, which in turn implies that the paths $\gamma \circ g_1$ and $\gamma \circ g_2$ are homotopic relative to $\{0,1\}$, by Lemma 6.3, (ii), and the claim follows.

□

7.3. Let X be a topological space and $x_0 \in X$ be a fixed point. Define the **space of pointed loops** as

$$\Omega(X, x_0) = \{\gamma : [0,1] \to X : \gamma \text{ is continuous and } \gamma(0) = \gamma(1) = x_0\}.$$

Consider the equivalence relation on $\Omega(X, x_0)$ given by $\gamma \simeq \gamma' \, (\mathsf{rel}\{0,1\})$ and define $\pi_1(X, x_0)$ as the corresponding set of equivalence classes. By Lemma 7.2, the concatenation of loops gives a group structure on $\pi_1(X, x_0)$, which is called the **fundamental group**. We will denote by $[\gamma] \in \pi_1(X, x_0)$ the equivalence class of a loop γ based at x_0, and the product of two elements in $[\gamma], [\gamma'] \in \pi_1(X, x_0)$ will be denoted by $[\gamma] \cdot [\gamma'] = [\gamma \cdot \gamma']$.

7.4. Exercise. *(i) Let $\mathscr{C}_0$ be the collection of all pairs (X, x), where X is a nonempty topological spaces and $x \in X$, $\mathsf{Hom}_{\mathscr{C}}((X,x),(Y,y))$ the set of all continuous maps f from X to Y such that $f(x) = y$, $\circ_{\mathscr{C}}$ the usual composition of maps and id_X is the identity map from X to itself. Prove that this forms a category, denoted by $\mathsf{Top}_\bullet$, called the category of* ***pointed topological spaces****.*

(ii) Let $\mathscr{C}_0$ be the collection of all pairs (X, x), where X is a nonempty topological spaces and $x \in X$, $\mathsf{Hom}_{\mathscr{C}}((X,x),(Y,y))$ the set of all equivalence classes of continuous maps f from X to Y such that $f(x) = y$ with respect to homotopy relative to $\{x\}$, $\circ_{\mathscr{C}}$ the usual composition of maps and id_X is the equivalence class of the identity map from X to itself. Prove that this forms a category, denoted by $\mathsf{hTop}_\bullet$.

(iii) Prove that the maps $(F_0, F_1) : \mathsf{Top}_\bullet \to \mathsf{hTop}_\bullet$ given by $F_0(X,x) = (X,x)$ and such that

$$F_{X,Y} : \mathsf{Hom}_{\mathsf{Top}_\bullet}\big((X,x),(Y,y)\big) \to \mathsf{Hom}_{\mathsf{hTop}_\bullet}\big((X,x),(Y,y)\big)$$

sends any continuous map $f : X \to Y$ satisfying that $f(x) = y$ to the unique equivalence class under the homotopy equivalence relation relative to $\{x\}$ including f, form a functor, called the ***pointed quotient*** *functor and denoted by $Q_\bullet$.*

7.5. Given two pointed topological spaces (X, x) and (Y, y), one defines the **wedge product** $(X,x) \wedge (Y,y)$ as the following pointed topological space. Consider $Z = X \sqcup Y$, endowed with the disjoint union topology. Let $\sim$ be the equivalence relation on Z generated by $x \sim y$. The underlying topological space $X \wedge Y$ of $(X,x) \wedge (Y,y)$ is $Z/\sim$ endowed with the quotient topology. The distinguished point of $X \wedge Y$ is the equivalence class $\{x, y\}$.

7.6. Exercise. *Consider the $X_\pm = \partial B_1(\pm 1, 0) = S^1 \pm (1,0) \subseteq \mathbb{R}^2$ and $X = X_+ \cup X_- \subseteq \mathbb{R}^2$ be endowed with the subspace topology. Prove that the pointed topological spaces $(X, 0)$ and $(S^1, 1) \wedge (S^1, 1)$ are isomorphic.*

7.7. Let $F = (F_0, F_1) : \mathsf{Top}_\bullet \to \mathsf{Grp}$ be defined as follows. For every pointed topological space (X, x), set $F(X) = \pi_1(X, x)$, and for every continuous map $f : X \to Y$ between topological spaces X and Y, define $F_1(f) : \pi_1(X,x) \to \pi_1(Y, f(x))$ as the map sending the equivalence class $[\gamma]$ of a loop based at x to $[f \circ \gamma]$. It is easy to see that this defines a functor, called π_1.

7.8. Exercise. *Prove that there exists a unique functor $\bar{\pi}_1 : \mathsf{hTop}_\bullet \to \mathsf{Grp}$ such that $\bar{\pi}_1 \circ Q_\bullet = \pi_1$. By abuse of notation, we will denote the functor $\bar{\pi}_1$ also by π_1.*

7.9. Lemma. *Let X be a path connected topological space and let $x, x' \in X$ be two different points. Then, there is an isomorphism of groups $\pi_1(X, x) \simeq \pi_1(X, x')$.*

Proof. Let $\eta : [0,1] \to X$ be a path satisfying that $\eta(0) = x'$ and $\eta(1) = x$, which exists by assumption. Define the map

$$\mathsf{Ad}_\eta : \pi_1(X,x) \to \pi_1(X,x')$$

sending $[\gamma] \in \pi_1(X,x)$, where γ is a loop based at x, to $[\eta \cdot \gamma \cdot \eta^-] = [\eta] \cdot [\gamma] \cdot [\eta^-]$. This is well defined by Lemma 7.2, (ii), and it is clearly a morphism of groups. Moreover, by Lemma 7.2, (iii), it is also a bijection, since its inverse is the map

$$\mathsf{Ad}_{\eta^-} : \pi_1(X,x') \to \pi_1(X,x)$$

sending $[\gamma'] \in \pi_1(X,x')$, where γ' is a loop based at x', to $[\eta^- \cdot \gamma' \cdot \eta] = [\eta^-] \cdot [\gamma'] \cdot [\eta]$. □

7.10. Exercise. *Let X_1 and X_2 be two topological spaces, and let $x_1 \in X_1$ and $x_2 \in X_2$ be two elements. Consider the canonical projections $p_i : X_1 \times X_2 \to X_i$ for $i = 1,2$. They induce morphisms of groups $\pi_1(p_i) : \pi_1(X_1 \times X_2, (x_1,x_2)) \to \pi_1(X_i, x_i)$ for $i = 1,2$.*

(i) Prove that the map

$$\pi_1\big(X_1 \times X_2, (x_1,x_2)\big) \to \pi_1(X_1,x_1) \times \pi_1(X_2,x_2)$$

given by $[\gamma] \mapsto (\pi_1(p_1)([\gamma]), \pi_1(p_2)([\gamma]))$ is a morphism of groups.

(ii) Given paths $\gamma_i : [0,1] \to X_i$ for $i = 1,2$, define the path $(\gamma_1,\gamma_2) : [0,1] \to X_1 \times X_2$ by means of $(\gamma_1,\gamma_2)(t) = (\gamma_1(t),\gamma_2(t))$, for all $t \in [0,1]$. Show that the map

$$\pi_1(X_1,x_1) \times \pi_1(X_2,x_2) \to \pi_1\big(X_1 \times X_2, (x_1,x_2)\big)$$

given by $([\gamma_1],[\gamma_2]) \mapsto [(\gamma_1,\gamma_2)]$ is well defined and it is the inverse of the map in the previous item. Conclude that they are isomorphisms of groups.

7.11. Exercise. *Recall that a* ***topological group*** *is a topological space G together with a group structure given by product $\mu : G \times G \to G$ that is also a continuous map, such that the map $i : G \to G$ sending g to g^{-1} is continuous. Let e_G denote the unit of G. By the previous exercise, $\pi_1(G \times G, (e_G,e_G)) \cong \pi_1(G,e_G) \times \pi_1(G,e_G)$, and $\pi_1(\mu) : \pi_1(G,e_G) \times \pi_1(G,e_G) \to \pi_1(G,e_G)$ is a morphism of groups. Prove that $\pi_1(\mu)$ coincides with the product of $\pi_1(G,e_G)$, and deduce that that $\pi_1(G,e_G)$ is an abelian group.*

7.12. A topological space X is called **simply connected** if it is path connected and $\pi_1(X,x) = 1$ for all $x \in X$ (or, equivalently, for one $x \in X$). We say that a topological space X is **locally simply connected** if every point $x \in X$ has a fundamental system of open simply connected neighborhoods.

7.13. Lemma. *Let $f_1, f_2 : X \to Y$ be two continuous maps and let $H : X \times [0,1] \to Y$ be a homotopy from f_1 to f_2. Let $x \in X$, and set $y_i = f_i(x)$, for $i = 1,2$. Let $\eta : [0,1] \to Y$ be the path given by $\eta(t) = H(x,t)$, for $t \in [0,1]$. Then, $\pi_1(f_1) = \mathsf{Ad}_\eta \circ \pi_1(f_2)$, i.e. the following triangle*

$$\begin{array}{ccc} & & \pi_1(Y,y_2) \\ & \overset{\pi_1(f_2)}{\nearrow} & \\ \pi_1(X,x) & & \Big\downarrow \mathsf{Ad}_\eta \\ & \underset{\pi_1(f_1)}{\searrow} & \\ & & \pi_1(Y,y_1) \end{array}$$

commutes.

Proof. Let γ be a loop based at x. We have to show that $(f_1 \circ \gamma) \cdot \eta \simeq \eta \cdot (f_2 \circ \gamma)\,(\mathsf{rel}\{0,1\})$. Define $h : [0,1] \times [0,1] \to [0,1]$ by $h(s,t) = H(\gamma(s), t)$, for $s, t \in [0,1]$. Consider the maps $h_i : [0,1] \to [0,1] \times [0,1]$ for $i = 1,2$ given by

$$h_1(t) = \begin{cases} (2t, 0), & \text{if } t \in [0, 1/2], \\ (1, 2t-1), & \text{if } t \in [1/2, 1], \end{cases} \qquad h_2(t) = \begin{cases} (0, 2t), & \text{if } t \in [0, 1/2], \\ (2t-1, 1), & \text{if } t \in [1/2, 1], \end{cases}$$

which are clearly continuous. It is a simple computation to verify that $h \circ h_1 = (f_1 \circ \gamma) \cdot \eta$ and $h \circ h_2 = \eta \cdot (f_2 \circ \gamma)$. By Exercise 6.4, the maps h_1 and h_2 are homotopic relative to $\{0,1\}$, which in turn implies that the paths $h \circ h_1$ and $h \circ h_2$ are homotopic relative to $\{0,1\}$, by Lemma 6.3, (ii), and the lemma follows. □

7.14. Corollary. *Let X and Y be two nonempty topological spaces, and let $x \in X$. If $f : X \to Y$ is a homotopy equivalence, then $\pi_1(f) : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism of groups. In particular, X is simply connected if and only if Y is.*

Proof. Let $g : Y \to X$ be a continuous map such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. By the previous lemma applied to $g \circ f$ and $f \circ g$, $\pi_1(g \circ f) = \pi_1(g) \circ \pi_1(f)$ and $\pi_1(f \circ g) = \pi_1(f) \circ \pi_1(g)$ are bijections, so *a fortiori* $\pi_1(g) \circ \pi_1(f)$ is an injection and $\pi_1(f) \circ \pi_1(g)$ is a surjection. This implies that $\pi_1(f)$ is injective and $\pi_1(f)$ is surjective, respectively, so $\pi_1(f)$ is an isomorphism as claimed. The last part is an immediate consequence. □

7.15. Exercise. *Let $Y = [-1,1] \times\,]-1,1[$ and $\sim$ be the equivalence relation on Y generated by $(-1, y) \sim (1, -y)$, for all $y \in\,]-1,1[$. Define $\mathbb{M} = Y/\sim$ the set of equivalence classes, provide with the quotient topology. It is usually called the (topological)* ***Möbius band***. *Let $X' \subseteq \mathbb{M}$ be the image of $[-1,1] \times \{0\}$ under the canonical projection $Y \to Y/\sim$.*

(i) Prove that X' is homeomorhic to S^1.

(ii) Prove that the continuous map $p : Y \to X'$ given by $(x, y) \mapsto (x, 0)$, for all $(x, y) \in Y$ induces a continuous map $p : \mathbb{M} \to X'$ such that the composition of p and the inclusion $X' \subseteq \mathbb{M}$ is homotopic to the identity map of $\mathbb{M}$. Deduce that $\pi_1(\mathbb{M}, \bar{\mathbf{0}}_{\mathbb{R}^2}) \cong \pi_1(S^1, 1)$.

7.16. Exercise. *(i) Let $n \in \mathbb{N}$ be a positive integer and $U \subseteq \mathbb{R}^n$ be a subset. A path $\gamma : [0,1] \to U$ is said to be* ***piecewise affine*** *if there exists a finite set $S = \{0 = t_0 < t_1 < \cdots < t_m = 1\} \subseteq [0,1]$ such that $\gamma|_{[t_i, t_{i+1}]}$ is an affine map for all $i \in [\![0, m-1]\!]$, i.e.*

$$\gamma(t) = \gamma(t_i) + (\gamma(t_{i+1}) - \gamma(t_i)) \frac{t - t_i}{t_{i+1} - t_i},$$

for all $t \in [t_i, t_{i+1}]$ and $i \in [\![0, m-1]\!]$. Prove that, given any path $\gamma : [0,1] \to U$ in an open subset of $\mathbb{R}^n$ there exists a piecewise affine path $\gamma' : [0,1] \to U$ such that $\gamma \simeq \gamma'\,(\mathsf{rel}\{0,1\})$.
Hint. Consider $\delta = \inf\{\mathsf{d}(\gamma(t), \mathbb{R}^n \setminus U) : t \in [0,1]\} \in \mathbb{R}_{>0}$, where $\mathsf{d}(x_0, A) = \inf\{\|x_0 - x\| : x \in A\}$ is the usual Euclidean distance between a point $x_0 \in \mathbb{R}^n$ and a subset $A \subseteq \mathbb{R}^n$, and, by uniform continuity of γ, let $m \in \mathbb{N}$ be such that $\|\gamma(t) - \gamma(s)\| \leq \mathsf{d}/2$ if $|t - s| \leq 1/m$ and $t, s \in [0,1]$. Define $t_i = i/m$ for $i \in [\![0, m]\!]$ and note that $\gamma(t_{i+1}) \in B_{\mathsf{d}}(\gamma(t_i)) \subseteq U$, for all $i \in [\![0, m-1]\!]$.

(ii) Let $n \in \mathbb{N}$ be a positive integer and $x \in \mathbb{R}^{n+1}$ be a nonzero element. Prove that $\mathbb{R}^{n+1} \setminus \{\lambda x : \lambda \in \mathbb{R}_{\geq 0}\}$ is star-shaped with star center $-x$ and in consequence it is contractible. Deduce that $S^n \setminus \{x\}$ is contractible for any $x \in S^n$.

(iii) Let $n \geq 2$ be an integer. Prove that given any piecewise affine loop $\gamma : [0,1] \to \mathbb{R}^{n+1} \setminus \{\mathbf{0}_{\mathbb{R}^{n+1}}\}$ there is a nonzero $x \in \mathbb{R}^{n+1}$ such that $\mathsf{Img}(\gamma) \subseteq \mathbb{R}^{n+1} \setminus \{\lambda x : \lambda \in \mathbb{R}_{\geq 0}\}$. Deduce that $\mathbb{R}^{n+1} \setminus \{\mathbf{0}_{\mathbb{R}^{n+1}}\}$ (and hence S^n) is simply connected. One can also prove that $\mathbb{R}^{n+1} \setminus \{\mathbf{0}_{\mathbb{R}^{n+1}}\}$ (and hence S^n) is not contractible, but it is outside the scope of these notes.

7.17. Since the singleton space $\{*\}$ is simply connected, the previous corollary tells us that any contractible space is simply connected. We have the following implications:

$$\text{convex} \Longrightarrow \text{star-shaped} \Longrightarrow \text{contractible} \Longrightarrow \text{simply connected} \Longrightarrow \text{path connected} \Longrightarrow \text{connected}$$

Note that none of the converses hold in general (see Exercises 4.6, 6.9, 7.16, 9.20 and 10.8).

§8. Algebraic interlude II: The category of transitive G-sets

8.1. Let G be a group, whose product we write by a dot or by a simple yuxtaposition, and whose unit we denote by e_G. We will denote its opposite group by G^{op}, whose underlying set is G and whose product is given by $x \cdot_{\mathrm{op}} x' = x' \cdot x$, for $x, x' \in G$. It is clear that G and G^{op} are isomorphic groups via $g \mapsto g^{-1}$. Recall that a **right** (resp., **left**) **group action** of G on an object X in a category $\mathscr{C}$ is an morphism of groups $\rho : G \to \mathsf{Aut}_{\mathscr{C}}(X)^{\mathrm{op}}$ (resp., $\rho : G \to \mathsf{Aut}_{\mathscr{C}}(X)$). The action is called **effective** if ρ is injective. In the case of Set, Top, or other concrete categories, we typically write $x \cdot g$ (resp., $g \cdot x$) instead of $\rho(g)(x)$, for $g \in G$ and $x \in X$. We will focus in this section on the category Set, but the reader can check that all the results (and exercises) in this section also hold in other concrete categories, such as Top.

8.2. If X is a set provided with a right (resp., left) action of G is typically called a **right** (resp., **left**) **G-set**. Given a nonempty right (resp., left) G-set X and $x \in X$ we denote by $x \cdot G = \{x \cdot g : g \in G\}$ (resp., $G \cdot x = \{g \cdot x : g \in G\}$) the **orbit** of x, and by X/G (resp., $G\backslash X$) the set of all orbits of all elements of X. For simplicity, we will mainly deal in this section with right G-sets, and will call them just G-sets for shortness. This assumption is harmless since any left G-set X is a right G-set via $x \cdot g = g^{-1} \cdot x$, for $g \in G$ and $x \in X$, and vice versa. Equivalently, a left G-set is equivalent to a right G^{op}-set.

8.3. Recall that a nonempty G-set (or the corresponding action) is **transitive** if there is only one orbit. Note that any G-set X is a disjoint union of transitive G-sets, the orbits of X. Moreover, if $H \leq G$ is a subgroup, then the product of G gives a left action of H on G and $H\backslash G$ has a right action of G given by $(H \cdot x) \cdot g = H \cdot (xg)$, for $x, g \in G$, which turns $H\backslash G$ into a transitive G-set.

8.4. Given x in a G-set X, the **isotropy group** of x is

$$G_x = \{g \in G : x \cdot g = x\}.$$

It is clearly a subgroup of G. Moreover, it is clear that

$$G_{x \cdot g} = g^{-1} G_x g, \tag{1}$$

for all $x \in X$ and $g \in G$. We recall that a group action of G on X is called **free** if the isotropy group G_x of every $x \in X$ is trivial.

8.5. Given two G-sets X and X', a **G-equivariant map** (or simply a **morphism of G-sets**) from X to X' is a map $f : X \to X'$ such that $f(x \cdot g) = f(x) \cdot g$, for all $x \in X$ and $g \in G$. Note that any transitive G-set X is isomorphic to a G-set of the form $H\backslash G$, for some subgroup $H \leq G$. Indeed, given $x \in X$, the surjective G-equivariant map $G \to X$ sending g to $x \cdot g$ induces an isomorphism $G_x\backslash G \to X$ of G-sets.

8.6. Lemma. *Let G be a group and X, X' two nonempty G-sets.*

(i) *If X is transitive and $f, f' : X \to X'$ are two G-equivariant maps such that $f(x_0) = f'(x_0)$ for some $x_0 \in X$, then $f = g$.*

(ii) *If X' is transitive and $f : X \to X'$ is a G-equivariant map, then it is surjective.*

(iii) *If X and X' are transitive, given $x \in X$ and $x' \in X'$, then there exists a (resp., bijective) G-equivariant map $f : X \to X'$ such that $f(x) = x'$ if and only if $G_x \subseteq G_{x'}$ (resp., $G_x = G_{x'}$).*

Proof. For item (i), note that, for every $x \in X$, there exists $g \in G$ such that $x_0 \cdot g = x$, so

$$f(x) = f(x_0 \cdot g) = f(x_0) \cdot g = f'(x_0) \cdot g = f'(x_0 \cdot g) = f'(x).$$

For item (ii), note that, since X is nonempty, there exists $x_0 \in X$. By transitivity of X', for every $x' \in X$, there ,there exists $g \in G$ such that $f(x_0) \cdot g = x'$, so $f(x_0 \cdot g) = f(x_0) \cdot g = x'$.

We now prove the first part of item (iii). If a G-equivariant map $f : X \to X'$ such that $f(x) = x'$ exists, then, given $g \in G_x$, $x' = f(x) = f(x \cdot g) = f(x) \cdot g = x' \cdot g$, so $g \in G_{x'}$. Conversely, assume that $G_x \subseteq G_{x'}$, and define $f : X \to X'$ by $f(x \cdot g) = x' \cdot g$, for all $g \in G$. This is well-defined since $x \cdot g = x \cdot g'$ if and only if $gg'^{-1} \in G_x \subseteq G_{x'}$, which implies that $x' \cdot g = x' \cdot g'$. Moreover, since X is transitive, every element of X is of the form $x \cdot g$, for some $g \in G$. The map f is clearly G-equivariant and $f(x) = x'$.

The statement concerning the bijective property in item (iii) follows directly. Indeed, a G-equivariant map f is bijective if and only if it has a G-equivariant inverse map. Hence, applying the first part of item (iii), we see that a bijective G-equivariant map f sending x to x' tells us that $G_x = G_{x'}$. Conversely, if $G_x = G_{x'}$, there exists a G-equivariant map f sending x to x' and another f' sending x' to x. Then, the G-equivariant map $f' \circ f$ send x to itself, whereas $f \circ f'$ sends x' to itself, so they must be the identity of X and of X' respectively by (i). □

8.7. Recall that, given a subgroup $H \leq G$ of a group G, the **normalizer** $N_G(H)$ of H in G is the subgroup of G given by the element $g \in G$ such that $g^{-1}Hg = H$. Equivalently, it is the largest subgroup of G including H such that H is normal in it.

8.8. Proposition. *Let G be a group, X a (nonempty) transitive G-set and $x_0 \in X$. Denote by $\mathsf{Aut}_G(X)$ the group of G-equivariant automorphisms of X. Define the map $e_{x_0} : N_G(G_{x_0}) \to \mathsf{Aut}_G(X)$ sending g to the unique G-equivariant automorphism of X satisfying that $e_{x_0}(g)(x_0) = x_0 \cdot g$. Then e_{x_0} is surjective morphism of groups with kernel G_{x_0}, so it induces an isomorphism of groups*

$$\bar{e}_{x_0} : N_G(G_{x_0})/G_{x_0} \to \mathsf{Aut}_G(X).$$

Proof. We first note that the map e_{x_0} is well defined. Let $g \in N_G(G_{x_0})$, so $g^{-1} \in N_G(G_{x_0})$. Then, $G_{x_0 \cdot g} = g^{-1}G_{x_0}g = G_{x_0}$, and there exists a unique G-equivariant automorphism $e_{x_0}(g) : X \to X$ sending x_0 to $x_0 \cdot g$ by Lemma 8.6, (iii). Moreover, e_{x_0} is a morphism of groups, since

$$\begin{aligned} e_{x_0}(gg')(x_0) &= x_0 \cdot (gg') = (x_0 \cdot g) \cdot g' = e_{x_0}(g)(x_0) \cdot g' = e_{x_0}(g)(x_0 \cdot g') \\ &= e_{x_0}(g)(e_{x_0}(g')(x_0)) = (e_{x_0}(g) \circ (e_{x_0}(g'))(x_0), \end{aligned}$$

for all $g, g' \in N_G(G_{x_0})$, so $e_{x_0}(gg') = e_{x_0}(g) \circ (e_{x_0}(g')$ for all $g, g' \in N_G(G_{x_0})$, by Lemma 8.6, (i).

The morphism of groups e_{x_0} is also surjective. Indeed, given $f \in \mathsf{Aut}_G(X)$, let $x'_0 = f(x_0)$. Since X is transitive there exists $g \in G$ such that $x'_0 = x_0 \cdot g$, and in particular $G_{x'_0} = g^{-1}G_{x_0}g$. By Lemma 8.6, (iii), $G_{x_0} = G_{x'_0}$, so $G_{x_0} = g^{-1}G_{x_0}g$, which implies that $g \in N_G(G_{x_0})$. Since $e_{x_0}(g)(x_0) = x_0 \cdot g = x'_0 = f(x_0)$, Lemma 8.6, (i) tells us thet $e_{x_0} = f$.

We will finally prove that the kernel of e_{x_0} is G_{x_0}. It is clear that $g \in \mathsf{Ker}(e_{x_0})$ if and only if $e_{x_0}(g) = \mathrm{id}_X$, which, by Lemma 8.6, (i), is tantamount to $e_{x_0}(g)(x_0) = x_0 \cdot g = x_0$, which is equivalent to $g \in G_{x_0}$. □

8.9. Exercise. *Let X be a right G-set and Y a left G-set. Consider the left action of G on $X \times Y$ given by $g \cdot (x, y) = (x \cdot g^{-1}, g \cdot y)$, for $g \in G$, $x \in X$ and $y \in Y$. Define $X \times^G Y$ to be $G \backslash (X \times Y)$.*

(i) *Prove that if* $X = \sqcup_{i\in I} Y_i$*, with* X_i *right* G*-sets, then the left* G*-set* $X \times Y$ *is isomorphic to the disjoint union left* G*-set* $\sqcup_{i\in I}(X \times Y_i)$*. Conclude that the sets* $X \times^G Y$ *and* $\sqcup_{i\in I}(X \times^G Y_i)$ *are in bijection.*

(ii) *Let* $X = H\backslash G$*.*

(a) *Prove that the map* $Y \to X \times^G Y$ *given by* $y \mapsto [(H, y)]$ *induces a map* $\iota : H\backslash Y \to X \times^G Y$*.*

(b) *Prove that the map* $G \times Y \to Y$ *given by* $(g, y) \mapsto g \cdot y$ *is* G*-equivariant, where* $G \times Y$ *has the regular left action given by* $g \cdot (g', y) = (gg', y)$*, for all* $g, g' \in G$ *and* $y \in Y$*. Prove that the previous mapping induces a map* $\mu : X \times^G Y \to H\backslash Y$*.*

(c) *Finally, prove that* ι *and* μ *are inverse to each other.*

8.10. Let G and H be two groups. Recall that a **G-H-biset** is a left $(G \times H^{\mathrm{op}})$-set, *i.e.* a set X provided with a left action of G and a right action of H such that $g \cdot (x \cdot h) = (g \cdot x) \cdot h$, for all $g \in G$, $h \in H$ and $x \in X$. Equivalently, a G-H-biset is a set X together with a morphism of groups $H \to \mathrm{Aut}_G(X)^{\mathrm{op}}$. Morphisms of G-H-bisets are just morphisms of $(G \times H^{\mathrm{op}})$-sets.

8.11. Assume that X is a G-H-biset such that its underlying left G-set structure is isomorphic to G. In other words, we assume that $X = G$ with the regular left action of G, and there is morphism of groups $H \to \mathrm{Aut}_G(G)^{\mathrm{op}}$. Since the map $G^{\mathrm{op}} \to \mathrm{Aut}_G(G)$ sending g to $x \mapsto x \cdot g$ for all $x, g \in G$ is an isomorphism of groups, we see that the right H-set structure on G provided with the regular left G-set structure is given by a morphism of groups $H \to G$.

§9. Covering spaces I: Basic properties

9.1. Let $p : \tilde{X} \to X$ be a continuous map of topological spaces. We say that p is a **covering** if for every $x \in X$ there are an open set $U \subseteq X$ such that $x \in U$ as well as decomposition

$$p^{-1}(U) = \bigsqcup_{i\in I} \tilde{U}_i$$

for some nonempty index set I (depending on U) such that $\tilde{U}_i \subseteq \tilde{X}$ is open and $p|_{\tilde{U}_i} : \tilde{U}_i \to U$ is a homeomorphism for all $i \in I$. Equivalently, p is a covering if for every $x \in X$ there are an open set $U \subseteq X$ such that $x \in U$ as well as a nonempty discrete topological space F (depending on U) and a homeomorphism

$$\tau : p^{-1}(U) \to F \times U$$

such that such that $\pi_U \circ \tau = p$, where $\pi_U : F \times U \to U$ is the canonical projection. Note that this implies that p is surjective. Moreover, Exercise 2.3 tells us that p is open. If the cardinal of the set I (or F) is independent of the open sets U considered, we will say the covering $p : \tilde{X} \to X$ has **degree** $\#(I) = \#(F)$ (see Exercise 9.3).

9.2. If $p : \tilde{X} \to X$ is a covering and $x \in X$, we call $F_x = p^{-1}(\{x\})$ the **fiber** of p at x. By definition, the subspace topology of $p^{-1}(\{x\})$ induced from that of $\tilde{X}$ is discrete. We will say that a covering $p : \tilde{X} \to X$ is **universal** if $\tilde{X}$ is simply connected.

9.3. Exercise. *Let* $p : \tilde{X} \to X$ *be a covering such that* X *is connected. Prove that* F_x *is homeomorphic to* $F_{x'}$*, for all* $x, x' \in X$*.*

9.4. Exercise. *Prove that the quotient topology of* $\mathbb{R}/\mathbb{Q}$ *coincides with the trivial topology. Deduce that the canonical projection* $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$ *is not a covering.*

9.5. The definitions in paragraphs 9.1 and 9.2 can also be carried out *mutatis mutandi* for pointed topological spaces, by replacing topological spaces and their morphisms by pointed ones and their morphisms, giving the notion of **pointed covering**.

9.6. Exercise. *(i) Let X be fixed topological space. Let $\mathscr{C}_0$ be the collection of all pairs $(\tilde{X}, p)$, where $\tilde{X}$ is a topological space and $p : \tilde{X} \to X$ is a covering, $\mathsf{Hom}_{\mathscr{C}}((\tilde{X}, p), (\tilde{X}', p'))$ the set of all continuous maps from $f : \tilde{X} \to \tilde{X}'$ such that $p' \circ f = p$, $\circ_{\mathscr{C}}$ the usual composition of maps and $\mathrm{id}_{(\tilde{X},p)}$ is the identity map from $\tilde{X}$ to itself. Prove that this forms a category, denoted by $\mathsf{Cov}(X)$.*

(ii) Suppose now that (X, x) is a fixed pointed topological space. Consider the same construction as before by replacing topological spaces and their morphisms by pointed topological spaces ones and their morphisms. Prove that this forms a category, denoted by $\mathsf{Cov}(X, x)$.

9.7. A covering $p : \tilde{X} \to X$ of X is called **trivial** if there exists a nonempty discrete topological space F such that $(\tilde{X}, p)$ is isomorphic in $\mathsf{Cov}(X)$ to the covering $p_I : X \times F \to X$, where $X \times F$ has the product topology and p_I sends $(x, f) \in X \times F$ to $x \in X$. The same definition makes perfect sense in the category of pointed coverings.

9.8. Exercise. *(i) Let $p : \mathbb{R} \to S^1$ be the map sending $t \in \mathbb{R}$ to $e^{i2\pi t}$, where $S^1 \subseteq \mathbb{R}^2 \cong \mathbb{C}$ has the subspace topology. Prove that*

$$p^{-1}(S^1 \setminus \{e^{i2\pi t_0}\}) = \bigsqcup_{n \in \mathbb{Z}}]t_0 + n, t_0 + n + 1[,$$

for all $t_0 \in \mathbb{R}$. Conclude that this gives a (universal) covering of S^1.

(ii) Let $p : \mathbb{C} \to \mathbb{C}^\times$ be the map sending z to e^z. Prove that this gives a (universal) covering of $\mathbb{C}^\times$.

(iii) Let $n \in \mathbb{N}$ and $p : \mathbb{C}^\times \to \mathbb{C}^\times$ be the map sending z to z^n. Prove that this gives a covering of $\mathbb{C}^\times$ of degree n.

(iv) Recall that the ***(real) projective space*** *of dimension $n \in \mathbb{N}$ is defined as $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{\mathbf{0}_{\mathbb{R}^{n+1}}\})/\sim$, where $\sim$ is the equivalence relation on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}_{\mathbb{R}^{n+1}}\}$ given by $x \sim y$ if and only if there exists $\lambda \in \mathbb{R}^\times$ such that $y = \lambda x$. Let $n \in \mathbb{N}$ be an integer and let $p : S^n \to \mathbb{RP}^n$ be the canonical projection sending $x \in S^n$ to the equivalence class including $x \in \mathbb{R}^{n+1}$. Show that this gives a covering of $\mathbb{RP}^n$ of degree 2, which is universal if $n \geq 2$. For $n = 1$, prove that $\mathbb{RP}^1$ is homeomorphic to S^1.*

(v) Let G and G' be topological groups and let $p : G \to G'$ be a continuous morphism of groups. Prove that p is a covering if and only if p is open, surjective and the kernel of p is a discrete subset of G.

9.9. Exercise. *Let $p : \tilde{X} \to X$ be a covering and let $f : Y \to X$ be a continuous map. Prove that $\hat{p} : \tilde{X} \times_X Y \to Y$, where $\tilde{X} \times_X Y = \{(\tilde{x}, y) \in \tilde{X} \times Y : p(\tilde{x}) = f(y)\} \subseteq \tilde{X} \times Y$ has the subspace topology, is a covering as well.*

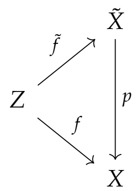
9.10. Let $p : \tilde{X} \to X$ be a covering, where X is locally path connected and path connected. Then, $\tilde{X}$ is also locally path connected, so its path connected components $\{\tilde{X}_i\}_{i \in I}$ are open in $\tilde{X}$. Define $p_i = p|_{\tilde{X}_i} : \tilde{X}_i \to X$. We claim that $p|_{\tilde{X}_i} : \tilde{X}_i \to X$ is a covering, for all $i \in I$, *i.e.* any covering over a locally path connected and path connected space X decomposes as a disjoint union of covering of path connected spaces over X. To prove it we proceed as follows. Since X is locally path connected, for every $x \in X$ we can pick an open neighborhood U of x that is path connected and that $p^{-1}(U) = \sqcup_{j \in J} \tilde{U}_j$, with $U_j \subseteq \tilde{X}$ open and $p|_{\tilde{U}_j} : \tilde{U}_j \to U$ a homeomorphism. Hence, $\tilde{U}_j$ is path connected, so for every $j \in J$, there is $i \in I$ such that $\tilde{U}_j \subseteq \tilde{X}_i$. If we prove that $p|_{\tilde{X}_i} : \tilde{X}_i \to X$ is surjective, then it is a covering. To do so, note that the image $V_i = p(\tilde{X}_i)$ is open, since p is open, and it coincides with $\{x \in X : p^{-1}(\{x\}) \cap \tilde{X}_i \neq \emptyset\}$. However, the subset $\{x \in X : p^{-1}(\{x\}) \cap \tilde{X}_i = \emptyset\} \subseteq X$ is also open. Since X is connected and $V_i \neq \emptyset$, we get that $X = V_i = p(\tilde{X}_i)$, as was to be shown. The covering $p : \tilde{X} \to X$ is then a disjoint union of coverings $p_i : \tilde{X}_i \to X$ of path connected topological spaces.

9.11. Let G be a group. A covering $p : \tilde{X} \to X$ of topological spaces is called a **G-covering** if $\tilde{X}$ is provided with a free left group action of G such that $p(g \cdot \tilde{x}) = p(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$ and $g \in G$, and the restriction of the action of G to the fiber $p^{-1}(\{x\})$ is transitive. Note in particular that the fiber of p at every point $x \in X$ is (noncanonically) isomorphic to G as left G-sets. A **morphism of G-coverings** from a G-covering $p : \tilde{X} \to X$ to another one $p' : \tilde{X}' \to X$ is a continuous map $f : \tilde{X} \to \tilde{X}'$ that is G-equivariant and $p' \circ f = p$.

9.12. Exercise. *(i) Let X be fixed topological space and G a group. Let $\mathscr{C}_0$ be the collection of all pairs $(\tilde{X}, p)$, where $\tilde{X}$ is a topological space and $p : \tilde{X} \to X$ is a G-covering, $\mathsf{Hom}_{\mathscr{C}}((\tilde{X}, p), (\tilde{X}', p'))$ the set of all continuous and G-equivariant maps $f : \tilde{X} \to \tilde{X}'$ such that $p' \circ f = p$, $\circ_{\mathscr{C}}$ the usual composition of maps and $\mathrm{id}_{(\tilde{X},p)}$ is the identity map from $\tilde{X}$ to itself. Prove that this forms a category, denoted by $\mathsf{Cov}_G(X)$.*

(ii) Suppose now that (X, x) is a fixed pointed topological space and G is a group. Consider the same construction as before by replacing topological spaces and their morphisms by pointed topological spaces ones and their morphisms. Prove that this forms a category, denoted by $\mathsf{Cov}_G(X, x)$.

9.13. Given morphisms $p : \tilde{X} \to X$ and $f : Z \to X$ in a category $\mathscr{C}$, a **lifting** of f along p is a morphism $\tilde{f} : Z \to \tilde{X}$ in $\mathscr{C}$ such that $p \circ \tilde{f} = f$, *i.e.* the diagram



commutes.

9.14. Proposition. *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering of pointed topological spaces and let (Z, z_0) be a connected pointed topological space together with a morphism $f : (Z, z_0) \to (X, x_0)$ of pointed topological spaces. Let $\tilde{f}_1, \tilde{f}_2 : (Z, z_0) \to (\tilde{X}, \tilde{x}_0)$ be two liftings of f along p, i.e. $p \circ \tilde{f}_i = f$, for $i = 1, 2$. Then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. Since $p \circ \tilde{f}_1 = f = p \circ \tilde{f}_2$, we get a continuous map

$$\tilde{f} : Z \to \tilde{X} \times_X \tilde{X} = \{(\tilde{x}, \tilde{x}') \in \tilde{X} \times \tilde{X} : p(\tilde{x}) = p(\tilde{x}')\}$$

given by $\tilde{f}(z) = (\tilde{f}_1(z), \tilde{f}_2(z))$, for all $z \in Z$. Since p is a covering, it is separated (using Exercise 2.5 and the fact that $p^{-1}(\{x\})$ is discrete so *a fortiori* relatively Hausdorff) and a local homeomorphism, so

$$\Delta_p = \{(\tilde{x}, \tilde{x}) \in \tilde{X} \times \tilde{X}\} \subseteq \tilde{X} \times_X \tilde{X}$$

is both closed and open in $\tilde{X} \times_X \tilde{X}$. Hence, $\tilde{f}^{-1}(\Delta_p)$ is both closed and open in Z. Since it contains z_0 and Z is connected we get that $\tilde{f}^{-1}(\Delta_p) = Z$, which proves the statement. □

9.15. Exercise. *Let $\tilde{X}$ be a topological space and let G be a group acting on the left on $\tilde{X}$ via a morphism $\rho : G \to \mathsf{Aut}_{\mathsf{Top}}(\tilde{X})$ of groups. We consider in this case the quotient space $G \backslash \tilde{X}$ provided with the quotient topology under the canonical projection $p_G : \tilde{X} \to G \backslash \tilde{X}$. We say that that ρ is a* ***covering group action*** *if for every $\tilde{x} \in \tilde{X}$ there is an open neighborhood U such that $U \cap \rho(g)(U) = \emptyset$ for all $g \in G \setminus \{e_G\}$, where e_G denotes the unit of G.* [2] *Note that this implies that the action is free, and, in particular, the action ρ is effective.*

(i) Assume that $\tilde{X}$ is connected. Let $p : \tilde{X} \to X$ be a covering and let $G = \mathsf{Aut}_{\mathsf{Cov}(X)}(\tilde{X}, p)$. Prove that the left action of G on $\tilde{X}$ given by $\rho(g)(\tilde{x}) = g(\tilde{x})$ for all $g \in G$ and $\tilde{x} \in \tilde{X}$ is a covering group action.

[2]This is called a properly discontinuous action in some classical textbooks, but we refrain from using this oxymoronic expression which simultaneously conveys that the action is continuous and discontinuous.

(ii) *Prove that, if $\rho : G \to \mathsf{Aut}_{\mathsf{Top}}(\tilde{X})$ is a covering group action and $H \leq G$ is a subgroup of G, then $\rho|_H : H \to \mathsf{Aut}_{\mathsf{Top}}(\tilde{X})$ is a covering group action.*

(iii) *Assume that we have an effective action of a group G on a connected topological space $\tilde{X}$. Prove that if p_G is a covering, then ρ is a covering group action.*

(iv) *Prove that if ρ is a covering group action, then p_G is a covering. Conclude in this case that $p_G : \tilde{X} \to G\backslash\tilde{X}$ is a G-covering.*

(v) *Prove that if $p : \tilde{X} \to X$ is a G-covering, then there is a homeomorphism $f : G\backslash\tilde{X} \to X$ such that $p = f \circ p_G$.*

9.16. Proposition. *Let $p : \tilde{X} \to X$ be a covering of topological spaces, Z a topological space together with a continuous map $Z \to \tilde{X}$, and $A \subseteq Z$ a subset. Let $H : Z \times [0,1] \to X$ be continuous map such that $H(z,0) = p \circ \tilde{f}(z)$ for all $z \in Z$, and $H(z,t) = p \circ \tilde{f}(z)$ for all $z \in A$ and $t \in [0,1]$. Then, there exists a unique continuous map $\tilde{H} : Z \times [0,1] \to \tilde{X}$ such that $p \circ \tilde{H} = H$, $\tilde{H}(z,0) = \tilde{f}(z)$ for all $z \in Z$, and $\tilde{H}(z,t) = \tilde{f}(z)$ for all $z \in A$ and $t \in [0,1]$.*

Proof. We first prove the uniqueness. Given two continuous maps $\tilde{H}_1, \tilde{H}_2 : Z \times [0,1] \to \tilde{X}$ as in the statement, then, for every $z \in Z$, $\tilde{H}_1(z,-), \tilde{H}_2(z,-) : ([0,1],0) \to (\tilde{X}, \tilde{f}(z))$ are two liftings of $H(z,-) : ([0,1],0) \to (X, p(\tilde{f}(z)))$ along p, which should coincide by Proposition 9.14.

Finally we prove the existence. We will prove that there is an open covering $\mathcal{U}$ of Z such that for all $U \in \mathcal{U}$ there is a continuous map $\tilde{H}_U : U \times [0,1] \to \tilde{X}$ such that $p \circ \tilde{H}_U = H|_{U\times[0,1]}$, $\tilde{H}(z,0) = \tilde{f}(z)$ for all $z \in U$, and $\tilde{H}(z,t) = \tilde{f}(z)$ for all $z \in A \cap U$ and $t \in [0,1]$. By the previous uniqueness result, $\tilde{H}_U$ is in fact unique. Moreover, we claim that the proposition follows directly from this local result. Indeed, given two open sets $U, U' \in \mathcal{U}$ and the corresponding maps $\tilde{H}_U$, $\tilde{H}_{U'}$, the previous uniqueness also tells us that $\tilde{H}_U|_{(U\cap U')\times[0,1]} = \tilde{H}_{U'}|_{(U\cap U')\times[0,1]}$, and in consequence the unique continuous map $\tilde{H} : Z \times [0,1] \to \tilde{X}$ such that $\tilde{H}|_{U\times[0,1]} = \tilde{H}_U$ satisfies the conditions of the statement.

It thus remains to prove that given any $z_0 \in Z$, there is an open set $U \subseteq Z$ including z_0 and a continuous map $\tilde{H}_U : U \times [0,1] \to \tilde{X}$ satisfying the previous conditions. Let $(V_j)_{j\in J}$ be an open covering of X such that $p^{-1}(V_j) = \sqcup_{i\in I_j} \tilde{V}_{i,j}$ for some nonempty set I_j, with $\tilde{V}_{i,j} \subseteq \tilde{X}$ open and $p|_{\tilde{V}_{i,j}} : \tilde{V}_{i,j} \to V_j$ a homeomorphism. Since H is continuous, for every $t \in [0,1]$, there exist a neighborhood U_t of z_0, an open connected neighborhood $J_t \subseteq [0,1]$ of t and $j_t \in J$ such that $H(U_t \times J_t) \subseteq V_{j_t}$. By the compactness of $[0,1]$, there is a finite set $T \subseteq [0,1]$ such that $\{J_t\}_{t\in T}$ is a finite covering of $[0,1]$. Define $U = \cap_{t\in T} U_t$. Suppose $T = \{t_1 < \cdots < t_n\}$ for some integer $n \geq 2$. To simplify, we will write j_k instead of j_{t_k}, for $k \in [\![1,n]\!]$. Pick $s_k \in J_{t_k} \cap J_{t_{k+1}}$ for all $k \in [\![1,n-1]\!]$. By defining $s_0 = 0$ and $s_n = 1$ we see that $H(U \times [s_k, s_{k+1}]) \subseteq V_{j_{k+1}}$, for all $k \in [\![0,n-1]\!]$. Assume that a continuous map $\tilde{H}_{U,k} : U \times [0,s_k] \to \tilde{X}$ satisfying the conditions of the proposition was constructed, for some $k \in [\![1,n-2]\!]$. We will prove that there is a continuous map $\tilde{H}_{U,k+1} : U \times [0,s_{k+1}] \to \tilde{X}$ also satisfying the conditions of the proposition and $\tilde{H}_{U,k+1}|_{U\times[0,s_k]} = \tilde{H}_{U,k}$. Since $H(U \times [s_k,s_{k+1}]) \subseteq V_{j_{k+1}}$, $H(z_0,s_k) \in V_{j_{k+1}}$. Let $i \in I_{j_{k+1}}$ such that $\tilde{H}_{U,k}(z_0,s_k) \in \tilde{V}_{i,j_{k+1}}$ and set $p_i = p|_{\tilde{V}_{i,j_{k+1}}} : \tilde{V}_{i,j_{k+1}} \to V_{j_{k+1}}$. Define now $\tilde{H}_{U,k+1}$ such that $\tilde{H}_{U,k+1}|_{U\times[0,s_k]} = \tilde{H}_{U,k}$ and $\tilde{H}_{U,k+1}|_{U\times[s_k,s_{k+1}]} = p_i^{-1} \circ H|_{U\times[s_k,s_{k+1}]}$. This gives the desired map. The proposition is thus proved. □

9.17. Remark. *The previous proposition tells us that, if $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering of pointed topological spaces, then $\pi_1(p) : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ in injective.*

9.18. Remark. *By taking $Z = \{*\}$ in the previous proposition we see that, given a continuous map $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$) and a pointed covering $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ of topological spaces, there exists a unique continuous map $\tilde{\gamma} : [0,1] \to \tilde{X}$ such that $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{x}_0$.*

9.19. Theorem. *Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering of pointed topological spaces and let (Z, z_0) be a path connected and locally path connected pointed topological space together with a morphism $f : (Z, z_0) \to (X, x_0)$ of* *pointed topological spaces. Then, there exists a lifting $\tilde{f} : (Z, z_0) \to (\tilde{X}, \tilde{x}_0)$ of f along p, i.e. $p \circ \tilde{f} = f$, if and only if $\pi_1(f)(\pi_1(Z, z_0)) \subseteq \pi_1(p)(\pi_1(\tilde{X}, \tilde{x}_0))$.*

Proof. Assume first that there exists a lifting $\tilde{f} : (Z, z_0) \to (\tilde{X}, \tilde{x}_0)$ of f along p, *i.e.* $p \circ \tilde{f} = f$. Then, since π_1 is a functor we get $\pi_1(p) \circ \pi_1(\tilde{f}) = \pi_1(f)$, which in particular gives us $\pi_1(f)(\pi_1(Z, z_0)) \subseteq \pi_1(p)(\tilde{X}, \tilde{x}_0)$.

Assume now that $\pi_1(f)(\pi_1(Z, z_0)) \subseteq \pi_1(p)(\tilde{X}, \tilde{x}_0)$. Define a map $\tilde{f} : Z \to \tilde{X}$ as follows. Given $z \in Z$, let $\eta : [0,1] \to Z$ be a path satisfying that $\eta(0) = z_0$ and $\eta(1) = z$, which exists since Z is path connected. By choosing $Z = \{*\}$ in Proposition 9.16, we see that the path $f \circ \eta$ in X has a lift $\tilde{\eta} : ([0,1], 0) \to (\tilde{X}, \tilde{x}_0)$ along p, *i.e.* $p \circ \tilde{\eta} = f \circ \eta$. Define now $\tilde{f}(z) = \tilde{\eta}(1)$. We claim that $\tilde{f}$ is well defined, *i.e.* it is independent of the choice of path η, and it is continuous.

We first prove the good definition of $\tilde{f}$. Let $\eta' : [0,1] \to Z$ be another path satisfying that $\eta'(0) = z_0$ and $\eta'(1) = z$, and let $\tilde{\eta}' : [0,1] \to \tilde{X}$ be the unique path such that $p \circ \tilde{\eta}' = f \circ \eta'$. We have to show that $\tilde{\eta}'(1) = \tilde{\eta}(1)$. Consider the loop $\gamma = \eta \cdot \eta'^{-}$ in Z based at z_0 and thus the loop $f \circ \gamma$ in X based at x_0. Since $\pi_1(f)(\pi_1(Z, z_0)) \subseteq \pi_1(p)(\pi_1(\tilde{X}, \tilde{x}_0))$, there exists a loop $\tilde{\gamma}$ in $\tilde{X}$ based at $\tilde{x}_0$ such that $p \circ \tilde{\gamma} \simeq f \circ \gamma\,(\mathsf{rel}\{0,1\})$ by means of a homotopy H. By Proposition 9.16, there exists thus a homotopy of paths $\tilde{H}$ such that $\tilde{H}(t, 0) = \tilde{\gamma}(t)$ for all $t \in [0,1]$ and $p \circ \tilde{H} = H$. In particular, $p \circ \tilde{H}(t, 1) = H(t, 1) = f \circ \gamma(t)$ for all $t \in [0,1]$, which means that the path $\tilde{\gamma}' : [0,1] \to \tilde{X}$ given by $\tilde{\gamma}'(t) = \tilde{H}(t, 1)$ is a lifting of $f \circ \gamma$. Since $\tilde{\gamma}$ is a loop and $\tilde{H}$ is a homotopy of paths, $\tilde{\gamma}'$ is a loop as well. As $p \circ \tilde{\gamma}' = f \circ (\eta \cdot \eta'^{-})$, is now clear that $\tilde{\gamma}' = \tilde{\eta} \cdot \tilde{\eta}'^{-}$, which in turn implies that $\tilde{\eta}'(1) = \tilde{\eta}(1)$.

We finally prove the continuity of $\tilde{f}$. Let $z \in Z$ be any point. Pick an open neighborhood U of $f(z)$ such that $p^{-1}(U) = \sqcup_{i \in I} \tilde{U}_i$, for a nonempty set I, $\tilde{U}_i \subseteq \tilde{X}$ open and $p_i = p|_{\tilde{U}_i} : \tilde{U}_i \to U$ is a homeomorphism for all $i \in I$. Since f is continuous and Z is locally path connected, there exists a path connected open neighborhood V of z such that $f(V) \subseteq U$. Assume that $\tilde{f}(z) \in \tilde{U}_{i_0}$. Note now that $\tilde{f}|_V = p_{i_0}^{-1} \circ f|_V$, which follows from uniqueness of liftings by Proposition 9.14. Then, $\tilde{f}|_V$ is clearly continuous, and in consequence $\tilde{f}$ is also continuous. □

9.20. Exercise. *Let G be a topological group that is locally path connected and path connected as a topological space and e_G be its unit. Let $p : \tilde{G} \to G$ be a covering of path connected topological spaces and pick $\tilde{e} \in \tilde{G}$ such that $p(\tilde{e}) = e_G$.*

(i) Prove that there exists a unique group structure on $\tilde{G}$ with unit $\tilde{e}$ such that it becomes a topological group for the given topology and p is a morphism of groups.

(ii) Prove that G is abelian if and only if $\tilde{G}$ is so.

(iii) Prove that the sequence of groups

$$1 \to \pi_1(\tilde{G}, \tilde{e}) \xrightarrow{\pi_1(p)} \pi_1(G, e_G) \xrightarrow{\partial_{\tilde{e}}} \mathsf{Ker}(p) \to 1$$

is exact, where $\partial_{\tilde{e}}([\gamma]) = \tilde{\gamma}(1)$, where $\tilde{\gamma} : ([0,1], 0) \to (\tilde{G}, \tilde{e})$ is the unique lifting of γ.

(iv) Prove that $\pi_1(S^1, 1) \cong \mathbb{Z}$ and thus $\pi_1(\mathbb{T}^2, (1,1)) \cong \mathbb{Z} \times \mathbb{Z}$, where $\mathbb{T}^2 = S^1 \times S^1$ is the 2-dimensional torus. Deduce that $\pi_1(\mathbb{M}, (0,0)) \cong \mathbb{Z}$ (see Exercise 7.15).

(v) Prove that there is no continuous map $r : \bar{B}_1(0,0) \to S^1$ such that $r|_{S^1} = \mathrm{id}_{S^1}$.

(vi) Prove that every continuous map $f : \bar{B}_1(0,0) \to \bar{B}_1(0,0)$ has a fixed point.

(vii) Prove that for any continuous function $f : S^2 \to \mathbb{R}^2$ there is a point $x \in S^2$ such that $f(x) = f(-x)$.
Hint. Consider $g : S^2 \to S^1$ given by $g(x) = (f(x) - (f(-x)))/\|(f(x) - (f(-x)))\|$ for $x \in S^2$, $\gamma : [0,1] \to S^2$ defined as $\gamma(t) = (\cos(2\pi t), \sin(2\pi t), 0)$ for $t \in [0,1]$ and $\gamma' = g \circ \gamma$. Note γ' is null homotopic, since γ is. Note on the other hand that $\gamma'(s + 1/2) = -\gamma'(s)$ for $s \in [0, 1/2]$ and deduce from this that any lifting $\tilde{\gamma}' : [0,1] \to \mathbb{R}$ of γ' is not a loop, so γ' is not null homotopic.

9.21. By Theorem 9.19, the existence of a lifting $f:(Z,z_0)\to(X,x_0)$ along a covering p is assured if Z is simply connected. Moreover, if (X,x_0) is a path connected and locally path connected topological space, then the previous theorem tells us that the map

$$\left\{\begin{array}{c}\text{isomorphism classes of pointed}\\ \text{path connected covering spaces}\\ p:(\tilde{X},\tilde{x}_0)\to(X,x_0)\end{array}\right\}\longrightarrow\left\{\begin{array}{c}\text{subgroups of}\\ \pi_1(X,x_0)\end{array}\right\} \tag{2}$$

sending a covering $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ of pointed path connected topological spaces to $\pi_1(p)(\pi_1(\tilde{X},\tilde{x}_0))$ is injective.

9.22. Exercise. *(i) Classify all coverings of S^1, up to isomorphism of coverings.*

(ii) Prove that every subgroup of $\mathbb{Z}\times\mathbb{Z}$ has at most two generators. Conclude that it coincides with one of the subgroups below:

(1) the trivial subgroup $\{(0,0)\}$ of $\mathbb{Z}\times\mathbb{Z}$;

(2) $\{\mathbb{Z}.(a,b):$ for all $a,b\in\mathbb{Z}$ such that $(a,b)\neq(0,0)\}$;

(3) $\{\mathbb{Z}.(a,b)\oplus\mathbb{Z}.(c,d):$ for all $a,b,c,d\in\mathbb{Z}$ such that $ad\neq bc\}$.

(iii) Classify all coverings of $\mathbb{T}^2=S^1\times S^1$, up to isomorphism of coverings.

Hint. Given $a,b,c,d\in\mathbb{Z}$, consider the maps $p:S^1\times\mathbb{R}\to S^1\times S^1$ given by $p(z,x)=(z^a,z^be^{2\pi ix})$ for $z\in S^1$ and $x\in\mathbb{R}$, as well as $p':S^1\times S^1\to S^1\times S^1$ given by $p'(z,z')=(z^az'^c,z^bz'^d)$ for $z,z'\in S^1$.

§10. Covering spaces II: Automorphisms of coverings

10.1. We will generalize the construction in paragraph 9.21 as follows. Let X be a path connected and locally path connected topological space and $p:\tilde{X}\to X$ a covering. For $x\in X$, consider the fiber $F_x=p^{-1}(\{x\})\subseteq\tilde{X}$, which is a discrete subspace. Fix $x_0\in X$ and define the map

$$\rho:F_{x_0}\times\pi_1(X,x_0)\to F_{x_0} \tag{3}$$

by $\rho(\tilde{x},[\gamma])=\tilde{\gamma}(1)$, where $\tilde{\gamma}:[0,1]\to\tilde{X}$ is a path satisfying that $p\circ\tilde{\gamma}\simeq\gamma\,(\mathsf{rel}\{0,1\})$ and $\tilde{\gamma}(0)=\tilde{x}$. It is well defined by Proposition 9.16. Then, ρ is a right action of $\pi_1(X,x_0)$ on F_{x_0}, called the **monodromy action** at x_0. If $p:\tilde{X}\to X$ and $p':\tilde{X}'\to X$ are two coverings and $f:\tilde{X}\to\tilde{X}'$ is a morphism of coverings, then it induces a map $f|_{p^{-1}(\{x_0\})}:p^{-1}(\{x_0\})\to p'^{-1}(\{x_0\})$ which is clearly equivariant with respect to the right action of $\pi_1(X,x_0)$. As a consequence, one obtains a functor

$$\mathsf{Fib}_{x_0}:\mathsf{Cov}(X)\to\mathsf{Set}\text{-}\pi_1(X,x_0)$$

sending the covering $p:\tilde{X}\to X$ to the fiber $F_{x_0}=p^{-1}(\{x_0\})$, and a morphism $f:\tilde{X}\to\tilde{X}'$ to the $\pi_1(X,x_0)$-equivariant map $f|_{p^{-1}(\{x_0\})}:p^{-1}(\{x_0\})\to p'^{-1}(\{x_0\})$.

10.2. Note that the monodromy action (3) at $x_0\in X$ is transitive if and only if $\tilde{X}$ is path connected. Indeed, if $\tilde{X}$ is path connected, given any two points $\tilde{x}$ and $\tilde{x}'$ of F_{x_0}, there is a path $\tilde{\eta}$ in $\tilde{X}$ starting at $\tilde{x}$ and ending at $\tilde{x}'$. Then, $\tilde{x}'=\rho(\tilde{x},[p\circ\tilde{\eta}])$. Conversely, fix a point $\tilde{x}\in F_{x_0}$ and let $\tilde{x}'\in\tilde{X}$ be a general point. It suffices to prove that the transitivity of the action at x_0 implies that there is path in $\tilde{X}$ starting at $\tilde{x}'$ and ending at $\tilde{x}$. Since X is connected, there is a path η in X starting at $p(\tilde{x}')$ and ending at $x_0=p(\tilde{x})$. By Proposition 9.16, there is a path $\tilde{X}$ starting at $\tilde{x}'$ and ending at a point $\tilde{x}''$ in F_{x_0}. Since the monodromy action at x_0 is transitive, then two points in F_{x_0} can be connected by a path in $\tilde{X}$, so there is a path $\tilde{\eta}'$ starting at $\tilde{x}''$ and ending at a point $\tilde{x}$. The path $\tilde{\eta}\cdot\tilde{\eta}'$ proves our claim.

10.3. Note also that, given $\tilde{x} \in \tilde{X}$ such that $p(\tilde{x}) = x_0$, then the isotropy group $\Gamma_{\tilde{x}}$ of $\tilde{x}$ for the monodromy action (3) is exactly $\pi_1(p)(\pi_1(\tilde{X}, \tilde{x}))$. Indeed, by definition $\Gamma_{\tilde{x}}$ is the subgroup of homotopy classes of loops at x_0 (relative to $\{0,1\}$) whose lift to $\tilde{X}$ (along p) are also loops at $\tilde{x}$. In this sense, the functor Fib_{x_0} also encodes the information given by the map (2).

10.4. Finally, notice that, if $\tilde{X}$ is path connected, the monodromy action (3) at $x_0 \in X$ is free if and only if $\tilde{X}$ is simply connected. Indeed, by paragraph 10.2, $\tilde{X}$ is path connected is equivalent to the monodromy action being transitive. Then, using (1), the action is free if and only if the isotropy group $\Gamma_{\tilde{x}}$ of any $\tilde{x} \in \tilde{X}$ is trivial, *i.e.* if $\pi_1(p)(\pi_1(\tilde{X}, \tilde{x}))$ is trivial. By Proposition 9.16 (or Remark 9.17), this is equivalent to $\pi_1(\tilde{X}, \tilde{x})$ being trivial, *i.e.* $\tilde{X}$ is simply connected.

10.5. The functoriality of Fib_{x_0} tells us that, if $f : \tilde{X} \to \tilde{X}$ is an automorphism of a covering $p : \tilde{X} \to X$, where X is a path connected and locally path connected topological space and $x_0 \in X$, then $\mathsf{Fib}_{x_0}(f)$ is an automorphism of the $\pi_1(X, x_0)$-set $F_{x_0} = \mathsf{Fib}_{x_0}(\tilde{X}, p)$. We have even the following stronger result.

10.6. Theorem. *Let X be a path connected and locally path connected topological space and $x_0 \in X$. Let $p : \tilde{X} \to X$ be a covering, where $\tilde{X}$ is path connected. Then, the functor Fib_{x_0} restricts to an isomorphism of groups*

$$\mathsf{Fib}_{x_0}|_{\mathsf{Aut}_{\mathsf{Cov}(X)}(\tilde{X})} : \mathsf{Aut}_{\mathsf{Cov}(X)}(\tilde{X}) \to \mathsf{Aut}_{\pi_1(X,x_0)}(F_{x_0}). \tag{4}$$

Proof. Note first that, by Theorem 9.19, $\mathsf{Fib}_x(f)(\tilde{x}) = \tilde{x}$ for some $\tilde{x} \in F_{x_0}$ if and only if f is the identity. This implies that the map (4) is injective. As explained in paragraph 10.2, the path connectedness assumption on $\tilde{X}$ implies that F_{x_0} is a transitive $\pi_1(X, x_0)$-set. Let $\bar{f} : F_{x_0} \to F_{x_0}$ be a $\pi_1(X, x_0)$-equivariant automorphism and pick $\tilde{x} \in F_{x_0}$. By Lemma 8.6, (iii), the isotropy groups $\Gamma_{\tilde{x}}$ and $\Gamma_{\bar{f}(\tilde{x})}$ coincide. Since $\Gamma_{\tilde{x}} = \pi_1(p)(\pi_1(\tilde{X}, \tilde{x}))$ and $\Gamma_{\bar{f}(\tilde{x})} = \pi_1(p)(\pi_1(\tilde{X}, \bar{f}(\tilde{x})))$, by paragraph 10.3, Theorem 9.19 tells us that there exists a morphism $f : \tilde{X} \to \tilde{X}$ of the covering $p : \tilde{X} \to X$ such that $f(\tilde{x}) = \bar{f}(\tilde{x})$. The same arguments apply to $\bar{f}^{-1}$, giving a morphism $f' : \tilde{X} \to \tilde{X}$ of the covering $p : \tilde{X} \to X$ such that $f'(\tilde{x}) = \bar{f}^{-1}(\tilde{x})$. Since $f' \circ f(\tilde{x}) = (\tilde{x})$ and $f \circ f'(\tilde{x}) = (\tilde{x})$, Proposition 9.14 implies that f and f' are inverses to each other. Moreover, since $\bar{f}(x_0) = \mathsf{Fib}_{x_0}(f)(x_0)$ and F_{x_0} is transitive, Lemma 8.6, (iii), tells us that $\bar{f} = \mathsf{Fib}_{x_0}(f)$. The theorem is proved. □

10.7. Combining Proposition 8.8 and Theorem 10.6, we get the main result of this section, namely that, given a $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ of pointed path connected topological spaces with X locally path connected, then there is an isomorphism of groups

$$\mathsf{Aut}_{\mathsf{Cov}(X)}(\tilde{X}) \cong N_{\pi_1(X,x_0)}\Big(\pi_1(p)\big(\pi_1(\tilde{X}, \tilde{x}_0)\big)\Big) / \pi_1(p)\big(\pi_1(\tilde{X}, \tilde{x}_0)\big).$$

In particular, if $\tilde{X}$ is simply connected, then

$$\mathsf{Aut}_{\mathsf{Cov}(X)}(\tilde{X}) \cong \pi_1(X, x_0), \tag{5}$$

which gives in particular a left action of $\pi_1(X, x_0)$ on $\tilde{X}$ by automorphisms of coverings.

10.8. Exercise. *(i) Give another proof that $\pi_1(S^1, 1) \cong \mathbb{Z}$.*

(ii) Prove that $\pi_1(\mathbb{RP}^n, *) \cong \mathbb{Z}/2\mathbb{Z}$, for $*$ any point in $\mathbb{RP}^n$ and $n \geq 2$.

10.9. We say that a covering $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ of pointed topological spaces, where X is a path connected and locally path connected topological space, is **normal** (or **regular**) if the action of $\mathsf{Aut}_{\mathsf{Cov}(X)}(\tilde{X})$ on F_{x_0} is transitive. If $\tilde{X}$ is path connected this is equivalent to $\pi_1(p)(\pi_1(\tilde{X}, \tilde{x}_0))$ being a normal subgroup of $\pi_1(X, x_0)$. Indeed, by Theorem 9.19, $\mathsf{Aut}_{\mathsf{Cov}(X)}(\tilde{X})$ acts transitively on F_{x_0} if and only if $\pi_1(p)(\pi_1(\tilde{X}, \tilde{x})) = \pi_1(p)(\pi_1(\tilde{X}, \tilde{x}'))$ for any two points $\tilde{x}, \tilde{x}' \in F_{x_0}$. However, since the latter subgroups are precisely the isotropy subgroups of $\tilde{x}, \tilde{x}' \in F_x$ under the monodromy action, by paragraph 10.3, then $\pi_1(p)(\pi_1(\tilde{X}, \tilde{x})) = \pi_1(p)(\pi_1(\tilde{X}, \tilde{x}'))$ for any two points $\tilde{x}, \tilde{x}' \in F_{x_0}$ is equivalent to $\pi_1(p)(\pi_1(\tilde{X}, \tilde{x}))$ being a normal subgroup of $\pi_1(X, x_0)$, by (1).

10.10. Exercise. *Let $\tilde{X}$ be a topological space and let G be a group acting (on the left) on $\tilde{X}$ (see Exercise 9.15). Assume that the action is covering and consider the quotient space $G\backslash\tilde{X}$ provided with the quotient topology under the canonical projection $p_G : \tilde{X} \to G\backslash\tilde{X}$.*

- *(i) Show that the morphism of groups $\iota_G : G \to \mathsf{Aut}_{\mathsf{Cov}(G\backslash\tilde{X})}(\tilde{X})$ sending g to $\tilde{x} \mapsto g \cdot \tilde{x}$ is injective.*
- *(ii) Let $G = \mathbb{Z}/n$, X a topological space and $\tilde{X} = G \times X$ be the product space, where G has the discrete topology. Endow $\tilde{X}$ with the regular action $g \cdot (g', x) = (gg', x)$. Prove that there is a homeomorphism $G\backslash\tilde{X} \cong X$, $p_G : \tilde{X} \to X$ is the projection on the second component under the previous homeomorphism, and $\mathsf{Aut}_{\mathsf{Cov}(G\backslash\tilde{X})}(\tilde{X}) \cong \mathbb{S}_n$, the group of permutations of n letters. Conclude that ι_G is not surjective in this case.*
- *(iii) Assume that $\tilde{X}$ is path connected and locally path connected. Prove that p_G is a normal covering and that ι_G is surjective giving an isomorphism $G \cong \mathsf{Aut}_{\mathsf{Cov}(G\backslash\tilde{X})}(\tilde{X})$ of groups.*

10.11. Exercise. *Let $S_i = S^1 + (2i-1, 0) = \partial B_1(2i-1, 0) \subseteq \mathbb{R}^2 \cong \mathbb{C}$ for $i \in [\![1, 4]\!]$. Let $X = S_2 \cup S_3 \subseteq \mathbb{R}^2$ and let*

$$\tilde{X} = S_1 \cup S_2 \cup S_3 \cup S_4 \subseteq \mathbb{R}^2 \cong \mathbb{C}$$

provided with the subspace topology of $\mathbb{R}^2 \cong \mathbb{C}$. Define $p : \tilde{X} \to X$ such that

$$p(z) = \begin{cases} 2 + z, & \text{if } z \in S_1, \\ 5 - (z-3)^2, & \text{if } z \in S_2, \\ 3 + (z-5)^2, & \text{if } z \in S_3, \\ -2 + z, & \text{if } z \in S_4. \end{cases}$$

- *(i) Prove that $p : \tilde{X} \to X$ is a covering of degree 3.*
- *(ii) Prove that the only automorphism of the covering $p : \tilde{X} \to X$ is the identity.*
  Hint. Prove that any automorphism $\tilde{f} : \tilde{X} \to \tilde{X}$ of the covering satisfies that $\tilde{f}(S_1) = S_1$ and $\tilde{f}(S_4) = S_4$, so $\tilde{f}(4) = 4$.
- *(iii) Using the previous item deduce that $\pi_1(p)(\pi_1(\tilde{X}, \tilde{x}))$ is not a normal subgroup of $\pi_1(X, x)$.*

§11. Covering spaces III: Simply connectedness

11.1. We say that a topological space X is **semilocally simply connected** if every $x \in X$ has an open neighborhood U such that the image of $\pi_1(\mathsf{inc}) : \pi_1(U, x) \to \pi_1(X, x)$ is trivial, where $\mathsf{inc} : U \to X$ is the inclusion.

11.2. Exercise. *(i) Let X be a topological space provided with an open covering $\mathcal{U}$ such that U is simply connected, for all $U \in \mathcal{U}$. Prove that X is semilocally simply connected.*

- *(ii) Define $X = \cup_{n \in \mathbb{N}} \partial B_{1/n}(1/n, 0) \subseteq \mathbb{R}^2$ with the subspace topology. This space is called the **Hawaiian earring**. Prove that X is not semilocally simply connected.*
- *(iii) Let $\bar{B}_1(0,0) \subseteq \mathbb{R}^2$ be provided with the subspace topology and let $\sim$ be the equivalence relation on $\bar{B}_1(0,0)$ generated by $(0,0) \sim (1,0)$. Define $X = \bar{B}_1(0,0)/\sim$, endowed with the quotient topology. Let $X_n = X \times \{n\}$ and denote the equivalence class of an element $(x,y) \in \bar{B}_1(0,0)$ by $[(x,y)]$. Define now $\hat{X} = \sqcup_{n \in \mathbb{N}_0} X_n$ and let $\sim'$ be the equivalence relation on $\hat{X}$ generated by*

$$([(x,0)], n) \sim' ([(\cos(2\pi x), \sin(2\pi x))], n+1),$$

for all $x \in [0,1]$ and $n \in \mathbb{N}_0$. Endow $\hat{X}$ with the topology given by

$$\mathcal{U} = \Big\{ \Big(\bigsqcup_{\substack{n \in \mathbb{N}_0 \\ n \leq n_0}} U_n \Big) \sqcup \Big(\bigsqcup_{\substack{n \in \mathbb{N}_0 \\ n > n_0}} X_n \Big) : \text{ for all } U_n \subseteq X_n \text{ open and } n_0 \in \mathbb{N}_0 \Big\}.$$

Define $X = \hat{X}/\sim'$ and endow it with the quotient topology. Prove that X is semilocally simply connected but no open set $U \subseteq X$ including the equivalence class of $([(1,0)],0) \in \hat{X}$ is simply connected.

11.3. Theorem. *Let X be a path connected and locally path connected topological space. The following conditions are equivalent:*

(i) *X is semilocally simply connected;*

(ii) *X has a universal covering.*

(iii) *the map (2) is bijective.*

Proof. It is clear that (iii) implies (ii), by taking the inverse image of the trivial subgroup under (2). Moreover, (ii) clearly implies (i). Indeed, let $p : \tilde{X} \to X$ be an universal and $x \in X$. Take an open neighborhood U of x such that $p^{-1}(U) = \sqcup_{i \in I} \tilde{U}_i$, for a nonempty set I, $\tilde{U}_i \subseteq \tilde{X}$ open and $p_i : p|_{\tilde{U}_i} : \tilde{U}_i \to U$ is a homeomorphism for all $i \in I$. Fix $i \in$ and let $\mathsf{inc} : U \to X$ and $\tilde{\mathsf{inc}} : \tilde{U}_i \to \tilde{X}$ be the inclusions. Then, $\pi_1(\mathsf{inc} \circ p_i) = \pi_1(p \circ \mathsf{inc}_i) = \pi_1(p) \circ \pi_1(\mathsf{inc}_i)$ together with the fact that $\pi_1(p_i)$ is an isomorphism tells us that

$$\mathsf{Img}\,(\pi_1(\mathsf{inc})) = \mathsf{Img}\,(\pi_1(p) \circ \pi_1(\mathsf{inc}_i)) \subseteq \mathsf{Img}\,(\pi_1(p)),$$

which is a trivial subgroup of $\pi_1(X, x_0)$, since $\pi_1(\tilde{X}, \tilde{x}_0)$ is trivial.

Let us finally prove that (i) implies (iii). Let $C \leq \pi_1(X, x_0)$ be a subgroup. Define $\hat{X}$ the set of homotopy classes relative to $\{0,1\}$ of all paths $\eta : [0,1] \to X$ starting at x_0, and let $\hat{p} : \hat{X} \to X$ be the map $\hat{p}([\eta]) = \eta(1)$, for every such path η. Let $\hat{x}_0$ be the homotopy class of the constant path with value x_0. Note that the map $\rho : \pi_1(X, x_0) \times \hat{X} \to \hat{X}$ given by $\rho([\gamma], [\eta]) = [\gamma] \cdot \eta$ gives a well defined left action. Let $\tilde{X} = C \backslash \hat{X}$ be the set of orbits of $\hat{X}$ under this action and let $\tilde{x}_0$ be the orbit containing $\hat{x}_0$. We will denote the orbit of a homotopy equivalence class $[\eta]$ also by $[\eta]$. It is clear that $p(\hat{x}) = p(\hat{x}')$ if $\hat{x}$ and $\hat{x}'$ are in the same orbit, so we have a well defined map $p : \tilde{X} \to X$.

Let $\mathcal{U}$ be the set of all open path connected subsets of X such that the image of the mapping $\pi_1(\mathsf{inc}_U) : \pi_1(U, x) \to \pi_1(X, x)$ is trivial for one (or, equivalently, for all) $x \in U$, where $\mathsf{inc}_U : U \to X$ is the inclusion. It is easy to see that $\mathcal{U}$ is a basis of the topology of X. Indeed, given $U, U' \in \mathcal{U}$ such that $U \cap U' \neq \emptyset$, we see that $U \cap U' \in \mathcal{U}$, since $\pi_1(\mathsf{inc}_{U \cap U'})$ factors through $\pi_1(\mathsf{inc}_U)$, so the image of the former is included in the image of the latter, which is trivial. Now, given $U \in \mathcal{U}$ and $\eta : [0,1] \to X$ a path such that $\eta(0) = x_0$ and $\eta(1) \in U$, define

$$\hat{U}([\eta]) = \big\{ [\eta] \cdot [\eta'] \in \hat{X} : \eta' : [0,1] \to U \text{ such that } \eta'(0) = \eta(1) \big\} \subseteq \hat{X}$$

The set

$$\hat{\mathcal{U}} = \big\{ \hat{U}([\eta]) : U \in \mathcal{U}, \eta : ([0,1], 0) \to (X, x_0), \eta(1) = x_1 \in U \big\}$$

defines a topology on $\hat{X}$. Indeed, given $\hat{U}([\eta]), \hat{U}'([\eta']) \in \hat{\mathcal{U}}$ such that $\hat{U}([\eta]) \cap \hat{U}'([\eta']) \neq \emptyset$, then $U'' = U \cap U' \neq \emptyset$ and there exists $\eta'' : ([0,1], 0) \to (X, x_0)$ such that $\eta(1) \in U \cap U'$. It is clear that $\hat{U}''([\eta'']) \subseteq \hat{U}([\eta]), \hat{U}'([\eta'])$. It is easy to see that $\hat{X}$ is path connected. Indeed, given $[\eta] \in \hat{X}$ and a representative $\eta : [0,1] \to X$, the map $\hat{\eta} : [0,1] \to \hat{X}$ given by $\hat{\eta}(s)(t) = [\eta(st)]$ is a path from $\hat{x}_0$ to $[\eta]$. We give $\tilde{X}$ the quotient topology for the canonical projection $\pi_C : \hat{X} \to \tilde{X}$. Since $\hat{X}$ is path connected, $\tilde{X}$ is also path connected. Define $\tilde{U}([\eta])$ to be the image of $\hat{U}([\eta])$ under the canonical projection π_C. Then,

$$\pi_C^{-1}(\tilde{U}([\eta])) = \bigcup_{[\gamma] \in C} \hat{U}([\gamma] \cdot [\eta]),$$

so it is an open subset of $\tilde{X}$. It is clearly a basis of the topology of $\tilde{X}$.

We claim that the map $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering. Given $x \in X$, take $U \in \mathcal{U}$ such that $x \in U$. Then,

$$p^{-1}(U) = \bigsqcup_{\substack{[\eta] \in \tilde{X} \\ \eta(1) \in U}} \tilde{U}([\eta]),$$

and $p|_{\tilde{U}([\eta])} : \tilde{U}([\eta]) \to U$ is clearly a homeomorphism, so p is a covering as claimed.

Finally, we will prove that $C = \pi_1(p)(\pi_1(\tilde{X}, \tilde{x}_0))$. Note first that $\hat{X}$ is simply connected. Indeed, if $\hat{\gamma} : [0,1] \to \hat{X}$ is a loop based at $\hat{x}_0$, consider $\gamma = p \circ \hat{\gamma}$ and define the map $\hat{\gamma}' : [0,1] \to \hat{X}$ given by $\hat{\gamma}(s)(t) = [\gamma(st)]$. By Proposition 9.14, $\hat{\gamma} = \hat{\gamma}'$. Since $\hat{\gamma}$ is a loop, $\hat{x}_0 = \hat{\gamma}(1) = \hat{\gamma}'(1) = [\gamma]$, *i.e.* $\gamma \simeq \varepsilon_{x_0}\ (\mathsf{rel}\{0,1\})$, which implies that $\hat{\gamma}' = \hat{\gamma} \simeq \varepsilon_{\hat{x}_0}\ (\mathsf{rel}\{0,1\})$, by Proposition 9.16. Finally, using Exercise 10.10 for the quotient $\hat{X} \to C\backslash\hat{X}$ as well as (5), we see that $\pi_1(p)(\pi_1(\tilde{X}, \tilde{x}_0)) = C$, as was to be shown. □

11.4. Exercise. *Let X be the topological space in Exercise 10.11. Prove that X is semilocally simply connected and compute a universal covering.*

§12. Covering spaces IV: Classification of coverings

12.1. Let X be a semilocally simply connected topological space with a point $x_0 \in X$. Fix a universal covering $p : \tilde{X} \to X$, which exists by Theorem 11.3. Recall from (5) that there is a left action of $\pi_1(X, x_0)$ on $\tilde{X}$ by automorphisms of coverings, given by $\tilde{x} \cdot [\gamma] = \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is a lift of γ to $\tilde{X}$ along p such that $\tilde{\gamma}(0) = \tilde{x}$. By Theorem 10.6, we can identify $\pi_1(X, x_0)$ with the fiber $p^{-1}(\{x_0\})$ once we fix a point $\tilde{x} \in F_{x_0}$, since the action of $\pi_1(X, x_0)$ on $p^{-1}(\{x_0\})$ is free and transitive (see 10.2 and 10.4). Given a set F provided with a right action of $\pi_1(X, x_0)$, define $\Gamma_p(F) = F \times^{\pi_1(X,x_0)} \tilde{X}$ as the quotient of $F \times \tilde{X}$ by the right action of $\pi_1(X, x_0)$ given by $(f, \tilde{x}') \cdot [\gamma] = (f \cdot [\gamma], [\gamma]^{-1} \cdot \tilde{x}')$, for $f \in F$, $\tilde{x}'$ and $[\gamma] \in \pi_1(X, x)$. Endow $F \times \tilde{X}$ with the product topology, where F has the discrete topology, and consider the quotient topology on $F \times^{\pi_1(X,x_0)} \tilde{X}$. It is easy to see that the canonical projection $F \times \tilde{X} \to X$ given by composing the projection on the second component and p induces a continuous map $p_F : F \times^{\pi_1(X,x_0)} \tilde{X} \to X$.

12.2. Proposition. *Let X be a semilocally simply connected topological space with a point $x \in X$ and let $p : \tilde{X} \to X$ be a universal covering. Then, the continuous map $p_F : F \times^{\pi_1(X,x_0)} \tilde{X} \to X$ constructed in the previous paragraph is a covering.*

Proof. Since $p : \tilde{X} \to X$ is a covering, there exists an open neighborhood U of x such that there is a homeomorphism $p^{-1}(U) \cong \pi_1(X, x_0) \times U$, where $\pi_1(X, x_0)$ has the discrete topology and p identifies with the projection on the second factor under this homeomorphism. Then, we get a homeomorphism $p_F^{-1}(U) \cong F \times U$, where F has the discrete topology and p_F identifies with the projection on the second factor under this homeomorphism. □

12.3. If $f : F \to F'$ is a $\pi_1(X, x_0)$-equivariant map, then $f \times \mathrm{id}_{\tilde{X}} : F \times \tilde{X} \to F' \times \tilde{X}$ induces a map $\Gamma_p(f) : F \times^{\pi_1(X,x_0)} \tilde{X} \to F' \times^{\pi_1(X,x_0)} \tilde{X}$ such that $p_{F'} \circ \Gamma_p(f) = p_F$. Moreover, $f \times \mathrm{id}_{\tilde{X}}$ is clearly continuous, for the corresponding topologies, which in turn implies that $\Gamma_p(f)$ is continuous.

12.4. Proposition. *Let X be a semilocally simply connected topological space with a point $x_0 \in X$ and let $p : \tilde{X} \to X$ be a universal covering. Then, the maps given by sending a right $\pi_1(X, x_0)$-set to $F \mapsto \Gamma_p(F) = F \times^{\pi_1(X,x_0)} \tilde{X}$ defined in paragraph 12.1, and a $\pi_1(X, x_0)$-equivariant map $f : F \to F'$ between right $\pi_1(X, x_0)$-sets to the morphism of coverings $\Gamma_p(f) : F \times^{\pi_1(X,x_0)} \tilde{X} \to F' \times^{\pi_1(X,x_0)} \tilde{X}$ defined in the previous paragraph, is a functor*

$$\Gamma_p : \mathsf{Set}\text{-}\pi_1(X, x_0) \to \mathsf{Cov}(X).$$

Moreover, Γ_p and the functor Fib_{x_0} defined in paragraph 10.1 are quasi-inverse to each other.

Proof. The fact that Γ_p is a functor is clear. We will prove that Γ_p and Fib_{x_0} are quasi-inverse functors. We first note that $\mathsf{Fib}_{x_0} \circ \Gamma_p$ is clearly the identity functor of $\mathsf{Set}\text{-}\pi_1(X, x_0)$. On the other hand, $\Gamma_p \circ \mathsf{Fib}_{x_0}$ is naturally isomorphic to the identity functor of $\mathsf{Cov}(X)$, which implies that Fib_{x_0} and Γ_p are quasi-inverse equivalences (see Exercise 5.15). Indeed, given any covering $\hat{p} : \hat{X} \to X$, it decomposes as the disjoint union of the coverings $\hat{p}_i = \hat{p}|_{\hat{X}_i} : \hat{X}_i \to X$ of path connected topological spaces by paragraph 9.10, where $\{\hat{X}_i\}_{i \in I}$ are the path connected components of $\hat{X}$. The result now follows from Exercise 8.9, using that every covering $\hat{p} : \hat{X} \to X$ of path connected topological spaces is isomorphic to one of the form $H\backslash\tilde{X} \to X$, where $H \leq \pi_1(X, x_0) \cong \mathsf{Aut}_{\mathsf{Cov}(X)}(\tilde{X})$, by Exercise 9.15 for the quotient $\tilde{X} \to H\backslash\tilde{X}$ as well as (5). □

12.5. Exercise. *Let G be a group and $\mathscr{C}_0$ be the collection of all pairs (X, x), where X is a nonempty right G-set and $x \in X$, $\mathsf{Hom}_{\mathscr{C}}((X, x), (Y, y))$ the set of all G-equivariant maps f from X to Y such that $f(x) = y$, $\circ_{\mathscr{C}}$ the usual composition of maps and id_X is the identity map from X to itself. Prove that this forms a category, denoted by $\mathsf{Set}_\bullet\text{-}G$, called the category of* ***pointed right G-sets****.*

12.6. Let X be a semilocally simply connected topological space, $x_0 \in X$ and let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a universal covering. The functors Γ_p and Fib_{x_0} considered in Proposition 12.4 can also be easily extended to functors

$$\Gamma_{p,\bullet} : \mathsf{Set}_\bullet\text{-}\pi_1(X, x_0) \to \mathsf{Cov}(X, x_0)$$

and

$$\mathsf{Fib}_{x_0,\bullet} : \mathsf{Cov}(X, x_0) \to \mathsf{Set}_\bullet\text{-}\pi_1(X, x_0)$$

as follows. Given a right $\pi_1(X, x_0)$-set F and $v \in F$, we will write $\hat{v}$ for the class in $F \times^{\pi_1(X,x_0)} \tilde{X}$ of $(v, \tilde{x}_0) \in F \times \tilde{X}$. We define $\Gamma_{p,\bullet}(F, v) = (\Gamma_p(F), \hat{v})$ for any right $\pi_1(X, x_0)$-set F with $v \in F$, and note that the covering map $p_F : F \times^{\pi_1(X,x_0)} \tilde{X} \to X$ sends $\hat{v}$ to x_0. If $f : (F, v) \to (F', v')$ is a morphism of pointed right $\pi_1(X, x_0)$-sets, then set $\Gamma_{p,\bullet}(f) = \Gamma_p(f)$ and note that $\Gamma_p(f)$ sends $\hat{v}$ to $\hat{v}'$. Analogously, given a covering $\hat{p} : (\hat{X}, \hat{x}) \to (X, x_0)$ of pointed topological spaces, we set $\mathsf{Fib}_{x_0,\bullet}(\hat{p}) = (\hat{p}^{-1}(\{x\}), \hat{x})$ and for any morphism $f : \hat{X} \to \hat{X}'$ from a covering $\hat{p} : (\hat{X}, \hat{x}) \to (X, x_0)$ of pointed topological spaces to another one $\hat{p}' : (\hat{X}, \hat{x}) \to (X, x_0)$ of pointed topological spaces to the $\pi_1(X, x_0)$-equivariant map $f|_{\hat{p}^{-1}(\{x_0\})} : \hat{p}^{-1}(\{x_0\}) \to \hat{p}'^{-1}(\{x_0\})$ that sends $\hat{x}$ to $\hat{x}'$.

12.7. Proposition. *Let X be a semilocally simply connected topological space with a point $x_0 \in X$ and let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a universal covering. Then the functors $\Gamma_{p,\bullet}$ and $\mathsf{Fib}_{x_0,\bullet}$ defined in paragraph 12.6 are quasi-inverse to each other.*

Proof. The proof is the exactly the same as that of Proposition 12.4. □

12.8. It is easy to verify that the functors $\Gamma_{p,\bullet}$ and $\mathsf{Fib}_{x_0,\bullet}$ in Proposition 12.7 restrict to functors

$$\Gamma_{p,\bullet} : G\text{-}^{\mathsf{ft}}\,\mathsf{Set}_\bullet\text{-}\pi_1(X, x_0) \to \mathsf{Cov}_G(X, x_0) \tag{6}$$

and

$$\mathsf{Fib}_{x_0,\bullet} : \mathsf{Cov}_G(X, x_0) \to G\text{-}^{\mathsf{ft}}\,\mathsf{Set}_\bullet\text{-}\pi_1(X, x_0), \tag{7}$$

which are quasi-inverse to other, where $G\text{-}^{\mathsf{ft}}\,\mathsf{Set}_\bullet\text{-}\pi_1(X, x_0)$ denotes the category of pointed G-$\pi_1(X, x_0)$-bisets such that the left action of G is free and transitive (see paragraph 8.10 and Exercise 12.5).

12.9. Proposition. *Let X be a semilocally simply connected topological space with a point $x_0 \in X$. Then there is a bijection*

$$\left\{ \begin{matrix} \textit{isomorphism classes of pointed} \\ G\textit{-covering spaces} \\ \hat{p} : (\hat{X}, \hat{x}_0) \to (X, x_0) \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \textit{morphisms of groups} \\ \pi_1(X, x_0) \to G \end{matrix} \right\}.$$

Proof. By Exercise 5.17, the equivalence given by the quasi-inverse functors (6) and (7) in paragraph 12.8 induces an isomorphism between the sets of isomorphism classes of the categories $G\text{-}^{\mathsf{ft}}\,\mathsf{Set}_{\bullet}\text{-}\pi_1(X, x_0)$ and $\mathsf{Cov}_G(X, x_0)$. By paragraph 8.11, a G-$\pi_1(X, x_0)$-biset structure on the left regular G-set G is exactly the same as a morphism of groups $\pi_1(X, x_0) \to G$. Moreover, it is easy to verify that there are no morphisms between two pointed G-$\pi_1(X, x_0)$-bisets corresponding to different morphisms of groups $\pi_1(X, x_0) \to G$. The proposition thus follows. □

§13. Algebraic interlude III: Free amalgamated products

13.1. Given a collection of objects $\{X_i\}_{i\in I}$ in a category $\mathscr{C}$, a **coproduct** is an object C in $\mathscr{C}$ together with morphisms $u : X_i \to C$ for all $i \in I$ such that for every other object C' in $\mathscr{C}$ together with morphisms $u'_i : X_i \to C'$ there exists a unique morphism $w : C \to C'$ such that $u'_i = w \circ u_i$ for all $i \in I$.

13.2. Exercise. *Let $\mathscr{C}$ be a category, and let $\{X_i\}_{i\in I}$ be a collection of objects. Assume that $(C, (u_i)_{i\in I})$ and $(C', (u'_i)_{i\in I})$ are two coproducts $\{X_i\}_{i\in I}$. Prove that there exists a unique isomorphism $w : C \to C'$ such that $u'_i = w \circ u_i$ for all $i \in I$. We denote the thus uniquely defined coproduct by $\coprod_{i\in I} X_i$.*

13.3. Let $\{G_i\}_{i\in I}$ be a collection of groups, with I nonempty. By replacing if necessary G_i by $G_i \times \{i\}$ for $i \in I$ we assume that the previous family is disjoint. For $i \in I$, set $G'_i = G_i \setminus \{e_{G_i}\}$, where e_{G_i} denotes the neutral element of G_i. Given $n \in \mathbb{N}$, define $\mathsf{Inj}_n = \{f : [\![1, n]\!] \to I : f \text{ injective}\}$ and

$$\mathsf{Gr}_n = \Big\{ g : [\![1, n]\!] \to \bigsqcup_{i\in I} G'_i : \text{ there exists } f \in \mathsf{Inj}_n \text{ such that } g(\ell) \in G'_{f(\ell)} \text{ for all } \ell \in [\![1, n]\!] \Big\}.$$

We will denote an element g of Gr_n by $(g(1), \dots, g(n))$. For $n = 0$, set $\mathsf{Gr}_n = \{*\}$. Define further

$$\mathsf{Gr} = \bigsqcup_{n\in\mathbb{N}_0} \mathsf{Gr}_n .$$

We define a map $\mu : \mathsf{Gr} \times \mathsf{Gr} \to \mathsf{Gr}$ as follows. First, $\mu(*, w) = \mu(w, *) = w$, for all $w \in \mathsf{Gr}$. Moreover, given $n, m \in \mathbb{N}$, $g = (g(1), \dots, g(n)) \in \mathsf{Gr}_n$ and $h = (h(1), \dots, h(m)) \in \mathsf{Gr}_m$ we define $\mu(g, h)$ recursively as follows. If $g(n) \in G_i$ and $h(1) \in G_j$ with $i \neq j$, we set

$$\mu(g, h) = \mu\Big(\big(g(1), \dots, g(n)\big), \big(h(1), \dots, h(m)\big)\Big) = \big(g(1), \dots, g(n), h(1), \dots, h(m)\big) \in \mathsf{Gr}_{n+m} .$$

If $g(n), h(1) \in G_i$ but $g(n)h(1) \neq e_{G_i}$, where , we set

$$\mu(g, h) = \mu\Big(\big(g(1), \dots, g(n)\big), \big(h(1), \dots, h(m)\big)\Big) = \big(g(1), \dots, g(n)h(1), \dots, h(m)\big) \in \mathsf{Gr}_{n+m-1} .$$

Finally, if $g(n), h(1) \in G_i$ and $g(n)h(1) = e_{G_i}$, we define $\mu(g, h)$ recursively by

$$\mu(g, h) = \mu\Big(\big(g(1), \dots, g(n)\big), \big(h(1), \dots, h(m)\big)\Big) = \mu\Big(\big(g(1), \dots, g(n-1)\big), \big(h(2), \dots, h(m)\big)\Big),$$

where we set $(g(1), \dots, g(n-1)) = *$ if $n = 1$ and $(h(2), \dots, h(m)) = *$ if $m = 1$.

13.4. Exercise. *Suppose the same assumptions as in the previous paragraph.*

(i) Prove that Gr endowed with the operation μ is a group.

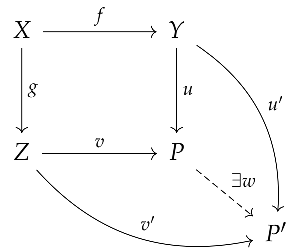
(ii) Given $i \in I$, let $u_i : G_i \to \mathsf{Gr}$ be the map sending e_{G_i} to $* \in \mathsf{Gr}_0$, and $x \in G'_i$ to the element $g \in \mathsf{Gr}_1$ satisfying that $g(1) = x$. Prove that u_i is a morphism of groups for all $i \in I$.

(iii) Let $\{v_i : G_i \to G : i \in I\}$ be a family of morphisms of groups. Prove that there exists a unique morphism of groups $v : \mathsf{Gr} \to G$ such that $v \circ u_i = v_i$ for all $i \in I$. Conclude that coproducts exist in the category of groups.

13.5. In case of only two groups G_1 and G_2, the coproduct of them is typically called the **free product** of groups, and is denoted by $G_1 * G_2$. Given a set X and the family of groups $\{G_x\}_{x\in X}$ with $G_x = \mathbb{Z}$ for every $x \in X$, the coproduct $\coprod_{x\in X} G_x$ is called the **free group** generated by X, and it is typically denoted by $\langle X\rangle$. A quotient of the free group $\langle X\rangle$ by the smallest normal subgroup including a subset $R \subseteq \langle X\rangle$ will be denoted by $\langle X|R\rangle$.

13.6. Exercise. *Let X be a set and $\langle X\rangle$ be the free group generated by X. Denote by $u_x : \mathbb{Z} = G_x \to \langle X\rangle$ be the morphism of groups given by the definition of coproduct. Let $i_X : X \to \langle X\rangle$ be the map sending $x \in X$ to $u_x(1)$, for all $x \in X$. Prove that, given any map $f : X \to G$, where G is a group, there exists a unique morphism of groups $F : \langle X\rangle \to G$ such that $F \circ i_X = f$.*

13.7. Given morphisms $f : X \to Y$ and $g : X \to Z$ in a category $\mathscr{C}$, a **push-out** is an object P in $\mathscr{C}$ together with morphisms $u : Y \to P$ and $v : Z \to P$ such that $u \circ f = v \circ g$, and such that for every other object P' in $\mathscr{C}$ together with morphisms $u' : Y \to P'$ and $v' : Z \to P'$ such that $u' \circ f = v' \circ g$ there is a unique morphism $w : P \to P'$ such that $u' = w \circ u$ and $v' = w \circ v$. Graphically,



13.8. Exercise. *Let $\mathscr{C}$ be a category, and let $f : Y \to X$ and $g : Z \to X$ be two morphisms. Assume that (P, u, v) and (P', u', v') are two push-outs for (f, g). Prove that there exists a unique isomorphism $w : P \to P'$ such that $u' = w \circ u$ and $v' = w \circ v$.*

13.9. By the previous exercise, if push-out exists, they are uniquely determined (up to unique isomorphism). On the other hand, push-outs also exist in the category Grp, as we now show. Indeed, given morphisms $f_1 : G \to G_1$ and $f_2 : G \to G_2$ of groups, then

$$P = \frac{G_1 * G_2}{N},$$

where $G_1 * G_2$ is the free product of the groups G_1 and G_2, and N is the normal closure of the set $X = \{f_1(x)f_2(x)^{-1} : x \in G\} \subseteq G_1 * G_2$, *i.e.* the smallest normal subgroup of $G_1 * G_2$ containing X. The push-out of groups is sometimes called **free product with amalgamation**.

§14. Homotopy III: The Seifert-Van Kampen theorem

14.1. In this section we give a very simple proof of the Seifert-Van Kampen theorem, following [Ful1995]. The latter reference attributes the proof to A. Grothendieck. See [Mas1977], Chap. IV, Section 2, for more general results.

14.2. Theorem. *Let X be a path connected, locally path connected and semilocally simply connected space. Let $U, V \subseteq X$ be two nonempty open subsets such that $U \cup V = X$, and U, V and $U \cap V$ are path connected and semilocally simply connected. Then for every $x \in U \cap V$ the diagram*

$$\begin{array}{ccc} \pi_1(U\cap V,x_0) & \xrightarrow{\pi_1(\mathsf{inc}_{U\cap V,u})} & \pi_1(U,x_0) \\ \Big\downarrow{\scriptstyle \pi_1(\mathsf{inc}_{U\cap V,V})} & & \Big\downarrow{\scriptstyle \pi_1(\mathsf{inc}_{U,X})} \\ \pi_1(V,x_0) & \xrightarrow{\pi_1(\mathsf{inc}_{V,X})} & \pi_1(X,x_0) \end{array}$$

is a push-out in the category of groups, where $\mathsf{inc}_{Y,Z} : Y \to Z$ *is the inclusion map of a subset* $Y \subseteq Z$.

Proof. We have to prove that given any group G and any commutative diagram

$$\begin{array}{cccc} \pi_1(U\cap V,x_0) & \xrightarrow{\pi_1(\mathsf{inc}_{U\cap V,u})} & \pi_1(U,x_0) & \\ \Big\downarrow{\scriptstyle \pi_1(\mathsf{inc}_{U\cap V,V})} & & \Big\downarrow{\scriptstyle \pi_1(\mathsf{inc}_{U,X})} & \searrow{\scriptstyle u} \\ \pi_1(V,x_0) & \xrightarrow{\pi_1(\mathsf{inc}_{V,X})} & \pi_1(X,x_0) & \\ & \searrow{\scriptstyle v} & & G \end{array}$$

of morphisms of groups, there exists a unique morphism of groups $w : \pi_1(X,x_0) \to G$ such that $u = w \circ \pi_1(\mathsf{inc}_{U,X})$ and $v = w \circ \pi_1(\mathsf{inc}_{V,X})$. By Proposition 12.9, there exist unique pointed G-coverings $p_U : (\tilde{U}, \tilde{x}_U) \to (U, x_0)$ and $p_V : (\tilde{V}, \tilde{x}_V) \to (V, x_0)$ corresponding to the morphisms of groups $\pi_1(U,x_0) \to G$ and $\pi_1(V,x_0) \to G$, respectively. Let $\tilde{U}' = p_U^{-1}(U \cap V)$ and $\tilde{V}' = p_V^{-1}(U \cap V)$, as well as the pointed G-coverings $p_U|_{\tilde{U}'} : (\tilde{U}', \tilde{x}_U) \to (U \cap V, x_0)$ and $p_V|_{\tilde{V}'} : (\tilde{V}', \tilde{x}_V) \to (U \cap V, x_0)$. Under the bijection in Proposition 12.9, the last pointed G-coverings correspond to the morphisms of groups given by the compositions of

$$\pi_1(U\cap V,x_0) \to \pi_1(U,x_0) \to G \qquad \text{and} \qquad \pi_1(U\cap V,x_0) \to \pi_1(V,x_0), \to G, \tag{8}$$

respectively. Since the composition of the maps in the left of (8) coincides with the composition of the maps in the right, $p_U|_{\tilde{U}'} : (\tilde{U}', \tilde{x}_U) \to (U \cap V, x_0)$ and $p_V|_{\tilde{V}'} : (\tilde{V}', \tilde{x}_V) \to (U \cap V, x_0)$ are isomorphic. In consequence, there exists a unique G-equivariant morphism of pointed topological spaces $f : (\tilde{U}', \tilde{x}_U) \to (\tilde{V}', \tilde{x}_V)$ such that $p_V|_{\tilde{V}'} \circ f = p_U|_{\tilde{U}'}$. Define $\tilde{X} = (\tilde{U} \sqcup \tilde{V})/\sim$, where $\sim$ is the equivalence relation on $\tilde{X}$ generated by $u \sim f(u)$ for $u \in \tilde{U}'$, and endow it with the quotient topology. Denote by $y^\sim \in (\tilde{U} \sqcup \tilde{V})/\sim$ the equivalence class of $y \in \tilde{U} \sqcup \tilde{V}$, and set $\tilde{x}_0 = \tilde{x}_V^\sim = \tilde{x}_U^\sim$. Let $p : \tilde{X} \to X$ be the unique map induced by the mapping $\tilde{U} \sqcup \tilde{V} \to X$ sending $u \in \tilde{U}$ to $p_U(u)$ and $v \in \tilde{V}$ to $p_V(v)$. It is clear that p is continuous map, sends $\tilde{x}_0$ to x_0 and it is a G-covering. By Proposition 12.9, there exists a unique morphism of groups $w : \pi_1(X,x_0) \to G$ such that the diagram

$$\begin{array}{cccc} \pi_1(U\cap V,x_0) & \xrightarrow{\pi_1(\mathsf{inc}_{U\cap V,u})} & \pi_1(U,x_0) & \\ \Big\downarrow{\scriptstyle \pi_1(\mathsf{inc}_{U\cap V,V})} & & \Big\downarrow{\scriptstyle \pi_1(\mathsf{inc}_{U,X})} & \searrow{\scriptstyle u} \\ \pi_1(V,x_0) & \xrightarrow{\pi_1(\mathsf{inc}_{V,X})} & \pi_1(X,x_0) & \\ & \searrow{\scriptstyle v} & \searrow{\scriptstyle w} & G \end{array}$$

commutes. The theorem follows. □

14.3. Exercise. *Let X be the topological space given in Exercise 10.11. Prove that* $\pi_1(X,4) = \mathbb{Z} * \mathbb{Z} = \langle x, y \rangle$, *the free group on two generators.*

14.4. Exercise. *Let* $Y = [-1,1] \times [-1,1]$ *and let* $\sim$ *be the equivalence relation on* Y *generated by* $(x,1) \sim (x,-1)$ *and* $(-1,y) \sim (1,-y)$, *for all* $x, y \in [-1,1]$. *Define* $\mathbb{K} = Y/\sim$ *the set of equivalence classes, provided with the* *quotient topology. It is usually called the (topological)* **Klein bottle**. *Let $U \subseteq \mathbb{K}$ (resp., $V \subseteq \mathbb{K}$) be the image of $[-1,1]\times\,]-1,1[$ (resp., $Y \setminus ([-1,1] \times \{0\})$) under the canonical projection $Y \to Y/\sim$. Let $p \in U \cap V$.*

(i) *Prove that U and V are homeomorphic to the Möbius band $\mathbb{M}$ (see Exercises 7.15 and 9.20).*

(ii) *Prove that $U \cap V$ is homeomorphic to a cylinder $S^1 \times\,]-1,1[$.*
Hint. Prove that the map $f : [-1,1] \times (\,]-1,1[\,\setminus\{0\}) \to S^1 \times\,]-1,1[$ given by

$$f(x,y) = \begin{cases} (e^{\pi i x/2}, 2y+1), & \text{if } (x,y) \in [-1,1]\times\,]-1,0[, \\ (e^{\pi i(2x+1)/2}, -2y+1), & \text{if } (x,y) \in [-1,1]\times\,]0,1[\end{cases}$$

induces a homeomorphism $U \cap V \to S^1 \times\,]-1,1[$.

(iii) *Prove that the group morphisms*

$$\mathbb{Z} \cong \pi_1(U \cap V, p) \to \pi_1(U,p) \cong \mathbb{Z} \qquad \text{and} \qquad \mathbb{Z} \cong \pi_1(U \cap V, p) \to \pi_1(V,p) \cong \mathbb{Z}$$

send 1 to 2.
Hint. Use Exercise 9.20, (iii).

(iv) *Deduce the isomorphism of groups $\pi_1(\mathbb{K}, p) \cong \langle x, y | x^2 y^{-2} \rangle$.*

14.5. Given topological spaces X and Y, a closed subspace $A \subseteq X$ and a continuous map $f : A \to Y$, consider the equivalence relation $\sim$ on $X \sqcup Y$ generated by $a \sim f(a)$ for all $a \in A$, and define $X \cup_f Y$ as the set of all equivalence classes endowed with the quotient topology. It is called the **adjunction space** given by attaching Y to X along f.

14.6. Exercise. *Let $n \in \mathbb{N}$, $X = S^1$ and $Y = \bar{B}_1(0,0) \subseteq \mathbb{R}^2$. Consider the map $f : X \to Y$ sending $z \in S^1$ to z^n. Let $Z = X \sqcup Y$ and $\sim$ the equivalence relation on Z defined in paragraph 14.5. Let $\tilde{U} = B_1(0,0)$ and $\tilde{V} = (\bar{B}_1(0,0) \setminus \{(0,0)\}) \sqcup X$. Define U and V as the image of $\tilde{U}$ and $\tilde{V}$ under the canonical projection $Z \to Z/\sim$. Let $p \in U \cap V$.*

(i) *Prove that the group morphism $\mathbb{Z} \cong \pi_1(U \cap V, p) \to \pi_1(V,p) \cong \mathbb{Z}$ sends 1 to n.*

(ii) *Deduce that $\pi_1(X \cup_f Y, p) \cong \mathbb{Z}/n\mathbb{Z}$.*

§15. Surfaces I: General theory

15.1. We will recall first some basic facts on the theory of surfaces. We refer the reader to the wonderful book [Lee2011], or the nice exposition [Mas1977], for a more comprehensive treatment.

15.2. We recall that a **(topological) manifold with boundary** of dimension $n \in \mathbb{N}_0$ (or n-manifold with boundary for short) is a Hausdorff second countable topological space M together with an open covering $\mathcal{U}$ such that there are homeomorhisms of $\phi : U \to V$ where V is an open subset of $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$, for all $U \in \mathcal{U}$. An element $p \in M$ is called a **boundary point** if there is a (equivalently, if for every) homeomorhism of $\phi : U \to V$ with $p \in U \in \mathcal{U}$ such that $\phi(p) \in \partial\mathbb{H}^n$. Denote by ∂M the set of boundary points. It is easy to see that ∂M is a closed subset. An n-manifold with boundary M is called a **(topological) manifold** of dimension n (or n-manifold for short) if $\partial M = \emptyset$.

15.3. A **coordinate ball** of an n-manifold M is a homeomorphism $\phi : B \to B_r(x_0) \subseteq \mathbb{R}^n$, where B is an open subset of M. We will typically denote a coordinate ball simply by B. We say that a coordinate ball $\phi : B \to B_r(x_0) \subseteq \mathbb{R}^n$ is **regular** if there exists an open set $B' \supseteq \bar{B}$ in M including the closure $\bar{B}$ of B, $r' > r$ and a homeomorphism $\phi' : B' \to B_{r'}(x_0) \subseteq \mathbb{R}^n$ such that $\phi'(B) = B_r(x_0)$. Note that the fact that ϕ' is a homeomorphism implies that $\phi'(\bar{B}) = \bar{B}_r(x_0)$. A **(topological) surface** is a manifold of dimension 2.

15.4. Remark. *We remark that there exist nonregular coordinate balls. For instance, there exists a subset $S \subseteq \mathbb{R}^3$, called the **Alexander horned sphere**, such that*

(i) *S is homeomorphic to the usual sphere $S^2 \subseteq \mathbb{R}^3$;*

(ii) *$\mathbb{R}^3 \setminus S$ is the disjoint union of two connected components, B and C, with B bounded;*

(iii) *$B \sqcup S$ is homeomorphic to the closed unit ball $\bar{B}_1(\mathbf{0}_{\mathbb{R}^3})$;*

(iv) *there exists $s \in S$ such that for every open set U including s, the image of the map*

$$\pi_1(U \cap C) \to \pi_1(C)$$

is not trivial.

The last condition implies that $B \subseteq \mathbb{R}^3$ is a nonregular coordinate ball of $\mathbb{R}^3$.

15.5. Exercise. *(i)* *Prove that S^2 is a surface.*

(ii) *Prove that the torus $\mathbb{T}^2 = S^1 \times S^1$ is a surface.*

(iii) *Prove that the (real) projective plane $\mathbb{RP}^2$ is a surface.*

15.6. Given M and N be two n-manifolds with nonempty boundaries and $f : \partial M \to \partial N$ a homeomorphism, one can prove that the adjunction $M \cup_f N$, recalled in paragraph 14.5, is an n-manifold (see [Lee2011], Thm. 3.79). Given two connected n-manifolds M_1 and M_2, regular coordinate balls $B_i \subsetneq M_i$ for $i = 1, 2$, then the subspaces $M_i' = M_i \setminus B_i$ are manifolds with boundaries. Given a homeomorphism $f : \partial M_1 \to \partial M_2$, by the previous comments, $M_1' \cup_f M_2'$ is an n-manifold, called **connected sum** of M_1 and M_2, and is denoted by $M_1 \# M_2$. This definition depends in principle on the choice of regular coordinate balls and the homeomorphism f between the boundaries, although one can prove that in fact at most two nonhomeomorphic manifolds can be obtained when picking different coordinate balls and homeomorphisms. This makes use of the so-called annulus theorem, a very hard theorem that states that if B is a regular coordinate ball embedded in $B_r(\mathbf{0}_{\mathbb{R}^n}) \subseteq \mathbb{R}^n$, then $\bar{B}_r(\mathbf{0}_{\mathbb{R}^n}) \setminus B$ is homeomorphic to $\bar{B}_2(\mathbf{0}_{\mathbb{R}^n}) \setminus B_1(\mathbf{0}_{\mathbb{R}^n})$. If $n = 2$, the two possible connected sums are in fact homeomorphic to each other (see [Lee2011], Problem 10-8, for the compact case). One can also show that $M_1 \# M_2$ is compact (resp., connected) if M_1 and M_2 are so (resp., and $n > 1$) (see [Lee2011], Problem 4-18).

15.7. We will describe a combinatorial way to construct surfaces. Let S be a set, $S_{\pm 1} = S \times \{\pm 1\}$, $S_{\pm} = S_{+1} \sqcup S_{-1}$ and $W = \sqcup_{k \in \mathbb{N}} S_{\pm}^k$, the set of all **words** in S. An element of $(a, 1)$ is written simply a, whereas $(a, -1)$ will be written a^{-1}. We define $(a^{-1})^{-1} = a$. An element $w = (w_1, \dots, w_k) \in S_{\pm}^k$ for $k \in \mathbb{N}$ will be simply written $w = w_1 \dots w_k$, whose **length** is $\ell(w) = k$. The set W has a natural structure of (associative) monoid (without unit), given by concatenation of words, that we denote by yuxtaposition, *i.e.* $(w_1 \dots w_k) \cdot (w_1' \dots w_{k'}') = w_1 \dots w_k w_1' \dots w_{k'}'$. A **polygonal presentation** $\mathcal{P}$, written $\langle S | w^1, \dots, w^k \rangle$, is a finite set S together with a finite tuple of words $w^1, \dots, w^k$, each of which has length at least 3, such that each element $a \in S$ appears in at least one word (as a or a^{-1}). A **surface presentation** is a polygonal presentation such that every letter $a \in S$ appear exactly twice in the whole list $w^1, \dots, w^k$ (as a or a^{-1}). We will also allow the surface presentations $\langle a | aa^{-1} \rangle$ and $\langle a | a^{-1}a \rangle$ (associated to the sphere S^2), as well as $\langle a | aa \rangle$ and $\langle a | a^{-1}a^{-1} \rangle$ (associated to the projective plane $\mathbb{RP}^2$).

15.8. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, define $\zeta_{k,n} = ie^{2\pi k i/n} \in \mathbb{C} \cong \mathbb{R}^2$. For an integer $n \geq 3$, we denote by $P_n \subseteq \mathbb{R}^2$ the regular convex polygon with n edges and set of vertices $V_n = \{\zeta_{k,n} : k \in [\![0, n-1]\!]\}$. It is endowed with the subspace topology of $\mathbb{R}^2$. Let $E_n = \{e_k = [\zeta_{k,n}, \zeta_{k+1,n}] : k \in [\![0, n-1]\!]\}$ the set edges of P_n and $\bar{E}_n = \{\bar{e}_k = (\zeta_{k,n}, \zeta_{k+1,n}) : k \in [\![0, n-1]\!]\}$ be the set of oriented edges of P_n. If e is an usual edge, $\bar{e}$ will thus denote the corresponding oriented edge. Given $x, y, x', y' \in \mathbb{R}^2$, denote by $L_{(x,y)}^{(x',y')}, \bar{L}_{(x,y)}^{(x',y')} : [x, y] \to [x', y']$ the maps

$$L_{(x,y)}^{(x',y')}(tx + (1-t)y) = (tx' + (1-t)y'),$$

for $t \in [0, 1]$, and $\bar{L}_{(x,y)}^{(x',y')} = L_{(x,y)}^{(y',x')}$.

15.9. Given a polygonal presentation $\langle S|w^1,\dots,w^k\rangle$, where $w^i = w^1_1 \dots w^i_{\ell(w^i)}$ for all $i \in [\![1,k]\!]$, consider $P = \sqcup_{i=1}^k P_{\ell(w^i)}$ as well as the maps $\phi : \sqcup_{i=1}^k E_{\ell(w^i)} \to S_\pm$ sending the j-th edge e^i_j of $P_{\ell(w^i)}$ to w^i_j, for $i \in [\![1,k]\!]$ and $j \in [\![1,\ell(w^i)]\!]$. The **geometric realization** $|\mathcal{P}|$ of $\mathcal{P} = \langle S|w^1,\dots,w^k\rangle$ is the quotient topological space of P, which is endowed with the disjoint union topology, by the equivalence relation $\sim$ on P generated by

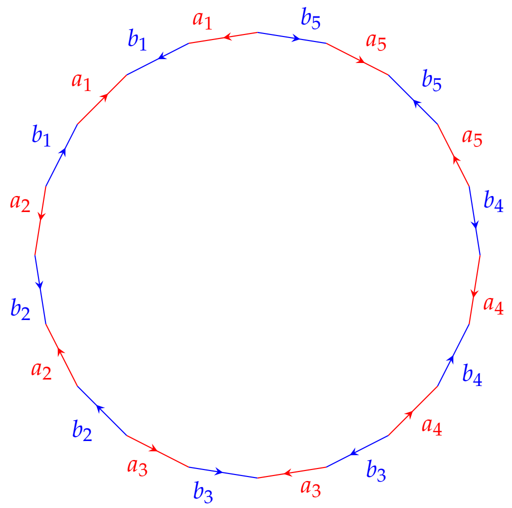
$$x \sim L^{\vec{e}'}_{\vec{e}}(x),$$

for all $x \in e$ and for all pairs of edges e, e' in P such that $\phi(e) = \phi(e')$, and by

$$x \sim \bar{L}^{\vec{e}'}_{\vec{e}}(x),$$

for all $x \in e$ and for all pairs of edges e, e' in P such that $\phi(e) = \phi(e')^{-1}$.

15.10. We will typically draw a polygonal presentations of the form $\langle S|w\rangle$ (or its geometric realizations) as follows. We draw the polygon $P_{\ell(w)}$. For every edge e in $P_{\ell(w)}$, we orient it adding an arrow $\vec{e}$ together with the label $\phi(e)$, if $\phi(e) \in S_{+1}$, and an arrow $-\vec{e}$ together with the label $\phi(e)^{-1}$, if $\phi(e) \in S_{-1}$. For example, you find below the corresponding graphical representation of $\langle a_1, b_1, \dots, a_5, b_5 | \prod_{i=1}^5 [a_i, b_i]\rangle$, where we recall that $[x,y] = xyx^{-1}y^{-1}$ is the commutator of the elements x, y in a group.



15.11. Exercise. *(i) Prove that S^2 is homeomorphic to the geometric realization of the surface presentation $\langle a,b|abb^{-1}a^{-1}\rangle$.*

(ii) Prove that $\mathbb{T}^2$ is homeomorphic to the geometric realization of the surface presentation $\langle a,b|aba^{-1}b^{-1}\rangle$.

(iii) Prove that $\mathbb{RP}^2$ is homeomorphic to the geometric realization of the surface presentation $\langle a,b|abab\rangle$.

(iv) Prove that Klein bottle $\mathbb{K}$ (see Exercise 14.4) is homeomorphic to the geometric realization of the surface presentation $\langle a,b|abab^{-1}\rangle$.

15.12. The following operations are called **elementary transformations** of the polygonal presentation.

(i) **Relabeling :** the result in $\mathcal{P}$ of applying a bijection $S \to S'$ or applying the bijection $S_\pm \to S_\pm$ sending $a_0^{\pm 1}$ to $a_0^{\mp 1}$ for a fixed $a_0 \in S$ and $a^{\pm 1}$ to itself, for $a \in S \setminus \{a_0\}$.

(ii) **Subdividing :** replacing every occurrence in $\mathcal{P}$ of some $a_0 \in S$ by $a_0 e$ and a_0^{-1} by $e^{-1}a_0^{-1}$, where e is a new element not in $S_\pm$.

(iii) **Consolidating :** if elements a and b of S always occur in the words of $\mathcal{P}$ either as ab or $b^{-1}a^{-1}$, replacing every occurrence of ab by a and every occurrence of $b^{-1}a^{-1}$ by a^{-1}, provided that the result is one or more words of length at least 3 or a single word of length 2.

(iv) **Reflecting :** $\langle S|x_1 \dots x_n, w^1, \dots, w^k\rangle$ transforms into $\langle S|x_n^{-1} \dots x_1^{-1}, w^1, \dots, w^k\rangle$, for $x_1, \dots, x_n \in S_\pm$.

(v) **Rotating :** $\langle S|x_1 \dots x_n, w^1, \dots, w^k\rangle$ transforms into $\langle S|x_2 \dots x_n x_1, w^1, \dots, w^k\rangle$, for $x_1, \dots, x_n \in S_\pm$.

(vi) **Cutting :** $\langle S|w^1 w^2, \dots, w^k\rangle$ transforms into $\langle S \sqcup \{e\}|w^1 e, e^{-1} w^2, \dots, w^k\rangle$, provided w^1 and w^2 have both length at least 2.

(vii) **Pasting :** $\langle S \sqcup \{e\}|w^1 e, e^{-1} w^2, \dots, w^k\rangle$ transforms into $\langle S|w^1 w^2, \dots, w^k\rangle$.

(viii) **Folding :** $\langle S \sqcup \{e\}|w^1 e e^{-1}, w^2, \dots, w^k\rangle$ transforms into $\langle S|w^1, w^2, \dots, w^k\rangle$, provided w^1 has length at least 3, or length 2 if $k = 1$.

(ix) **Unfolding :** $\langle S|w^1, w^2, \dots, w^k\rangle$ transforms into $\langle S \sqcup \{e\}|w^1 e e^{-1}, w_2, \dots, w^k\rangle$.

15.13. A lengthy but elementary verification of topological invariance of each of the previous elementary transformations provides the following result.

15.14. Proposition ([Lee2011], Prop. 6.10)**.** *Let $\mathcal{P}$ and $\mathcal{P}'$ be two polygonal presentations that can be linked through a finite sequence of elementary transformations. Then, $|\mathcal{P}|$ is homeomorphic to $|\mathcal{P}'|$.*

15.15. Using the previous result for clever representations of the surfaces and some technical results on triangulations of surfaces we get the following result.

15.16. Proposition ([Lee2011], Prop. 6.12)**.** *Let M be a surface homeomorphic to $|\mathcal{P}|$ for a surface presentation $\mathcal{P} = \langle S|w\rangle$, and M' be a surface homeomorphic to $|\mathcal{P}'|$ for a surface presentation $\mathcal{P}' = \langle S'|w'\rangle$. Then $M\#M'$ is homeomorphic to the geometric realization of the surface presentation $\langle S \sqcup S'|ww'\rangle$.*

15.17. Exercise. *(i) Let M be a connected sum of $g \in \mathbb{N}$ copies of the torus $\mathbb{T}^2 = S^1 \times S^1$. Prove that M is homeomorphic to the geometric realization of the surface presentation $\langle a_1, b_1, \dots, a_g, b_g| \prod_{i=1}^{g} [a_i, b_i]\rangle$.*

(ii) Let M be a connected sum of $n \in \mathbb{N}$ copies of the torus $\mathbb{RP}^2$. Prove that M is homeomorphic to the geometric realization of the surface presentation $\langle c_1, \dots, c_n| \prod_{i=1}^{n} c_i^2\rangle$.

15.18. Theorem ([Lee2011], Prop. 6.14, Thm. 6.15, or [Mas1977], Thm. I.5.1)**.** *Given a any compact surface M there exists a surface presentation $\mathcal{P} = \langle S|w^1, \dots, w^k\rangle$ such that M is homeomorphic to $|\mathcal{P}|$. Moreover, any compact connected surface is homeomorphic to one of the following*

(i) the sphere $\mathbb{S}^2$;

(ii) a connected sum of one or more copies of the torus $\mathbb{T}^2 = S^1 \times S^1$;

(iii) a connected sum of one or more copies of the projective plane $\mathbb{RP}^2$.

15.19. Exercise. *(i) Prove that the Klein bottle $\mathbb{K}$ (see Exercise 14.4) is homeomorphic to $\mathbb{RP}^2\#\mathbb{RP}^2$.*

(ii) Prove that the $\mathbb{T}^2\#\mathbb{RP}^2$ is homeomorphic to $\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$.

§16. Surfaces II: Homotopy groups

16.1. We conclude these short notes with the statement of the fundamental group of any compact connected surface M. Let $\mathcal{P} = \langle x_1, \dots, x_m|w\rangle$ be its surface presentation, according to Theorem 15.18, and Exercises 15.11 and 15.17. Define $\hat{M} = \bigvee_{i=1}^{m}(S^1, 1)$, the wedge product of m copies of $(S^1, 1)$. By the previous results, there is a continuous map $f : \partial\bar{B}_1(\mathbf{0}_{\mathbb{R}^2}) \to \hat{M}$ such that M is homeomorphic to $\bar{B}_1(\mathbf{0}_{\mathbb{R}^2}) \cup_f \hat{M}$. Denote $\bar{M} = \bar{B}_1(\mathbf{0}_{\mathbb{R}^2}) \sqcup \hat{M}$. Take then $U, V \subseteq \bar{B}_1(\mathbf{0}_{\mathbb{R}^2}) \cup_f \hat{M}$ to be the image under the canonical projection $\bar{M} \to \bar{B}_1(\mathbf{0}_{\mathbb{R}^2}) \cup_f \hat{M}$ of $(\bar{B}_1(\mathbf{0}_{\mathbb{R}^2}) \setminus \{\mathbf{0}_{\mathbb{R}^2}\}) \sqcup \hat{M}$ and $B_1(\mathbf{0}_{\mathbb{R}^2})$, respectively. Then V is contractible, so simply connected, and using Theorem 14.2 we get the following result.

16.2. Theorem ([Lee2011], Cor. 10.17, or [Mas1977], Chap. IV, Section 5). *Omitting the general point of the following compact connected surfaces, the corresponding fundamental group is given by*

(i) $\pi_1(\mathbb{S}^2) = 1$;

(ii) $\pi_1(\mathbb{T}^2 \# \dots \# \mathbb{T}^2) \cong \langle a_1, b_1, \dots, a_g, b_g \,|\, \prod_{i=1}^{g} [a_i, b_i] \rangle$, *where we are considering* g *factors in the connected sum and we recall that* $[x, y] = xyx^{-1}y^{-1}$ *is the commutator of the elements* x, y *in a group;*

(iii) $\pi_1(\mathbb{RP}^2 \# \dots \# \mathbb{RP}^2) \cong \langle c_1, \dots, c_n \,|\, \prod_{i=1}^{n} c_i^2 \rangle$, *where we are considering* n *factors in the connected sum.*

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