Lectures on basic homological algebra

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Basic conventions

0.1. We will denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of (strictly) positive integers, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ the set of nonnegative integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ the set of all integers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{R}$ the set of all real numbers. Given two integers $m \leq n$, we define $[\![m, n]\!] = \{x \in \mathbb{Z} : m \leq x \leq n\}$. If I is a set, $\#(I)$ denotes its cardinal. For a map $f : X \to Y$, the image of f will be denoted by $\mathsf{Img}(f)$. Moreover, given a set X, $\mathsf{id}_X : X \to X$ will usually denote the identity on X.

0.2. Unless otherwise stated, all of our rings A have a unit 1_A, which is different from the zero element $\mathbf{0}_A$, and all of the morphisms of rings preserve the unit. If A is a ring, $A^\times$ denotes the group of invertible elements of A. We denote the category of all left (resp., right) A-modules with A-linear morphisms by ${}_A\,\mathsf{Mod}$ (resp., Mod_A). If M is a left (resp., right) module, we will denote $\mathbf{0}_M$ its zero element, whereas 0 will denote the trivial module having only one element. We will work with left modules unless otherwise stated. We will also use the basic notions of category theory given in the Appendix.

§1. Lecture I : Chain complexes

§1.1. Graded modules

1.1. Definition. *Let A be a ring. A ($\mathbb{Z}$-)**graded A-module** is an A-module M together with a direct sum decomposition $M = \oplus_{n\in\mathbb{Z}} M_n$ of A-modules. Given two graded A-modules $M = \oplus_{n\in\mathbb{Z}} M_n$ and $N = \oplus_{n\in\mathbb{Z}} N_n$, a **morphism** of graded A-modules from M to N is a morphism of A-modules $f : M \to N$ such that $f(M_n) \subseteq N_n$, for all $n \in \mathbb{Z}$. We denote by $\mathsf{Hom}_{\mathsf{gr}\,A}(M, N)$ the set of all morphisms of graded A-modules from M to N. It is clear that $\mathsf{Hom}_{\mathsf{gr}\,A}(M, N)$ is an abelian group via $(f + g)(m) = f(m) + g(m)$, for $m \in M$ and $f, g \in \mathsf{Hom}_{\mathsf{gr}\,A}(M, N)$. This defines a category that we will denote by ${}_A\,\mathsf{GMod}$.*

1.2. Even though it does not represent anything relevant at the moment, we will consider in the sequel two different ways of indicating a grading on a module M. The one we presented in the previous definition, with lower indices $M = \oplus_{n\in\mathbb{Z}} M_n$, is called **homological**. There is another one, called **cohomological**, denoted with upper indices $M = \oplus_{n\in\mathbb{Z}} M^n$. The relation between the two is $M^n = M_{-n}$, for $n \in \mathbb{Z}$.

1.3. A (homological) graded A-module $M = \oplus_{n\in\mathbb{Z}} M_n$ is said to be **bounded below** (resp., **bounded above**) if there exists $n_0 \in \mathbb{Z}$ such that $M_n = 0$ for all integers $n < n_0$ (resp., $n > n_0$). It is **bounded** if it is bounded below and bounded above.

1.4. If $f : M \to N$ is a morphism of graded A-modules, one denotes by f_n the map $f|_{M_n} : M_n \to N_n$ induced by f, for $n \in \mathbb{Z}$. Note that an A-module M can be considered as a graded A-module, that we denote $\iota(M)$, such that $\iota(M)_0 = M$ and $\iota(M)_n = 0$, for all $n \in \mathbb{Z} \setminus \{0\}$. Moreover, if $f : M \to N$ is a morphism of A-modules, then we define $\iota(f) : \iota(M) \to \iota(N)$ as the unique morphism of A-modules such that $\iota(f)|_{\iota(M)_n} = 0$ if $n \in \mathbb{Z} \setminus \{0\}$ and $\iota(f)|_{\iota(M)_0}$ is just f. It is clear that that this defines a fully faithful functor

$$\iota : {}_A\,\mathsf{Mod} \to {}_A\,\mathsf{GMod}.$$

This allows us to see an A-module as a graded A-module concentrated in degree zero. To ease the burden of the notation we will usually not write ι in the sequel.

1.5. Exercise. *Show that there is natural isomorphism $\mathsf{Hom}_{\mathsf{gr}\,A}(\iota(A), M) \cong M_0$ of abelian groups, where A is the regular module over itself.*

1.6. If $M = \oplus_{n\in\mathbb{Z}} M_n$ is a graded A-module, a nonzero element $m \in M_n$ will be called **homogeneous** of degree n. We will usually denote the degree of a homogeneous element m by $\deg m$ or $|m|$. A **graded submodule** of a graded A-module $M = \oplus_{n\in\mathbb{Z}} M_n$ is an A-submodule N of M such that $N = \sum_{n\in\mathbb{Z}}(N \cap M_n)$. In particular, N is also a graded A-module, where $N_n = N \cap M_n$, for $n \in \mathbb{Z}$, and the inclusion map $N \to M$ is a morphism of graded A-modules. If N is a graded submodule of a graded A-module M, then we define the usual quotient M/N, defined as the set of equivalence classes under the equivalence relation $\sim_N$ on M given by $m \sim_N m'$ if and only if $m - m' \in N$, together with the operations

$$[m] + [m'] = [m + m'] \text{ and } a.[m] = [a.m]$$

for $m, m' \in M$ and $a \in A$, where $[m]$ denotes the equivalence class in M/N including m, is also a graded A-module, where $(M/N)_n = \{[m] : m \in M_n\}$ for all $n \in \mathbb{Z}$. Note that the projection map $M \to M/N$ sending m to $[m]$ is a morphism of graded A-modules.

1.7. Given a morphism of graded A-modules $f : M \to N$, we define its **kernel** of f as the subset of M given by all elements $m \in M$ such that $f(m) = \mathbf{0}_N$, and it will be denoted by $\mathsf{Ker}(f)$. Analogously, we define the **image** of f as the subset of N given by all elements $f(m) \in N$ for $m \in M$, and it will be denoted by $\mathsf{Img}(f)$. It is easy to verify that $\mathsf{Ker}(f)$ is a graded submodule of the graded A-module M, whereas $\mathsf{Img}(f)$ is a graded submodule of the graded A-module N. Furthermore, define the **cokernel** $\mathsf{Coker}(f)$ of f as the graded A-module $N/\mathsf{Img}(f)$.

1.8. Given a graded A-module $M = \oplus_{n\in\mathbb{Z}} M_n$ and $k \in \mathbb{Z}$, we define its **shift** $M[k]$ as the graded module satisfying that $M[k]_n = M_{n+k}$, for all $n \in \mathbb{Z}$. [1] Moreover, we denote by $s_{M,k} : M \to M[k]$ the morphism of A-modules whose underlying set-theoretic map is the identity of M. If $k = 1$, we will usually write s_M instead of $s_{M,1}$, or even simply s if the graded A-module M is clear from the context. Note that $s_M$is not a morphism of graded A-modules. If $f : M \to N$ is a morphism of graded A-modules, define $f[k] : M[k] \to N[k]$ as the morphism of graded A-modules sending $s_{M,k}(m)$ to $s_{N,k}(f(m))$, for all $m \in M$. It is clear that that this defines a functor

$$(-)[k] : {}_A\mathsf{GMod} \to {}_A\mathsf{GMod}\,.$$

By abuse of language **we usually do not indicate the shift on the morphisms**, unless we want to emphasize its role.

1.9. Exercise. *Show that there is a natural isomorphism of functors $(-)[k] \circ (-)[\ell] \cong (-)[k+\ell]$, for all $k, \ell \in \mathbb{Z}$.*

1.10. We denote by $\mathscr{H}\!om_{\mathsf{gr}\,A}(M,N)$ the graded $\mathbb{Z}$-module

$$\mathscr{H}\!om_{\mathsf{gr}\,A}(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathsf{Hom}_{\mathsf{gr}\,A}(M, N[n]).$$

Note that the map $s_{M,n}$ defined in paragraph 1.8 belongs to $\mathscr{H}\!om_{\mathsf{gr}\,A}(M, M[n])_{-n} \subseteq \mathscr{H}\!om_{\mathsf{gr}\,A}(M, M[n])$. We will usually say that a morphism of graded A-modules $f : M \to N[k]$ is a **homogeneous homomorphism of degree k** from M to N. Moreover, given graded A-modules M', M, M'', we consider the $\mathbb{Z}$-bilinear map

$$\circ_{\mathsf{gr}} : \mathscr{H}\!om_{\mathsf{gr}\,A}(M, M'') \times \mathscr{H}\!om_{\mathsf{gr}\,A}(M', M) \longrightarrow \mathscr{H}\!om_{\mathsf{gr}\,A}(M', M'')$$

sending $(g, f) \in \mathsf{Hom}_{\mathsf{gr}\,A}(M, M''[m]) \times \mathsf{Hom}_{\mathsf{gr}\,A}(M', M[n])$ to $g[n] \circ f \in \mathsf{Hom}_{\mathsf{gr}\,A}(M', M''[n+m])$.

1.11. Exercise. *Consider the morphism*

$$\mathscr{H}\!om_{\mathsf{gr}\,A}(M, N[n]) \to \mathscr{H}\!om_A(M,N)[n]$$

of graded $\mathbb{Z}$-modules sending $f \in \mathscr{H}\!om_{\mathsf{gr}\,A}(M, N[n])$ to $s_{\mathscr{H}\!om_A(M,N),n}(s_{N,n}^{-1} \circ f)$. Show that it is an isomorphism of graded $\mathbb{Z}$-modules.

[1] We follow the convention of [Wei1994] for the shift. We remark however that this is the opposite to the one followed in other references.

1.12. Given a family $\{M_i\}_{i\in I}$ of graded A-modules we define its **product** as the graded A-module P such that $P_n = \prod_{i\in I} M_{i,n}$ is the usual product of A-modules, for $n \in \mathbb{Z}$. We will denote the product graded A-module P by $\prod_{i\in I} M_i$. Analogously, their **coproduct** (or **direct sum**) is the graded A-module C such that $C_n = \oplus_{i\in I} M_{i,n}$ is the usual direct sum of A-modules, for $n \in \mathbb{Z}$. We will denote the direct sum graded A-module C by $\coprod_{i\in I} M_i$.

1.13. Exercise. *Show that the previous product and coproduct of graded A-modules are precisely the categorical product and coproduct in the category ${}_A\,\mathsf{GMod}$, respectively.*

§1.2. Complexes of modules

1.14. Definition. *A **(homological) complex of A-modules** (also called, a **(homological) differential graded A-module**) is a graded A-module $M = \oplus_{n\in\mathbb{Z}} M_n$ together with a morphism of graded A-modules $d_M : M \to M[-1]$, called **differential**, such that $d_M \circ_{\mathrm{gr}} d_M = 0$, where we recall that $d_M \in \mathcal{H}om_{\mathrm{gr}\,A}(M, M)_{-1}$ (see paragraph 1.10). We will typically denote a complex of A-modules by (M, d_M). Given two complexes of A-modules (M, d_M) and (N, d_N), a **morphism of complexes of A-modules** from (M, d_M) to (N, d_N) is a morphism of graded A-modules $f : M \to N$ such that $f[-1] \circ d_M = d_N \circ f$. We denote by $\mathrm{Hom}_{\mathrm{dg}\,A}(M, N)$ the abelian subgroup of $\mathrm{Hom}_{\mathrm{gr}\,A}(M, N)$ formed by all morphisms of complexes of A-modules. This defines a category ${}_A\,\mathsf{DGMod}$.*

1.15. According to our convention of not writting the shift on the morphisms, we will write from now on the previous identity simply as $d_M \circ d_M = 0$, or $d_M^2 = 0$. Note that, by definition, $d_M(M_n) \subseteq M_{n-1}$, for all $n \in \mathbb{Z}$. Let us denote this induced map $d_M|_{M_n} : M_n \to M_{n-1}$, simply by d_n, for all $n \in \mathbb{Z}$. It is clear that $d_n : M_n \to M_{n-1}$ is a morphism of A-modules. Hence, we can define a complex of A-modules as an infinite sequence of A-modules and morphisms of A-modules

$$\dots \xrightarrow{d_{n+2}} M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \dots$$

such that $d_{n-1} \circ d_n = 0$, for all $n \in \mathbb{Z}$. Moreover, a morphism from (M, d_M) to $(M', d_{M'})$ is equivalently given by given by a diagram

$$\begin{tikzcd} \dots \arrow[r, "d_{n+2}"] & M_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & M_n \arrow[r, "d_n"] \arrow[d, "f_n"] & M_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & \dots \\ \dots \arrow[r, "d'_{n+2}"] & M'_{n+1} \arrow[r, "d'_{n+1}"] & M'_n \arrow[r, "d'_n"] & M'_{n-1} \arrow[r, "d'_{n-1}"] & \dots \end{tikzcd}$$

such that all squares are commutative, *i.e.* $d'_n \circ f_n = f_{n-1} \circ d_n$ for $n \in \mathbb{Z}$, where we are writting d'_n for the restriction of $d_{M'}$ to M'_n for $n \in \mathbb{Z}$.

1.16. A (homological) complex of A-modules (M, d_M) is said to be **bounded below** (resp., **bounded above**) if the underlying graded module M is so. It is **bounded** if it is bounded below and bounded above.

1.17. Exercise. *Given a graded A-module, define $\iota'(M)$ as the complex of A-modules whose underlying graded A-module is M and with zero differential. Moreover, if $f : M \to N$ is a morphism of graded A-modules, let $\iota'(f) : \iota'(M) \to \iota'(N)$ be the morphismf of complexes of A-modules whose underlying set-theoretic map is f. Prove that these definitions give a fully faithful functor*

$$\iota' : {}_A\,\mathsf{GMod} \to {}_A\,\mathsf{DGMod}.$$

This allows us to see a graded A-module as a complex of A-modules with zero differential.

1.18. A **subcomplex** of a complex of A-modules (M, d_M) is a graded A-submodule N of M such that $d_M(N) \subseteq N$. In particular, $(N, d_M|_N)$ is also a complex of A-modules and the inclusion map $N \to M$ is a morphism of complexes of A-modules. If N is a subcomplex of a complex of A-modules (M, d_M), then we graded A-module M/N is naturally a complex of A-modules for the differential induced by d_M, *i.e.* the one sending $[m]$ to $[d_M(m)]$ for $m \in M$, where $[m]$ denotes the equivalence class in M/N including m. Note that the projection map $M \to M/N$ sending m to $[m]$ is a morphism of complexes of A-modules.

1.19. Given a morphism of complexes of A-modules $f : M \to N$ its **kernel** $\mathsf{Ker}(f)$ is the kernel of the underlying morphism $f : M \to N$ of graded A-modules. The fact that f commutes with the differentials implies that $\mathsf{Ker}(f)$ is a subcomplex of (M, d_M). Analogously, we define the **image** of f as the image of the underlying morphism $f : M \to N$ of graded A-modules. Again, the fact that f commutes with the differentials implies that $\mathsf{Img}(f)$ is a subcomplex of (N, d_N). Furthermore, define the **cokernel** $\mathsf{Coker}(f)$ of f as the complex of A-modules $N/\mathsf{Img}(f)$.

1.20. Given a family $\{(M_i, d_i)\}_{i\in I}$ of complexes A-modules we define its **product** as the graded A-module $P = \prod_{i\in I} M_i$ together with the differential sending $(m_{i,n})_{i\in I} \in P_n$ to $(d_i(m_{i,n}))_{i\in I}$, for all $n \in \mathbb{Z}$. We will denote the product complex of A-modules also by $\prod_{i\in I} M_i$. Analogously, their **coproduct** (or **direct sum**) is the graded A-module $C = \coprod_{i\in I} M_i$ with the differential sending $\sum_{i\in I} m_{i,n}$ to $\sum_{i\in I} d_i(m_{i,n})$, for $n \in \mathbb{Z}$. We will denote the direct sum complex of A-modules also by $\coprod_{i\in I} M_i$.

1.21. Exercise. *Show that the previous product and coproduct of complexes of A-modules are precisely the categorical product and coproduct in the category ${}_A\mathsf{DGMod}$, respectively.*

1.22. Let (M, d_M) and (N, d_N) be complexes of A-modules. Recall the graded $\mathbb{Z}$-module $\mathscr{H}om_{\mathsf{gr}\,A}(M, N)$ defined in paragraph 1.10 of the underlying graded A-modules M and N. We endow it with the unique morphism

$$d : \mathscr{H}om_{\mathsf{gr}\,A}(M, N) \to \mathscr{H}om_{\mathsf{gr}\,A}(M, N)[-1]$$

of graded $\mathbb{Z}$-modules sending $f \in \mathscr{H}om_{\mathsf{gr}\,A}(M, N)_k = \mathsf{Hom}_{\mathsf{gr}\,A}(M, N[k])$ to $d_{N[k]} \circ f - f[-1] \circ d_M$. It is easy to see that $d[-1] \circ d = 0$. We will denote this complex by $\mathscr{H}om_{\mathsf{dg}\,A}(M, N)$. Note that a morphism of complexes of A-modules from (M, d_M) to (N, d_N) is precisely an element of $\mathsf{Ker}(d) \cap \mathscr{H}om_{\mathsf{dg}\,A}(M, N)_0$.

1.23. Given a complex of A-modules (M, d_M) and $n \in \mathbb{Z}$, we define its ***n*-th homology group** $H_n(M, d_M)$ as $\mathsf{Ker}(d_n)/\mathsf{Img}(d_{n+1})$. The graded A-module $H(M, d_M) = \oplus_{n\in\mathbb{Z}} H_n(M, d_M)$ is equivalently given as $\mathsf{Ker}(d_M)/\mathsf{Img}(d_M[1])$. It is called the **total homology group**.

1.24. Fact. *Let (M, d_M) and (N, d_N) be two complexes of A-modules and let $f : (M, d_M) \to (N, d_N)$ be a morphism of complexes of A-modules. Then, $f(\mathsf{Ker}(d_M)) \subseteq \mathsf{Ker}(d_N)$ and $f(\mathsf{Img}(d_M)) \subseteq \mathsf{Img}(d_N)$. As a consequence, f induces a morphism of graded A-modules from $H(M, d_M)$ to $H(N, d_N)$, sending $[m]$ to $[f(m)]$, for $m \in \mathsf{Ker}(d_M)$, where the brackets denotes the class in homology.*

Proof. The two inclusions follow directly from $f[-1] \circ d_M = d_N \circ f$. The last statement is a direct consequence of the first. □

1.25. We denote the morphism of graded A-modules $H(M, d_M) \to H(N, d_N)$ is denoted by $H(f)$, and its restriction $H_n(M, d_M) \to H_n(N, d_N)$ is denoted by $H_n(f)$ for $n \in \mathbb{Z}$. It is easy to see that this defines a functor

$$H : {}_A\mathsf{DGMod} \to {}_A\mathsf{GMod} \tag{1}$$

sending a complex of A-modules (M, d_M) to $H(M, d_M)$ and a morphism of complexes of A-modules f to $H(f)$.

1.26. Exercise. *Let H be the functor defined in (1). Show that H preserves products and coproducts. We will further prove in Exercise 4.106 that H preserves filtered colimits.*

1.27. A morphism f of complexes of A-modules is called a **quasi-isomorphism** if $H(f)$ is an isomorphism of graded A-modules. On the other hand, a complex (M, d_M) is said to be **exact** (or **acyclic**) if $H(M, d_M) = 0$ is the zero graded A-module. Equivalently, (M, d_M) is exact if the zero morphism

$(0,0) \to (M, d_M)$ is a quasi-isomorphism. We say that complex of A-modules (M, d_M) is **split** if there is a homogeneous element $h \in \mathscr{H}om_{\mathsf{dgA}}(M, M)_1$ such that $d_M = d_M \circ_{\mathsf{gr}} h \circ_{\mathsf{gr}} d_M$. If the complex is also acyclic, we say that it is **split exact**.

1.28. Exercise. *Consider the complex of $\mathbb{Z}$-modules (M, d_M) where $M_n = \mathbb{Z}/4.\mathbb{Z}$ and d_n is the multiplication by 2 for all $n \in \mathbb{Z}$. Show that (M, d_M) is acyclic but not split exact.*

1.29. Given a complex of A-modules (M, d_M) and $k \in \mathbb{Z}$, we define its **shift** whose underlying graded A-module is the shift $M[k]$ of the underlying graded A-module of M, together with the differential $d_{M[k]}$ sending $s_{M,k}(m)$ to $(-1)^k s_{M[-1],k}(d(m))$, for $m \in M$. If $f : M \to N$ is a morphism of complexes of A-modules, define the morphism of complexes of A-modules $f[k] : M[k] \to N[k]$ whose underlying morphism of graded of A-modules is the shift $f[k]$ of the morphism of underlying graded A-modules (see paragraph 1.8). It is clear that that this defines a functor

$$(-)[k] : {}_A\,\mathsf{DGMod} \to {}_A\,\mathsf{DGMod}\,.$$

As we explained before, **we usually do not indicate the shift on the morphisms**.

1.30. Exercise. *Show that the functor H defined in (1) commutes with shift, i.e. there is a natural isomorphism $H \circ (-)[k] \cong (-)[k] \circ H$ of functors.*

1.31. Given an integer $n \geq 3$, a **(finite) sequence** of complexes A-modules is the data

$$M^1 \xrightarrow{f^1} M^2 \xrightarrow{f^2} M^3 \xrightarrow{f^3} \dots \xrightarrow{f^{n-2}} M^{n-1} \xrightarrow{f^{n-1}} M^n, \tag{2}$$

where (M^i, d^i) is a complex of A-modules for $i \in [\![1, n]\!]$ and f^i is a morphism of complexes A-modules for $i \in [\![1, n-1]\!]$. We say that the finite sequence is **exact** at the position $i \in [\![2, n-1]\!]$ if $\mathsf{Img}(f^{i-1}) = \mathsf{Ker}(f^i)$. Furthermore, we say that is **exact** if it is exact at every $i \in [\![2, n-1]\!]$. If we restrict the previous definition to complexes with zero differentials (resp., concentrated in degree zero) we get the notion of exact sequence of graded A-modules (resp., A-modules) (see Exercise 1.17). A finite sequence (2) is called **short** if $n = 5$ and $M^1 = M^5 = 0$. We usually write this simply as

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0.$$

1.32. Exercise (Cones)**.** *Let $f : (M, d_M) \to (N, d_N)$ be a morphism of complexes of A-modules. Consider the graded A-module $M[-1] \oplus N$ endowed with the differential d given by*

$$d\big(s_{M,-1}(m), n\big) = \Big(- s_{M[-1],-1}\big(d_M(m)\big), d_N(n) - s_{N,-1}\big(f(m)\big)\Big),$$

*for $m \in M$ and $n \in N$. Show that d is indeed a differential. The complex $(M[-1] \oplus N, d)$ is called the **cone** of f, and it is denoted by $\mathsf{cone}(f)$. Prove that the sequence of complexes of A-modules*

$$0 \longrightarrow N \xrightarrow{i} \mathsf{cone}(f) \xrightarrow{\delta} M[-1] \longrightarrow 0,$$

where $i(n) = (\mathbf{0}_M, n)$ and $\delta(s_{M,-1}(m), n) = -s_{M,-1}(m)$ for $m \in M$ and $n \in N$, is exact.

1.33. Exercise (Cones of modules)**.** *Assume that M and N are complexes of A-modules concentrated in degree zero. Show that $C = \mathsf{cone}(f)$ is the complex of A-modules such that $C_1 = M$, $C_0 = N$, $C_n = 0$ for all $n \in \mathbb{Z} \setminus \{0, 1\}$ whose differential satisfies that $d_1 : C_1 = M \to N = C_0$ is $-f$.*

1.34. Exercise (Cone of identity)**.** *Let (M, d_M)be a complex of A-modules and let $C = \mathsf{cone}(\mathrm{id}_M)$. Let $h \in \mathscr{H}om_{\mathsf{grA}}(C, C)_1 = \mathsf{Hom}_{\mathsf{grA}}(C, C[1])$ be the morphism of graded A-modules sending $(s_{M,-1}(m), m')$ to $(-m', \mathbf{0}_{M[1]})$, for $m, m' \in M$. Show that $d_C = d_C \circ_{\mathsf{gr}} h \circ_{\mathsf{gr}} d_C$, so C is split.*

1.35. Given $n_0 \in \mathbb{Z}$, let ${}_A\mathsf{DGMod}_{\geq n_0}$ be the full subcategory of ${}_A\mathsf{DGMod}$ given by all complexes of A-modules (M, d_M) satisfying that $M_n = 0$ for all $n < n_0$. Consider the assignements

$$\sigma_{\geq n_0}, \tau_{\geq n_0} : {}_A\mathsf{DGMod} \to {}_A\mathsf{DGMod}_{\geq n_0} \tag{3}$$

defined as follows. Given a complex of A-modules (M, d_M), we set $\sigma_{\geq n_0}(M, d_M)$ (resp., $\tau_{\geq n_0}(M, d_M)$) to be the complex of A-modules such that

$$\sigma_{\geq n_0}(M, d_M)_n = \begin{cases} M_n, & \text{if } n \geq n_0, \\ 0, & \text{if } n < n_0, \end{cases} \quad \left(\text{resp., } \tau_{\geq n_0}(M, d_M)_n = \begin{cases} M_n, & \text{if } n > n_0, \\ \mathsf{Ker}(d_{M,n_0}), & \text{if } n = n_0, \\ 0, & \text{if } n < n_0, \end{cases} \right)$$

for $n \in \mathbb{Z}$, with the differential

$$d_n : \sigma_{\geq d}(M, d_M)_n \to \sigma_{\geq d}(M, d_M)_{n-1} \big(\text{resp., } d_n : \tau_{\geq n_0}(M, d_M)_n \to \tau_{\geq n_0}(M, d_M)_{n-1}\ \big)$$

given by

$$d_n = \begin{cases} d_{M,n}, & \text{if } n > n_0, \\ 0, & \text{if } n \leq n_0, \end{cases}$$

for $n \in \mathbb{Z}$. Note that $d_{n_0+1} : \tau_{\geq n_0}(M, d_M)_{n_0+1} \to \tau_{\geq n_0}(M, d_M)_{n_0}$ is well defined, since we have the inclusion $\mathsf{Img}(d_{M,n_0+1}) \subseteq \mathsf{Ker}(d_{M,n_0})$.

1.36. Moreover, if $f : (M, d_M) \to (N, d_N)$ is a morphism of complexes of A-modules, define

$$\sigma_{\geq n_0}(f) : \sigma_{\geq n_0}(M, d_M) \to \sigma_{\geq n_0}(N, d_N) \ \big(\text{resp., } \tau_{\geq n_0}(f) : \tau_{\geq n_0}(M, d_M) \to \tau_{\geq n_0}(N, d_N)\big)$$

as the unique morphism of complexes of A-modules satisfying that

$$\sigma_{\geq n_0}(f)_n = \begin{cases} f_n, & \text{if } n \geq n_0, \\ 0, & \text{if } n < n_0, \end{cases} \quad \left(\text{resp., } \tau_{\geq n_0}(f)_n = \begin{cases} f_n, & \text{if } n > n_0, \\ f|_{\mathsf{Ker}(d_{M,n_0})}, & \text{if } n = n_0, \\ 0, & \text{if } n < n_0. \end{cases} \right)$$

Note that $\tau_{\geq n_0}(f)_{n_0} : \tau_{\geq n_0}(M, d_M)_{n_0} \to \tau_{\geq n_0}(M, d_M)_{n_0}$ is well defined, since we have the inclusion $f_{n_0}(\mathsf{Ker}(d_{M,n_0})) \subseteq \mathsf{Ker}(d_{N,n_0})$.

1.37. Exercise. *Prove that the assignements (3) defined in the previous paragraphs are functors.*

1.38. The functor $\sigma_{\geq n_0}$ is called the **brutal truncation**, whereas the functor $\tau_{\geq n_0}$ is called the **good truncation**.

1.39. Exercise. *Let*

$${}_A\mathsf{DGMod}_{\geq n_0} \to {}_A\mathsf{DGMod} \tag{4}$$

be the inclusion functor.

(i) Prove that the canonical inclusion $i : \tau_{\geq n_0}(M, d_M) \to (M, d_M)$ is a morphism of complexes of A-modules such that $H_n(i)$ is an isomorphism for all $n \geq n_0$.

(ii) Prove that the map $p : (M, d_M) \to \tau_{\geq n_0}(M, d_M)$ given as id_{M_n} if $n \geq n_0$ and zero if $n < n_0$ is a morphism of complexes of A-modules, such that $H_n(p)$ is an isomorphism for all $n > n_0$.

(iii) Prove that the functor $\sigma_{\geq n_0}$ is left adjoint to the inclusion functor (4), which is in turn left adjoint to the functor $\tau_{\geq n_0}$.

§1.3. Snake Lemma and long exact sequences

1.40. Lemma (Snake Lemma)**.** *Consider a commutative diagram of A-modules*

$$\begin{tikzcd} & M' \arrow[r, "i"] \arrow[d, "f'"] & M \arrow[r, "p"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "j"] & N \arrow[r, "q"] & N'' & \end{tikzcd} \tag{5}$$

such that the rows are exact. Then, there exists an exact sequence

$$\mathsf{Ker}(f') \longrightarrow \mathsf{Ker}(f) \longrightarrow \mathsf{Ker}(f'') \xrightarrow{\partial} \mathsf{Coker}(f') \longrightarrow \mathsf{Coker}(f) \longrightarrow \mathsf{Coker}(f'') \tag{6}$$

of A-modules, where the first two maps are induced by i and p, respectively, and the last two maps are induced by j and q, respectively. Moreover, the map $\partial : \mathsf{Ker}(f'') \to \mathsf{Coker}(f)$ sends $m'' \in \mathsf{Ker}(f'')$ to the unique element $[n']$ for $n' \in N'$ such that there exist $m \in M$ satisfying that $j(n') = f(m)$ and $p(m) = m''$. Finally, if i is injective, so is $\mathsf{Ker}(f') \to \mathsf{Ker}(f)$, and if q is sujective, so is $\mathsf{Coker}(f') \to \mathsf{Coker}(f'')$.

Proof. The verification is straightforward, but we are going to explain the most important parts. In particular, we are going to construct the dashed arrow in the following commutative diagram

$$\begin{tikzcd} & \mathsf{Ker}(f') \arrow[r, "i|_{\mathsf{Ker}(f')}"] \arrow[d, "\mathsf{inc}"] & \mathsf{Ker}(f) \arrow[r, "p|_{\mathsf{Ker}(f)}"] \arrow[d, "\mathsf{inc}"] & \mathsf{Ker}(f'') \arrow[d, "\mathsf{inc}"] \arrow[ddll, dashed, rounded corners] & \\ & M' \arrow[r, "i"] \arrow[d, "f'"] & M \arrow[r, "p"] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "j"] \arrow[d, "\mathsf{proj}"] & N \arrow[r, "q"] \arrow[d, "\mathsf{proj}"] & N'' \arrow[d, "\mathsf{proj}"] & \\ & \mathsf{Coker}(f) \arrow[r, "\bar{j}"] & \mathsf{Coker}(f') \arrow[r, "\bar{q}"] & \mathsf{Coker}(f'') & \end{tikzcd} \tag{7}$$

where $\bar{j}$ and $\bar{q}$ are the maps induced by j and q, respectively. The shape of this diagram gives the name to the lemma.

Let us now show that the map ∂ is well defined. We first prove that, given $m'' \in \mathsf{Ker}(f'')$ and two elements $n'_1, n'_2 \in N'$ such that there exist $m_1, m_2 \in M$ satisfying that $j(n'_i) = f(m_i)$ and $p(m_i) = m''$ for $i = 1, 2$, then $[n'_1] = [n'_2]$, *i.e.* $n'_1 - n'_2 \in \mathsf{Img}(f')$. Since $p(m_1) = m'' = p(m_2)$ and the upper row of (5) is exact, $m_1 - m_2 \in \mathsf{Img}(i)$, *i.e.* there exists $m' \in M'$ such that $i(m') = m_1 - m_2$. Hence, $j(n'_1 - n'_2) = f(m_1 - m_2) = f(i(m')) = j(f'(m'))$, which implies that $n'_1 - n'_2 = f'(m')$, since j is injective. We will now prove that, given $m'' \in \mathsf{Ker}(f'')$ there exist $n' \in N'$ and $m \in M$ satisfying that $j(n') = f(m)$ and $p(m) = m''$. Indeed, since p is surjective, there exists $m \in M$ such that $p(m) = m''$. The identity $f'' \circ p = q \circ f$ tells us that $q(f(m)) = \mathbf{0}_{N''}$, which, by the exactness of the lower row of (5), gives us that there exists $n' \in N'$ such that $j(n') = f(m)$, as was to be shown.

We also remark that the upper and lower rows of (7) are exact, *i.e.* that

$$\mathsf{Ker}(f') \longrightarrow \mathsf{Ker}(f) \longrightarrow \mathsf{Ker}(f'') \quad \text{and} \quad \mathsf{Coker}(f') \longrightarrow \mathsf{Coker}(f) \longrightarrow \mathsf{Coker}(f'')$$

are exact sequences. Let us prove the first one and leave the other to the reader. Since $p \circ i = 0$, we see that $p|_{\mathsf{Ker}(f)} \circ i|_{\mathsf{Ker}(f')}$ vanishes too, so $\mathsf{Ker}(p|_{\mathsf{Ker}(f)}) \supseteq \mathsf{Img}(i|_{\mathsf{Ker}(f')})$. Given $m \in \mathsf{Ker}(f)$ such that $p|_{\mathsf{Ker}(f)}(m) = p(m) = \mathbf{0}_{M''}$, the exactness of the upper row of (5) tells us that there exists $m' \in M'$ such that $i(m') = m$. It suffices to show that $m' \in \mathsf{Ker}(f')$. Since $f \circ i = j \circ f'$, $j(f'(m')) = f(i(m')) = f(m) = \mathbf{0}_N$, which implies that $f'(m') = \mathbf{0}_{N'}$, as j is injective. As a consequence, $m' \in \mathsf{Ker}(f')$, as was to be shown.

To prove the exactness of (6) we need to show that $\mathsf{Ker}(\partial) = \mathsf{Img}(p|_{\mathsf{Ker}(f)})$ and $\mathsf{Img}(\partial) = \mathsf{Ker}(\bar{j})$. Let us prove the first one and leave the other to the reader. By definition of ∂, if $m'' \in \mathsf{Img}(p|_{\mathsf{Ker}(f)})$, let us write $m'' = p(m)$ with $m \in \mathrm{Ker}(f)$, then we see that $n' = \mathbf{0}_{N'}$ satisfies that $j(n') = f(m)$, so we have that $\partial(m'') = \mathbf{0}_{\mathrm{Coker}(f)}$, *i.e.* $\mathrm{Ker}(\partial) \supseteq \mathrm{Img}(p|_{\mathrm{Ker}(f)})$. Conversely, if $m'' \in \mathrm{Ker}(\partial)$, we get by definition of ∂ that there exist $m \in M$ and $n' \in \mathrm{Img}(f')$ such that $j(n') = f(m)$ and $p(m) = m''$. Let $n' = f'(m')$, with $m' \in M'$. Then, $f(m) = j(f'(m')) = f(i(m'))$, *i.e.* $m - i(m') \in \mathrm{Ker}(f)$. Since $p(m - i(m')) = p(m) = m''$, and $m - i(m') \in \mathrm{Ker}(f)$, we get that $m'' \in \mathrm{Img}(p|_{\mathrm{Ker}(f)})$, as was to be shown.

The last statement is immediate, since the restriction of an injective map is injective and the induced map to a quotient of a surjective map is surjective. □

1.41. Exercise (5-Lemma). *Consider the commutative diagram*

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d, "f_1'"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & M_4 \arrow[r] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\
N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5
\end{tikzcd}$$

of A-modules, with exact rows. Show that

(Five.1) if f_1 is surjective, and f_2 and f_4 are injective, then f_3 is injective;

(Five.2) if f_5 is injective, and f_2 and f_4 are surjective, then f_3 is surjective.

Conclude that, if f_1, f_2, f_4, f_5 are isomorphisms, so is f_3.

1.42. Moreover, the map ∂ constructed in Lemma 1.40 is natural in the following sense.

1.43. Lemma (Snake Lemma (cont.)). *Consider the commutative diagram of A-modules*

$$\begin{tikzcd}
 & M_1' \arrow[rr, "i_1"] \arrow[dr, "g'"] \arrow[dd, "f_1'"'] & & M_1 \arrow[rr, "p_1"] \arrow[dr, "g"] \arrow[dd, "f_1"] & & M_1'' \arrow[rr] \arrow[dr, "g''"] \arrow[dd, "f_1''"] & & 0 & \\
 & & M_2' \arrow[rr, "i_2"] \arrow[dd, "f_2'"] & & M_2 \arrow[rr, "p_2"] \arrow[dd, "f_2"] & & M_2'' \arrow[rr] \arrow[dd, "f_2''"] & & 0 \\
0 \arrow[r] & N_1' \arrow[rr, "j_1"] \arrow[dr, "h'"] & & N_1 \arrow[rr, "q_1"] \arrow[dr, "h"] & & N_1'' \arrow[dr, "h''"] & & & \\
 & 0 \arrow[r] & N_2' \arrow[rr, "j_2"] & & N_2 \arrow[rr, "q_2"] & & N_2'' & &
\end{tikzcd}$$

such that the rows are exact. Then, the exact sequences in Lemma 1.40 fit into a commutative diagram

$$\begin{tikzcd}
\mathrm{Ker}(f_1') \arrow[r] \arrow[d, "g'|_{\mathrm{Ker}(f_1')}"] & \mathrm{Ker}(f_1) \arrow[r] \arrow[d, "g|_{\mathrm{Ker}(f_1)}"] & \mathrm{Ker}(f_1'') \arrow[r, "\partial"] \arrow[d, "g''|_{\mathrm{Ker}(f_1'')}"] & \mathrm{Coker}(f_1') \arrow[r] \arrow[d, "\bar{h}'"] & \mathrm{Coker}(f_1) \arrow[r] \arrow[d, "\bar{h}"] & \mathrm{Coker}(f_1'') \arrow[d, "\bar{h}''"] \\
\mathrm{Ker}(f_2') \arrow[r] & \mathrm{Ker}(f_2) \arrow[r] & \mathrm{Ker}(f_2'') \arrow[r, "\partial"] & \mathrm{Coker}(f_2') \arrow[r] & \mathrm{Coker}(f_2) \arrow[r] & \mathrm{Coker}(f_2'')
\end{tikzcd}$$

of A-modules with exact rows.

Proof. We leave the verification to the reader. □

1.44. Theorem. *Let*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \tag{8}$$

be a short exact sequence of complexes of A-modules. Then, for every $n \in \mathbb{Z}$ there exists a natural map $\partial_n : H_n(M'', d'') \to H_{n-1}(M', d')$ of A-modules, called the ***connecting morphism****, such that the sequence*

$$\begin{tikzcd}
H_n(M', d_{M'}) \arrow[r, "H_n(f)"] & H_n(M, d_M) \arrow[r, "H_n(g)"] & H_n(M'', d_{M''}) \arrow[dll, rounded corners, "\partial_n"] \\
H_{n-1}(M', d_{M'}) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(M, d_M) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(M'', d_{M''})
\end{tikzcd} \tag{9}$$

is exact for all $n \in \mathbb{Z}$.

Moreover, the map ∂_n is natural in the following sense. Given a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & M'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N' \arrow[r, "p"] & N \arrow[r, "q"] & N'' \arrow[r] & 0
\end{tikzcd}$$

of complexes of A-modules with exact rows, the following diagram

$$\begin{tikzcd}[column sep=small]
H_n(M',d_{M'}) \arrow[rr, "H_n(f)"] \arrow[dr, "H_n(\alpha)"] & & H_n(M,d_M) \arrow[rr, "H_n(g)"] \arrow[dr, "H_n(\beta)"] & & H_n(M'',d_{M''}) \arrow[dr, "H_n(\gamma)"] \arrow[ddllll, rounded corners, "\partial_n"'] & \\
& H_n(N',d_{N'}) \arrow[rr, "H_n(p)"] & & H_n(N,d_N) \arrow[rr, "H_n(q)"] & & H_n(N'',d_{N''}) \arrow[ddllll, rounded corners, "\partial_n"'] \\
H_{n-1}(M',d_{M'}) \arrow[rr, "H_{n-1}(f)"] \arrow[dr, "H_{n-1}(\alpha)"] & & H_{n-1}(M,d_M) \arrow[rr, "H_{n-1}(g)"] \arrow[dr, "H_{n-1}(\beta)"] & & H_{n-1}(M'',d_{M''}) \arrow[dr, "H_{n-1}(\gamma)"] & \\
& H_{n-1}(N',d_{N'}) \arrow[rr, "H_{n-1}(p)"] & & H_{n-1}(N,d_N) \arrow[rr, "H_{n-1}(q)"] & & H_{n-1}(N'',d_{N''})
\end{tikzcd}$$

commutes for all $n \in \mathbb{Z}$.

Proof. Given $n \in \mathbb{Z}$, consider the diagram of A-modules

$$\begin{tikzcd}
& M'_n/\operatorname{Img}(d'_{n+1}) \arrow[r, "\bar{f}_n"] \arrow[d, "\bar{d}'_n"] & M_n/\operatorname{Img}(d_{n+1}) \arrow[r, "\bar{g}_n"] \arrow[d, "\bar{d}_n"] & M''_n/\operatorname{Img}(d''_{n+1}) \arrow[r] \arrow[d, "\bar{d}''_n"] & 0 \\
0 \arrow[r] & \operatorname{Ker}(d'_{n-1}) \arrow[r, "f_{n-1}|_{\operatorname{Ker}(d'_{n-1})}"] & \operatorname{Ker}(d_{n-1}) \arrow[r, "g_{n-1}|_{\operatorname{Ker}(d_{n-1})}"] & \operatorname{Ker}(d''_{n-1}) &
\end{tikzcd} \tag{10}$$

induced by the short exact sequence (8), where the maps with a bar over them are those directly obtained from the maps without the bar. The careful reader can check that the rows of (10) are exact. The exact sequence (9) is thus obtained from Lemma 1.40 applied to (10). The last part of the statement, namely the naturality of the maps ∂_n, follows from Lemma 1.43. □

1.45. Exercise (Cones (cont.)). *Let $f : (M, d_M) \to (N, d_N)$ be a morphism of complexes of A-modules, and let $\operatorname{cone}(f)$ be the cone of f defined in Exercise 1.32. Recall the short exact sequence of complexes of A-modules*

$$0 \longrightarrow N \xrightarrow{\ i\ } \operatorname{cone}(f) \xrightarrow{\ \delta\ } M[-1] \longrightarrow 0,$$

where $i(n) = (\mathbf{0}_M, n)$ and $\delta(s_{M,-1}(m), n) = -s_{M,-1}(m)$ for $m \in M$ and $n \in N$.

(Co.1) Prove that the composition of the natural isomorphism $H_{n-1}(M, d_M) \to H_n(M[-1], d_{M[-1]})$ in Exercise 1.30 and the connecting morphism $\partial_n : H_n(M[-1], d_{M[-1]}) \to H_{n-1}(M', d_{M'})$ is $H_{n-1}(f)$ for all $n \in \mathbb{Z}$.

(Co.2) Prove that f is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is acyclic.

(Co.3) Show that $\operatorname{cone}(\mathrm{id}_M)$ is a split exact complex, for any complex of A-modules (M, d_M) (see Exercise 1.34).

1.46. Exercise (3 × 3-Lemma). *Consider the commutative diagram*

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & M'_1 \arrow[r] \arrow[d] & M_1 \arrow[r] \arrow[d] & M''_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M'_2 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M''_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M'_3 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & M''_3 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

of A-modules, with exact colums. Show that

(Three.1) if the bottom two rows are exact, so is the top row;

(Three.2) if the top two rows are exact, so is the bottom row;

(Three.3) if the top and bottom rows are exact, and the composition $M'_2 \to M_2 \to M''_2$ vanishes, the middle row is exact.

1.47. Definition. *Let (M, d_M) and $(M', d_{M'})$ be two complexes of A-modules and $f, g : M \to M'$ two morphisms of complexes of A-modules. We say that f is* ***homotopic*** *to g if there is a homogeneous element $h \in \mathscr{H}\!om_{\mathsf{dgA}}(M, M')_1$, called a* ***homotopy***, *such that $d(h) = f - g$. In more concrete terms, a homotopy from f to g is a family of morphisms $h_n : M_n \to M'_{n+1}$ of A-modules for $n \in \mathbb{Z}$ such that $d'_{n+1} \circ h_n + h_{n-1} \circ d_n = f_n - g_n$, for $n \in \mathbb{Z}$, where we denote by d'_n the n-th differential induced by $d_{M'}$. It is clear that homotopy is an equivalence relation on $\mathsf{Hom}_{\mathsf{dg}\,A}(M, M')$.*

On the other hand, we say that two morphisms $f : M \to M'$ and $g : M' \to M$ are ***homotopy inverses*** *if $g \circ f$ is homotopic to id_M and $f \circ g$ is homotopic to $\mathsf{id}_{M'}$. We say in this case that f (or g) is a* ***homotopy equivalence***.

1.48. We can represent a homotopy by a diagram

$$\begin{tikzcd} \dots \arrow[r, "d_{n+2}"] & M_{n+1} \arrow[r, "d_{n+1}"] \arrow[dl, "h_{n+1}"'] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & M_n \arrow[r, "d_n"] \arrow[dl, "h_n"'] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] & M_{n-1} \arrow[r, "d_{n-1}"] \arrow[dl, "h_{n-1}"'] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] & \dots \\ \dots \arrow[r, "d'_{n+2}"'] & M'_{n+1} \arrow[r, "d'_{n+1}"'] & M'_n \arrow[r, "d'_n"'] & M'_{n-1} \arrow[r, "d'_{n-1}"'] & \dots \end{tikzcd}$$

1.49. Exercise. *Show that a complex (M, d_M) of A-modules is split exact if and only if the identity map id_M is homotopic to the zero map.*

1.50. Exercise (Cones and homotopies). *Let (M, d_M) and $(M', d_{M'})$ be two complexes of A-modules and let $f, g : M \to M'$ be two morphisms of complexes of A-modules. Show that f and g are homotopic if and only if there exists a morphism of complexes of A-modules $\tilde{h} : \mathsf{cone}(\mathsf{id}_M) \to M'$ such that $\tilde{h} \circ i = f - g$, where $i : M \to \mathsf{cone}(\mathsf{id}_M)$ is the canonical inclusion.*

1.51. Lemma. *Let (M, d_M) and $(M', d_{M'})$ be two complexes of A-modules and let $f, g : M \to N$ be two morphisms of complexes of A-modules. If f and g are homotopic, then $H(f) = H(g) : H(M, d_M) \to (M', d_{M'})$.*

Proof. Let $(h_n)_{n \in \mathbb{Z}}$ be the homotopy satisfying that $d'_{n+1} \circ h_n + h_{n-1} \circ d_n = f_n - g_n$, for $n \in \mathbb{Z}$, where we denote by d'_n the n-th differential induced by $d_{M'}$. Given $m \in \mathsf{Ker}(d_n)$, $d'_{n+1} \circ h_n(m) = f_n(m) - g_n(m)$, so $H(f)([m]) = H(g)([m])$, as was to be shown, where $[m]$ denotes the homology class of m. □

1.52. Exercise (Cylinder). *Let $f : (M, d_M) \to (N, d_N)$ be a morphism of complexes of A-modules. Recall the cone of f introduced in Exercise 1.32, and the short exact sequence of complexes of A-modules*

$$0 \longrightarrow N \xrightarrow{i} \mathsf{cone}(f) \xrightarrow{\delta} M[-1] \longrightarrow 0, \tag{11}$$

where $i(n) = (\mathbf{0}_M, n)$ and $\delta(s_{M,-1}(m), n) = -s_{M,-1}(m)$ for $m \in M$ and $n \in N$. Consider the morphism $\delta[1] : \mathsf{cone}(f)[1] \to M$ of complexes of A-modules and define the ***cylinder*** *$\mathsf{cyl}(f)$ of f as $\mathsf{cone}(\delta[1])$. More concretely, the underlying graded A-module of $\mathsf{cyl}(f)$ is $M \oplus M[-1] \oplus N$ and its differential is given by*

$$d(m', s_{M,-1}(m), n) = \Big(d_M(m') + m, -s_{M[1],-1}\big(d_M(m)\big), d_N(n) - s_{N,-1}\big(f(m)\big)\Big),$$

for $m', m \in M$ and $n \in N$.

(Cyl.1) *Let $\alpha : N \to \mathsf{cyl}(f)$ be the map sending $n \in N$ to $(\mathbf{0}_M, \mathbf{0}_{M[-1]}, n)$. Prove that the short sequence of complexes of A-modules*

$$0 \longrightarrow N \xrightarrow{\alpha} \mathsf{cyl}(f) \xrightarrow{\pi} \mathsf{cone}(-\mathsf{id}_M) \longrightarrow 0,$$

where π sends $(m', s_{M,-1}(m), n)$ to $(m', s_{M,-1}(m))$, for $m', m \in M$ and $n \in N$, is exact. Deduce that α is a quasi-isomorphism of complexes of A-modules (see Exercise 1.45).

(Cyl.2) *Let $\beta : \mathsf{cyl}(f) \to N$ be the map sending $(m', s_{M,-1}(m), n)$ to $f(m') + n$, for $m', m \in M$ and $n \in N$. Prove that β is a morphism of complexes of A-modules such that $\beta \circ \alpha = \mathsf{id}_N$, and that $\mathsf{id}_{\mathsf{cyl}(f)}$ is homotopic to $\alpha \circ \beta$ by means of the homotopy h sending $(m', s_{M,-1}(m), n)$ to $(\mathbf{0}_{M[1]}, m', \mathbf{0}_{N[1]})$, for $m', m \in M$ and $n \in N$.*

(Cyl.3) *Consider more generally a short exact sequence*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} N' \longrightarrow 0$$

of complexes of A-modules. Prove that the map $\gamma : \mathsf{cone}(f) \to N'$ sending $(s_{M,-1}(m), n)$ to $g(n)$ for $m \in M$ and $n \in N$ is a morphism of complexes of A-modules.

(Cyl.4) *Following with the previous item, prove that the diagram*

$$\begin{tikzcd}
 & 0 \arrow[r] & N \arrow[r, "i"] \arrow[d, "\alpha"] & \mathsf{cone}(f) \arrow[r, "\delta"] \arrow[d, equal] & M[-1] \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "i'"] \arrow[d, equal] & \mathsf{cyl}(f) \arrow[r, "\delta'"] \arrow[d, "\beta"] & \mathsf{cone}(f) \arrow[r] \arrow[d, "\gamma"] & 0 & \\
0 \arrow[r] & M \arrow[r, "f"] & N \arrow[r, "g"] & N' \arrow[r] & 0 &
\end{tikzcd}$$

is commutative, where i' and δ' are the same morphisms as i and δ for the short exact sequence (11) with f replaced by $\delta[1]$. Deduce that γ is a quasi-isomorphism.

(Cyl.5) *Assume that M and N are complexes of A-modules concentrated in degree zero (see Exercise 1.33). Show that γ is a homotopy equivalence if and only if $f : M \to N$ has a retraction of A-modules, i.e. there exists a morphism of A-modules $r : N \to M$ such that $r \circ f = \mathsf{id}_M$.*

1.53. Exercise. *Prove that all the statements of this subsection, as well as their proofs, also hold if we work more generally with an abelian category $\mathscr{A}$ instead with a category of A-modules over a ring (see Exercises 4.101 and 4.102).*

§2. Lecture II : Derived functors

§2.1. Projective, injective and flat modules

2.1. We have the following basic result on the homomorphism group, which also follows from Exercises 4.85 and 4.104. We give however a direct proof for the reader's convenience.

2.2. Fact. *Let M be a module over a ring A and*

$$0 \longrightarrow N' \xrightarrow{\ f\ } N \xrightarrow{\ g\ } N'' \longrightarrow 0 \tag{12}$$

be a short exact sequence of A-modules. Then, the sequences

$$0 \longrightarrow \mathrm{Hom}_A(M, N') \xrightarrow{\ f_*\ } \mathrm{Hom}_A(M, N) \xrightarrow{\ g_*\ } \mathrm{Hom}_A(M, N'')$$

and

$$0 \longrightarrow \mathrm{Hom}_A(N'', M) \xrightarrow{\ g^*\ } \mathrm{Hom}_A(N, M) \xrightarrow{\ f^*\ } \mathrm{Hom}_A(N'', M)$$

of $\mathbb{Z}$-modules are exact, where $h_*(k) = h \circ k$ and $h^*(k) = k \circ h$, for $h \in \{f, g\}$ and k in the corresponding space.

Proof. Since f is injective then $0 = f_*(h) = f \circ h$ for $h \in \mathrm{Hom}_A(M, N')$ implies that $h = 0$, *i.e.* f_* is injective. Analogously, since g is surjective, $0 = g^*(h) = h \circ g$ for $h \in \mathrm{Hom}_A(N'', M)$ implies that $h = 0$, *i.e.* g^* is injective. Since $g \circ f = 0$, then $g_* \circ f_* = 0$ and $f^* \circ g^* = 0$, *i.e.* $\mathrm{Ker}(g_*) \supseteq \mathrm{Img}(f_*)$ and $\mathrm{Img}(g^*) \subseteq \mathrm{Ker}(f^*)$. It remains to prove the converse inclusions. We will only prove the inclusion $\mathrm{Ker}(g_*) \subseteq \mathrm{Img}(f_*)$ and leave the other to the reader. Given $h \in \mathrm{Hom}_A(M, N)$ such that $g_*(h) = g \circ h = 0$, we see that $\mathrm{Img}(h) \subseteq \mathrm{Ker}(g) = \mathrm{Img}(f)$. Since f is injective, we conclude that, given $m \in M$ there exists a unique $n' \in N'$ such that $f(n') = h(m)$, *i.e.* there exists a map $k : M \to N'$ such that $f \circ k = h$. Moreover,

$$f\big(k(m) + ak(m')\big) = f\big(k(m)\big) + af\big(k(m')\big) = h(m) + ah(m') = h(m + am') = f\big(k(m + am')\big),$$

for $m, m' \in M$ and $a \in A$, and the injectivity of f imply that k is a morphism of A-modules. □

2.3. We assume for this lecture that the reader is already familiar with the notions of projective and injective modules. We only recall that a module M over a ring A is called **projective** (resp., **injective**) if, given any short exact sequence (12) of A-modules, the short sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, N') \xrightarrow{\ f_*\ } \mathrm{Hom}_A(M, N) \xrightarrow{\ g_*\ } \mathrm{Hom}_A(M, N'') \longrightarrow 0$$

$$\Big(\text{resp., } 0 \longrightarrow \mathrm{Hom}_A(N'', M) \xrightarrow{\ g^*\ } \mathrm{Hom}_A(N, M) \xrightarrow{\ f^*\ } \mathrm{Hom}_A(N', M) \longrightarrow 0\Big)$$

of $\mathbb{Z}$-modules is exact (see Fact 2.2). By replacing ${}_A\mathrm{Mod}$ by an arbitrary abelian category $\mathscr{C}$ (see paragraph 4.99), we obtain the corresponding notions of projective and injective objects in $\mathscr{C}$.

2.4. Exercise (Projectives and injectives). *(i) Prove that the morphism $\mathrm{Hom}_A(A, M) \to M$ of abelian groups sending $f \in \mathrm{Hom}_A(A, M)$ to $f(1_A)$ is bijective, with inverse $M \to \mathrm{Hom}_A(A, M)$ sending $m \in M$ to the unique map $f \in \mathrm{Hom}_A(A, M)$ such that $f(a) = a.m$. Show that this isomorphism is natural, so it induces a natural isomorphism between the functor $\mathrm{Hom}_A(A, -)$ and the forgetful functor ${}_A\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$ sending an A-module to its underlying abelian group. Deduce that free A-modules are projective.*

(ii) *Let $\{M_i\}_{i\in I}$ be an arbitrary family of A-modules. Let $u_i : M_i \to \oplus_{i\in I} M_i$ be the canonical inclusion and $p_i : \prod_{i\in I} M_i \to M_i$ be the canonical projection. Prove that the morphisms of abelian groups*

$$\mathsf{Hom}_A(\oplus_{i\in I} M_i, M) \to \prod_{i\in I} \mathsf{Hom}_A(M_i, M) \text{ and } \mathsf{Hom}_A(M, \prod_{i\in I} M_i) \to \prod_{i\in I} \mathsf{Hom}_A(M, M_i)$$

given by sending f to $(f \circ u_i)_{i\in I}$, and g to $(p_i \circ g)_{i\in I}$, respectively, are natural isomorphisms. Deduce that direct sums and direct summands of projective A-modules are projective, and products and direct summands of injective A-modules are injective.

(iii) *Show that an A-module is projective if and only if it is a direct summand of a free A-module.*

2.5. Exercise (Baer's criterion). *Let M be an A-module satisfying that given an left ideal I of A and a morphism of A-modules $f : I \to M$, there exists a morphism of A-modules $\hat{f} : A \to M$ such that $\hat{f}|_I = f$. It is clear that any injective A-module satisfies this property.*

(In.1) *Given an inclusion of A-modules $N' \subsetneq N$, a morphism of A-modules $g' : N' \to M$, and $n_0 \in N \setminus N'$, let $I = \{a \in A : a.n_0 \in N'\}$. Show that I is a left ideal and that there exists a morphisms of A-modules $g'' : N'' \to M$ with $N'' = \{an_0 + n' : a \in A, n' \in N'\} \subseteq N$, such that $g''|_{N'} = g'$.*

(In.2) *Prove that there exists a morphism of A-modules $g : N \to M$ such that $g''|_{N'} = g'$, i.e. M is injective.*

(In.3) *Using this characterization of injectivity, prove that a module M over a **principal ideal domain** (**PID**) A is injective if and only if it is **divisible**, i.e. given $a \in A \setminus \{\mathbf{0}_A\}$ and $m \in M$, there exists $m' \in M$ such that $am' = m$. In particular, $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module.*

(In.4) *Using the previous item show that the quotient of an injective module over a PID A is injective.*

(In.5) *Let G be a divisible abelian group. Prove that the A-module $\mathsf{Hom}_{\mathbb{Z}}(A, G)$, whose structure is given by $(a \cdot f)(b) = f(ba)$, for all $a, b \in A$ and $f \in \mathsf{Hom}_{\mathbb{Z}}(A, G)$, is injective.*

2.6. Exercise (Enough projectives and injectives). *Let M be an A-module.*

(En.1) *Let $F(M) = \oplus_{m\in M} A_m$ with $A_m = A$ the free regular A-module, where the index $m \in M$ is just a bookkeping device. Define the unique morphism of A-modules $\pi_M : F(M) \to M$ sending $a \in A_m$ to $a.m \in M$. Prove that π_M is surjective. Note that $F(M)$ is a projective A-module.*

(En.2) *Let $I(M) = \mathsf{Hom}_{\mathbb{Z}}(A, \prod_{f\in\hat{M}} G_f)$, where G_f is the abelian group $\mathbb{Q}/\mathbb{Z}$, we set $\hat{M} = \mathsf{Hom}_{\mathbb{Z}}(M, \mathbb{Q}, \mathbb{Z})$ and $I(M)$ is endowed with the structure of A-module given in Exercise 2.5, and let $i_M : M \to I(M)$ be the map given by $i_M(m)(a) = (f(a.m))_{f\in\hat{M}}$. Prove that i_M is an injective morphism of A-modules. Note that $I(M)$ is an injective A-module.*

2.7. Exercise. (i) *Let $n \in \mathbb{N}$. Prove that the morphism*

$$j_n : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$$

of abelian groups given by sending $m \in \mathbb{Z}$ to the class of $m/n \in \mathbb{Q}$ in $\mathbb{Q}/\mathbb{Z}$ induces an injective morphism of abelian groups $\bar{j}_n : \mathbb{Z}/n.\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$.

(ii) *Prove that the morphism*

$$j : \mathbb{Z} \to \prod_{n\in\mathbb{N}} \mathbb{Q}/\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})^{\mathbb{N}}$$

of abelian groups given by sending $m \in \mathbb{Z}$ to the tuple $(j_n(m))_{n\in\mathbb{N}}$ in $(\mathbb{Q}/\mathbb{Z})^{\mathbb{N}}$ is injective.

(iii) *Let M be an abelian group. Given $m \in M \setminus \{\mathbf{0}_M\}$ consider the abelian group $\langle m\rangle \subseteq M$ generated by m, and $i_m : \langle m\rangle \to (\mathbb{Q}/\mathbb{Z})^{I_m}$ be the injective morphism of abelian groups given by the previous items, where I_m is a singleton or $\mathbb{N}$. Deduce that there exists a morphism of abelian groups $\hat{i}_m : M \to (\mathbb{Q}/\mathbb{Z})^{I_m}$ such that $\hat{i}_m|_{\langle m\rangle} = i_m$. Deduce that $\hat{M} = \mathsf{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \neq 0$ if $M \neq 0$.*

(iv) Prove that the morphism of abelian groups

$$\hat{\imath} : M \to \prod_{m \in M \setminus \{0_M\}} (\mathbb{Q}/\mathbb{Z})^{I_m} = (\mathbb{Q}/\mathbb{Z})^{\cup_{m \in M \setminus \{0_M\}} I_m}$$

sending $n \in M$ to the tuple $(\hat{\imath}_m(n))_{m \in M \setminus \{0_M\}}$ is injective.

2.8. Exercise. *Let*

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

be a not necessarily exact sequence of abelian groups. Prove that it is exact if and only if the sequence

$$0 \longrightarrow \mathsf{Hom}_{\mathbb{Z}}(N'', \mathbb{Q}/\mathbb{Z}) \xrightarrow{g^*} \mathsf{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{f^*} \mathsf{Hom}_{\mathbb{Z}}(N', \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

is exact.

2.9. We have the following basic result on tensor products, which also follows from Exercises 4.85 and 4.104. We give however a direct proof for the reader's convenience.

2.10. Fact. *Let M be a left module over a ring A and*

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0 \tag{13}$$

be a short exact sequence of right A-modules. Then, the sequence

$$N' \otimes_A M \xrightarrow{f \otimes_A \mathsf{id}_M} N \otimes_A M \xrightarrow{g \otimes_A \mathsf{id}_M} N'' \otimes_A M \longrightarrow 0$$

of $\mathbb{Z}$-modules is exact.

The analogous result holds for a right A-module and a short exact sequence of left A-modules.

Proof. Let $\sum_{i \in I} n''_i \otimes_A m_i$ be a finite sum with $n''_i \in N''$ and $m_i \in M$ for $i \in I$. Since g is surjective, there exists $n_i \in N$ such that $g(n_i) = n''_i$ for $i \in I$. Hence,

$$(g \otimes_A \mathsf{id}_M)(\sum_{i \in I} n_i \otimes_A m_i) = \sum_{i \in I} g(n_i) \otimes_A m_i = \sum_{i \in I} n''_i \otimes_A m_i,$$

which means that the map $g \otimes_A \mathsf{id}_M$ is surjective. Since $g \circ f = 0$, $(g \otimes_A \mathsf{id}_M) \circ (f \otimes_A \mathsf{id}_M) = 0$, *i.e.* $\mathsf{Ker}(g \otimes_A \mathsf{id}_M) \supseteq \mathsf{Img}(f \otimes_A \mathsf{id}_M)$. It remains to prove the converse inclusion. Given an element $\sum_{i \in I} n_i \otimes_A m_i \in \mathsf{Ker}(g \otimes_A \mathsf{id}_M)$, consider the abelian group

$$(N \otimes_A M)/\,\mathsf{Img}(f \otimes_A \mathsf{id}_M) = (N \otimes_A M)/\big\langle \{f(n') \otimes_A m : n' \in N', m \in M\} \big\rangle,$$

where $\langle S \rangle$ denotes the abelian subgroup generated by the set S. Define

$$h : N'' \times M \to (N \otimes_A M)/\,\mathsf{Img}(f \otimes_A \mathsf{id}_M)$$

by $h(n'', m) = [n \otimes_A m]$ for $n'' \in N''$ and $m \in M$, where $n \in N$ satisfies that $g(n) = n''$ and the brackets denote the class in the quotient. It is clear that h is well defined since, given $n_2 \in N$ such that $g(n_2) = n''$, there exists a unique $n' \in N'$ such that $n = n_2 + f(n')$, so $n \otimes_A m = n_2 \otimes_A m + f(n') \otimes_A m$, *i.e.* $[n \otimes_A m] = [n_2 \otimes m]$. Since h is bilinear and $h(n'', am) = h(n''a, m)$ for $a \in A$, $n'' \in N''$ and $m \in M$, it induces a unique morphism of abelian groups $\bar{h} : N'' \otimes_A M \to (N \otimes_A M)/\,\mathsf{Img}(f \otimes_A \mathsf{id}_M)$. By definition, $\bar{h}(g(n) \otimes_A m) = [n \otimes_A m]$, for $n \in N$ and $m \in M$. Hence, $0 = \bar{h}(\sum_{i \in I} g(n_i) \otimes_A m_i) = \sum_{i \in I} [n_i \otimes_A m_i]$, which implies that $\sum_{i \in I} n_i \otimes_A m_i$ belongs to $\mathsf{Img}(f \otimes_A \mathsf{id}_M)$, as was to be shown. ☐

2.11. We recall that an A-module is called **flat** if, given any short exact sequence of right A-modules (13), the short sequence

$$0 \longrightarrow N' \otimes_A M \xrightarrow{f \otimes_A \mathrm{id}_M} N \otimes_A M \xrightarrow{g \otimes_A \mathrm{id}_M} N'' \otimes_A M \longrightarrow 0$$

of $\mathbb{Z}$-modules is exact. The analogous definition holds for right A-modules.

2.12. Using the natural isomorphism

$$A \otimes_A M \to M$$

sending $a \otimes_A m$ to m, for $a \in A$ and $m \in M$, whose inverse sends m to $1_A \otimes_A m$, and the fact that tensor products commute with direct sums on each side, we easily get that projectives modules are flat. We also have the following important result. The analogous result for the category of right modules also holds.

2.13. Proposition. *Let A be a ring and let $F : \mathscr{C} \to {}_A\mathsf{Mod}$ be a functor defined on a filtered small category $\mathscr{C}$ such that $F(X)$ is a flat module for every $X \in \mathscr{C}_0$. Then, the filtered colimit of F is a flat module.*

Proof. Let C be the filtered colimit of F. It suffices to show that given an injection $f : M' \to M$ of right A-modules, $f \otimes_A \mathrm{id}_C : M' \otimes_A C \to M \otimes_A C$ is injective. Consider the functors $G', G : {}_A\mathsf{Mod} \to \mathsf{Ab}$ given by $M' \otimes_A (-)$ and $M \otimes_A (-)$, respectively. We first note that, by Exercise 4.85 and Theorem 4.86, G and G' preserve arbitrary colimits, so

$$M' \otimes_A C \cong G'(\underset{\to\mathscr{C}}{\mathrm{colim}} F) \cong \underset{\to\mathscr{C}}{\mathrm{colim}} G' \circ F \text{ and } M \otimes_A C \cong G(\underset{\to\mathscr{C}}{\mathrm{colim}} F) \cong \underset{\to\mathscr{C}}{\mathrm{colim}} G \circ F. \tag{14}$$

On the other hand, since $f : M' \to M$ is an injection of right modules and every $F(X)$ is flat, the corresponding morphism $f \otimes_A \mathrm{id}_{F(X)} : M' \otimes_A F(X) \to M \otimes_A F(X)$ is injective for all $X \in \mathscr{C}_0$. It is easy to see that the family of morphisms $\hat{f} = \{f \otimes_A \mathrm{id}_{F(X)} : X \in \mathscr{C}_0\}$ is thus a natural transformation from $G' \circ F$ to $G \circ F$, and it is in fact a monomorphism in the category $\mathsf{Fun}(\mathscr{C}, \mathsf{Ab})$. Since the functor

$$\underset{\to\mathscr{C}}{\mathrm{colim}} : \mathsf{Fun}(\mathscr{C}, \mathsf{Ab}) \to \mathsf{Ab}$$

is exact, by Exercise 4.106,

$$\underset{\to\mathscr{C}}{\mathrm{colim}}(\hat{f}) : \underset{\to\mathscr{C}}{\mathrm{colim}} G' \circ F \to \underset{\to\mathscr{C}}{\mathrm{colim}} G \circ F$$

is injective. The previous map coincides with $f \otimes_A \mathrm{id}_C$ under the isomorphisms in (14), and the proposition follows. □

2.14. Let M be a left (resp., right) module over a ring A. Define the **Pontrjagin dual** $\hat{M}$ of M as the right (resp., left) A-module $\hat{M} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ for the action $(f \cdot a)(m) = f(am)$ (resp., $(a \cdot f)(m) = f(ma)$), for $a \in A$, $m \in M$ and $f \in \hat{M}$. The following result shows that injective and flat modules are very strongly related.

2.15. Proposition. *Let A be a ring and M be a left (resp., right) module. The following conditions are equivalent:*

(i) *M is flat;*

(ii) *$\hat{M}$ is injective.*

Proof. We will prove the case where M is a left module, since the other is analogous. By Exercise 4.85 we see that there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}((-) \otimes_A M, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_A(-, \hat{M})$$

between the previous two functors from $\mathsf{Mod}_A^{\mathrm{op}}$ to the category of abelian groups. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, because it is divisible, if M is flat then $\hat{M}$ is injective. The converse follows from Exercise 2.8. □

§2.2. Resolutions

2.16. Definition. *Let M be an A-module. A **left resolution** of M is a complex of A-modules (P, d_P) such that $P_n = 0$ for all $n < 0$ and $H_n(P, d_P) = 0$ for all $n > 0$, together with a surjective morphism of A-modules $\varepsilon : P_0 \to M$, called **augmentation**, whose kernel is the image of $d_1 : P_1 \to P_0$. We present a left resolution as an exact complex*

$$\dots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

*and we shall denote a left resolution by $\varepsilon : P_\bullet \to M$. A **projective resolution** (resp., **flat resolution**) of M is a left resolution of M such that P_n is projective (resp., flat) for all $n \in \mathbb{Z}$.*

*Analogously, a **right resolution** of M is a complex of A-modules (I, d_I) such that $I_n = 0$ for all $n > 0$ and $H_n(I, d_I) = 0$ for all $n < 0$, together with a injective morphism of A-modules $\eta : M \to I_0$, **coaugmentation**, whose image is the kernel of $d_0 : I_0 \to I_{-1}$. We present a right resolution as an exact complex*

$$0 \longrightarrow M \xrightarrow{\eta} I_0 \xrightarrow{d_0} I_{-1} \xrightarrow{d_{-1}} I_{-2} \xrightarrow{d_{-2}} \dots$$

*and we shall denote a right resolution by $\eta : M \to I_\bullet$. An **injective resolution** of M is a right resolution of M such that I_n is injective for all $n \in \mathbb{Z}$.*

2.17. Remark. *In the case of right resolutions one usually utilizes cohomological notation instead of homological notation for the complexes, that was explained in paragraph 1.2. Since we will mostly work with left resolutions, we will not use the cohomological notation very much in these notes. It will appear however when dealing with* Ext *groups.*

2.18. Lemma. *Let M be an A-module. Then, there exists a projective resolution $\varepsilon_M : P(M) \to M$ and an injective resolution $\eta_M : M \to E(M)$. If $f : M \to N$ is a morphism of A-modules there exist morphisms of complexes of A-modules $P(f) : P(M) \to P(N)$ and $E(f) : E(M) \to E(N)$ such that $\varepsilon_N \circ P(f)_0 = f \circ \varepsilon_M$ and $\eta_N \circ f = E(f)_0 \circ \eta_M$. Moreover, they further satisfy that $P(\mathrm{id}_M) = \mathrm{id}_{P(M)}$ and $P(f \circ g) = P(f) \circ P(g)$, and $E(\mathrm{id}_M) = \mathrm{id}_{E(M)}$ and $E(f \circ g) = E(f) \circ E(g)$, for any pair of morphisms of A-modules $f : M' \to M$ and $g : M \to M''$.*

Proof. We will prove the case of projective resolutions and leave the case of injective resolutions to the reader, which follows *mutatis mutandi*. Define $P_0 = F(M)$ and $\varepsilon = \pi_M$ in Exercise 2.6. Assume that you have constructed a finite exact sequence

$$P_n \xrightarrow{d_{P,n}} \dots \xrightarrow{d_{P,3}} P_2 \xrightarrow{d_{P,2}} P_1 \xrightarrow{d_{P,1}} P_0 \xrightarrow{\varepsilon_M} M \longrightarrow 0$$

such that P_i is projective for all $i \in [\![0, n]\!]$, for some $n \in \mathbb{N}_0$. Let $N = \mathrm{Ker}(d_{P,n})$, and set $P_{n+1} = F(N)$ and $d_{P,n+1}$ the composition of $\pi_N : P_{n+1} \to N$ and the inclusion of N inside of P_n. Then, the sequence

$$P_{n+1} \xrightarrow{d_{P,n+1}} P_n \xrightarrow{d_{P,n}} \dots \xrightarrow{d_{P,3}} P_2 \xrightarrow{d_{P,2}} P_1 \xrightarrow{d_{P,1}} P_0 \xrightarrow{\varepsilon_M} M \longrightarrow 0$$

is also exact. The full projective resolution of M then follows by induction.

Let us now assume that $f : M \to N$ is a morphism of A-modules. Define the unique morphism of A-modules $F(f) : F(M) \to F(N)$ sending $a \in A_m$ to $a \in A_{f(m)}$, for all $m \in M$. Note that $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$ and $F(g \circ f) = F(g) \circ F(f)$, for any pair of morphisms of A-modules $f : M' \to M$ and $g : M \to M''$. It is easy to see that $\pi_N \circ F(f) = f \circ \pi_M$. Let $P(f)_0 = F(f)$. Assume that you have constructed a finite sequence of vertical maps in the following diagram

$$\begin{tikzcd}
\dots \arrow[r, "d_{P,n+2}"] & P_{n+1} \arrow[r, "d_{P,n+1}"] & P_n \arrow[r, "d_{P,n}"] \arrow[d, "P(f)_n"] & \dots \arrow[r, "d_{P,3}"] & P_2 \arrow[r, "d_{P,2}"] \arrow[d, "P(f)_2"] & P_1 \arrow[r, "d_{P,1}"] \arrow[d, "P(f)_1"] & P_0 \arrow[r, "\varepsilon_M"] \arrow[d, "P(f)_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\dots \arrow[r, "d_{Q,n+2}"] & Q_{n+1} \arrow[r, "d_{Q,n+1}"] & Q_n \arrow[r, "d_{Q,n}"] & \dots \arrow[r, "d_{Q,3}"] & Q_2 \arrow[r, "d_{Q,2}"] & Q_1 \arrow[r, "d_{Q,1}"] & Q_0 \arrow[r, "\varepsilon_N"] & N \arrow[r] & 0
\end{tikzcd}$$

such that $f \circ \varepsilon_M = \varepsilon_N \circ P(f)_0$ and $d_{Q,i} \circ P(f)_i = P(f)_{i-1} \circ d_{P,i}$ for all $i \in [\![1, n]\!]$. Taking into account that $d_{Q,n} \circ P(f)_n = P(f)_{n-1} \circ d_{P,n}$, $P(f)_n(\mathsf{Ker}(d_{P,n})) \subseteq \mathsf{Ker}(d_{Q,n})$. Set $P(f)_{n+1} = F(P(f)_n|_{\mathsf{Ker}(d_{P,n})})$. It is easy to see that $d_{Q,n+1} \circ P(f)_{n+1} = P(f)_{n-1} \circ d_{P,n}$. The full morphism of complexes then follows by induction.

The properties $P(\mathsf{id}_M) = \mathsf{id}_{P(M)}$ and $P(g \circ f) = P(g) \circ P(f)$, for any pair of morphisms of A-modules $f : M' \to M$ and $g : M \to M''$, follow directly from the fact that $F(\mathsf{id}_M) = \mathsf{id}_{F(M)}$ and $F(g \circ f) = F(g) \circ F(f)$, for any pair of morphisms of A-modules $f : M' \to M$ and $g : M \to M''$. The lemma is thus proved. $\square$

2.19. Exercise. *Let A be a ring, $\mathscr{C}$ a filtered small category, and $F : \mathscr{C} \to {}_A\mathsf{Mod}$ be a functor.*

(i) Using the previous Lemma, prove that there exists a functor $\hat{F} : \mathscr{C} \to {}_A\mathsf{DGMod}$ such that $\hat{F}(X)$ is a complex $\varepsilon_X : P_X \to F(X)$ giving a projective resolution of $F(X)$, where we regard $F(X)$ sitting in degree -1, such $\hat{F}(f)_{-1} = F(f)$, for all morphisms f in $\mathscr{C}$.

(ii) Using Exercise 4.106, prove that $\mathsf{colim}_{\mathscr{C}} \hat{F}$ is a left resolution of $\mathsf{colim}_{\mathscr{C}} F$.

2.20. Lemma (Comparison Lemma for projectives)**.** *Let (P, d_P) be a complex of A-modules such that $P_n = 0$ for all $n < 0$ and P_n is projective for all $n \in \mathbb{Z}$, together with a morphism of A-modules $\varepsilon_P : P_0 \to M$. Let $f : M \to N$ be a morphism of A-modules and let $\varepsilon_Q : Q_\bullet \to N$ be a left resolution of N. Then, there exists a morphism of complexes of A-modules $F : (P, d_P) \to (Q, d_Q)$ such that $f \circ \varepsilon_P = \varepsilon_Q \circ F_0$. Moreover, any two such morphisms F and F' satisfy that $F - F'$ is homotopic to zero. We can present this graphically by the following commutative diagram*

$$\begin{tikzcd}
\dots \arrow[r, "d_{P,n+2}"] & P_{n+1} \arrow[r, "d_{P,n+1}"] \arrow[d, "F_{n+1}"] & P_n \arrow[r, "d_{P,n}"] \arrow[d, "F_n"] & \dots \arrow[r, "d_{P,3}"] & P_2 \arrow[r, "d_{P,2}"] \arrow[d, "F_2"] & P_1 \arrow[r, "d_{P,1}"] \arrow[d, "F_1"] & P_0 \arrow[r, "\varepsilon_P"] \arrow[d, "F_0"] & M \arrow[d, "f"] & \\
\dots \arrow[r, "d_{Q,n+2}"] & Q_{n+1} \arrow[r, "d_{Q,n+1}"] & Q_n \arrow[r, "d_{Q,n}"] & \dots \arrow[r, "d_{Q,3}"] & Q_2 \arrow[r, "d_{Q,2}"] & Q_1 \arrow[r, "d_{Q,1}"] & Q_0 \arrow[r, "\varepsilon_Q"] & N \arrow[r] & 0
\end{tikzcd}$$

Proof. The existence of the morphism of A-modules F_0 follows from the diagram

$$\begin{tikzcd}
 & P_0 \arrow[dl, dashed, bend right, "\exists F_0"'] \arrow[d, "\varepsilon_P \circ f"] & \\
Q_0 \arrow[r, "\varepsilon_Q"] & N \arrow[r] & 0
\end{tikzcd}$$

since P_0 is projective and ε_Q is surjective. Moreover, if F'_0 is another morphism of A-modules such that $\varepsilon_Q \circ F'_0 = f \circ \varepsilon_P$, then $\varepsilon_Q \circ (F_0 - F'_0)$ vanishes, *i.e.* $\mathsf{Img}(F_0 - F'_0) \subseteq \mathsf{Ker}(\varepsilon_Q) = \mathsf{Img}(d_{Q,1})$. Then, we have the commutative diagram

$$\begin{tikzcd}
 & P_0 \arrow[dl, dashed, bend right, "\exists h_0"'] \arrow[d, "F_0 - F'_0"] & \\
Q_1 \arrow[r, "d_{Q,1}"] & \mathsf{Img}(d_{Q,1}) \arrow[r] & 0
\end{tikzcd}$$

since P_0 is projective and the lower horizontal map is surjective, *i.e.* $d_{Q,1} \circ h_0 = F_0 - F'_0$.

Assume now that we have constructed $F_0, \dots, F_n$ such that $\varepsilon_Q \circ F_0 = f \circ \varepsilon_P$ and $F_{i-1} \circ d_{P,i} = d_{Q,i-1} \circ F_i$ for all $i \in [\![0, n-1]\!]$, for some $n \in \mathbb{N}_0$. We also assume that, if $F'_0, \dots, F'_n$ is such that $\varepsilon_Q \circ F'_0 = f \circ \varepsilon_P$ and $F'_{i-1} \circ d_{P,i} = d_{Q,i-1} \circ F'_i$ for all $i \in [\![0, n-1]\!]$, there exist morphisms of A-modules $h_0, \dots, h_n$ such $h_i : P_i \to Q_{i+1}$ such that $d_{Q,i+1} \circ h_i + h_{i-1} \circ d_{P,i} = F_i - F'_i$ for all $i \in [\![0, n]\!]$. We will prove it for $n+1$. If $n = 0$ we have $\varepsilon_Q \circ F_0 \circ d_{P,1} = f \circ \varepsilon_P \circ d_{P,1} = 0$, whereas if $n > 0$ we get the identity $d_{Q,n} \circ F_n \circ d_{P,n+1} = F_{n-1} \circ d_{P,n} \circ d_{P,n+1} = 0$. Hence $\mathsf{Img}(F_n \circ d_{P,n+1}) \subseteq \mathsf{Ker}(\varepsilon_Q) = \mathsf{Img}(d_{Q,1})$ if $n = 0$ and $\mathsf{Img}(F_n \circ d_{P,n+1}) \subseteq \mathsf{Ker}(d_{Q,n}) = \mathsf{Img}(d_{Q,n+1})$ if $n > 0$, by exactness of (Q, d_Q). The existence of F_{n+1}

follows from the diagram

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dashed, bend right, "\exists F_{n+1}"'] \arrow[d, "F_n \circ d_{P,n+1}"] & \\ Q_{n+1} \arrow[r, "d_{Q,n+1}"] & \mathsf{Img}(d_{Q,n+1}) \arrow[r] & 0 \end{tikzcd}$$

since P_{n+1} is projective and the lower horizontal map is surjective. We get that $F_n \circ d_{P,n+1} = d_{Q,n+1} \circ F_{n+1}$, as was to be shown.

It remains to prove that, if there exists a morphism of A-modules $F'_{n+1} : P_{n+1} \to Q_{n+1}$ such that $F'_n \circ d_{P,n+1} = d_{Q,n+1} \circ F'_{n+1}$, there exists a morphism of A-modules $h_{n+1} : P_{n+1} \to Q_{n+2}$ such that $d_{Q,n+2} \circ h_{n+1} + h_n \circ d_{P,n+1} = F_{n+1} - F'_{n+1}$. To prove it, note that

$$\begin{aligned} d_{Q,n+1} \circ (F_{n+1} - F'_{n+1}) &= (F_n - F'_n) \circ d_{P,n+1} = h_{n-1} \circ d_{P,n} \circ d_{P,n+1} + d_{Q,n+1} \circ h_n \circ d_{P,n+1} \\ &= d_{Q,n+1} \circ h_n \circ d_{P,n+1}, \end{aligned}$$

which implies that $\mathsf{Img}(F_{n+1} - F'_{n+1} - h_n \circ d_{P,n+1}) \subseteq \mathsf{Ker}(d_{Q,n+1}) = \mathsf{Img}(d_{Q,n+2})$. The existence of h_{n+1} follows from the diagram

$$\begin{tikzcd} & P_{n+1} \arrow[dl, dashed, bend right, "\exists h_{n+1}"'] \arrow[d, "F_{n+1} - F'_{n+1} - h_n \circ d_{P,n+1}"] & \\ Q_{n+2} \arrow[r, "d_{Q,n+2}"] & \mathsf{Img}(d_{Q,n+2}) \arrow[r] & 0 \end{tikzcd}$$

since P_{n+1} is projective and the lower horizontal map $d_{Q,n+2}$ is surjective. We get thus the identity $h_n \circ d_{P,n+1} + d_{Q,n+2} \circ h_{n+1} = F_{n+1} - F'_{n+1}$, as was to be shown. □

2.21. We present the analogous result to Lemma 2.20 for injectives. The proof is just the dual.

2.22. Lemma (Comparison Lemma for injectives). *Let (I, d_I) be a complex of A-modules such that $I_n = 0$ for all $n > 0$ and I_n is injective for all $n \in \mathbb{Z}$, together with a morphism of A-modules $\eta_I : M \to I_0$. Let $f : N \to M$ be a morphism of A-modules and let $\eta_J : N \to J_\bullet$ be a right resolution of N. Then, there exists a morphism of complexes of A-modules $F : (J, d_J) \to (I, d_I)$ such that $\eta_I \circ f = F_0 \circ \eta_J$. Moreover, any two such morphisms F and F' satisfy that $F - F'$ is homotopic to zero. We can present this graphically by the following commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\eta_J"] \arrow[d, "f"] & J_0 \arrow[r, "d_{J,0}"] \arrow[d, "F_0"] & J_{-1} \arrow[r, "d_{J,-1}"] \arrow[d, "F_{-1}"] & J_2 \arrow[r, "d_{J,-2}"] \arrow[d, "F_{-2}"] & \dots \arrow[r, "d_{J,-n+2}"] & J_{-n+1} \arrow[r, "d_{J,-n+1}"] \arrow[d, "F_{-n+1}"] & J_{-n} \arrow[r, "d_{J,-n}"] \arrow[d, "F_{-n}"] & \dots \\ & M \arrow[r, "\eta_I"] & I_0 \arrow[r, "d_{I,0}"] & I_{-1} \arrow[r, "d_{I,-1}"] & I_{-2} \arrow[r, "d_{I,-2}"] & \dots \arrow[r, "d_{I,-n+2}"] & I_{-n+1} \arrow[r, "d_{I,-n+1}"] & I_{-n} \arrow[r, "d_{I,-n}"] & \dots \end{tikzcd}$$

2.23. Lemma (Horseshoe Lemma for projectives (part 1)). *Let*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a short exact sequence of A-modules and let $\varepsilon_{P'} : (P', d_{P'}) \to M'$ and $\varepsilon_{P''} : (P'', d_{P''}) \to M''$ be projective resolutions. Then, there exist morphisms of A-modules $\varepsilon_2 : P''_0 \to M$ and $\lambda_n : P''_n \to P'_{n-1}$ for $n \in \mathbb{N}$ such that $g \circ \varepsilon_2 = \varepsilon_{P''}$, $f \circ \varepsilon_{P'} \circ \lambda_1 = \varepsilon_2 \circ d_{P'',1}$ and $d_{P',n-1} \circ \lambda_n = \lambda_{n-1} \circ d_{P'',n}$ for integers $n \geq 2$.

Define $P_n = P'_n \oplus P''_n$, $F_n : P'_n \to P_n$ as the canonical inclusion and $G_n : P_n \to P''_n$ as the canonical projection, for every $n \in \mathbb{N}_0$. Moreover, let $\varepsilon_P : P_0 \to M$ be the morphism of A-modules sending (p', p'') to $f(\varepsilon_{P'}(p')) + \varepsilon_2(p'')$ for $p' \in P'_0$ and $p'' \in P''_0$, and, for $n \in \mathbb{N}$, define the morphism of A-modules $d_{P,n} : P_n \to P_{n-1}$ by

$$d_{P,n}(p', p'') = \big(d_{P',n}(p') + (-1)^n \lambda_n(p''), d_{P'',n}(p'')\big),$$

for $p' \in P'_n$ and $p'' \in P''_n$. Then $\varepsilon_P : (P, d_P) \to M$ is a projective resolution that fits in the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\dots \arrow[r, "d_{P',n+2}"] & P'_{n+1} \arrow[r, "d_{P',n+1}"] \arrow[d, "F_{n+1}"] & P'_n \arrow[r, "d_{P',n}"] \arrow[d, "F_n"] & \dots \arrow[r, "d_{P',3}"] & P'_2 \arrow[r, "d_{P',2}"] \arrow[d, "F_2"] & P'_1 \arrow[r, "d_{P',1}"] \arrow[d, "F_1"] & P'_0 \arrow[r, "\varepsilon_{P'}"] \arrow[d, "F_0"] & M' \arrow[r] \arrow[d, "f"] & 0 \\
\dots \arrow[r, "d_{P,n+2}"] & P_{n+1} \arrow[r, "d_{P,n+1}"] \arrow[d, "G_{n+1}"] & P_n \arrow[r, "d_{P,n}"] \arrow[d, "G_n"] & \dots \arrow[r, "d_{P',3}"] & P_2 \arrow[r, "d_{P,2}"] \arrow[d, "G_2"] & P_1 \arrow[r, "d_{P,1}"] \arrow[d, "G_1"] & P_0 \arrow[r, "\varepsilon_P"] \arrow[d, "G_0"] & M \arrow[r] \arrow[d, "g"] & 0 \\
\dots \arrow[r, "d_{P'',n+2}"] & P''_{n+1} \arrow[r, "d_{P'',n+1}"] \arrow[d] & P''_n \arrow[r, "d_{P'',n}"] \arrow[d] & \dots \arrow[r, "d_{P'',3}"] & P''_2 \arrow[r, "d_{P'',2}"] \arrow[d] & P''_1 \arrow[r, "d_{P'',1}"] \arrow[d] & P''_0 \arrow[r, "\varepsilon_{P''}"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows.

Proof. From the following diagram

$$\begin{tikzcd}
 & P''_0 \arrow[d, "\varepsilon_{P''}"] \arrow[dl, dashed, bend right, "\exists \varepsilon_2"'] & \\
M \arrow[r, "g"] & M'' \arrow[r] & 0
\end{tikzcd}$$

and the fact that P''_0 is projective and g surjective we obtain the map $\varepsilon_2 : P''_0 \to M$ such that $g \circ \varepsilon_2 = \varepsilon_{P''}$. It is clear that ε_P is surjective, $\varepsilon_P \circ F_0 = f \circ \varepsilon_{P'}$ and $g \circ \varepsilon_P = \varepsilon_{P''} \circ G_0$.

We will construct morphisms of A-modules $\lambda_i : P''_i \to P'_{i-1}$ for $i \in \mathbb{N}$ such that $f \circ \varepsilon_{P'} \circ \lambda_1 = \varepsilon_2 \circ d_{P'',1}$ and $d_{P',i-1} \circ \lambda_i = \lambda_{i-1} \circ d_{P'',i}$ for $i \in \mathbb{N}$. We first construct $\lambda_1 : P''_1 \to P'_0$ such that $f \circ \varepsilon_{P'} \circ \lambda_1 = \varepsilon_2 \circ d_{P'',1}$. Since $g \circ \varepsilon_2 \circ d_{P'',1} = \varepsilon_{P''} \circ d_{P'',1} = 0$, and $\mathsf{Ker}(g) = \mathsf{Img}(f) = \mathsf{Img}(f \circ \varepsilon_{P'})$, we see that the image of $\varepsilon_2 \circ d_{P'',1}$ is included in $\mathsf{Img}(f \circ \varepsilon_{P'})$. Hence, from the following diagram

$$\begin{tikzcd}
 & P''_1 \arrow[d, "\varepsilon_2 \circ d_{P'',1}"] \arrow[dl, dashed, bend right, "\exists \lambda_1"'] & \\
P'_0 \arrow[r, "f \circ \varepsilon_{P'}"] & \mathsf{Img}(f \circ \varepsilon_{P'}) \arrow[r] & 0
\end{tikzcd}$$

there exists $\lambda_1 : P''_1 \to P'_0$ such that $f \circ \varepsilon_{P'} \circ \lambda_1 = \varepsilon_2 \circ d_{P'',1}$, since P''_1 is projective and the horizontal map is surjective.

Assume we have further constructed $\lambda_i : P''_i \to P'_{i-1}$ for $i \in [\![1, n]\!]$ and some $n \in \mathbb{N}_0$ such that $f \circ \varepsilon_{P'} \circ \lambda_1 = \varepsilon_2 \circ d_{P'',1}$ and $d_{P',i-1} \circ \lambda_i = \lambda_{i-1} \circ d_{P'',i}$ for $i \in [\![2, n]\!]$. If $n = 1$, consider the composition $f \circ \varepsilon_{P'} \circ \lambda_1 \circ d_{P'',2} = \varepsilon_2 \circ d_{P'',1} \circ d_{P'',2} = 0$. Since f is injective this implies that $\varepsilon_{P'} \circ \lambda_1 \circ d_{P'',2} = 0$, *i.e.* $\mathsf{Img}(\lambda_1 \circ d_{P'',2}) \subseteq \mathsf{Ker}(\varepsilon_{P'}) = \mathsf{Img}(d_{P',1})$. Hence, from the following diagram

$$\begin{tikzcd}
 & P''_2 \arrow[d, "\lambda_1 \circ d_{P'',2}"] \arrow[dl, dashed, bend right, "\exists \lambda_2"'] & \\
P'_1 \arrow[r, "d_{P',1}"] & \mathsf{Img}(d_{P',1}) \arrow[r] & 0
\end{tikzcd}$$

there exists $\lambda_2 : P''_2 \to P'_1$ such that $d_{P',1} \circ \lambda_2 = \lambda_1 \circ d_{P'',2}$, since P''_2 is projective and the horizontal map is surjective. If $n > 1$, consider the composition $d_{P',n-1} \circ \lambda_n \circ d_{P'',n+1} = \lambda_{n-1} \circ d_{P'',n} \circ d_{P'',n+1} = 0$, *i.e.*

$\mathsf{Img}(\lambda_n \circ d_{P'',n+1}) \subseteq \mathsf{Ker}(d_{P',n-1}) = \mathsf{Img}(d_{P',n})$. The diagram

$$\begin{tikzcd}
 & P''_{n+1} \arrow[dl, dashed, bend right, "\exists\lambda_{n+1}"'] \arrow[d, "\lambda_n \circ d_{P'',n+1}"] & \\
P'_n \arrow[r, "d_{P',n}"] & \mathsf{Img}(d_{P',n}) \arrow[r] & 0
\end{tikzcd}$$

there exists $\lambda_{n+1} : P''_{n+1} \to P'_n$ such that $d_{P',n} \circ \lambda_{n+1} = \lambda_n \circ d_{P'',n+1}$, since P''_{n+1} is projective and the horizontal map is surjective.

For $n \in \mathbb{N}$, we now define the morphism of A-modules $d_{P,n} : P_n \to P_{n-1}$ as in the statement. It is easy to see that it satisfies that $\varepsilon_P \circ d_{P,1} = 0$, $d_{P,n} \circ d_{P,n+1} = 0$, as well as $d_{P,n} \circ F_n = F_{n-1} \circ d_{P',n}$ and $d_{P'',n} \circ G_n = G_{n-1} \circ d_{P,n}$. It only remains to prove that $(P, d_P) \to M$ is a left resoltuion. This follows directly from Theorem 1.44. □

2.24. Remark. *Note that the complex of A-modules (P, d_P) in the previous proof is given as the cone of a morphism of complexes $(P'', d_{P''}) \to (P', d_{P'})[-1]$.*

2.25. We present the analogous result to Lemma 2.23 for injectives. The proof is just the dual.

2.26. Lemma (Horseshoe Lemma for injectives (part 1))**.** *Let*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a short exact sequence of A-modules and let $\eta_{I'} : M' \to (I', d_{I'})$ and $\eta_{I''} : M'' \to (P'', d_{P''})$ be injective resolutions. Then, there exist morphisms of A-modules $\eta_1 : M \to I'_0$ and $\lambda_n : I''_n \to I'_{n-1}$ for $-n \in \mathbb{N}$ such that $\eta_1 \circ f = \eta_{I'}$, $\lambda_0 \circ \eta_{I''} \circ g = d_{I',0} \circ \eta_1$ and $d_{I',n-1} \circ \lambda_n = \lambda_{n-1} \circ d_{I'',n}$ for $-n \in \mathbb{N}_0$.

Define $I_n = I'_n \oplus I''_n$, $F_n : I'_n \to I_n$ as the canonical inclusion and $G_n : I_n \to I''_n$ as the canonical projection, for every $-n \in \mathbb{N}_0$. Moreover, let $\eta : M \to I_0$ be the morphism of A-modules sending $m \in M$ to $(\eta_1(m), \eta_{I''}(g(m)))$, and, for $-n \in \mathbb{N}$, define the morphism of A-modules $d_{I,n} : I_n \to I_{n-1}$ by

$$d_{I,n}(i', i'') = \big(d_{I',n}(i') + (-1)^n \lambda_n(i''), d_{I'',n}(i'')\big),$$

for $i' \in I'_n$ and $i'' \in I''_n$. Then $\eta_I : M \to (I, d_I)$ is a projective resolution that fits in the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "\eta_{I'}"] \arrow[d, "f"] & I'_0 \arrow[r, "d_{I',0}"] \arrow[d, "F_0"] & I'_{-1} \arrow[r, "d_{I',-1}"] \arrow[d, "F_{-1}"] & I'_{-2} \arrow[r, "d_{I',-2}"] \arrow[d, "F_{-2}"] & \dots \arrow[r, "d_{I',-n+1}"] & I'_{-n} \arrow[r, "d_{I',-n}"] \arrow[d, "F_{-n}"] & I'_{-n-1} \arrow[r, "d_{I',-n-1}"] \arrow[d, "F_{-n-1}"] & \dots \\
0 \arrow[r] & M \arrow[r, "\eta_I"] \arrow[d, "g"] & I_0 \arrow[r, "d_{I,0}"] \arrow[d, "G_0"] & I_{-1} \arrow[r, "d_{I,-1}"] \arrow[d, "G_{-1}"] & I_{-2} \arrow[r, "d_{I,-2}"] \arrow[d, "G_{-2}"] & \dots \arrow[r, "d_{I,-n+1}"] & I_{-n} \arrow[r, "d_{I,-n}"] \arrow[d, "G_{-n}"] & I_{-n-1} \arrow[r, "d_{I,-n-1}"] \arrow[d, "G_{-n-1}"] & \dots \\
0 \arrow[r] & M'' \arrow[r, "\eta_{I''}"] \arrow[d] & I''_0 \arrow[r, "d_{I'',0}"] \arrow[d] & I''_{-1} \arrow[r, "d_{I'',-1}"] \arrow[d] & I''_{-2} \arrow[r, "d_{I'',-2}"] \arrow[d] & \dots \arrow[r, "d_{I'',-n+1}"] & I''_{-n} \arrow[r, "d_{I'',-n}"] \arrow[d] & I''_{-n-1} \arrow[r, "d_{I'',-n-1}"] \arrow[d] & \dots \\
 & 0 & 0 & 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

with exact rows.

2.27. Lemma (Horseshoe Lemma for projectives (part 2))**.** *Let*

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & M'' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & N' \arrow[r, "h"] & N \arrow[r, "k"] & N'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram of A-modules with exact rows. Let $\varepsilon_{P'} : (P', d_{P'}) \to M'$, $\varepsilon_{P''} : (P'', d_{P''}) \to M''$, $\varepsilon_{Q'} : (Q', d_{Q'}) \to N'$ *and* $\varepsilon_{Q''} : (Q'', d_{Q''}) \to N''$ *be projective resolutions. Let* $A : (P', d_{P'}) \to (Q', d_{Q'})$ *and* $C : (P'', d_{P''}) \to (Q'', d_{Q''})$ *be morphisms of complexes such that* $\varepsilon_{Q'} \circ A_0 = \alpha \circ \varepsilon_{P'}$ *and* $\varepsilon_{Q''} \circ C_0 = \gamma \circ \varepsilon_{P''}$, *which exist by the Comparison Lemma 2.20. Let* $\varepsilon_P : (P, d_P) \to M$ *and* $\varepsilon_Q : (Q, d_Q) \to N$ *be the projective resolutions obtained from applying Lemma 2.23. Let us denote by* $\varepsilon_{P,2} : P''_0 \to M$ *and* $\lambda_{P,n} : P''_n \to P'_{n-1}$ *for* $n \in \mathbb{N}$ *the morphisms produced in* (P, d_P), *and* $\varepsilon_{Q,2} : Q''_0 \to N$ *and* $\lambda_{Q,n} : Q''_n \to Q'_{n-1}$ *for* $n \in \mathbb{N}$ *the morphisms produced from* (Q, d_Q).

Then, there exist morphisms of A-modules $\mu_n : P''_n \to Q'_n$ *for* $n \in \mathbb{N}_0$ *such that*

$$d_{Q',n} \circ \mu_n - \mu_{n-1} \circ d_{P'',n} = (-1)^{n+1}\big(A_{n-1} \circ \lambda_{P,n} - \lambda_{Q,n} \circ C_n\big)$$

for $n \in \mathbb{N}$ *and* $h \circ \varepsilon_{Q'} \circ \mu_0 = \varepsilon_{Q,2} \circ C_0 - \beta \circ \varepsilon_{P,2}$. *Given* $n \in \mathbb{N}_0$, *define the morphism of A-modules* $B_n : P_n \to Q_n$ *by*

$$B_n(p', p'') = \big(A_n(p') - \mu_n(p''), C_n(p'')\big),$$

for $p' \in P'_n$ *and* $p'' \in P''_n$. *Then* $B : (P, d_P) \to (Q, d_Q)$ *is a morphism of complexes such that* $\varepsilon_Q \circ B_0 = \beta \circ \varepsilon_P$, $B_n \circ F_n = H_n \circ A_n$, $K_n \circ B_n = C_n \circ G_n$ *for all* $n \in \mathbb{N}_0$. *We may represent this by the commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r, "d_{P',2}"] & P'_1 \arrow[rrr, "d_{P',1}"] \arrow[dr, "F_1"] \arrow[ddd, "A_1"] & & & P'_0 \arrow[rrr, "\varepsilon_{P'}"] \arrow[dr, "F_0"] \arrow[ddd, "A_0"] & & & M' \arrow[rrr] \arrow[dr, "f"] \arrow[ddd, "\alpha"] & & & 0 & & \\
\cdots \arrow[rr, "d_{P,2}"] & & P_1 \arrow[rrr, "d_{P,1}"] \arrow[dr, "G_1"] \arrow[ddd, "B_1"] & & & P_0 \arrow[rrr, "\varepsilon_P"] \arrow[dr, "G_0"] \arrow[ddd, "B_0"] & & & M \arrow[rrr] \arrow[dr, "g"] \arrow[ddd, "\beta"] & & & 0 & \\
\cdots \arrow[rrr, "d_{P'',2}"] & & & P''_1 \arrow[rrr, "d_{P'',1}"] \arrow[ddd, "C_1"] & & & P''_0 \arrow[rrr, "\varepsilon_{P''}"] \arrow[ddd, "C_0"] & & & M'' \arrow[rrr] \arrow[ddd, "\gamma"] & & & 0 \\
\cdots \arrow[r, "d_{Q',2}"] & Q'_1 \arrow[rrr, "d_{Q',1}"] \arrow[dr, "H_1"] & & & Q'_0 \arrow[rrr, "\varepsilon_{Q'}"] \arrow[dr, "H_0"] & & & N' \arrow[rrr] \arrow[dr, "h"] & & & 0 & & \\
\cdots \arrow[rr, "d_{Q,2}"] & & Q_1 \arrow[rrr, "d_{Q,1}"] \arrow[dr, "K_1"] & & & Q_0 \arrow[rrr, "\varepsilon_Q"] \arrow[dr, "K_0"] & & & N \arrow[rrr] \arrow[dr, "k"] & & & 0 & \\
\cdots \arrow[rrr, "d_{Q'',2}"] & & & Q''_1 \arrow[rrr, "d_{Q'',1}"] & & & Q''_0 \arrow[rrr, "\varepsilon_{Q''}"] & & & N'' \arrow[rrr] & & & 0
\end{tikzcd}$$

with exact rows and slanted lines.

Proof. We first note that

$$k \circ \varepsilon_{Q,2} \circ C_0 = \varepsilon_{Q''} \circ C_0 = \gamma \circ \varepsilon_{P''} = \gamma \circ g \circ \varepsilon_{P,2} = k \circ \beta \circ \varepsilon_{P,2},$$

which implies that the image of $\varepsilon_{Q,2} \circ C_0 - \beta \circ \varepsilon_{P,2}$ is included in $\mathsf{Ker}(k) = \mathsf{Img}(h) = \mathsf{Img}(h \circ \varepsilon_{Q'})$. We have the following commutative diagram

$$\begin{tikzcd}
& P''_0 \arrow[d, "\varepsilon_{Q,2}\circ C_0 - \beta\circ\varepsilon_{P,2}"] \arrow[dl, dashed, bend right, "\exists\mu_0"'] & \\
Q'_0 \arrow[r, "h\circ\varepsilon_{Q'}"] & \mathsf{Img}(h) \arrow[r] & 0
\end{tikzcd}$$

i.e. there exists $\mu_0 : P''_0 \to Q'_0$ such that $\varepsilon_{Q,2} \circ C_0 - \beta \circ \varepsilon_{P,2} = h \circ \varepsilon_{Q'} \circ \mu_0$, since P''_0 is projective and the horizontal map is surjective. This is tantamount to $\varepsilon_Q \circ B_0 = \beta \circ \varepsilon_P$.

Assume now we have constructed morphisms of A-modules $\mu_i : P''_i \to Q'_i$ for $i \in [\![0, n-1]\!]$ for some $n \in \mathbb{N}$ such that

$$d_{Q',i} \circ \mu_i - \mu_{i-1} \circ d_{P'',i} = (-1)^{i+1}\big(A_{i-1} \circ \lambda_{P,i} - \lambda_{Q,i} \circ C_i\big)$$

for $i \in [\![1, n-1]\!]$ and $h \circ \varepsilon_{Q'} \circ \mu_0 = \varepsilon_{Q,2} \circ C_0 - \beta \circ \varepsilon_{P,2}$. If $n = 1$, then

$$\begin{aligned} h \circ \varepsilon_{Q'} \circ \mu_0 \circ d_{P'',1} &= \varepsilon_{Q,2} \circ C_0 \circ d_{P'',1} - \beta \circ \varepsilon_{P,2} \circ d_{P'',1} = \varepsilon_{Q,2} \circ d_{Q'',1} \circ C_1 - \beta \circ f \circ \varepsilon_{P'} \circ \lambda_{P,1} \\ &= h \circ \varepsilon_{Q'} \circ \lambda_{Q,1} \circ C_1 - h \circ \alpha \circ \varepsilon_{P'} \circ \lambda_{P,1} = h \circ \varepsilon_{Q'} \circ \lambda_{Q,1} \circ C_1 - h \circ \varepsilon_{Q'} \circ A_0 \circ \lambda_{P,1}, \end{aligned}$$

where we used that $\varepsilon_{P,2} \circ d_{P'',1} = f \circ \varepsilon_{P'} \circ \lambda_{P,1}$ and $\varepsilon_{Q,2} \circ d_{Q'',1} = h \circ \varepsilon_{Q'} \circ \lambda_{Q,1}$. This implies that the image of $\mu_0 \circ d_{P'',1} + A_0 \circ \lambda_{P,1} - \lambda_{Q,1} \circ C_1$ is included in $\mathsf{Ker}(\varepsilon_{Q'}) = \mathsf{Img}(d_{Q',1})$, since h is injective. We have the following commutative diagram

$$\begin{tikzcd} & P''_1 \arrow[dl, dashed, bend right, "\exists \mu_1"'] \arrow[d, "\mu_0 \circ d_{P'',1} + A_0 \circ \lambda_{P,1} - \lambda_{Q,1} \circ C_1"] & \\ Q'_1 \arrow[r, "d_{Q',1}"] & \mathsf{Img}(d_{Q',1}) \arrow[r] & 0 \end{tikzcd}$$

i.e. there exists $\mu_1 : P''_1 \to Q'_1$ such that $d_{Q',1} \circ \mu_1 = \mu_0 \circ d_{P'',1} + A_0 \circ \lambda_{P,1} - \lambda_{Q,1} \circ C_1$, since P''_1 is projective and the horizontal map is surjective. This is tantamount to $d_{Q,1} \circ B_1 = B_0 \circ d_{P,1}$.

If $n > 1$,

$$\begin{aligned} d_{Q',n-1} \circ \mu_{n-1} \circ d_{P'',n} &= \mu_{n-2} \circ d_{P'',n-1} \circ d_{P'',n} + (-1)^n \big(A_{n-2} \circ \lambda_{P,n-1} \circ d_{P'',n} - \lambda_{Q,n-1} \circ C_{n-1} \circ d_{P'',n}\big) \\ &= (-1)^n A_{n-2} \circ \lambda_{P,n-1} \circ d_{P'',n} - (-1)^n \lambda_{Q,n-1} \circ C_{n-1} \circ d_{P'',n} \\ &= (-1)^n d_{Q',n-1} \circ A_{n-1} \circ \lambda_{P,n} - (-1)^n d_{Q',n-1} \circ \lambda_{Q,n} \circ C_n, \end{aligned}$$

which implies that the image of the map $\mu_{n-1} \circ d_{P'',n} + (-1)^{n+1}(A_{n-1} \circ \lambda_{P,n} - \lambda_{Q,n} \circ C_n)$ is included in $\mathsf{Ker}(d_{Q',n-1}) = \mathsf{Img}(d_{Q',n})$. We have the following commutative diagram

$$\begin{tikzcd} & P''_n \arrow[dl, dashed, bend right, "\exists \mu_n"'] \arrow[d, "\mu_{n-1} \circ d_{P'',n} + (-1)^{n+1}(A_{n-1} \circ \lambda_{P,n} - \lambda_{Q,n} \circ C_n)"] & \\ Q'_n \arrow[r, "d_{Q',n}"] & \mathsf{Img}(d_{Q',n}) \arrow[r] & 0 \end{tikzcd}$$

i.e. there exists $\mu_n : P''_n \to Q'_n$ such that $d_{Q',n} \circ \mu_n = \mu_{n-1} \circ d_{P'',n} + (-1)^{n+1}(A_{n-1} \circ \lambda_{P,n} - \lambda_{Q,n} \circ C_n)$, since P''_n is projective and the horizontal map is surjective. This is tantamount to $d_{Q,n} \circ B_n = B_{n-1} \circ d_{P,n}$. Moreover, it is clear that $B_n \circ F_n = H_n \circ A_n$, $K_n \circ B_n = C_n \circ G_n$ for all $n \in \mathbb{N}_0$. The lemma is thus proved. □

2.28. We present the analogous result to Lemma 2.27 for injectives. The proof is just the dual.

2.29. Lemma (Horseshoe Lemma for injectives (part 2))**.** *Let*

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & M'' \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "h"] & N \arrow[r, "k"] & N'' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram of A-modules with exact rows. Let $\eta_{I'} : M' \to (I', d_{I'})$, $\eta_{I''} : M'' \to (I'', d_{I''})$, $\varepsilon_{J'} : N' \to (J', d_{J'})$ and $\eta_{J''} : N'' \to (J'', d_{J''})$ be injective resolutions. Let $A : (I', d_{I'}) \to (J', d_{J'})$ and $C : (I'', d_{I''}) \to (J'', d_{J''})$ be morphisms of complexes such that $\eta_{J'} \circ \alpha = A_0 \circ \eta_{I'}$ and $\eta_{J''} \circ \gamma = C_0 \circ \eta_{I''}$, which exist by the Comparison Lemma 2.22. Let $\eta_I : M \to (I, d_I)$ and $\eta_J : N \to (J, d_J)$ be the injective resolutions obtained from applying Lemma 2.26. Let us denote by $\eta_{I,2} : M \to I''_0$ and $\lambda_{I,n} : I''_n \to I'_{n-1}$ for $-n \in \mathbb{N}_0$ the morphisms produced in (I, d_I), and $\eta_{J,2} : N \to J''_0$ and $\lambda_{J,n} : J''_n \to J'_{n-1}$ for $-n \in \mathbb{N}_0$ the morphisms produced from (J, d_J).

Then, there exist morphisms of A-modules $\mu_n : I''_n \to J'_n$ for $n \in \mathbb{N}_0$ such that

$$d_{I',n} \circ \mu_n = \mu_{n-1} \circ d_{I'',n} + (-1)^{n+1}\big(A_{n-1} \circ \lambda_{I,n} - \lambda_{J,n} \circ C_n\big)$$

for $-n \in \mathbb{N}$ *and* $\mu_0 \circ \eta_{I''} \circ g = \eta_{J,2} \circ \beta - A_0 \circ \varepsilon_{I,2}$. *Given* $-n \in \mathbb{N}_0$, *define the morphism of A-modules* $B_n : I_n \to J_n$ *by*

$$B_n(i', i'') = \big(A_n(i') - \mu_n(i''), B_n(i'')\big),$$

for $i' \in I'_n$ *and* $i'' \in I''_n$. *Then* $B : (I, d_I) \to (J, d_J)$ *is a morphism of complexes such that* $B_0 \circ \eta_P = \eta_Q \circ \beta$, $B_n \circ F_n = H_n \circ A_n$, $K_n \circ B_n = C_n \circ G_n$ *for all* $-n \in \mathbb{N}_0$. *We may represent this by the commutative diagram*

$$\begin{tikzcd}
\cdots & & & I'_1 \arrow[lll, "d_{I',2}"] \arrow[dr, "F_1"] \arrow[ddd, "A_1"] & & & I'_0 \arrow[lll, "d_{I',1}"] \arrow[dr, "F_0"] \arrow[ddd, "A_0"] & & & M' \arrow[lll, "\eta_{I'}"] \arrow[dr, "f"] \arrow[ddd, "\alpha"] & & & 0 \arrow[lll] & & \\
& \cdots & & & I_1 \arrow[lll, "d_{I,2}"] \arrow[dr, "G_1"] \arrow[ddd, "B_1"] & & & I_0 \arrow[lll, "d_{I,1}"] \arrow[dr, "G_0"] \arrow[ddd, "B_0"] & & & M \arrow[lll, "\eta_I"] \arrow[dr, "g"] \arrow[ddd, "\beta"] & & & 0 \arrow[lll] & \\
& & \cdots & & & I''_1 \arrow[lll, "d_{I'',2}"] \arrow[ddd, "C_1"] & & & I''_0 \arrow[lll, "d_{I'',1}"] \arrow[ddd, "C_0"] & & & M'' \arrow[lll, "\eta_{I''}"] \arrow[ddd, "\gamma"] & & & 0 \arrow[lll] \\
\cdots & & & J'_1 \arrow[lll, "d_{J',2}"] \arrow[dr, "H_1"] & & & J'_0 \arrow[lll, "d_{J',1}"] \arrow[dr, "H_0"] & & & N' \arrow[lll, "\eta_{J'}"] \arrow[dr, "h"] & & & 0 \arrow[lll] & & \\
& \cdots & & & J_1 \arrow[lll, "d_{J,2}"] \arrow[dr, "K_1"] & & & J_0 \arrow[lll, "d_{J,1}"] \arrow[dr, "K_0"] & & & N \arrow[lll, "\eta_Q"] \arrow[dr, "k"] & & & 0 \arrow[lll] & \\
& & \cdots & & & J''_1 \arrow[lll, "d_{J'',2}"] & & & J''_0 \arrow[lll, "d_{J'',1}"] & & & N'' \arrow[lll, "\eta_{J''}"] & & & 0 \arrow[lll]
\end{tikzcd}$$

with exact rows and slanted lines.

2.30. The following exercise provides one of the most significant properties of bounded below (resp., above) complexes of projective (resp., injective) modules.

2.31. Exercise. *Let A be a ring.*

(i) Let (P, d) *be bounded below (homological) complex of A-modules such that* P_n *is projective for all* $n \in \mathbb{Z}$.

(1) Let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a short exact sequence of complexes of A-modules. Prove that the short sequence

$$0 \longrightarrow \mathscr{H}om_{\mathsf{dgA}}(P, M') \xrightarrow{f_*} \mathscr{H}om_{\mathsf{dgA}}(P, M) \xrightarrow{g_*} \mathscr{H}om_{\mathsf{dgA}}(P, M'') \longrightarrow 0$$

of complexes of abelian groups is exact, where we recall that $f_*(h) = f \circ_{\mathsf{gr}} h$ *and* $g_*(k) = g \circ_{\mathsf{gr}} k$, *for* $h \in \mathscr{H}om_{\mathsf{grA}}(P, M')$ *and* $k \in \mathscr{H}om_{\mathsf{grA}}(P, M)$.

(2) By the same arguments as in the proof of Lemma 2.20, prove that the complex of abelian groups $\mathscr{H}om_{\mathsf{dgA}}(P, M)$ *is acyclic if M is acyclic.*

(3) Using the previous items together with Exercise 1.32, prove that, given a quasi-isomorphism of complexes $f : M \to N$ *of A-modules, then*

$$\mathscr{H}om_{\mathsf{dgA}}(P, M) \xrightarrow{f_*} \mathscr{H}om_{\mathsf{dgA}}(P, N)$$

is a quasi-isomorphism of complexes of abelian groups.

(ii) Let (I, d) *be bounded above (homological) complex of A-modules such that* I_n *is injective for all* $n \in \mathbb{Z}$.

(1) Let

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

be a short exact sequence of complexes of A-modules. Prove that the short sequence

$$0 \longrightarrow \mathscr{H}om_{\mathsf{dgA}}(M'', I) \xrightarrow{g^*} \mathscr{H}om_{\mathsf{dgA}}(M, I) \xrightarrow{f^*} \mathscr{H}om_{\mathsf{dgA}}(M', I) \longrightarrow 0$$

of complexes of abelian groups is exact, where we recall that $f^*(h) = h \circ_{\mathsf{gr}} f$ and $g^*(k) = k \circ_{\mathsf{gr}} g$, for $h \in \mathscr{H}om_{\mathsf{grA}}(M, I)$ and $k \in \mathscr{H}om_{\mathsf{grA}}(M'', I)$.

(2) *By the same arguments as in the proof of Lemma 2.22, prove that the complex of abelian groups $\mathscr{H}om_{\mathsf{dgA}}(M, I)$ is acyclic if M is acyclic.*

(3) *Using the previous items together with Exercise 1.32, prove that, given a quasi-isomorphism of complexes $f : M \to N$ of A-modules, then*

$$\mathscr{H}om_{\mathsf{dgA}}(N, I) \xrightarrow{f^*} \mathscr{H}om_{\mathsf{dgA}}(M, I)$$

is a quasi-isomorphism of complexes of abelian groups.

(iii) (1) *Let (C, d) be any split exact complex. Prove that $\mathscr{H}om_{\mathsf{dgA}}(C, M)$ and $\mathscr{H}om_{\mathsf{dgA}}(M, C)$ are acyclic for any complex M. Deduce that, given a quasi-isomorphism of complexes $f : M \to N$ of A-modules, then*

$$\mathscr{H}om_{\mathsf{dgA}}(C, M) \xrightarrow{f_*} \mathscr{H}om_{\mathsf{dgA}}(C, N) \quad \text{and} \quad \mathscr{H}om_{\mathsf{dgA}}(N, C) \xrightarrow{f^*} \mathscr{H}om_{\mathsf{dgA}}(M, C)$$

are quasi-isomorphisms of complexes of abelian groups.

(2) *Given any bounded complex (C', d') formed by modules which are neither projective nor injective, consider the cone $C = \mathsf{cone}(\mathrm{id}_{C'})$ of the identity morphism. Using the previous item and Exercise 1.45 deduce that the converse of either item (i), (3), or (ii), (3), stated before, does not hold.*

§2.3. Universal δ-functors

2.32. From now on we assume that $\mathscr{A}$ and $\mathscr{B}$ are two abelian categories (see paragraph 4.99). The reader can assume instead that that a $\mathscr{A}$ and $\mathscr{B}$ are a full subcategories of a category of modules for some ring, such that they are closed under isomorphic objects, finite products and coproducts, and kernels and cokernels.

2.33. Definition. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories. A* ***homological (resp., cohomological) δ-functor*** *from $\mathscr{A}$ to $\mathscr{B}$ is a family of additive functors $T_n : \mathscr{A} \to \mathscr{B}$ (resp., $T^n : \mathscr{A} \to \mathscr{B}$) for $n \in \mathbb{Z}$ such that $T_n = 0$ (resp., $T^n = 0$) for all integers $n < 0$, together with morphisms*

$$\delta_n : T_n(M'') \to T_{n-1}(M') \qquad \left(\text{resp., } \delta^n : T^n(M'') \to T^{n+1}(M') \right)$$

given for any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{A}$. We assume that, given a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[r, "g"] \arrow[d, "\beta"] & M'' \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & N' \arrow[r, "h"] & N \arrow[r, "k"] & N'' \arrow[r] & 0 \end{tikzcd}$$

in $\mathscr{A}$, with exact rows, the diagram

$$\begin{tikzcd} \dots \arrow[r] & T_{n+1}(M'') \arrow[r, "\delta_{n+1}"] \arrow[d, "T_{n+1}(\gamma)"] & T_n(M') \arrow[r, "T_n(f)"] \arrow[d, "T_n(\alpha)"] & T_n(M) \arrow[r, "T_n(g)"] \arrow[d, "T_n(\beta)"] & T_n(M'') \arrow[r, "\delta_n"] \arrow[d, "T_n(\gamma)"] & T_{n-1}(M') \arrow[r, "T_{n-1}(f)"] \arrow[d, "T_{n-1}(\alpha)"] & \dots \\ \dots \arrow[r] & T_{n+1}(N'') \arrow[r, "\delta_{n+1}"] & T_n(N') \arrow[r, "T_n(h)"] & T_n(N) \arrow[r, "T_n(k)"] & T_n(N'') \arrow[r, "\delta_n"] & T_{n-1}(N') \arrow[r, "T_{n-1}(h)"] & \dots \end{tikzcd} \tag{15}$$

$$\left(\text{resp.,}\quad \begin{tikzcd} \dots \arrow[r] & T^{n-1}(M'') \arrow[r, "\delta^{n-1}"] \arrow[d, "T^{n-1}(\gamma)"] & T^{n}(M') \arrow[r, "T^{n}(f)"] \arrow[d, "T^{n}(\alpha)"] & T^{n}(M) \arrow[r, "T^{n}(g)"] \arrow[d, "T^{n}(\beta)"] & T^{n}(M'') \arrow[r, "\delta^{n}"] \arrow[d, "T^{n}(\gamma)"] & T^{n+1}(M') \arrow[r, "T^{n+1}(f)"] \arrow[d, "T^{n+1}(\alpha)"] & \dots \\ \dots \arrow[r] & T^{n-1}(N'') \arrow[r, "\delta^{n-1}"] & T^{n}(N') \arrow[r, "T^{n}(h)"] & T^{n}(N) \arrow[r, "T^{n}(k)"] & T^{n}(N'') \arrow[r, "\delta^{n}"] & T^{n+1}(N') \arrow[r, "T^{n+1}(h)"] & \dots \end{tikzcd}\right)$$

is commutative, with exact rows. In particular, this means that T_0 (resp., T^0) is a right (resp., left) exact functor.

2.34. Exercise. *Let A be a ring, $a \in A$ a fixed element. Define $T_0 = T^1, T_1 = T^0 : {}_A\mathsf{Mod} \to {}_{\mathbb{Z}}\mathsf{Mod}$ be the functors given by $T_0(M) = M/a.M$ and $T_1(M) = \{m \in M : a.m = \mathbf{0}_M\}$ for every A-module M. Moreover, $T_0(f) = \bar{f}$ is the induced map whereas $T_1(f) = f|_{T_1(M)}$ is the restriction, for every morphism of A-modules $f : M \to N$. Define $T_n = T^n = 0$ if $n \in \mathbb{Z} \setminus \{0,1\}$. Prove that this gives a (co)homological δ-functor.*

2.35. Definition. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, and let $T_n, U_n : \mathscr{A} \to \mathscr{B}$ (resp., $T^n, U^n : \mathscr{A} \to \mathscr{B}$) for $n \in \mathbb{N}$ be two homological (resp., cohomological) δ-functors with connecting morphisms $\delta_{T,n}$ and $\delta_{U,n}$ (resp., $\delta^{T,n}$ and $\delta^{U,n}$). A* ***morphism of homological (resp., cohomological) δ-functors*** *is a family of natural transformations $t_n : T_n \to U_n$ (resp., $t^n : T^n \to U^n$) for $n \in \mathbb{Z}$ such that given for any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{A}$ the diagram*

$$\begin{tikzcd} T_n(M'') \arrow[r, "\delta_{T,n}"] \arrow[d, "t_n(M'')"] & T_{n-1}(M') \arrow[d, "t_{n-1}(M')"] \\ U_n(M'') \arrow[r, "\delta_{U,n}"] & U_{n-1}(M') \end{tikzcd} \qquad \left(\text{resp.,}\quad \begin{tikzcd} T^n(M'') \arrow[r, "\delta^{T,n}"] \arrow[d, "t^n(M'')"] & T^{n+1}(M') \arrow[d, "t^{n-1}(M')"] \\ U^n(M'') \arrow[r, "\delta^{U,n}"] & U^{n+1}(M') \end{tikzcd}\right)$$

is commutative.

We say that a homological (resp., cohomological) δ-functor $\{T_n\}_{n\in\mathbb{Z}}$ (resp., $\{T^n\}_{n\in\mathbb{Z}}$) is ***universal*** *if, given a homological (resp., cohomological) δ-functor $\{U_n\}_{n\in\mathbb{Z}}$ (resp., $\{U^n\}_{n\in\mathbb{Z}}$) and a natural transformation $t_0 : U_0 \to T_0$ (resp., $t^0 : T^0 \to U^0$) there exists a unique morphism of homological (resp., cohomological) δ-functors $\{t_n : U_n \to T_n\}_{n\in\mathbb{Z}}$ (resp., $\{t^n : T^n \to U^n\}_{n\in\mathbb{Z}}$).*

2.36. Exercise. *Let $T : \mathscr{A} \to \mathscr{B}$ be an additive functor that is exact. Prove that the family of functors given by $T_0 = T^0 = T$ and $T_n = T^n = 0$ gives a universal δ-functor.*

2.37. Definition. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, and let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. We say that F is* ***coerasable*** *(resp.,* ***erasable****) if for very object $M \in \mathscr{A}$ there exists an epimorphism $\varepsilon : P \to M$ (resp., a monomorphism $\eta : M \to I$) such that $F(\varepsilon) = 0$ (resp., $F(\eta) = 0$).*

2.38. Proposition. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, and let $\{T_n\}_{n\in\mathbb{Z}}$ (resp., $\{T^n\}_{n\in\mathbb{Z}}$) be a homological (resp., cohomological) δ-functor such that T_n is coerasable (resp., T^n is erasable) for all $n \in \mathbb{N}$. Then, $\{T_n\}_{n\in\mathbb{Z}}$ (resp., $\{T^n\}_{n\in\mathbb{Z}}$) is universal.*

Proof. We will prove the case of homological δ-functor and leave the case of the cohomological δ-functor to the reader, which follows *mutatis mutandi*. Let $\{U_n\}_{n\in\mathbb{Z}}$ be a homological δ-functor and $t_0 : U_0 \to T_0$ be a a natural transformation. Assume we have constructed natural transformations $t_i : U_i \to T_i$ for $i \in [\![0, n-1]\!]$ for some $n \in \mathbb{N}$ such that given for any short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{A}$ the diagram

$$\begin{tikzcd} U_i(M'') \arrow[r, "\delta_{U,i}"] \arrow[d, "t_i(M'')"] & U_{i-1}(M') \arrow[d, "t_{i-1}(M')"] \\ T_i(M'') \arrow[r, "\delta_{T,i}"] & T_{i-1}(M') \end{tikzcd}$$

is commutative for $i \in [\![0, n-1]\!]$. Given N in $\mathscr{A}$, since T_n is coerasable, there exists a surjection $g : P \to N$ such that $T_n(g) = 0$. Consider the short exact sequence $0 \to K \to P \to N \to 0$ in $\mathscr{A}$, where $K = \mathsf{Ker}(g)$

and $K \to P$ is the inclusion map, that we denote by i. We then have the commutative diagram

$$\begin{tikzcd}
& U_n(N) \arrow[r, "\delta_{U,n}"] & U_{n-1}(K) \arrow[r, "U_{n-1}(i)"] \arrow[d, "t_{n-1}(K)"] & U_{n-1}(P) \arrow[d, "t_{n-1}(P)"] \\
T_n(P) \arrow[r, "T_n(g)"] & T_n(N) \arrow[r, "\delta_{T,n}"] & T_{n-1}(K) \arrow[r, "T_{n-1}(i)"] & T_{n-1}(P)
\end{tikzcd}$$

with exact rows. Since $0 = t_{n-1}(P) \circ U_{n-1}(i) \circ \delta_{U,n} = T_{n-1}(i) \circ t_{n-1}(K) \circ \delta_{U,n}$, we see that the image of $t_{n-1}(K) \circ \delta_{U,n}$ is included in $\mathsf{Ker}(T_{n-1}(i)) = \mathsf{Img}(\delta_{T,n})$. On the other hand, the identity $T_n(g) = 0$ tells us that $\delta_{T,n}$ is injective, so there exists a unique morphism $t_n(N) : U_n(N) \to T_n(N)$ in $\mathscr{B}$ such that $\delta_{T,n} \circ t_n(N) = t_{n-1}(K) \circ \delta_{U,n}$. The careful reader will verify that this morphism $t_n(N)$ is independent of the choice of surjection $g : P \to N$ such that $T_n(g) = 0$ (this follows from the argument in the next paragraph for $f = \mathsf{id}_N$). We will now prove that this gives a natural transformation $t_n : U_n \to T_n$.

Assume thus that $f : N' \to N$ is a morphism in $\mathscr{A}$. Let $0 \to K \to P \to N \to 0$ be as before. Take $P \times_N N' = \{(p, n') \in P \times N' : g(p) = f(n')\}$ be the pull-back, $\pi_2 : P' \to N'$ the projection onto the second component and $\pi_1 : P' \to P$ is the projection onto the first component. The careful reader will check that $P' \in \mathscr{A}$, since it is the kernel of the morphism $P \times N' \to N$ in $\mathscr{A}$ sending (p, n') to $g(p) - f(n')$. Since T_n is coerasable there exists P' in $\mathscr{A}$ and a surjection $\varepsilon' : P' \to P \times_N N'$ such that $T_n(\varepsilon') = 0$. Let $g' : P' \to N'$ be $\pi_2 \circ \varepsilon'$ and $f' : P' \to P$ be $\pi_1 \circ \varepsilon'$, and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K' \arrow[r, "i'"] \arrow[d, "f'|_{K'}"] & P' \arrow[r, "g'"] \arrow[d, "f'"] & N' \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K \arrow[r, "i"] & P \arrow[r, "g"] & N \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where $K' = \mathsf{Ker}(g')$ and $K' \to P'$ is the inclusion map, that we denote by i'. Hence we get the diagram

$$\begin{tikzcd}
& & U_n(N') \arrow[rr, "\delta_{U,n}"] \arrow[dr, "U_n(f)"] \arrow[dd, dashed, "t_n(N')"] & & U_{n-1}(K') \arrow[rr, "U_{n-1}(i')"] \arrow[dr, "U_{n-1}(f'|_{K'})"] \arrow[dd, "t_{n-1}(K')"] & & U_{n-1}(P') \arrow[dr, "U_{n-1}(f')"] \arrow[dd, "t_{n-1}(P')"] & \\
& & & U_n(N) \arrow[rr, "\delta_{U,n}"] \arrow[dd, dashed, "t_n(N)"] & & U_{n-1}(K) \arrow[rr, "U_{n-1}(i)"] \arrow[dd, "t_{n-1}(K)"] & & U_{n-1}(P) \arrow[dd, "t_{n-1}(P)"] \\
T_n(P') \arrow[rr, "T_n(g')=0"] \arrow[dr, "T_n(f')"] & & T_n(N') \arrow[rr, "\delta_{T,n}"] \arrow[dr, "T_n(f)"] & & T_{n-1}(K') \arrow[rr, "T_{n-1}(i')"] \arrow[dr, "T_{n-1}(f'|_{K'})"] & & T_{n-1}(P') \arrow[dr, "T_{n-1}(f')"] & \\
& T_n(P) \arrow[rr, "T_n(g)=0"] & & T_n(N) \arrow[rr, "\delta_{T,n}"] & & T_{n-1}(K) \arrow[rr, "T_{n-1}(i)"] & & T_{n-1}(P)
\end{tikzcd}$$

Note that $T_n(g') = T_n(\pi_2) \circ T_n(\varepsilon') = 0$. Since all the faces in the previous diagram commute with possible exception to the one including the two dashed arrows, we see that

$$\begin{aligned}
\delta_{T,n} \circ t_n(N) \circ U_n(f) &= t_{n-1}(K) \circ \delta_{U,n} \circ U_n(f) = t_{n-1}(K) \circ U_{n-1}(f'|_{K'}) \circ \delta_{U,n} \\
&= T_{n-1}(f'|_{K'}) \circ t_{n-1}(K') \circ \delta_{U,n} = T_{n-1}(f'|_{K'}) \circ \delta_{T,n} \circ t_n(N') = \delta_{T,n} \circ T_n(f) \circ t_n(N').
\end{aligned}$$

From the injectivity of $\delta_{T,n}$ we get that $t_n(N) \circ U_n(f) = T_n(f) \circ t_n(N')$, as was to be shown.

We will finally prove that the natural transformation $t_n : U_n \to T_n$ satisfies that, given a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{A}$, the diagram

$$\begin{tikzcd}
T_n(M'') \arrow[r, "\delta_{T,n}"] \arrow[d, "t_n(M'')"] & T_{n-1}(M') \arrow[d, "t_{n-1}(M')"] \\
U_n(M'') \arrow[r, "\delta_{U,n}"] & U_{n-1}(M')
\end{tikzcd} \tag{16}$$

is commutative. Since T_n is coerasable there exists P in $\mathscr{A}$ and a surjection $\varepsilon : P \to M$ such that $T_n(\varepsilon) = 0$. We have the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, "i"] \arrow[d, "\tilde{\varepsilon}"] & P \arrow[r, "g\circ\varepsilon"] \arrow[d, "\varepsilon"] & M'' \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0 \end{tikzcd} \tag{17}$$

with exact rows, where $K = \mathsf{Ker}(g \circ \varepsilon)$, $K \to P'$ is the inclusion map, which we denote by i, and $\tilde{\varepsilon}$ is the composition of $\varepsilon|_K$ and the inverse of the isomorphism $f : M' \to \mathsf{Img}(f)$. We further note that $T_n(g \circ \varepsilon) = T_n(g) \circ T_n(\varepsilon) = 0$. Hence, we have the commutative diagram

$$\begin{tikzcd} U_n(M'') \arrow[r, "\delta_{U,n}"] \arrow[d, "t_n(M'')"] & U_{n-1}(K) \arrow[r, "U_{n-1}(\tilde{\varepsilon})"] \arrow[d, "t_{n-1}(K)"] & U_{n-1}(M') \arrow[d, "t_{n-1}(M')"] \\ T_n(M'') \arrow[r, "\delta_{T,n}"] & T_{n-1}(K) \arrow[r, "T_{n-1}(\tilde{\varepsilon})"] & T_{n-1}(M') \end{tikzcd}$$

where the first commutative square follows from the definition of t_n applied to the upper short exact sequence of (17), whereas the second follows from the naturality of t_{n-1}. Since $\{U_n\}_{n\in\mathbb{Z}}$ (resp., $\{T_n\}_{n\in\mathbb{Z}}$) is a δ-functor, applying (15) to (17), we see that $U_{n-1}(\tilde{\varepsilon}) \circ \delta_{U,n}$ (resp., $T_{n-1}(\tilde{\varepsilon}) \circ \delta_{T,n}$) in the previous diagram are precisely $\delta_{U,n}$ (resp., $\delta_{T,n}$) associated to the lower short exact sequence of (17). The commutativity of (16) follows. □

2.39. Exercise. *Let A be a ring and let $\mathscr{A}$ be the subcategory of ${}_A\mathsf{DGMod}$ given by the complexes of A-modules (M, d_M) such that $M_n = 0$ for all $n < 0$.*

- *(i) Prove that $\mathscr{A}$ is abelian.*
- *(ii) Let $H_n : \mathscr{A} \to {}_A\mathsf{Mod}$ be an additive functor sending (M, d_M) to $H_n(M, d_M)$, for $n \in \mathbb{Z}$.*
  - *(1) Prove that the family $\{H_n\}_{n\in\mathbb{Z}}$ is a δ-functor.*
  - *(2) Given (M, d_M) in $\mathscr{A}$, prove that the morphism of complexes of A-modules*

    $$\sigma_{\geq 0}(\delta[1]) : \sigma_{\geq 0}\big(\mathsf{cone}(\mathsf{id}_M)[1]\big) \to \sigma_{\geq 0}(M) = M$$

    is an epimorphism in $\mathscr{A}$, where $\sigma_{\geq 0}$ was introduced in (3) and δ was defined in Exercise 1.32.
  - *(3) Using Exercise 1.39, prove that $H_n(\sigma_{\geq 0}(\mathsf{cone}(\mathsf{id}_M)[1])) = 0$ for all $n \in \mathbb{N}$. Deduce that H_n is coerasable for all $n \in \mathbb{N}$ and, in consequence, the family $\{H_n\}_{n\in\mathbb{Z}}$ is a universal δ-functor.*

§2.4. Derived functors

2.40. All along this section we assume that $\mathscr{A}$ and $\mathscr{B}$ are abelian categories. Moreover, when dealing with left (resp., right) derived functors we will assume that, given $M \in \mathscr{A}$ there exists a projective (resp., injective) object M' in $\mathscr{A}$ and an epimorphism (resp., monomorphism) $M' \to M$ (resp., $M \to M'$). One says in this case that the abelian category $\mathscr{A}$ **has enough projectives** (resp., **injectives**).

2.41. Let $F : \mathscr{A} \to \mathscr{B}$ be a right (resp., left) exact additive functor. Let $\varepsilon_P : (P, d_P) \to M$ (resp., $\eta_I : M \to (I, d_I)$) be a projective (resp., injective) resolution of M such that $P_n \in \mathscr{A}$ (resp., $I_n \in \mathscr{A}$) for all $n \in \mathbb{N}_0$, which exists by the assumption in the previous paragraph. We define the **left** (resp., **right**) **derived functors** $L_nF(M)$ (resp., $R^nF(M)$) at M as $H_n(F(P), F(d_P))$ (resp., $H^n(F(I), F(d_I))$), for $n \in \mathbb{N}_0$. We still have to show that this notion is well defined, which is done in Lemma 2.43. We will prove that they are indeed (additive) functors in Proposition 2.48.

2.42. Remark. *Since the sequence $P_1 \to P_0 \to M \to 0$ (resp., $0 \to M \to I^0 \to I^1$) is exact and F is right (resp., left) exact, then ε_P (resp., η_I) induces an isomorphism $L_0F(M) \cong M$ (resp., $R^0F(M) \cong M$), for all $M \in \mathscr{A}$.*

2.43. Lemma. *Let $F : \mathscr{A} \to \mathscr{B}$ be a right (resp., left) exact additive functor of abelian categories. Consider another projective (resp., injective) resolution $\varepsilon_Q : (Q, d_Q) \to M$ (resp., $\eta_J : M \to (J, d_J)$) of M such that $Q_n \in \mathscr{A}$ (resp., $J_n \in \mathscr{A}$) for all $n \in \mathbb{N}_0$. Then, there exist canonical isomorphisms*

$$H_n\big(F(P), F(d_P)\big) \cong H_n\big(F(Q), F(d_Q)\big) \qquad \Big(\text{resp., } H^n\big(F(I), F(d_I)\big) \cong H^n\big(F(J), F(d_J)\big)\Big)$$

in $\mathscr{B}$ for all $n \in \mathbb{N}_0$.

Proof. By the Comparison Lemma 2.20 (resp., 2.22) there exists a morphisms of complexes $f : P \to Q$ and $g : Q \to P$ (resp., $f : J \to I$ and $g : I \to J$) commuting with the augmentations (resp., coaugmentations), *i.e.* $\varepsilon_Q \circ f_0 = \varepsilon_P$ and $\varepsilon_P \circ g_0 = \varepsilon_Q$ (resp., $f_0 \circ \eta_J = \eta_I$ and $g_0 \circ \eta_I = \eta_J$), unique up to homotopy. Since $\mathrm{id}_P : P \to P$ and $\mathrm{id}_Q : Q \to Q$ (resp., $\mathrm{id}_I : I \to I$ and $\mathrm{id}_J : J \to J$) are morphisms of of complexes commuting with the augmentations (resp., coaugmentations), we conclude that $f \circ g$ is homotopic to id_Q (resp., id_I) and $g \circ f$ is homotopic to id_P (resp., id_J). Since F is additive, we then have that $F(f) \circ F(g) = F(f \circ g)$ is homotopic to $F(\mathrm{id}_Q) = \mathrm{id}_{F(Q)}$ (resp., $F(\mathrm{id}_I) = \mathrm{id}_{F(I)}$) and $F(g) \circ F(f) = F(g \circ f)$ is homotopic to $F(\mathrm{id}_P) = \mathrm{id}_{F(P)}$ (resp., $F(\mathrm{id}_J) = \mathrm{id}_{F(J)}$). Lemma 1.51 tells us that

$$H_h(F(f)) \circ H_n(F(g)) = H_n(F(f) \circ F(g))) = H_n(\mathrm{id}_{F(Q)}) = \mathrm{id}_{H_n(F(Q))}$$

and

$$H_h(F(g)) \circ H_n(F(f)) = H_n(F(g) \circ F(f)) = H_n(\mathrm{id}_{F(P)}) = \mathrm{id}_{H_n(F(P))}$$

(resp., we have the identity

$$H_h(F(f)) \circ H_n(F(g)) = H_n(F(f) \circ F(g)) = H_n(\mathrm{id}_{F(I)}) = \mathrm{id}_{H_n(F(I))}$$

as well as the equality

$$H_h(F(g)) \circ H_n(F(f)) = H_n(F(g) \circ F(f)) = H_n(\mathrm{id}_{F(J)}) = \mathrm{id}_{H_n(F(J))}),$$

which proves the claim. □

2.44. Corollary. *If $M \in \mathscr{A}$ is a projective (resp., injective) object in $\mathscr{A}$, then $L_nF(M) = 0$ (resp., $R^nF(M) = 0$) for all $n \in \mathbb{N}$.*

2.45. Given a right (resp., left) exact additive functor $F : \mathscr{A} \to \mathscr{B}$, we say that an object $M \in \mathscr{A}$ is ***F*-acyclic** if $L_nF(M) = 0$ (resp., $R^nF(M) = 0$) for all $n \in \mathbb{N}$. A left (resp., right) resolution $\varepsilon_P : (P, d_P) \to M$ (resp., $\eta_I : M \to (I, d_I)$) in $\mathscr{A}$ is called ***F*-acyclic** if P_n (resp., I^n) is F-acyclic for all $n \in \mathbb{N}_0$.

2.46. Lemma. *Let $F : \mathscr{A} \to \mathscr{B}$ be a right (resp., left) exact additive functor. Let $f : M \to N$ be a morphism in $\mathscr{A}$. Then, there exists a natural map*

$$L_nF(f) : L_nF(M) \to L_nF(N) \qquad \Big(\text{resp.,} R^nF(f) : R^nF(M) \to R^nF(N)\Big)$$

for all $n \in \mathbb{N}_0$.

Proof. Let $\varepsilon_P : (P, d_P) \to M$ and $\varepsilon_Q : (Q, d_Q) \to N$ (resp., $\eta_I : M \to (I, d_I)$ and $\eta_J : N \to (J, d_J)$) be two projective (resp., injective) resolutions. Then, by the Comparison Lemma 2.20 (resp., 2.22), there is a morphism of complexes $\hat{f} : (P, d_P) \to (Q, d_Q)$ (resp., $\hat{f} : (I, d_I) \to (J, d_J)$) such that $\varepsilon_Q \circ \hat{f}_0 = f \circ \varepsilon_P$ (resp., $\hat{f}_0 \circ \eta_I = \eta_J \circ f$), unique up to homotopy. Then, the map $H_n(F(\hat{f})) : H_n(F(P), F(d_P)) \to H_n(F(Q), F(d_Q))$ (resp., $H^n(F(\hat{f})) : H^n(F(I), F(d_I)) \to H^n(F(J), F(d_J)))$ is independent of $\hat{f}$, and it defines $L_nF(f)$ (resp., $R^nF(f)$). □

2.47. Exercise. *Following with the comments in Remark 2.42, prove that $L_0F(f)$ (resp., $R^0F(f)$) identifies with $F(f)$, by means of the isomorphism $L_0F(M) \cong M$ (resp., $R^0F(M) \cong M$), for all $M \in \mathscr{A}$ in Remark 2.42.*

2.48. Proposition. *Let $F : \mathscr{A} \to \mathscr{B}$ be a right (resp., left) exact additive functor. The correspondence $M \mapsto L_nF(M)$ (resp., $M \mapsto R^nF(M)$) for $M \in \mathscr{A}$ and $f \mapsto L_nF(f)$ (resp., $M \mapsto R^nF(f)$) for a morphism in $\mathscr{A}$, constructed in paragraph 2.41 and Lemma 2.46, is an additive functor for all $n \in \mathbb{N}_0$.*

Proof. Let $f, f' : M \to N$ and $g : N \to O$ be morphisms in $\mathscr{A}$. Let $\varepsilon_P : (P, d_P) \to M$, $\varepsilon_Q : (Q, d_Q) \to N$ and $\varepsilon_R : (R, d_R) \to O$ (resp., $\eta_I : M \to (I, d_I)$, $\eta_J : N \to (J, d_J)$ and $\eta_K : O \to (K, d_K)$) be projective (resp., injective) resolutions.

It is clear that $L_nF(\mathrm{id}_M) = \mathrm{id}_{L_nF(M)}$ for all $n \in \mathbb{N}_0$, since id_P is a morphism of complexes satisfying that $\varepsilon_P \circ \mathrm{id}_{P_0} = \mathrm{id}_M \circ \varepsilon_P$ (resp., $\mathrm{id}_{I_0} \circ \eta_I = \eta_I \circ \mathrm{id}_M$).

Moreover, if we consider morphisms of complexes $\hat{f} : (P, d_P) \to (Q, d_Q)$ and $\hat{g} : (Q, d_Q) \to (R, d_R)$ (resp., $\hat{f} : (I, d_I) \to (J, d_J)$ and $\hat{g} : (J, d_J) \to (K, d_K)$) such that $\varepsilon_Q \circ \hat{f}_0 = f \circ \varepsilon_P$ and $\varepsilon_R \circ \hat{g}_0 = g \circ \varepsilon_Q$ (resp., $\hat{f}_0 \circ \eta_I = \eta_J \circ f$ and $\hat{g}_0 \circ \eta_J = \eta_K \circ g$), then $\hat{g} \circ \hat{f} : (P, d_P) \to (R, d_R)$ (resp., $\hat{g} \circ \hat{f} : (I, d_I) \to (K, d_K)$) satisfies that $\varepsilon_R \circ \hat{g}_0 \circ \hat{f}_0 = g \circ f \circ \varepsilon_P$ (resp., $\hat{g}_0 \circ \hat{f}_0 \circ \eta_I = \eta_K \circ g \circ f$). This implies that $L_nF(g \circ f) = L_nF(g) \circ L_nF(f)$ (resp., $R^nF(g \circ f) = R^nF(g) \circ R^nF(f)$), for all $n \in \mathbb{N}_0$.

Finally, if $\hat{f}' : (P, d_P) \to (Q, d_Q)$ (resp., $\hat{f}' : (I, d_I) \to (J, d_J)$) satisfies that $\varepsilon_Q \circ \hat{f}'_0 = f' \circ \varepsilon_P$ (resp., $\hat{f}'_0 \circ \eta_I = \eta_J \circ f'$), we get that $\hat{f} + \hat{f}' : (P, d_P) \to (Q, d_Q)$ (resp., $\hat{f} + \hat{f}' : (I, d_I) \to (J, d_J)$) satisfies that $\varepsilon_Q \circ (\hat{f}_0 + \hat{f}'_0) = (f + f') \circ \varepsilon_P$ (resp., $(\hat{f}_0 + \hat{f}'_0) \circ \eta_I = \eta_J \circ (f + f')$), so $L_nF(f + f') = L_nF(f) + L_nF(f')$ (resp., $R^nF(f + f') = R^nF(f) + R^nF(f')$), for all $n \in \mathbb{N}_0$. The proposition is thus proved. □

2.49. Exercise. *Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be abelian categories and $F : \mathscr{A} \to \mathscr{B}$ be a right (resp., left) exact additive functor. Let $G : \mathscr{B} \to \mathscr{C}$ be an exact additive functor. Prove that $G \circ F$ is right (resp., left) exact, additive, and there is a natural isomorphism $L_n(G \circ F) \cong G \circ L_nF$ (resp., $R^n(G \circ F) \cong G \circ R^nF$) for all $n \in \mathbb{N}_0$.*

2.50. Theorem. *Let $F : \mathscr{A} \to \mathscr{B}$ be a right (resp., left) exact additive functor. The derived functors $L_nF : \mathscr{A} \to \mathscr{B}$ (resp., $R^nF : \mathscr{A} \to \mathscr{B}$) for $n \in \mathbb{N}_0$ form a universal homological (resp., cohomological) δ-functor.*

Proof. We will prove the case where F is right exact. The fact that $\{L_nF\}_{n \in \mathbb{N}_0}$ is a homological δ-functor follows directly from applying Theorem 1.44 to the morphism of short exact sequences of complexes obtained from applying F to the diagram of projective resolutions constructed in Lemma 2.27. Since the functor L_nF is coerasable for all $n \in \mathbb{N}$, by Corollary 2.44, the homological δ-functor $\{L_nF\}_{n \in \mathbb{N}_0}$ is universal. □

2.51. Exercise. *Let $F : \mathscr{A} \to \mathscr{B}$ be an right (resp., left) exact additive functor. Using Exercise 2.36 and Theorem 2.50, prove that L_nF and R^nF vanish for all $n \in \mathbb{N}$ if F is exact. Conversely, using the exactness of the rows in (15), prove that F is exact if L_nF (resp., R^nF) vanish for all $n \in \mathbb{N}$.*

2.52. Exercise (Dimension shifting)**.** *Let $F : \mathscr{A} \to \mathscr{B}$ be a right (resp., left) exact additive functor, and let*

$$0 \longrightarrow M' \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \dots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence such that X_i is an F-acyclic object of $\mathscr{A}$ for all $i \in [\![0, n]\!]$, for some $n \in \mathbb{N}_0$.

(i) Asume that $n = 0$. Using the diagram (15) with exact rows coming from the fact that the derived functors are δ-functors (see Theorem 2.50), prove that $L_iF(M) \cong L_{i-1}F(M')$ (resp., $R^iF(M') \cong R^{i-1}F(M)$) for all integers $i \geq 2$ and that $L_1F(M)$ (resp., $R^1F(M')$) is the kernel (resp., cokernel) of the morphism $F(M') \to F(X_0)$ (resp., $F(X_0) \to F(M)$).

(ii) Let $n \in \mathbb{N}_0$. Using the previous item, prove that $L_iF(M) \cong L_{i-n-1}F(M')$ (resp., $R^iF(M') \cong R^{i-n-1}F(M)$) for all integers $i \geq n+2$ and that $L_{n+1}F(M)$ (resp., $R^{n+1}F(M')$) is the kernel (resp., cokernel) of the morphism $F(M') \to F(X_n)$ (resp., $F(X_0) \to F(M)$).

(iii) Let $\varepsilon_X : (X, d) \to M$ (resp., $\eta_X : M \to (X, d)$) be any left (resp., right) resolution of M. As in Remark 2.42, notice that $L_0F(M) \cong H_0(F(X), F(d))$ (resp., $R^0F(M) \cong H_0(F(X), F(d))$).

(iv) *Let $\varepsilon_X : (X, d) \to M$ (resp., $\eta_X : M \to (X, d)$) be an F-acyclic left (resp., right) resolution of M. Using the characterisation of $L_{n+1}F(M)$ (resp., $R^{n+1}F(M')$) in the second item for $n \in \mathbb{N}_0$, prove that $L_nF(M) \cong H_n(F(X), F(d))$ (resp., $R^nF(M) \cong H_{-n}(F(X), F(d))$) for $n \in \mathbb{N}$.*

The object M' (resp., M) is called the ***n-th syzygy*** *(resp.,* ***n-th cosyzygy****) of M (resp., M').*

2.53. Exercise. *Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories, and let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. Assume that F has a right (resp., left) adjoint. Prove that there exist natural isomorphisms of functors*

$$L_nF\Big(\coprod_{i\in I} M_i\Big) \cong \coprod_{i\in I} L_nF(M_i)\Bigg(\text{ resp., } R^nF\Big(\prod_{i\in I} M_i\Big) \cong \prod_{i\in I} R^nF(M_i)\Bigg),$$

for all $n \in \mathbb{N}_0$, provided the coproduct (resp., product) on the left exists.

2.54. Exercise. *Let $\mathscr{A}$ and $\mathscr{B}$ be the abelian categories of modules over rings A and B. Let $F : \mathscr{A} \to \mathscr{B}$ be a right exact additive functor that commutes with filtered colimits and such that filtered colimits of projective objects of $\mathscr{A}$ are F-acyclic. Let $G : \mathscr{C} \to \mathscr{A}$ be functor from a filtered small category. Using Exercises 4.106, 2.19, and 2.52, prove that there exist a natural isomorphism*

$$L_nF\Big(\operatorname*{colim}_{\to\mathscr{C}} G\Big) \cong \operatorname*{colim}_{\to\mathscr{C}} L_n(F \circ G),$$

for every $n \in \mathbb{N}_0$.

§3. Lecture III : Tor and Ext groups

§3.1. Basic definitions

3.1. Let A be a ring and M be a right A-module. Consider the functor $F = M \otimes_A (-) : {}_A\mathsf{Mod} \to {}_{\mathbb{Z}}\mathsf{Mod}$. It is easy to see that F is right exact (see Fact 2.10). For $n \in \mathbb{N}_0$, define

$$\mathsf{Tor}_n^A(M, N) = L_nF(N) \tag{18}$$

for all N in ${}_A\mathsf{Mod}$.

3.2. Analogously, if N be a left A-module, we can consider the functor $F' = (-) \otimes_A N : \mathsf{Mod}_A \to {}_{\mathbb{Z}}\mathsf{Mod}$. It is easy to see that F' is right exact (see Fact 2.10). For $n \in \mathbb{N}_0$, define

$$\mathsf{Tor}_n'^A(M, N) = L_nF'(M) \tag{19}$$

for all M in Mod_A. We will prove however that $\mathsf{Tor}_n^A(M, N) \cong \mathsf{Tor}_n'^A(M, N)$, for all $n \in \mathbb{N}_0$ (see Proposition 3.12).

3.3. Let A be a ring and M be a left A-module. Consider the functor $G = \mathsf{Hom}_A(M, -) : {}_A\mathsf{Mod} \to {}_{\mathbb{Z}}\mathsf{Mod}$. Recall that G is left exact (see Fact 2.2). For $n \in \mathbb{N}_0$, define

$$\mathsf{Ext}_A^n(M, N) = R^nG(N) \tag{20}$$

for all N in ${}_A\mathsf{Mod}$.

3.4. Analogously, if N be a left A-module, and consider the functor $G' = \mathsf{Hom}_A(-, N) : \mathsf{Mod}_A^{\mathrm{op}} \to {}_{\mathbb{Z}}\,\mathsf{Mod}$. Recall that G' is left exact (see Fact 2.2). For $n \in \mathbb{N}_0$, define

$$\mathsf{Ext}'^{n}_A(M, N) = R^n G'(M) \tag{21}$$

for all M in ${}_A\,\mathsf{Mod}$. We will prove however that $\mathsf{Ext}^n_A(M, N) \cong \mathsf{Ext}'^{n}_A(M, N)$, for all $n \in \mathbb{N}_0$ (see Proposition 3.13).

3.5. Remark. *Note that if A a commutative ring and M and N are (symmetric bi)modules over A, then both $\mathsf{Tor}^A_n(M, N)$ and $\mathsf{Ext}^n_A(M, N)$ are naturally endowed with a unique structure of (symmetric bi)module over A.*

§3.2. Balancing Tor and Ext groups

3.6. Let $\mathscr{A}$ be an abelian category. A **(homological) double complex** (also called **bicomplex**) is a family $M = \{M_{m,n}\}_{m,n\in\mathbb{Z}}$ of objects in $\mathscr{A}$ together with morphisms $d^h_{m,n} : M_{m,n} \to M_{m-1,n}$ and $d^v_{m,n} : M_{m,n} \to M_{m,n-1}$ in $\mathscr{A}$ satisfying the identities $d^h_{m-1,n} \circ d^h_{m,n} = 0 = d^v_{m,n-1} \circ d^v_{m,n}$ as well as $d^h_{m,n-1} \circ d^v_{m,n} + d^v_{m-1,n} \circ d^h_{m,n} = 0$, for all $m, n \in \mathbb{Z}$ (see Exercise 4.102). The morphisms $d^h_{m,n}$ are called the **horizontal differential**, whereas the morphisms $d^v_{m,n}$ are called the **vertical differential** of the double complex. We will usually denote a double complex by (M, d^h, d^v), and will represent it as follows.

$$\begin{tikzcd}
 & \vdots \arrow[d, "d^v_{m-1,n+2}"] & \vdots \arrow[d, "d^v_{m,n+2}"] & \vdots \arrow[d, "d^v_{m+1,n+2}"] & \\
\cdots & M_{m-1,n+1} \arrow[l, "d^h_{m-1,n+1}"'] \arrow[d, "d^v_{m-1,n+1}"] & M_{m,n+1} \arrow[l, "d^h_{m,n+1}"'] \arrow[d, "d^v_{m,n+1}"] & M_{m+1,n+1} \arrow[l, "d^h_{m+1,n+1}"'] \arrow[d, "d^v_{m+1,n-1}"] & \cdots \arrow[l] \\
\cdots & M_{m-1,n} \arrow[l, "d^h_{m-1,n}"'] \arrow[d, "d^v_{m-1,n}"] & M_{m,n} \arrow[l, "d^h_{m,n}"'] \arrow[d, "d^v_{m,n}"] & M_{m+1,n} \arrow[l, "d^h_{m+1,n}"'] \arrow[d, "d^v_{m+1,n}"] & \cdots \arrow[l] \\
\cdots & M_{m-1,n-1} \arrow[l, "d^h_{m-1,n-1}"'] \arrow[d, "d^v_{m-1,n-1}"] & M_{m,n-1} \arrow[l, "d^h_{m,n-1}"'] \arrow[d, "d^v_{m,n-1}"] & M_{m+1,n-1} \arrow[l, "d^h_{m+1,n-1}"'] \arrow[d, "d^v_{m+1,n-1}"] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd} \tag{22}$$

A double complex M is said to be **bounded** if the set $\{m \in \mathbb{Z} : M_{m,N-m} \neq 0\}$ is finite for every $N \in \mathbb{Z}$.

3.7. Given two double complexes (M, d^h, d^v) and (M', d'^h, d'^v), a **morphism of double complexes** is a family $f = (f_{m,n})_{m,n\in\mathbb{Z}}$ of morphisms in $\mathscr{A}$ with $f_{m,n} : M_{m,n} \to M'_{m,n}$ such that $d'^h_{m,n} \circ f_{m,n} = f_{m-1,n} \circ d^h_{m,n}$ and $d'^v_{m,n} \circ f_{m,n} = f_{m,n-1} \circ d^v_{m,n}$ for all $m, n \in \mathbb{Z}$. Moreover, if f is a morphism of double complexes from M' to M and g is a morphism of double complexes from M to M'', we define their composition $g \circ f$ as the morphism of double complexes given by $(g_{m,n} \circ f_{m,n})_{m,n\in\mathbb{Z}}$.

3.8. Exercise. *Let $\mathscr{A}$ be an abelian category.*

(i) Prove that the class of all double complexes together with morphisms of double complexes and their composition forms a category, that we denote by $\mathsf{BiCh}(\mathscr{A})$.

(ii) Let $F = (F_0, F_1) : \mathsf{Ch}(\mathsf{Ch}(\mathscr{A})) \to \mathsf{BiCh}(\mathscr{A})$ be the assignement given as follows. If $M = \{M_m\}_{m\in\mathbb{Z}}$ is a complex with differential $\partial_m : M_m \to M_{m-1}$ such that $M_m = \{M_{m,n}\}_{n\in\mathbb{Z}}$ is a complex with differential $\partial'_{m,n} : M_{m,n} \to M_{m,n-1}$ for all $m \in \mathbb{Z}$, define $F_0(M)$ as the bicomplex $\{M_{m,n}\}_{m,n\in\mathbb{Z}}$ with differentials $d^h_{m,n} = \partial_m|_{M_{m,n}}$ and $d^v_{m,n} = (-1)^m \partial'_{m,n}$. Moreover, if $f : M \to N$ is a morphism of complexes of objects in $\mathsf{Ch}(\mathscr{A})$, define $F(f)$ as the morphism of bicomplexes $\{f_m|_{M_{m,n}}\}_{m,n\in\mathbb{Z}}$. Prove that F is a functor equivalence.

3.9. Exercise. *Let $\mathscr{A}$ be the abelian category of modules over a ring.*

(i) *Given a double complex* (M, d^h, d^v) *in* $\mathsf{BiCh}(\mathscr{A})$ *define the object* $\mathsf{Tot}^{\prod}(M)$ *in* $\mathsf{Ch}(\mathscr{A})$ *given by*

$$\mathsf{Tot}^{\prod}(M)_N = \prod_{n \in \mathbb{Z}} M_{n,N-n}$$

with the differential sending $(x_{n,N-n})_{n\in\mathbb{Z}}$ *to* $(d^h(x_{n+1,N-1-n}) + d^v(x_{n,N-n}))_{n\in\mathbb{Z}}$, *for all* $N \in \mathbb{Z}$. *Prove that this gives indeed a complex.*

(ii) *Given a morphism* $(M, d^h, d^v) \to (M', d'^h, d'^v)$ *of double complexes, define the morphism of complexes*

$$\mathsf{Tot}^{\prod}(f) : \mathsf{Tot}^{\prod}(M) \to \mathsf{Tot}^{\prod}(M')$$

sending $(x_{n,N-n})_{n\in\mathbb{Z}}$ *to* $(f(x_{n,N-n}))_{n\in\mathbb{Z}}$, *for all* $N \in \mathbb{Z}$. *Prove that this gives indeed a morphism of complexes.*

(iii) *Prove that the previous construction defines a functor*

$$\mathsf{Tot}^{\prod} : \mathsf{BiCh}(\mathscr{A}) \to \mathsf{Ch}(\mathscr{A}).$$

(iv) *Given a double complex* (M, d^h, d^v) *in* $\mathsf{BiCh}(\mathscr{A})$ *define the object* $\mathsf{Tot}^{\oplus}(M)$ *in* $\mathsf{Ch}(\mathscr{A})$ *given by*

$$\mathsf{Tot}^{\oplus}(M)_N = \bigoplus_{n \in \mathbb{Z}} M_{n,N-n}$$

with the differential sending $\sum_{n\in\mathbb{Z}} x_{n,N-n})_{n\in\mathbb{Z}}$ *to* $\sum_{n\in\mathbb{Z}}(d^h(x_{n+1,N-1-n}) + d^v(x_{n,N-n}))$, *for all* $N \in \mathbb{Z}$, *where the sums have finite support. Prove that this gives indeed a complex.*

(v) *Given a morphism* $(M, d^h, d^v) \to (M', d'^h, d'^v)$ *of double complexes, define the morphism of complexes*

$$\mathsf{Tot}^{\oplus}(f) : \mathsf{Tot}^{\oplus}(M) \to \mathsf{Tot}^{\oplus}(M')$$

sending $\sum_{n\in\mathbb{Z}} x_{n,N-n})_{n\in\mathbb{Z}}$ *to* $\sum_{n\in\mathbb{Z}} f(x_{n,N-n}))_{n\in\mathbb{Z}}$, *for all* $N \in \mathbb{Z}$, *where the sums have finite support. Prove that this gives indeed a morphism of complexes.*

(vi) *Prove that the previous construction defines a functor*

$$\mathsf{Tot}^{\oplus} : \mathsf{BiCh}(\mathscr{A}) \to \mathsf{Ch}(\mathscr{A}).$$

The careful reader can verify that the previous definitions make sense in any abelian category having products and coproducts.

3.10. Lemma (Acyclic Assembly Lemma). *Let* (M, d^h, d^v) *be a double complex with entries in the category* ${}_A\mathsf{Mod}$, *for some ring* A. *Recall that a double complex has exact rows (resp., columns) if* $\mathsf{Ker}(d^h_{m,n}) = \mathsf{Im}(d^h_{m+1,n})$ *(resp.,* $\mathsf{Ker}(d^v_{m,n}) = \mathsf{Im}(d^v_{m,n+1})$*), for all* $m, n \in \mathbb{Z}$. *If one of the following two conditions is satisfied:*

(DC.1) *M is an essentially upper half-plane double complex (i.e. there exists* $n_0 \in \mathbb{Z}$ *such that* $M_{m,n} = 0$ *for all* $n < n_0$*) with exact columns (resp., exact rows),*

(DC.2) *M is an essentially right half-plane double complex (i.e. there exists* $m_0 \in \mathbb{Z}$ *such that* $M_{m,n} = 0$ *for all* $m < m_0$*) with exact rows (resp., exact columns),*

then $\mathsf{Tot}^{\prod}(M)$ *(resp.,* $\mathsf{Tot}^{\oplus}(M)$*) is acyclic.*

Proof. It is clear that a symmetry along the diagonal axis in (22) interchanges conditions (DC.1) and (DC.2) for either the case of products or direct sums.

Moreover, we claim that it suffices to prove the statement under the assumption (DC.1) for the case of direct products. Indeed, assume that if a double complex N satisfies (DC.1) for the case of direct products, then $\mathsf{Tot}^{\prod}(N)$ is acyclic. Let M be an essentially right half-plane double complex with exact columns. Given $N \in \mathbb{Z}$ define the double complex $\tau_N(M)$ given by

$$\tau_N(M)_{m,n} = \begin{cases} M_{m,n}, & \text{if } n > N, \\ \mathsf{Ker}(d^v_{m,N}), & \text{if } n = N, \\ 0, & \text{if } n < N, \end{cases}$$

with the differentials given by those of M restricted to $\tau_N(M)$. It is clear that $\tau_N(M)$ is an essentially upper half-plane double complex, so $\tau_N(M)$ is bounded, which implies that $\mathsf{Tot}^{\oplus}(\tau_N(M)) = \mathsf{Tot}^{\prod}(\tau_N(M))$. Moreover, it is direct that $\tau_N(M)$ has exact columns. By our assumption, $\mathsf{Tot}^{\prod}(\tau_N(M)) = \mathsf{Tot}^{\oplus}(\tau_N(M))$ is acyclic, for all $N \in \mathbb{Z}$. The careful reader will check that the map sending $N \in \mathbb{Z}$ to $\mathsf{Tot}^{\oplus}(\tau_N(M))$ lifts to a functor from the category associated with the partially ordered set $\mathbb{Z}$ with the order reversed to ${}_A\,\mathsf{DGMod}$. Since $\mathsf{Tot}^{\oplus}(M)$ is the filtered colimit of the previous functor, and taking homology commutes with filtered colimits (see Exercise 4.106), we conclude that $\mathsf{Tot}^{\oplus}(M)$ is acyclic, as was to be shown.

Let us now prove the statement for the case of direct products under the assumption (DC.1). We can assume without loss of generality that M is an upper half double complex, *i.e.* $M_{m,n} = 0$ for all $n < 0$. Let $x = (x_{N-n,n})_{n \in \mathbb{N}_0}$ be an element in the kernel of differential of $\mathsf{Tot}^{\prod}(M)_N$. We will construct an element $y = (y_{N+1-n,n})_{n \in \mathbb{N}_0}$ in $\mathsf{Tot}^{\prod}(M)_{N+1}$ whose image under the differential is exactly x. Set $y_{N+1,0} = \mathbf{0}_{M_{N+1,0}}$. Assume we have constructed $\{y_{N+1-i,i} : i \in [\![0, \ell-1]\!]\}$ for some $\ell \in \mathbb{N}$ such that $y_{N+1,0} = \mathbf{0}_{M_{N+1,0}}$, $y_{N+1-i,i} \in M_{N+1-i,i}$ and

$$d^h_{N+1-i,i}(y_{N+1-i,i}) + d^v_{N-i,i+1}(y_{N-i,i+1}) = x_{N-i,i}$$

for all $i \in [\![0, \ell-2]\!]$. Taking into account that

$$\begin{aligned} d^v_{N-\ell,\ell}\big(x_{N-\ell,\ell} - d^h_{N+1-\ell,\ell}(y_{N+1-\ell,\ell})\big) &= d^v_{N-\ell,\ell}(x_{N-\ell,\ell}) - d^v_{N-\ell,\ell}\big(d^h_{N+1-\ell,\ell}(y_{N+1-\ell,\ell})\big) \\ &= d^v_{N-\ell,\ell}(x_{N-\ell,\ell}) + d^h_{N-\ell,\ell-1}\big(d^v_{N+1-\ell,\ell}(y_{N+1-\ell,\ell})\big) \\ &= d^v_{N-\ell,\ell}(x_{N-\ell,\ell}) + d^h_{N-\ell,\ell-1}\big(x_{N+1-\ell,\ell-1} - d^h_{N+2-\ell,\ell-1}(y_{N+2-\ell,\ell-1})\big) \\ &= d^v_{N-\ell,\ell}(x_{N-\ell,\ell}) + d^h_{N-\ell,\ell-1}(x_{N+1-\ell,\ell-1}) = \mathbf{0}_{M_{N-\ell,\ell-1}} \end{aligned}$$

and the fact that columns are exact, we see there exists $y_{N+1-\ell,\ell} \in M_{N+1-\ell,\ell}$ such that

$$d^h_{N+1-\ell,\ell}(y_{N+1-\ell,\ell}) + d^v_{N-\ell,\ell+1}(y_{N-\ell,\ell+1}) = x_{N-\ell,\ell}.$$

The result thus follows. □

3.11. Exercise. *Let M be the double complex with entries in ${}_{\mathbb{Z}}\,\mathsf{Mod}$ such that $M_{m,n} = \mathbb{Z}/4.\mathbb{Z}$ for all $(m,n) \in \mathbb{Z}^2$, and with the differentials $d^h_{m,n}$ and $d^v_{m,n}$ given by multiplication by 2 for all $(m,n) \in \mathbb{Z}$. Let M' be the double complex given by $M'_{m,n} = \mathbb{Z}/4.\mathbb{Z}$ if $n \geq 0$ and $M'_{m,n} = 0$ if $n < 0$, together with the differentials given by the restriction of those pf M to M'.*

(i) Prove that $H_N(\mathsf{Tot}^{\prod}(M')) = \mathbb{Z}/2.\mathbb{Z}$, for all $N \in \mathbb{Z}$.

(ii) Prove that $\mathsf{Tot}^{\oplus}(M')$ and $\mathsf{Tot}^{\oplus}(M)$ are acyclic.

(iii) Prove that $\mathsf{Tot}^{\prod}(M)$ is not acyclic.

3.12. Proposition. *Concerning the definitions given in (18) and (19), there exists a natural isomorphism*

$$\mathsf{Tor}^A_n(M,N) \cong \mathsf{Tor}'^A_n(M,N)$$

in M and N, for all $n \in \mathbb{N}_0$.

Proof. Let $\varepsilon_P : (P, d_P) \to M$ and $\varepsilon_Q : (Q, d_Q) \to N$ be projective resolutions of right and left A-modules, respectively. Define C as the double complex given by $C_{m,n} = P_m \otimes_A Q_n$ for $m, n \in \mathbb{N}_0$, and $C_{m,n} = 0$ if $m < 0$ or $n < 0$, with the differentials $d^h_{m,n} = d_{P,m} \otimes_A \mathsf{id}_{Q_n}$ and $d^v_{m,n} = (-1)^m \mathsf{id}_{P_m} \otimes_A d_{Q,n}$ for $(m, n) \in \mathbb{N}_0^2$ such that $m + n > 0$. We also consider the complex C' given by $C'_n = M \otimes_A Q_n$ for $n \in \mathbb{N}_0$ endowed with the differential $\mathsf{id}_M \otimes_A d_{Q,n}$ for $n \in \mathbb{N}$, as well as the complex C'' given by $C''_m = P_m \otimes_A N$ for $m \in \mathbb{N}_0$ endowed with the differential $d_{P,m} \otimes_A \mathsf{id}_N$ for $m \in \mathbb{N}$. We may depict this as follows.

$$
\begin{tikzcd}
\vdots \arrow[d, "\mathsf{id}_M \otimes_A d_{Q,4}"] & \vdots \arrow[d, "\mathsf{id}_{P_0} \otimes_A d_{Q,4}"] & \vdots \arrow[d, "-\mathsf{id}_{P_1} \otimes_A d_{Q,4}"] & \vdots \arrow[d, "\mathsf{id}_{P_2} \otimes_A d_{Q,4}"] & \vdots \arrow[d, "-\mathsf{id}_{P_3} \otimes_A d_{Q,4}"] & \\
M \otimes_A Q_3 \arrow[d, "\mathsf{id}_M \otimes_A d_{Q,3}"] & P_0 \otimes_A Q_3 \arrow[l, "\varepsilon_P \otimes_A \mathsf{id}_{Q_3}"'] \arrow[d, "\mathsf{id}_{P_0} \otimes_A d_{Q,3}"] & P_1 \otimes_A Q_3 \arrow[l, "d_{P,1} \otimes_A \mathsf{id}_{Q_3}"'] \arrow[d, "-\mathsf{id}_{P_1} \otimes_A d_{Q,3}"] & P_2 \otimes_A Q_3 \arrow[l, "d_{P,2} \otimes_A \mathsf{id}_{Q_3}"'] \arrow[d, "\mathsf{id}_{P_2} \otimes_A d_{Q,3}"] & P_3 \otimes_A Q_3 \arrow[l, "d_{P,3} \otimes_A \mathsf{id}_{Q_3}"'] \arrow[d, "-\mathsf{id}_{P_3} \otimes_A d_{Q,3}"] & \cdots \arrow[l, "d_{P,4} \otimes_A \mathsf{id}_{Q_3}"'] \\
M \otimes_A Q_2 \arrow[d, "\mathsf{id}_M \otimes_A d_{Q,2}"] & P_0 \otimes_A Q_2 \arrow[l, "\varepsilon_P \otimes_A \mathsf{id}_{Q_2}"'] \arrow[d, "\mathsf{id}_{P_0} \otimes_A d_{Q,2}"] & P_1 \otimes_A Q_2 \arrow[l, "d_{P,1} \otimes_A \mathsf{id}_{Q_2}"'] \arrow[d, "-\mathsf{id}_{P_1} \otimes_A d_{Q,2}"] & P_2 \otimes_A Q_2 \arrow[l, "d_{P,2} \otimes_A \mathsf{id}_{Q_2}"'] \arrow[d, "\mathsf{id}_{P_2} \otimes_A d_{Q,2}"] & P_3 \otimes_A Q_2 \arrow[l, "d_{P,3} \otimes_A \mathsf{id}_{Q_2}"'] \arrow[d, "-\mathsf{id}_{P_3} \otimes_A d_{Q,2}"] & \cdots \arrow[l] \\
M \otimes_A Q_1 \arrow[d, "\mathsf{id}_M \otimes_A d_{Q,1}"] & P_0 \otimes_A Q_1 \arrow[l, "\varepsilon_P \otimes_A \mathsf{id}_{Q_1}"'] \arrow[d, "\mathsf{id}_{P_0} \otimes_A d_{Q,1}"] & P_1 \otimes_A Q_1 \arrow[l, "d_{P,1} \otimes_A \mathsf{id}_{Q_1}"'] \arrow[d, "-\mathsf{id}_{P_1} \otimes_A d_{Q,1}"] & P_2 \otimes_A Q_1 \arrow[l, "d_{P,2} \otimes_A \mathsf{id}_{Q_1}"'] \arrow[d, "\mathsf{id}_{P_2} \otimes_A d_{Q,1}"] & P_3 \otimes_A Q_1 \arrow[l, "d_{P,3} \otimes_A \mathsf{id}_{Q_1}"'] \arrow[d, "-\mathsf{id}_{P_3} \otimes_A d_{Q,1}"] & \cdots \arrow[l, "d_{P,4} \otimes_A \mathsf{id}_{Q_1}"'] \\
M \otimes_A Q_0 & P_0 \otimes_A Q_0 \arrow[l, "\varepsilon_P \otimes_A \mathsf{id}_{Q_0}"'] \arrow[d, "\mathsf{id}_{P_0} \otimes_A \varepsilon_Q"] & P_1 \otimes_A Q_0 \arrow[l, "d_{P,1} \otimes_A \mathsf{id}_{Q_0}"'] \arrow[d, "\mathsf{id}_{P_1} \otimes_A \varepsilon_Q"] & P_2 \otimes_A Q_0 \arrow[l, "d_{P,2} \otimes_A \mathsf{id}_{Q_0}"'] \arrow[d, "\mathsf{id}_{P_2} \otimes_A \varepsilon_Q"] & P_3 \otimes_A Q_0 \arrow[l, "d_{P,3} \otimes_A \mathsf{id}_{Q_0}"'] \arrow[d, "\mathsf{id}_{P_3} \otimes_A \varepsilon_Q"] & \cdots \arrow[l, "d_{P,3} \otimes_A \mathsf{id}_{Q_0}"'] \\
& P_0 \otimes_A N & P_1 \otimes_A N \arrow[l, "d_{P,1} \otimes_A \mathsf{id}_N"'] & P_2 \otimes_A N \arrow[l, "d_{P,2} \otimes_A \mathsf{id}_N"'] & P_3 \otimes_A N \arrow[l, "d_{P,3} \otimes_A \mathsf{id}_N"'] & \cdots \arrow[l, "d_{P,4} \otimes_A \mathsf{id}_N"']
\end{tikzcd}
$$

It is easy to verify that $\varepsilon_P \otimes_A \mathsf{id}_Q$ induces a morphism of complexes $\mathsf{Tot}^{\oplus}(C) \to C'$, that we denote by ε', and $\mathsf{id}_Q \otimes_A \varepsilon_Q$ induces a morphism of complexes $\mathsf{Tot}^{\oplus}(C) \to C''$, that we denote by ε''. To prove the statement, it suffices to show that ε' and ε'' are quasi-isomorphisms. By Exercise 1.45, (C0.2), it suffices to show that $\mathsf{cone}(\varepsilon')$ and $\mathsf{cone}(\varepsilon'')$ are acyclic complexes. We will prove it for ε' and leave the case of ε'' to the reader, since it follows the exact same pattern.

Consider the double complex D given by $D_{m,n} = P_m \otimes_A Q_n$ for $m, n \in \mathbb{N}_0$, $D_{-1,n} = M \otimes_A Q_n$ if $n \in \mathbb{N}_0$, and zero otherwise, together with the differentials $d^h_{m,n} = d_{P,m} \otimes_A \mathsf{id}_{Q_n}$ for $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$, $d^h_{0,n} = \varepsilon_P \otimes_A \mathsf{id}_{Q_n}$ for $n \in \mathbb{N}_0$, and $d^v_{m,n} = (-1)^m \mathsf{id}_{P_m} \otimes_A d_{Q,n}$ $m \in \mathbb{N}_0$ and $n \in \mathbb{N}$, and $d^v_{-1,n} = -\mathsf{id}_M \otimes_A d_{Q,n}$ for $n \in \mathbb{N}$. The careful reader can check that the cone of ε' is precisely $\mathsf{Tot}^{\oplus}(D)[1]$. We also note that the rows of the double complex D are exact. Indeed, the n-th row of D for $n \in \mathbb{N}_0$ is obtained by applying $(-) \otimes_A Q_n$ to the exact complex $\mathsf{cone}(\varepsilon_P)$ obtained from $\varepsilon_P : (P, d_P) \to M$, seen as a morphism of complexes where M is regarded to be concentrated in degree zero. Since Q_n is projective, the functor $(-) \otimes_A Q_n$ is exact, so the n-th row of D is exact. By the Acyclic Assembly Lemma 3.10, $\mathsf{Tot}^{\oplus}(D)[1]$ is exact, so ε' is a quasi-isomorphism, as was to be shown. □

3.13. Proposition. *Concerning the definitions given in* (20) *and* (21)*, there exists a natural isomorphism*

$$\mathsf{Ext}^n_A(M, N) \cong \mathsf{Ext}'^n_A(M, N)$$

in M and N, for all $n \in \mathbb{N}_0$.

Proof. The proof follows the same pattern as in the previous case, but it contains some minor differences, so we explain the main ingredients. Let $\varepsilon_P : (P, d_P) \to M$ be a projective resolution of left A-modules and let $\eta_I : N \to (Q, d_Q)$ be an injective resolutions of left A-modules. Define C as the double complex given by $C_{-m,-n} = \mathsf{Hom}_A(P_m, I_{-n})$ for $m, n \in \mathbb{N}_0$, and $C_{-m,-n} = 0$ if $m < 0$ or $n < 0$, with the differentials

$d^h_{-m,-n}(f) = f \circ d_{P,m+1}$ and $d^v_{-m,-n}(f) = (-1)^{m+n+1} d_{I,-n} \circ f$ for $(-m,-n) \in \mathbb{N}_0^2$. It is easy to verify that this indeed defines a double complex. We also consider the complex C' given by $C'_n = \mathrm{Hom}_A(M, I_n)$ for $-n \in \mathbb{N}_0$ endowed with the differential $d'_{-n}(f) = (-1)^n d_{I,-n} \circ f$ for $n \in \mathbb{N}_0$, as well as the complex C'' given by $C''_{-m} = \mathrm{Hom}_A(P_m, N)$ for $m \in \mathbb{N}_0$ endowed with the differential $d_{-m}(f) = f \circ d_{P,m+1}$ for $m \in \mathbb{N}_0$. We depict it as follows.

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
\mathrm{Hom}_A(M,I_3) \arrow[u, "d'_{-3}"] \arrow[r, "\varepsilon_P^*"] & \mathrm{Hom}_A(P_0,I_{-3}) \arrow[u, "d^v_{0,-3}"] \arrow[r, "d^h_{0,-3}"] & \mathrm{Hom}_A(P_1,I_{-3}) \arrow[u, "d^v_{-1,-3}"] \arrow[r, "d^h_{-1,-3}"] & \mathrm{Hom}_A(P_2,I_{-3}) \arrow[u, "d^v_{-2,-3}"] \arrow[r, "d^h_{-2,-3}"] & \mathrm{Hom}_A(P_3,I_{-3}) \arrow[u, "d^v_{-3,-3}"] \arrow[r, "d^h_{-3,-3}"] & \dots \\
\mathrm{Hom}_A(M,I_{-2}) \arrow[u, "d'_{-2}"] \arrow[r, "\varepsilon_P^*"] & \mathrm{Hom}_A(P_0,I_{-2}) \arrow[u, "d^v_{0,-2}"] \arrow[r, "d^h_{0,-2}"] & \mathrm{Hom}_A(P_1,I_{-2}) \arrow[u, "d^v_{-1,-2}"] \arrow[r, "d^h_{-1,-2}"] & \mathrm{Hom}_A(P_2,I_{-2}) \arrow[u, "d^v_{-2,-2}"] \arrow[r, "d^h_{-2,-2}"] & \mathrm{Hom}_A(P_3,I_{-2}) \arrow[u, "d^v_{-3,-2}"] \arrow[r, "d^h_{-3,-2}"] & \dots \\
\mathrm{Hom}_A(M,I_{-1}) \arrow[u, "d'_{-1}"] \arrow[r, "\varepsilon_P^*"] & \mathrm{Hom}_A(P_0,I_{-1}) \arrow[u, "d^v_{0,-1}"] \arrow[r, "d^h_{0,-1}"] & \mathrm{Hom}_A(P_1,I_{-1}) \arrow[u, "d^v_{-1,-1}"] \arrow[r, "d^h_{-1,-1}"] & \mathrm{Hom}_A(P_2,I_{-1}) \arrow[u, "d^v_{-2,-1}"] \arrow[r, "d^h_{-2,-1}"] & \mathrm{Hom}_A(P_3,I_{-1}) \arrow[u, "d^v_{-3,-1}"] \arrow[r, "d^h_{-3,-1}"] & \dots \\
\mathrm{Hom}_A(M,I_0) \arrow[u, "d'_0"] \arrow[r, "\varepsilon_P^*"] & \mathrm{Hom}_A(P_0,I_0) \arrow[u, "d^v_{0,0}"] \arrow[r, "d^h_{0,0}"] & \mathrm{Hom}_A(P_1,I_0) \arrow[u, "d^v_{-1,0}"] \arrow[r, "d^h_{-1,0}"] & \mathrm{Hom}_A(P_2,I_0) \arrow[u, "d^v_{-2,0}"] \arrow[r, "d^h_{-2,0}"] & \mathrm{Hom}_A(P_3,I_0) \arrow[u, "d^v_{-3,0}"] \arrow[r, "d^h_{-3,0}"] & \dots \\
 & \mathrm{Hom}_A(P_0,N) \arrow[u, "(\eta_I)_*"] \arrow[r, "d''_0"] & \mathrm{Hom}_A(P_1,N) \arrow[u, "(\eta_I)_*"] \arrow[r, "d''_{-1}"] & \mathrm{Hom}_A(P_2,N) \arrow[u, "(\eta_I)_*"] \arrow[r, "d''_{-2}"] & \mathrm{Hom}_A(P_3,N) \arrow[u, "(\eta_I)_*"] \arrow[r, "d''_{-3}"] & \dots
\end{tikzcd}
$$

It is easy to verify that ε_P induces a morphism of complexes $C' \to \mathrm{Tot}^{\prod}(C)$, that we denote by η', and η_I induces a morphism of complexes $C'' \to \mathrm{Tot}^{\prod}(C)$, that we denote by η''. To prove the statement, it suffices to show that η' and η'' are quasi-isomorphisms. By Exercise 1.45, (Co.2), it suffices to show that $\mathrm{cone}(\eta')$ and $\mathrm{cone}(\eta'')$ are acyclic complexes. We will prove it for η' and leave the case of η'' to the reader, since it follows the exact same pattern.

Consider the double complex D given by $D_{-m,-n} = \mathrm{Hom}_A(P_m, I_{-n})$ for $m,n \in \mathbb{N}_0$, together with $D_{1,n} = \mathrm{Hom}_A(M, I_n)$ if $-n \in \mathbb{N}_0$, and zero otherwise. The differentials are $d^h_{-m,-n}(f) = f \circ d_{P,m+1}$ and $d^v_{-m,-n}(f) = (-1)^{m+n+1} d_{I,-n} \circ f$ for $-m,-n \in \mathbb{N}_0$, together with $d^v_{1,-n}(f) = (-1)^n d_{I,-n} \circ f$ and $d^h_{1,-n}(f) = f \circ \varepsilon_P$ for $-n \in \mathbb{N}_0$. The careful reader can check that the cone of η' is precisely $\mathrm{Tot}^{\oplus}(D)$. We also note that the rows of the double complex D are exact. Indeed, the n-th row of D for $n \in \mathbb{N}_0$ is obtained by applying $\mathrm{Hom}_A(-, I_n)$ to the exact complex $\mathrm{cone}(\varepsilon_P)$ obtained from $\varepsilon_P : (P, d_P) \to M$, seen as a morphism of complexes where M is regarded to be concentrated in degree zero. Since I_n is injective, the functor $\mathrm{Hom}_A(-, I_n)$ is exact, so the n-th row of D is exact. By the Acyclic Assembly Lemma 3.10, $\mathrm{Tot}^{\oplus}(D)$ is exact, so η' is a quasi-isomorphism, as was to be shown. □

3.14. Exercise. *Let A be a ring and let M be a right A-module and N a left A-module. Let $F : {}_A\mathrm{Mod} \to \mathrm{Ab}$ be the functor $M \otimes_A (-)$ and $G : \mathrm{Mod}_A \to \mathrm{Ab}$ be the functor $(-) \otimes_A N$. Using the natural isomorphism $L_n F(N) \cong L_n G(M)$ for $n \in \mathbb{N}_0$ proved in Proposition 3.12 and Exercise 2.51, prove that N is F-acyclic if it is flat.*

3.15. Combining Exercises 2.54 and 3.14 together with the fact tensor products commute with artbitrary colimits on each side by Exercise 4.85 and Theorem 4.86, we get the following result. The analogous result with a colimit on the second argument also holds.

3.16. Proposition. *Let A be a ring and let $F : \mathscr{C} \to {}_A\mathrm{Mod}$ be a functor whose domain is a filtered small category. We have the natural isomorphism*

$$\mathrm{Tor}_n^A \left(\underset{\to \mathscr{C}}{\mathrm{colim}} F, M \right) \cong \underset{\to \mathscr{C}}{\mathrm{colim}}\, \mathrm{Tor}_n^A \left(F(-), M \right),$$

in M, for every left A-module M.

§3.3. Some calculations and properties

3.17. The name of the group (18) comes from the fact that this group measures the torsion of certain modules, as the following exercise shows.

3.18. Let A be a commutative ring and let $a \in A$ be a fixed nonzero element such that a is a nonzerodivisor, *i.e.* $\{b \in A : ab = \mathbf{0}_A\} = \{\mathbf{0}_A\}$. For a left A-module M, recall that ${}_aM = \{m \in M : am = \mathbf{0}_M\}$ is the **a-torsion** of M. Moreover, if A is a domain recall that

$$\mathsf{t}(M) = \{m \in M : \text{ there exists } b \in A \setminus \{0\} \text{ such that } bm = \mathbf{0}_M\} = \bigcup_{a \in A\setminus\{\mathbf{0}_A\}} {}_aM$$

is the **torsion** of M. Note that the torsion of $M/\mathsf{t}(M)$ vanishes. We say that a module is **torsion** if $M = \mathsf{t}(M)$. For the following exercises, recall that the torsion groups are naturally A-modules, as noted in Remark 3.5.

3.19. Exercise (More on torsion). *Let M be an abelian group and $p \in \mathbb{N}$ a prime integer. We will denote by $\mathscr{P} \subseteq \mathbb{N}$ the set of primes of $\mathbb{Z}$. Define*

$$\mathsf{t}_p(M) = \{m \in M : \text{ there exists } n \in \mathbb{N} \text{ such that } p^n m = \mathbf{0}_M\} = \bigcup_{n \in \mathbb{N}} {}_{p^n}M.$$

(i) *Prove that the family of abelian subgroups $\{\mathsf{t}_p(M) : p \in \mathscr{P}\}$ of M is independent, i.e. given any finite set $I \subseteq \mathscr{P}$ and elements $m_p \in \mathsf{t}_p(M)$ for all $p \in I$, the condition $\sum_{p \in I} m_p = \mathbf{0}_M$ implies that $m_p = \mathbf{0}_M$ for all $p \in I$.*

(ii) *Prove that given a nonzero element $m \in \mathsf{t}(M)$, there is a finite nonempty set $I \subseteq \mathscr{P}$ and a map $r : I \to \mathbb{N}$ such that $n = \prod_{p \in I} p^{r(p)} \in \mathbb{N}$, $n.m = \mathbf{0}_M$ and $\prod_{p \in I'} p^{r(p)} m \neq \mathbf{0}_M$ for all strict subsets $I' \subsetneq I$.*

(iii) *Deduce from the previous items that*

$$\mathsf{t}(M) = \bigoplus_{p \in \mathscr{P}} \mathsf{t}_p(M).$$

(iv) *For $M = \mathbb{Q}/\mathbb{Z}$, prove that $\mathsf{t}_p(M)$ is the image of the canonical map $\mathbb{Z}[1/p]/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ induced by the inclusion $\mathbb{Z}[1/p] \to \mathbb{Q}$, for all $p \in \mathscr{P}$.*

(v) *Using the two previous items and the last item of Exercise 4.88, prove that*

$$\hat{\hat{\mathbb{Z}}} = \widehat{\mathbb{Q}/\mathbb{Z}} = \mathsf{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \prod_{p \in \mathscr{P}} \mathbb{Z}_p.$$

In particular, note that the Pontrjagin duality is not involutive, since $\mathbb{Z}$ is countable but $\mathbb{Z}_p$ is not.

3.20. Exercise. *Let A be a commutative ring and let $a \in A$ be a fixed nonzero element such that a is a nonzerodivisor. Let M be a left A-module.*

(i) *Prove that $\mathsf{Tor}_0^A(A/(a), M) \cong M/a.M$, $\mathsf{Tor}_1^A(A/(a), M) \cong {}_aM$, and $\mathsf{Tor}_n^A(A/(a), M) = 0$, for $n \geq 2$.*

(ii) *Assume for this item and the next one that A is a PID. Using that any finitely generated module N is of the form $A^{n_0} \oplus A/(n_1) \oplus \cdots \oplus A/(n_\ell)$, for $n_0, \ldots, n_\ell \in \mathbb{N}_0$, as well as the previous item, show that $\mathsf{Tor}_1^A(N, M)$ is a torsion module, and $\mathsf{Tor}_n^A(N, M) = 0$ for $n \geq 2$ and all modules M and N, with N finitely generated.*

(iii) *Using Proposition 3.16 as well as the previous item, prove that $\mathsf{Tor}_1^A(N, M)$ is a torsion A-module, and $\mathsf{Tor}_n^A(N, M) = 0$ for $n \geq 2$ and all modules N and M.*

(iv) *Assume from now on that A is a domain, whose field of fractions is $Q(A)$. Prove that, given any nonzero $a \in A$, the submodule of $Q(A)/A$ generated by $1/a$ is isomorphic to $A/(a)$. Show moreover that the set of all of the previous submodules of $Q(A)/A$ form a partially ordered set for the inclusion satisfying the Moore-Smith condition and that their union is $Q(A)/A$.*

(v) Using the first and the previous items with Proposition 3.16 prove that there is a natural isomorphism $\mathsf{Tor}_1^A(Q(A)/A, M) \cong \mathsf{t}(M)$.

3.21. Exercise. *Let $n \in \mathbb{Z} \setminus \{0\}$ and let A be the commutative ring $\mathbb{Z}/n.\mathbb{Z}$. Let $d \in \mathbb{Z}$ be a divisor of n, $d' = n/d$ and let $M = \mathbb{Z}/d.\mathbb{Z}$ be the A-module with the obvious action. Given any A-module N, prove that $\mathsf{Tor}_0^A(M,N) \cong N/d.N$, $\mathsf{Tor}_{2i+1}^A(M,N) \cong {}_dN/d'.N$ and $\mathsf{Tor}_{2i+2}^A(M,N) \cong {}_{d'}N/d.N$ for $i \in \mathbb{N}_0$.*

3.22. Exercise. *Let A be a ring and let M be an A-module. Prove that the following conditions are equivalent.*

(i) M is flat;

(ii) $\mathsf{Tor}_1^A(N, M) = 0$ for all right A-modules N;

(iii) $\mathsf{Tor}_n^A(N, M) = 0$ for all right A-modules N and all $n \in \mathbb{N}$.

The analogous result holds for right A-modules.

3.23. Exercise. *Prove that a $\mathbb{Z}$-module is flat if and only if it has zero torsion.*

3.24. Exercise. *Prove that a an A-module M over a ring A is flat if and only if, given any right ideal $I \subseteq A$, the canonical morphism of abelian groups $I \otimes_A M \to M$ sending $a \otimes_A m$ to $a \cdot m$ for $a \in I$ and $m \in M$ is injective.*

3.25. Exercise. (i) Let A be a commutative ring and let $a \in A$ be a nonzerodivisor element. Prove that $\mathsf{Ext}_A^0(A/(a), M) \cong {}_aM$ and $\mathsf{Ext}_A^1(A/(a), M) \cong M/a.M$, and $\mathsf{Ext}_A^n(A/(a), M) = 0$, for $n \geq 2$.

(ii) Let A be a PID. Using Exercise 2.5, (In.4), prove that any A-module N has an injective resolution of the form

$$0 \longrightarrow N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow 0$$

Deduce that $\mathsf{Ext}_{\mathbb{Z}}^n(M, N) = 0$ for all abelian groups M and N, and every integer $n \geq 2$.

(iii) Assume that $A = \mathbb{Z}$. Let M be an abelian torsion group, i.e. $M = \mathsf{t}(M)$. Prove that $\mathsf{Ext}_{\mathbb{Z}}^0(M, \mathbb{Z}) = 0$ and $\mathsf{Ext}_{\mathbb{Z}}^1(M, \mathbb{Z}) \cong \hat{M}$, where $\hat{M} = \mathsf{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

3.26. Exercise. *Let M be an abelian group.*

(i) Given a nonzero $n \in \mathbb{Z}$, assume that the n-torsion ${}_nM$ of M vanishes. Applying the cohomological δ-functor $\{\mathsf{Ext}_{\mathbb{Z}}^\bullet(-, \mathbb{Q}/\mathbb{Z})\}_{\bullet \in \mathbb{N}_0}$ to the short exact sequence

$$0 \longrightarrow M \xrightarrow{\;n\;} M \longrightarrow M/n.M \longrightarrow 0 \tag{23}$$

of abelian groups, where the first nonzero map is the multiplication by n and the second one is the canonical projection, prove that the map $\hat{M} \to \hat{M}$ given by mutiplication by n is surjective. Deduce that the Pontrjagin dual $\hat{M}$ of M is divisible if M has zero torsion.

(ii) Given a nonzero $n \in \mathbb{Z}$, assume that the map $\hat{M} \to \hat{M}$ given by mutiplication by n is surjective. Using Exercise 2.8, prove that the map $M \to M$ given by multiplication by n is injective. Deduce that M has zero torsion if the Pontrjagin dual $\hat{M}$ of M is divisible.

(iii) Given a nonzero $n \in \mathbb{Z}$, assume that the n-torsion ${}_nM$ of M vanishes. Applying the the cohomological δ-functor $\{\mathsf{Ext}_{\mathbb{Z}}^\bullet(-, \mathbb{Z}/n.\mathbb{Z})\}_{\bullet \in \mathbb{N}_0}$ to (23), prove that $\mathsf{Ext}_{\mathbb{Z}}^0(M, \mathbb{Z}/n.\mathbb{Z}) = \mathsf{Hom}_{\mathbb{Z}}(M/n.M, \mathbb{Z}/n.\mathbb{Z})$ and $\mathsf{Ext}_{\mathbb{Z}}^1(M, \mathbb{Z}/n.\mathbb{Z}) = 0$.

3.27. Exercise. *Given an abelian group M and a nonzero integer $n \in \mathbb{Z}$, consider the commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathsf{t}(M) \arrow[r, "\mathsf{inc}"] \arrow[d, "n"] & M \arrow[r] \arrow[d, "n"] & M/\mathsf{t}(M) \arrow[r] \arrow[d, "n"] & 0 \\
0 \arrow[r] & \mathsf{t}(M) \arrow[r, "\mathsf{inc}"] & M \arrow[r] & M/\mathsf{t}(M) \arrow[r] & 0
\end{tikzcd}$$

of abelian groups with exact rows, where the first nonzero horizontal maps are the inclusion and the second are the canonical projection, and the vertical maps are given by multiplication by n. Applying the functor $\mathsf{Hom}_{\mathbb{Z}}(-,\mathbb{Q}/\mathbb{Z})$, and using Exercises 2.8 and 3.26 together with Lemma 1.40, prove that the inclusion map $\mathsf{t}(M) \to M$ induces an isomorphism

$$\widehat{\mathsf{t}(M)}/n.\widehat{\mathsf{t}(M)} \cong \hat{M}/n.\hat{M}$$

of abelian groups.

3.28. Exercise. *Let $p \in \mathbb{Z}$ be a prime integer.*

(i) Prove that $\mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p],\mathbb{Z}) = 0$.

(ii) Applying the δ-functor $\{\mathsf{Ext}^{\bullet}_{\mathbb{Z}}(-,\mathbb{Z})\}_{\bullet\in\mathbb{N}_0}$ to the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\mathsf{inc}} \mathbb{Z}[1/p] \longrightarrow \mathbb{Z}[1/p]/\mathbb{Z} \longrightarrow 0$$

of abelian groups, where the first nonzero map is the inclusion and the second is the canonical projection, together with the last item of Exercise 3.25 as well as Exercise 4.88, prove that $\mathsf{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[1/p],\mathbb{Z}) \cong \mathbb{Z}_p/\mathbb{Z}$.

(iii) Let M be an abelian group and let $N = \prod_{n\in\mathbb{N}} \mathbb{Z}/n.\mathbb{Z}$. Applying the δ-functor $\{\mathsf{Ext}^{\bullet}_{\mathbb{Z}}(-,N)\}_{\bullet\in\mathbb{N}_0}$ to the short exact sequence

$$0 \longrightarrow \mathsf{t}(M) \xrightarrow{\mathsf{inc}} M \longrightarrow M/\mathsf{t}(M) \longrightarrow 0$$

of abelian groups, where the first nonzero map is the inclusion and the second one is the canonical projection, using the last items of Exercises 3.25 and 3.26, as well as Exercise 3.27, prove that

$$\mathsf{Ext}^1_{\mathbb{Z}}(M,N) \cong \prod_{n\in\mathbb{N}} \hat{M}/n.\hat{M},$$

where $\hat{M} = \mathsf{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$. Deduce that $\mathsf{Ext}^1_{\mathbb{Z}}(M,N)$ vanishes if and only if $\hat{M}$ is divisible, i.e. if M has zero torsion by Exercise 3.26.

§3.4. Universal coefficients theorems

3.29. The following results are very useful in algebraic topology.

3.30. Theorem (Künneth formula). *Let (M,d_M) be a chain complex of flat A-modules such that $\mathsf{Img}(d_{M,n})$ is flat for all $n \in \mathbb{Z}$. Then, given a right A-module N we have the short exact sequence*

$$0 \longrightarrow N \otimes_A H_n(M,d_M) \longrightarrow H_n\big(N \otimes_A (M,d_M)\big) \longrightarrow \mathsf{Tor}_1^A\big(N, H_{n-1}(M,d_M)\big) \longrightarrow 0 \tag{24}$$

of $\mathbb{Z}$-modules for every $n \in \mathbb{Z}$.

Proof. Given $n \in \mathbb{Z}$, the short exact sequence

$$0 \longrightarrow \mathsf{Ker}(d_{M,n}) \longrightarrow M_n \xrightarrow{d_{M,n}} \mathsf{Img}(d_{M,n}) \longrightarrow 0 \tag{25}$$

of A-modules is exact, since $\mathsf{Img}(d_{M,n})$ is flat, which implies that

$$0 \longrightarrow N \otimes_A \mathsf{Ker}(d_{M,n}) \longrightarrow N \otimes_A M_n \longrightarrow N \otimes_A \mathsf{Img}(d_{M,n}) \longrightarrow 0$$

is an exact sequence of abelian groups. Consider the complexes $N \otimes_A \mathsf{Ker}(d_M)$ and $N \otimes_A \mathsf{Img}(d_M)$ with zero differentials. Hence, the previous short exact sequences assemble into a short exact sequence

$$0 \longrightarrow N \otimes_A \mathsf{Ker}(d_M) \longrightarrow N \otimes_A (M, d_M) \longrightarrow N \otimes_A \mathsf{Img}(d_M) \longrightarrow 0$$

of complexes of A-modules, where the middle complex has differential $\mathsf{id}_N \otimes_A d_M$. The long exact sequence in homology thus gives

$$\begin{tikzcd} \dots \arrow[r] & N \otimes_A \mathsf{Img}(d_{M,n+1}) \arrow[r, "\partial_{n+1}"] & N \otimes_A \mathsf{Ker}(d_{M,n}) \arrow[r] & H_n(N \otimes_A M_n, \mathsf{id}_N \otimes_A d_{M,n}) \arrow[dll, rounded corners] \\ & N \otimes_A \mathsf{Img}(d_{M,n}) \arrow[r, "\partial_n"] & N \otimes_A \mathsf{Ker}(d_{M,n-1}) \arrow[r] & \dots \end{tikzcd}$$

and we clearly see that ∂_n is just $\mathsf{id}_N \otimes_A i_n$ for $i_n : \mathsf{Img}(d_{M,n}) \to \mathsf{Ker}(d_{M,n-1})$ the canonical inclusion, for all $n \in \mathbb{Z}$. Using that

$$0 \longrightarrow \mathsf{Img}(d_{M,n}) \xrightarrow{i_n} \mathsf{Ker}(d_{M,n-1}) \longrightarrow H_{n-1}(M, d_M) \longrightarrow 0$$

is a flat resolution of $H_{n-1}(M, d_M)$ for $n \in \mathbb{Z}$, since $\mathsf{Ker}(d_{M,n-1})$ is flat by the long exact sequence of Tor groups coming from (25), we conclude that the homology of

$$0 \longrightarrow N \otimes_A \mathsf{Img}(d_{M,n}) \xrightarrow{\partial_n} N \otimes_A \mathsf{Ker}(d_{M,n-1}) \longrightarrow 0$$

gives exactly $\mathsf{Tor}_i^A\left(N, H_{n-1}(M, d_M)\right)$ for $i = 0, 1$. Combining this with the long exact sequence in homology we obtain the short exact in the statement of the theorem. $\square$

3.31. Remark. *Assume that A is a commutative domain that is also a PID (or more generally a hereditary ring) and that (M, d_M) is a complex of free modules (or more generally, projective) in the previous theorem. This applies in particular to $A = \mathbb{Z}$. By a well-known theorem of Kaplansky, every submodule of a free module over a PID is also free. Then, the short exact sequence (24) of abelian groups is split. Indeed, this follows from the fact that the short exact sequence (25) is split, since the submodule $\mathsf{Img}(d_{M,n})$ of the free module M_n is free.*

3.32. Theorem (Universal coefficient formula for cohomology)**.** *Let (M, d_M) be a chain complex of projective A-modules such that $\mathsf{Img}(d_{M,n})$ is projective for all $n \in \mathbb{Z}$. Then, given an A-module N we have the short exact sequence*

$$0 \to \mathsf{Ext}_A^1\left(H_{n-1}(M, d_M), N\right) \to H^n\left(\mathsf{Hom}_A(M, N)\right) \to \mathsf{Hom}_A\left(H_n(M, d_M), N\right) \to 0 \tag{26}$$

of $\mathbb{Z}$-modules for every $n \in \mathbb{Z}$, where we are writting $\mathsf{Hom}_A(M, N)$ instead of $\mathscr{H}om_{\mathsf{dgA}}(M, \iota(N))$ introduced in paragraph 1.22. Moreover, the previous short exact sequence is split.

Proof. The proof is analogous to that of (24). Given $n \in \mathbb{Z}$, the short exact sequence

$$0 \longrightarrow \mathsf{Ker}(d_{M,n}) \longrightarrow M_n \xrightarrow{d_{M,n}} \mathsf{Img}(d_{M,n}) \longrightarrow 0 \tag{27}$$

of A-modules is split exact, since $\mathsf{Img}(d_{M,n})$ is projective. This implies that

$$0 \longrightarrow \mathsf{Hom}_A\left(\mathsf{Img}(d_{M,n}), N\right) \longrightarrow \mathsf{Hom}_A(M_n, N) \longrightarrow \mathsf{Hom}_A\left(\mathsf{Ker}(d_{M,n}), N\right) \longrightarrow 0$$

is a split exact sequence of abelian groups. We consider the complexes $\mathsf{Hom}_A(\mathsf{Img}(d_M), N)$ and $\mathsf{Hom}_A(\mathsf{Ker}(d_M), N)$ with zero differentials. Hence, the previous short exact sequences assemble into a short exact sequence

$$0 \longrightarrow \mathsf{Hom}_A\left(\mathsf{Img}(d_M), N\right) \longrightarrow \mathsf{Hom}_A(M, N) \longrightarrow \mathsf{Hom}_A\left(\mathsf{Ker}(d_M), N\right) \longrightarrow 0$$

of complexes of A-modules, where the middle complex has the differential sending $h \in \mathsf{Hom}_A(M_n, N)$ to $(-1)^{n+1} h \circ d_{M,n}$. The long exact sequence in cohomology thus gives

$$\begin{tikzcd} \dots \arrow[r] & \mathsf{Hom}_A\big(\mathsf{Ker}(d_{M,n-1}), N\big) \arrow[r, "\partial^{n-1}"] & \mathsf{Hom}_A\big(\mathsf{Img}(d_{M,n}), N\big) \arrow[r] & H^n\big(\mathsf{Hom}_A(M,N)\big) \arrow[dll, rounded corners] \\ & \mathsf{Hom}_A\big(\mathsf{Ker}(d_{M,n}), N\big) \arrow[r, "\partial^{n}"] & \mathsf{Hom}_A\big(\mathsf{Img}(d_{M,n+1}), N\big) \arrow[r] & \dots \end{tikzcd}$$

and we clearly see that ∂^n isends h to $h \circ i_{n+1}$ for $i_{n+1} : \mathsf{Img}(d_{M,n+1}) \to \mathsf{Ker}(d_{M,n})$ the canonical inclusion, for all $n \in \mathbb{Z}$. Using that

$$0 \longrightarrow \mathsf{Img}(d_{M,n}) \xrightarrow{i_n} \mathsf{Ker}(d_{M,n-1}) \longrightarrow H_{n-1}(M, d_M) \longrightarrow 0$$

is a projective resolution of $H_{n-1}(M, d_M)$ for $n \in \mathbb{Z}$, since $\mathsf{Ker}(d_{M,n-1})$ is projective by applying the long exact sequence of Ext groups to (27), we conclude that the cohomology of

$$0 \longrightarrow \mathsf{Hom}_A\big(\mathsf{Ker}(d_{M,n-1}), N\big) \xrightarrow{\partial^{n-1}} \mathsf{Hom}_A\big(\mathsf{Img}(d_{M,n}), N\big) \longrightarrow 0$$

gives exactly $\mathsf{Ext}^i_A(H_{n-1}(M, d_M), N)$ for $i = 0, 1$. Combining this with the long exact sequence in cohomology we obtain the short exact in the statement of the theorem. The fact (26) is split follows from the fact that (27) is aplit. $\square$

3.33. Remark. *Let X be a topological space and G an abelian group. The singular homology of X with coefficients in $\mathbb{Z}$ is defined as the homology of the complex of abelian groups $S(X) = (S_n(X), d_n)_{n \in \mathbb{N}_0}$, formed by all simplices of X. On the other hand, the singular cohomology of X with coefficients in G is defined as the cohomology of the complex $\mathsf{Hom}_{\mathbb{Z}}(S(X), G)$, which is computed by means of Theorem 3.32 in terms of the singular homology of X.*

§4. Lecture IV : Basics on group homology and cohomology

§4.1. Basic definitions

4.1. In this section we fix a commutative ring k with unit and G a group. We identify ${}_k\mathsf{Mod}$ with the category of symmetric k-bimodules. Recall that the **group algebra** kG is the free k-module $\oplus_{g \in G} k.g$ generated by G, together with the product extending the product of G. To avoid confusion between the elements of k, specially if $k = \mathbb{Z}$, and those of G, we usually use the multiplicative notation for the operation of G.

4.2. Given a module M over kG, define the k-modules $M^G = \{m \in M : g.m = m, \text{ for all } g \in G\}$ and $M_G = M/\langle\{g.m - m \in M : m \in M, g \in G\}\rangle$. Moreover, if $f : M \to N$ is a morphism of kG-modules, $f(M^G) \subseteq N^G$ and

$$f(\langle\{g.m - m : m \in M, g \in G\}\rangle) \subseteq \langle\{g.n - n : n \in N, g \in G\}\rangle.$$

As a consequence, f induces morphisms of k-modules $f^G : M^G \to N^G$ and $f_G : M_G \to N_G$.

4.3. Exercise. *Prove that the constructions in the previous paragraph give functors of the form*

$$(-)^G, (-)_G : {}_{kG}\mathsf{Mod} \to {}_k\mathsf{Mod}.$$

4.4. The k-module M^G is called the **space of invariants** of M, and M_G is the **space of coinvariants**.

4.5. Let M be a kG-module. Consider the canonical inclusion $\jmath_M : M^G \to M$ and the canonical projection $p_M : M \to M/\langle\{g.m - m \in M : m \in M, g \in G\}\rangle$. Define the map $\imath_M : M^G \to M_G$ as the composition of the canonical inclusion $\jmath_M$ and the canonical projection p_M.

4.6. Exercise. *Prove that the map $\imath_M : M^G \to M_G$ defined in paragraph 4.5 gives a natural transformation from the functor $(-)^G$ to the functor $(-)_G$.*

4.7. Exercise. *Let*

$$\mathsf{Triv} : {}_k\mathsf{Mod} \to {}_{kG}\mathsf{Mod}.$$

be the functor sending a k-module M to the unique kG-module $\mathsf{Triv}(M)$ such that $\mathsf{Triv}(M) = \mathsf{Triv}(M)^G = M$, and a morphism of k-modules $f : M \to N$ to the morphism of kG-modules $\mathsf{Triv}(M) \to \mathsf{Triv}(N)$ whose underlying set theoretic map is f. Prove that $(-)_G$ is left adjoint to Triv, which is in turn left adjoint to $(-)^G$. Using Exercise 4.104, deduce that $(-)_G$ is right exact and that $(-)^G$ is left exact.

4.8. By the previous exercise the functor $(-)_G$ is right exact and the functor $(-)^G$ is left exact. We define the **group homology** $H_\bullet(G, M)$ of G with coefficients in the kG-module M as the left derived functor of $(-)_G$ evaluated at M, and the **group cohomology** $H^\bullet(G, M)$ of G with coefficients in M as the right derived functor of $(-)^G$ evaluated at M.

4.9. Define the **augmentation map** $\varepsilon : kG \to k$ as the unique morphism of k-algebras sending g to 1, for all $g \in G$. The kernel of ε is called the **augmentation ideal** of kG, and it denoted by $J(G)$. It is easy to prove that $J(G)$ is a free k-module with basis $\{g - e : g \in G\}$, where e is the unit element of G. Indeed, given $x = \sum_{g \in G} x_g g \in J(G)$, we have

$$x = \Big(\sum_{g \neq e} x_g g\Big) + x_e e = \Big(\sum_{g \neq e} x_g g\Big) - \Big(\sum_{g \neq e} x_g\Big) e = \sum_{g \neq e} x_g (g - e).$$

4.10. By means of the augmentation map $\varepsilon : kG \to k$ we get a canonical structure of kG-bimodule on k. More explicitly, $g \cdot \lambda = \lambda = \lambda \cdot g$, for all $g \in G$ and $\lambda \in k$.

4.11. Exercise. *(i) Consider k as a $k \otimes_k (kG)^{\mathrm{op}}$-module (resp., $kG \otimes_k k^{\mathrm{op}}$-module). Prove that the functor*

$$\mathsf{Triv} : {}_k\mathsf{Mod} \to {}_{kG}\mathsf{Mod}.$$

defined in Exercise 4.7 is naturally isomorphic to the functor

$$\mathsf{Hom}_k(k, -) : {}_k\mathsf{Mod} \to {}_{kG}\mathsf{Mod} \ \Big(\text{resp., } k \otimes_k (-) : {}_k\mathsf{Mod} \to {}_{kG}\mathsf{Mod}\Big)$$

(see Exercise 4.85).

(ii) Using Exercise 4.7 and the previous item, together with Exercises 4.84 and 4.85, prove that there exist natural isomorphisms of functors

$$(-)_G \cong k \otimes_{kG} (-) \text{ and } (-)^G \cong \mathsf{Hom}_{kG}(k, -).$$

4.12. By Exercise 4.11, we get that $H_n(G, M) = \mathsf{Tor}_n^{kG}(k, M)$ and $H^n(G, M) = \mathsf{Ext}^n_{kG}(k, M)$, for all $n \in \mathbb{N}_0$.

4.13. Exercise. *(i) Assume $G = \{e\}$ is the group with one element. Prove that $M^G = M_G = M$ and*

$$H_n(G, M) = H^n(G, M) = 0$$

for all $n \in \mathbb{N}$ and all kG-modules M.

(ii) Let $G = \mathbb{Z}$.

(1) Prove that the k-algebra kG is isomorphic to the algebra of Laurent polynomials $k[X, X^{-1}]$.

(2) *Prove that the sequence of $k[X, X^{-1}]$-modules*

$$0 \longrightarrow k[X, X^{-1}] \xrightarrow{X-1} k[X, X^{-1}] \xrightarrow{\varepsilon} k \longrightarrow 0$$

is exact, where $\varepsilon : k[X, X^{-1}] \to k$ is the unique morphism of $k[X, X^{-1}]$-modules sending X to 1.

(3) *Prove that $M^G \cong H_1(G, M)$, $M_G \cong H^1(G, M)$ and $H_n(G, M) = H^n(G, M) = 0$ for all integers $n \geq 2$ and all kG-modules M.*

4.14. Exercise. *Prove that $M_G = H_0(G, M) \cong M/J(G).M$, for all kG-modules M. Deduce that the augmentation map ε induces an isomorphism $(kG)_G \cong k$. Moreover, $J(G)_G = H_0(G, J(G)) \cong J(G)/J(G)^2$.*

4.15. Let G be a finite group. We define its **norm** $N(G)$ as $\sum_{g \in G} g \in kG$.

4.16. Fact. *Let G be a finite group. Then, $N(G)$ is a central element of kG, and $(kG)^G$ is a (two-sided) ideal of kG which satisfies $(kG)^G = \{c.N(G) : c \in k\}$. Moreover, the k-linear map $\imath_{kG} : (kG)^G \to (kG)_G \cong k$ defined in paragraph 4.5 sends $N(G)$ to the cardinal $\#(G)$ of G.*

Proof. It is clear that $g.N(G) = N(G) = N(G).g$ for all $g \in G$, which implies that $N(G)$ lies in the center of kG. The identity $(kG)^G = \{c.N(G) : c \in k\}$ implies that $(kG)^G$ is the two-sided ideal generated by $N(G)$, so it suffices to prove the former identity. It is clear that $\{c.N(G) : c \in k\} \subseteq (kG)^G$. Let $x = \sum_{g \in G} x_g g \in (kG)^G$, with $x_g \in k$ for all $g \in G$. Hence, $h.x = x$ implies that $x_e = x_h$, for all $h \in G$, where e is the unit of G. In consequence, $x = x_e.N(G)$, *i.e.* $x \in \{c.N(G) : c \in k\}$. The last assertion is clear. □

4.17. Exercise. *Assume that G is an infinite group. Prove that $(kG)^G = H^0(G, kG) = 0$.*

4.18. Proposition. *Assume that G is a finite group of order $d = \#(G)$. Let k be a field such that d is invertible in k. Then, $\imath_M$ induces an isomorphism $M^G \cong M_G$ and*

$$H_n(G, M) = H^n(G, M) = 0$$

for all $n \in \mathbb{N}$ and all kG-modules M.

Proof. Let $\mathscr{e} = d^{-1}.N(G) = N(G).d^{-1} \in kG$. We first note that $N(G)^2 = \sum_{g,h \in G} gh = d(\sum_{g \in G} g) = dN(G)$ tells us that $\mathscr{e}^2 = \mathscr{e}$. Moreover,

$$g.\mathscr{e} = g.N(G).d^{-1} = N(G).d^{-1} = \mathscr{e} = d^{-1}.N(G) = d^{-1}.N(G).g = \mathscr{e}.g,$$

for all $g \in G$, and in particular $\mathscr{e}$ is a central element of kG. This induces the direct sum

$$M = \mathscr{e}.M \oplus (\mathscr{e} - 1).M$$

of k-modules.

We also claim that $M^G = \mathscr{e}.M$. Indeed, the inclusion $\mathscr{e}.M \subseteq M^G$ follows from the fact that

$$g.(\mathscr{e}.m) = (g.\mathscr{e}).m = \mathscr{e}.m,$$

for all $g \in G$ and $m \in M$, whereas the inclusion $M^G \subseteq \mathscr{e}.M$ follows from $M^G = \mathscr{e}.M^G \subseteq \mathscr{e}.M$. Moreover, we further claim that $\langle \{g.m - m : m \in M, g \in G\} \rangle = (\mathscr{e} - 1).M$. Indeed, the inclusion $\langle \{g.m - m : m \in M, g \in G\} \rangle \subseteq (\mathscr{e} - 1).M$ follows from

$$\mathscr{e}.(g.m - m) = (\mathscr{e}.g - \mathscr{e}).m = (\mathscr{e} - \mathscr{e}).m = \mathbf{0}_M,$$

for all $g \in G$ and $m \in M$, since $(\mathscr{e} - 1).M$ is the kernel of the k-linear map $M \to M$ given by multiplication by $\mathscr{e}$ on the left, whereas the inclusion $(\mathscr{e} - 1).M \subseteq \langle \{g.m - m : m \in M, g \in G\} \rangle$ follows from

$$\begin{aligned}(\mathscr{e} - 1).m &= \Big((N(G) - d).d^{-1} \Big).m = \Big(\sum_{g \in G} (g - e) \Big).(d^{-1}.m) \\ &= \sum_{g \in G} \big(g.(d^{-1}.m) - (d^{-1}.m) \big) \in \langle \{g.m - m : m \in M, g \in G\} \rangle,\end{aligned}$$

for all $g \in G$ and $m \in M$. As a consequence, $M_G = M/(\mathscr{e}-1).M$. We now see that the map $\imath_M : M^G \to M_G$ sends $\mathscr{e}.m$ to its class in $M/(\mathscr{e}-1).M$ modulo $(\mathscr{e}-1).M$. By the previous description of M^G and M_G, together with the direct sum decomposition $M = \mathscr{e}.M \oplus (\mathscr{e}-1).M$, we conclude that $\imath_M$ is an isomorphism. The last statement follows from the first one together with Exercise 2.51, since in this case both functors $(-)^G$ and $(-)_G$ are exact. $\square$

4.19. Exercise. *Let G be a group.*

(i) Consider the map $\theta : G \to J(G)/J(G)^2$ sending $g \in G$ to the class of $g - e \in J(G)$. Prove that θ is a morphism of groups. Deduce that the kernel of θ includes the commutator subgroup $[G,G]$ of G, i.e. the subgroup of G generated by $\{ghg^{-1}h^{-1} : g,h \in G\}$.

(ii) Assume that $k = \mathbb{Z}$. Define $\rho : J(G) \to G/[G,G]$ as the unique morphism of abelian groups sending $g - e$ to the class of g, for all $g \in G$. Prove that $J(G)^2$ is in the kernel of ρ, and that θ and ρ induce inverse group isomorphisms between $J(G)/J(G)^2$ and $G/[G,G]$.

4.20. Proposition. *Let G be a group. Then,*

$$H_1(G,k) \cong J(G)/J(G)^2.$$

Moreover, if $k = \mathbb{Z}$, then the previous spaces are naturally isomorphic to $G/[G,G]$, where $[G,G]$ denotes the commutator subgrop of G.

Proof. Consider the short exact sequence

$$0 \longrightarrow J(G) \longrightarrow kG \xrightarrow{\varepsilon} k \longrightarrow 0$$

of kG-modules. It gives a long exact sequence

$$H_1(G,kG) \longrightarrow H_1(G,k) \longrightarrow J(G)_G \longrightarrow (kG)_G \xrightarrow{\bar{\varepsilon}} k_G \longrightarrow 0$$

of k-modules. Using that $\bar{\varepsilon} : (kG)_G \to k$ is an isomorphism and that $H_1(G,kG) = 0$ since kG is a projective kG-module, we obtain the isomorphism $H_1(G,k) \cong J(G)_G \cong J(G)/J(G)^2$, where we have used Exercise 4.14. The last part follows from Exercise 4.19. $\square$

§4.2. Some basic calculations

4.21. Given $n \in \mathbb{N}$, let $C_n = \mathbb{Z}/n.\mathbb{Z}$ be the cyclic group of n elements. Let $\sigma \in C_n$ be the generator given by the class of $1 \in \mathbb{Z}$. As explained in the first paragraph of this section we will use the multiplicative notation, so the elements of C_n are denoted in the form σ^m, for $m \in [\![1,n]\!]$, where σ^0 is the unit of C_n. We have thus $N(C_n) = \sum_{m=0}^{n-1} \sigma^m$ and $(\sigma - e)N(C_n) = N(C_n)(\sigma - e) = 0$.

4.22. Given $n \in \mathbb{N}$, define the complex $(F_\bullet, d_\bullet)_{\bullet \in \mathbb{Z}}$ given by $F_n = kC_n$ for $n \in \mathbb{N}_0$ and $F_n = 0$ for $n < 0$, such that d_{2k} is the multiplication by $N(C_n)$ and d_{2k-1} is the multiplication by $\sigma - e$ for all $k \in \mathbb{N}$. This is clearly a complex, since $(\sigma - e)N(C_n) = 0$ and $N(C_n)(\sigma - e) = 0$. Let $F_0 = kC_n \to k$ be the augmentation map ε. We can represent it as follows

$$\dots \xrightarrow{N(C_n)} kC_n \xrightarrow{\sigma - e} kC_n \xrightarrow{N(C_n)} kC_n \xrightarrow{\sigma - e} kC_n \xrightarrow{\varepsilon} k \longrightarrow 0$$

4.23. Fact. *The complex $(F_\bullet, d_\bullet)_{\bullet \in \mathbb{Z}}$ together with the augmentation map $F_0 = kC_n \to k$ is a free left resolution of the trivial kC_n-module k.*

Proof. It is clear that $\varepsilon \circ (\sigma - e) = 0$. Moreover, $\sigma^m - e = \sum_{k=0}^{m-1} \sigma^k(\sigma - e)$ tells us that $\mathsf{Ker}(\varepsilon) \subseteq \mathsf{Img}(\sigma - e)$. Analogously, Fact 4.16 tells us that $\mathsf{Ker}(\sigma - e) \subseteq \mathsf{Img}(N(C_n))$. Finally, given $x = \sum_{k=0}^{n-1} x_k \sigma^k$, $N(C_n)x = 0$ is tantamount to $\sum_{k=0}^{n-1} x_k = 0$. We conclude that $\mathsf{Ker}(N(C_n)) \subseteq J(C_n) = \mathsf{Img}(\sigma - e)$. $\square$

4.24. The previous projective resolution gives the following immediate computation.

4.25. Proposition. *Let M be a module over kC_n, for $n \in \mathbb{N}$. Then,*

$$H_\ell(C_n, M) \cong \begin{cases} M/(\sigma - e)M, & \text{if } \ell = 0, \\ M^{C_n}/N(C_n)M, & \text{if } \ell \in \mathbb{N} \text{ is odd}, \\ \{m \in M : N(C_n).m = 0\}/(\sigma - e)M, & \text{if } \ell \in \mathbb{N} \text{ is even}, \end{cases}$$

and

$$H^\ell(C_n, M) \cong \begin{cases} M^{C_n}, & \text{if } \ell = 0, \\ \{m \in M : N(C_n).m = 0\}/(\sigma - e)M, & \text{if } \ell \in \mathbb{N} \text{ is odd}, \\ M^{C_n}/N(C_n)M, & \text{if } \ell \in \mathbb{N} \text{ is even}. \end{cases}$$

4.26. Exercise. *Assume that $k = M = \mathbb{Z}$. Prove that*

$$H_\ell(C_n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } \ell = 0, \\ \mathbb{Z}/n.\mathbb{Z}, & \text{if } \ell \in \mathbb{N} \text{ is odd}, \\ 0, & \text{if } \ell \in \mathbb{N} \text{ is even}, \end{cases}$$

and

$$H^\ell(C_n, M) \cong \begin{cases} \mathbb{Z}, & \text{if } \ell = 0, \\ 0, & \text{if } \ell \in \mathbb{N} \text{ is odd}, \\ \mathbb{Z}/n.\mathbb{Z}, & \text{if } \ell \in \mathbb{N} \text{ is even}. \end{cases}$$

4.27. Exercise. *Let G be the free group generated by a nonempty set S. Recall that, given any element $g \in G \setminus \{e\}$, there exist unique $\ell \in \mathbb{N}$ and $(g_1, \dots, g_\ell) \in (S \sqcup S^{-1})^\ell$ such that $g = g_1 \dots g_\ell$ and $g_i g_{i+1} \neq e$ for all $i \in [\![1, \ell - 1]\!]$, where $S^{-1} = \{s^{-1} : s \in S\}$. The positive integer ℓ is called the* ***length*** *of g and it is denoted by $\ell(g)$, whereas the element $g_\ell \in S \sqcup S^{-1}$ is called* ***right-most term*** *of g and it is denoted by $\mathtt{rt}(g)$.*

(i) *Given $x \in S \sqcup S^{-1}$, define*

$$W(x) = \{g \in G \setminus \{e\} : \mathtt{rt}(g) = x\}.$$

Note that

$$G \setminus \{e\} = \Big(\bigsqcup_{s \in S} W(s)\Big) \sqcup \Big(\bigsqcup_{s \in S} W(s^{-1})\Big).$$

Prove that

$$gs^{-1} - e = -(gs^{-1})(s - e) + (g - e)$$

if $gs^{-1} \in W(s^{-1})$, and

$$gs - e = g(s - e) + (g - e)$$

if $gs \in W(s)$.

(ii) *Using the previous item, induction on the length and the fact that $\{g - e : g \in G\}$ is a basis of the k-module $J(G)$, prove that*

$$\{g(s - e) : s \in S \text{ and } g \in G\}$$

is a basis of the k-module $J(G)$. Deduce that $J(G)$ is a free kG-module generated by the set $\{s - e : s \in S\}$.

(iii) *Prove that $H_n(G, M) = H^n(G, M) = 0$ for all integers $n \geq 2$ and kG-modules M.*

§4.3. Shapiro's Lemma

4.28. Let $H \leq G$ be a subgroup. Note the inclusion induces a morphism of k-algebras $kH \to kG$, so kG is a left and right kH-module. Given a kH-module M, recall the kG-module $kG \otimes_{kH} M$, called the **induced module**, also denoted by $\mathrm{Ind}_H^G(M)$. We recall that the module structure of $\mathrm{Ind}_H^G(M)$ is given by $g \cdot (g' \otimes_{kH} m) = gg' \otimes_{kH} m$, for $g, g' \in G$ and $m \in M$.

4.29. Analogously, we consider the kG-module $\mathrm{Hom}_{kH}(kG, M)$, called the **coinduced module**, also denoted by $\mathrm{Coind}_H^G(M)$. We recall that the module structure of $\mathrm{Coind}_H^G(M)$ is given by $(g \cdot f)(g') = f(g'g)$, for $g, g' \in G$ and $f \in \mathrm{Hom}_{kH}(kG, M)$.

4.30. Exercise. *Let $H \leq G$ be a subgroup of a group G. Let $S \subseteq G$ be a set of representatives of G/H.*

(i) Prove that S is a basis of the right kH-module kG.

(ii) Prove that $\{s^{-1} : s \in S\}$ is a set of representatives of $H\backslash G$, so it is a basis of of the left kH-module kG.

4.31. Lemma. *Let $H \leq G$ be a subgroup of a group G of finite index, i.e. G/H is finite. Then, there is an isomorphism $\mathrm{Coind}_H^G(M) \cong \mathrm{Ind}_H^G(M)$ of kG-modules.*

Proof. Let $S \subseteq G$ be a set of representatives of G/H, so S is a basis of the right kH-module kG, by the previous exercise. The same exercise tells us that $\{s^{-1} : s \in S\}$ is a basis of the left kH-module kG. Given $m \in M$ and $s \in S$, define $\phi(s, m) \in \mathrm{Coind}_H^G(M)$ to be the unique kH-linear map sending s^{-1} to m and t^{-1} to $\mathbf{0}_M$ if $t \in S \setminus \{s\}$. Consider the unique k-linear map

$$\phi : \mathrm{Ind}_H^G(M) \to \mathrm{Coind}_H^G(M)$$

sending $s \otimes_{kH} m$ to $\phi(s, m)$, for $s \in S$ and $m \in M$. Given $g \in G$ and $s \in S$, there exists $t \in S$ and $h \in H$ such that $gs = th$. In consequence, $hs^{-1} = t^{-1}g$ and $\phi(gs, m) = \phi(t, hm)$ sends t^{-1} to hm and u^{-1} to $\mathbf{0}_M$ for $u \in S \setminus \{t\}$. Analogously, $g \cdot \phi(s, m)$ sends t^{-1} to

$$(g \cdot \phi(s,m))(t^{-1}) = \phi(s,m)(t^{-1}g) = \phi(s,m)(hs^{-1}) = h\phi(s,m)(s^{-1}) = hm$$

and u^{-1} to $\mathbf{0}_M$ for $u \in S \setminus \{t\}$. As a consequence, ϕ is kG-linear.

We will finally prove that ϕ is bijective. Note the isomorphism $\mathrm{Ind}_H^G(M) \cong \oplus_{s \in S} M$ of k-modules, as well as

$$\mathrm{Coind}_H^G(M) = \mathrm{Hom}_{kH}\left(\bigoplus_{s \in S} kH, M\right) \cong \prod_{s \in S} \mathrm{Hom}_{kH}(kH, M) \cong \prod_{s \in S} M.$$

Under the previous identifications, the map ϕ defined in the previous paragraph sends $m \in M$ in the homogeneous component indexed by $s \in S$ to the same element m. Since S is finite, the map ϕ is bijective. □

4.32. Theorem (Shapiro's Lemma)**.** *Let $H \leq G$ be a subgroup of a group G. Given a kH-module M, we have the natural isomorphisms*

$$H_n(G, \mathrm{Ind}_H^G(M)) \cong H_n(H, M) \text{ and } H^n(G, \mathrm{Coind}_H^G(M)) \cong H^n(H, M)$$

for all $n \in \mathbb{N}_0$.

Proof. Note first that kG is a free kH-module. Given a projective resolution $(P, d) \to M$ of the kH-module M, then, since the functor $kG \otimes_{kH} (-)$ is exact we get that $kG \otimes_{kH} (P, d) \to kG \otimes_{kH} M = \mathrm{Ind}_H^G(M)$ is a projective resolution of the kG-module $\mathrm{Ind}_H^G(M)$. Moreover, the associativity isomorphism for the tensor product gives an isomorphism

$$k \otimes_{kG} \Big(kG \otimes_{kH} (P, d)\Big) \cong \Big(k \otimes_{kG} kG\Big) \otimes_{kH} (P, d) \cong k \otimes_{kH} (P, d)$$

of complexes of k-modules, which in turn gives the first isomorphisms of the statement.

Analogously, if $(Q,d) \to k$ is a projective resolution of the kG-module k, then it is *a fortiori* a projective resolution in the category of kH-modules. The adjunction between the tensor product and the homomorphism gives an isomorphism

$$\mathrm{Hom}_{kG}\Big((Q,d), \mathrm{Hom}_{kH}(kG,M)\Big) \cong \mathrm{Hom}_{kH}\Big((Q,d), M\Big)$$

of complexes of k-modules, which in turn gives the second isomorphisms of the statement. □

4.33. Exercise. *Let M be a k-module and G be a finite group. Prove that*

$$H_n(G, kG \otimes_k M) = H^n(G, kG \otimes_k M) = 0$$

for all $n \in \mathbb{N}$.

4.34. Exercise (Hilbert's theorem 90 (additive version))**.** *Let $K \subseteq E$ be a finite Galois extension of fields, i.e. a normal and separable finite extension, and let G be its Galois group, i.e. the group of all ring automorphisms of E that are the identity on K. Prove that the KG-module E is isomorphic to $KG \otimes_K K$. Deduce that $E^G \cong E_G \cong K$ and that*

$$H_n(G, E) = H^n(G, E) = 0$$

for all $n \in \mathbb{N}$.

§4.4. The Bar complex

4.35. Given $n \in \mathbb{N}_0$, define $\mathrm{Bar}(kG)_n = (kG)^{\otimes_k (n+1)}$. Equivalently, $\mathrm{Bar}(kG)_0 = kG$, and if $n \in \mathbb{N}$, $\mathrm{Bar}(kG)_n$ is the free kG-module generated by the set $\{[g_1|\dots|g_n] : g_1, \dots, g_n \in G\} \subseteq G^n$. We will usually write the elements of $\mathrm{Bar}(kG)_0 = kG$ by $x[]$, for $x \in kG$. We consider the augmentation map $\varepsilon : \mathrm{Bar}(kG)_0 = kG \to k$, and if $n \in \mathbb{N}$, we define

$$d_{n,p} : \mathrm{Bar}(kG)_n \to \mathrm{Bar}(kG)_{n-1}$$

for $p \in [\![0, n]\!]$ as the unique kG-linear map such that

$$d_{n,p}([g_1|\dots|g_n]) = \begin{cases} g_1[g_2|\dots|g_n], & \text{if } p = 0, \\ [g_1|\dots|g_{p-1}|g_p g_{p+1}|g_{p+2}|\dots|g_n], & \text{if } p \in [\![1, n-1]\!], \\ [g_1|\dots|g_{n-1}], & \text{if } p = n, \end{cases}$$

for all $g_1, \dots, g_n \in G$. We then define

$$d_n : \mathrm{Bar}(kG)_n \to \mathrm{Bar}(kG)_{n-1}$$

as

$$d_n = \sum_{p=0}^{n} (-1)^p d_{n,p}.$$

4.36. Exercise. *(i) Prove that the image of d_1 is precisely the kernel of ε.*

(ii) Prove that $d_{n,q} \circ d_{n+1,p} = d_{n,p-1} \circ d_{n+1,q}$, for all $n \in \mathbb{N}$ and all $0 \le q \le p-1$. Deduce that $d_n \circ d_{n+1} = 0$, for all $n \in \mathbb{N}$. In consequence, $(\mathrm{Bar}(kG)_\bullet, d_\bullet)_{\bullet \in \mathbb{N}_0}$ is a complex of kG-modules, called the ***(unnormalized) Bar complex****.*

(iii) *Given $n \in \mathbb{N}$, let $S_n \subseteq G^n$ be the subset given by the n-tuples $[g_1|\dots|g_n]$ satisfying that there exists $i \in [\![1, n]\!]$ such that $g_i = e_G$, and let $\mathsf{S}(kG)_n$ the kG-submodule of $\mathsf{Bar}(kG)_n$ generated by S_n. Define $\mathsf{S}(kG)_0$ as the zero submodule. Prove that $d_n(S_n) \subseteq \mathsf{S}(kG)_{n-1}$ for all $n \in \mathbb{N}$. Deduce that $(\mathsf{S}(kG)_\bullet, d_\bullet)_{\bullet\in\mathbb{N}_0}$ is a subcomplex of $(\mathsf{Bar}(kG)_\bullet, d_\bullet)_{\bullet\in\mathbb{N}_0}$.*

4.37. Theorem. *The homology of the Bar complex $(\mathsf{Bar}(kG)_\bullet, d_\bullet)_{\bullet\in\mathbb{N}_0}$ is trivial at every $n \in \mathbb{N}$. By the previous exercise, the augmentation $\varepsilon : kG \to k$ implies that the Bar complex $(\mathsf{Bar}(kG)_\bullet, d_\bullet)_{\bullet\in\mathbb{N}_0}$ is a left resolution of k in the category of kG-modules.*

Proof. Define $s_{-1} : k \to kG$ as the unique k-linear map sending 1 to $e[] = []$. Moreover, for $n \in \mathbb{N}_0$, define the unique k-linear map

$$s_n : \mathsf{Bar}(kG)_n \to \mathsf{Bar}(kG)_{n+1}$$

sending $g_0[g_1|\dots|g_n]$ to $[g_0|g_1|\dots|g_n]$, for all $g_0, \dots, g_n \in G$. It is easy to prove that $\varepsilon \circ s_{-1} = \mathrm{id}_k$, $d_1 \circ s_0 + s_{-1} \circ \varepsilon = \mathrm{id}_{\mathsf{Bar}(kG)_0}$ and $d_{n+1} \circ s_n + s_{n-1} \circ d_n = \mathrm{id}_{\mathsf{Bar}(kG)_n}$ for all $n \in \mathbb{N}$. This implies that the complex $(\mathsf{Bar}(kG)_\bullet, d_\bullet)_{\bullet\in\mathbb{N}_0}$ is acyclic at every $n \in \mathbb{N}$, as was to be shown. □

Appendix : Basics on category theory

Categories, objects and morphisms

4.38. We present the following definition, which provides a very useful general language to deal with the different situations we will encounter. For a more detailed and comprehensive exposition, see [Mac1971]. For a discussion on how to safely frame category theory within axiomatic set theory (or first order logic) see [Mur2006] and the references therein. For the reader who knows something about axiomatic set theory, we are assuming Zermelo-Fraenkel axiomatics together with the axiom of choice, a situation typically abbreviated as ZFC, as well as the axiom of existence of Grothendieck universes. In all of our definitions, a **set** is an element of a fixed nonempty universe $\mathscr{U}$, whereas a **class** is a subset of the universe $\mathscr{U}$.

4.39. Definition. *A* ***category*** *$\mathscr{C}$ is a tuple $(\mathscr{C}_0, \mathscr{C}_1, s_{\mathscr{C}}, t_{\mathscr{C}}, \circ_{\mathscr{C}}, i_{\mathscr{C}})$ where*

(i) $\mathscr{C}_0$ is a class of elements, called the ***objects*** *of $\mathscr{C}$ and typically denoted by $X, Y, Z, \dots$;*

(ii) $\mathscr{C}_1$ is a class of elements, called the ***morphisms*** *of $\mathscr{C}$ and typically denoted by $f, g, h, \dots$;*

(iii) $s_{\mathscr{C}}, t_{\mathscr{C}} : \mathscr{C}_1 \to \mathscr{C}_0$ are two maps, called the ***source map*** *and the* ***target map****, and whose inverse image*

$$s_{\mathscr{C}}^{-1}(\{X\}) \cap t_{\mathscr{C}}^{-1}(\{Y\})$$

is a set denoted by $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ and is called the ***space of morphisms*** *from X to Y;*

(iv) $\circ_{\mathscr{C}} : \mathscr{C}_1 \times_{s,t} \mathscr{C}_1 \to \mathscr{C}_1$ is a map called the ***composition****, sending a pair (f, g) to $f \circ_{\mathscr{C}} g$, where*

$$\mathscr{C}_1 \times_{s,t} \mathscr{C}_1 = \{(f, g) \in \mathscr{C}_1 \times \mathscr{C}_1 : s_{\mathscr{C}}(f) = t_{\mathscr{C}}(g)\};$$

(v) $i_{\mathscr{C}} : \mathscr{C}_0 \to \mathscr{C}_1$ is a map, called the ***identity****, sending every element X to a distinguished morphism $i_{\mathscr{C}}(X) = \mathsf{id}_X$, called the* ***identity morphism****;*

satisfying that

(CAT.1) given $f, g, h \in \mathscr{C}_1$ such that $s_{\mathscr{C}}(f) = t_{\mathscr{C}}(g)$ and $s_{\mathscr{C}}(g) = t_{\mathscr{C}}(h)$, then

$$(f \circ_{\mathscr{C}} g) \circ_{\mathscr{C}} h = f \circ_{\mathscr{C}} (g \circ_{\mathscr{C}} h).$$

(CAT.2) For every $X \in \mathscr{C}_0$ and every $f, g \in \mathscr{C}_1$ such that $s_{\mathscr{C}}(f) = X = t_{\mathscr{C}}(g)$, we have

$$f \circ_{\mathscr{C}} \mathsf{id}_X = f \text{ and } \mathsf{id}_X \circ_{\mathscr{C}} g = g.$$

A category is said to be ***small*** *if $\mathscr{C}_0$ is a set.*

4.40. Remark. *More intuitively, a category can be defined as a class of objects $\mathscr{C}_0$ and for every $(X, Y) \in \mathscr{C}_0^2$ a set $\mathsf{Hom}_{\mathscr{C}}(X, Y)$, called the space of morphisms from X to Y, satisfying that $\mathsf{Hom}_{\mathscr{C}}(X, Y) = \mathsf{Hom}_{\mathscr{C}}(X', Y')$ if and only if $X = X'$ and $Y = Y'$, as well as maps*

$$\circ_{X,Y,Z} = \circ_{\mathscr{C}}|_{\mathsf{Hom}_{\mathscr{C}}(Y,Z) \times \mathsf{Hom}_{\mathscr{C}}(X,Y)} : \mathsf{Hom}_{\mathscr{C}}(Y, Z) \times \mathsf{Hom}_{\mathscr{C}}(X, Y) \to \mathsf{Hom}_{\mathscr{C}}(X, Z)$$

for every triple of objects $(X, Y, Z) \in \mathscr{C}_0^3$, and a distinguished morphism $\mathsf{id}_X \in \mathsf{Hom}_{\mathscr{C}}(X, X)$ for every object $X \in \mathscr{C}_0$ satisfying the axioms (CAT.1) and (CAT.2). If the category is clear, we will denote its composition $\circ_{\mathscr{C}}$ simply by $\circ$. Moreover, we typically denote a morphism $f \in \mathsf{Hom}_{\mathscr{C}}(X, Y)$ simply by writting $f : X \to Y$.

4.41. Exercise. *(i) Let $\mathscr{C}_0$ be the class of all sets, $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ the set of all maps from X to Y, $\circ_{\mathscr{C}}$ the usual composition of maps and id_X is the identity map from X to itself. Prove that this forms a category, denoted by* Set.

(ii) Let $\mathscr{C}_0$ be the class of all groups, $\mathsf{Hom}_{\mathscr{C}}(X, Y)$ the set of all morphisms of groups from X to Y, $\circ_{\mathscr{C}}$ the usual composition of morphisms of groups and id_X is the identity map from X to itself. Prove that this forms a category, denoted by Grp.

(iii) *Let $\mathscr{C}_0$ be the class of all topological spaces, $\mathsf{Hom}_{\mathscr{C}}(X,Y)$ the set of all continuous maps from X to Y, $\circ_{\mathscr{C}}$ the usual composition of maps and id_X is the identity map from X to itself. Prove that this forms a category, denoted by* Top.

(iv) *Let G be a group and $\mathscr{C}_0$ be the class of all sets provided with a right action of G, $\mathsf{Hom}_{\mathscr{C}}(X,Y)$ the set of all G-linear (or G-equivariant) maps from X to Y, $\circ_{\mathscr{C}}$ the usual composition of maps and id_X is the identity map from X to itself. Prove that this forms a category, denoted by* Set-G.

4.42. Exercise. *Let $\mathscr{C} = (\mathscr{C}_0, \mathscr{C}_1, s_{\mathscr{C}}, t_{\mathscr{C}}, \circ_{\mathscr{C}}, i_{\mathscr{C}})$ be a category. Define $\mathscr{C}^{\mathrm{op}} = (\mathscr{C}_0, \mathscr{C}_1, s_{\mathscr{C}^{\mathrm{op}}}, t_{\mathscr{C}^{\mathrm{op}}}, \circ_{\mathscr{C}^{\mathrm{op}}}, i_{\mathscr{C}^{\mathrm{op}}})$, where $s_{\mathscr{C}^{\mathrm{op}}} = t_{\mathscr{C}}$, $t_{\mathscr{C}^{\mathrm{op}}} = s_{\mathscr{C}}$, $i_{\mathscr{C}^{\mathrm{op}}} = i_{\mathscr{C}}$ and $\circ_{\mathscr{C}^{\mathrm{op}}}(f,g) = \circ_{\mathscr{C}}(g,f)$, for*

$$(f,g) \in \mathscr{C}_1 \times_{t,s} \mathscr{C}_1 = \{(f,g) \in \mathscr{C}_1 \times \mathscr{C}_1 : t_{\mathscr{C}}(f) = s_{\mathscr{C}}(g)\} = \{(f,g) \in \mathscr{C}_1 \times \mathscr{C}_1 : s_{\mathscr{C}^{\mathrm{op}}}(f) = s_{\mathscr{C}^{\mathrm{op}}}(g)\}.$$

Prove that $\mathscr{C}^{\mathrm{op}}$ is a category. It is called the ***opposite category***.

4.43. Exercise. *Let $\mathscr{C} = (\mathscr{C}_0, \mathscr{C}_1, s_{\mathscr{C}}, t_{\mathscr{C}}, \circ_{\mathscr{C}}, i_{\mathscr{C}})$ and $\mathscr{D} = (\mathscr{D}_0, \mathscr{D}_1, s_{\mathscr{D}}, t_{\mathscr{D}}, \circ_{\mathscr{D}}, i_{\mathscr{D}})$ be two categories. Consider the tuple $\mathscr{C} \times \mathscr{D} = (\mathscr{C}_0 \times \mathscr{D}_0, \mathscr{C}_1 \times \mathscr{D}_1, s_{\mathscr{C}\times\mathscr{D}}, t_{\mathscr{C}\times\mathscr{D}}, \circ_{\mathscr{C}\times\mathscr{D}}, i_{\mathscr{C}\times\mathscr{D}})$, where $s_{\mathscr{C}\times\mathscr{D}}(f,g) = (s_{\mathscr{C}}(f), s_{\mathscr{D}}(g))$, $t_{\mathscr{C}\times\mathscr{D}}(f,g) = (t_{\mathscr{C}}(f), t_{\mathscr{D}}(g))$, $i_{\mathscr{C}\times\mathscr{D}}(X,Y) = (i_{\mathscr{C}}(X), i_{\mathscr{D}}(Y))$ and*

$$\circ_{\mathscr{C}\times\mathscr{D}}((f,g),(h,k)) = (\circ_{\mathscr{C}}(f,h), \circ_{\mathscr{D}}(g,k)),$$

for all $f,h \in \mathscr{C}_1 \times_{s,t} \mathscr{C}_1$, $g,k \in \mathscr{D}_1 \times_{s,t} \mathscr{D}_1$, $X \in \mathscr{C}_0$ and $Y \in \mathscr{D}_0$. Prove that $\mathscr{C} \times \mathscr{D}$ is a category. It is called the ***product category***.

4.44. A category $\mathscr{C}$ is called **filtered** if $\mathscr{C}_0 \neq \emptyset$,

(Fil.1) given objects X and Y in $\mathscr{C}$, there exists an object Z and morphisms $f : X \to Z$ and $g : Y \to Z$;

(Fil.2) given a pair of morphisms $h,k : U \to V$ in $\mathscr{C}$, there exists a morphism $j : V \to W$ such that $j \circ h = j \circ k$.

4.45. Exercise. *Let I be a nonempty set endowed with a partial order $\succeq$.*

(i) *Prove that there exists a unique category $\mathscr{C}$ such that $\mathscr{C}_0 = I$, $\mathsf{Hom}_{\mathscr{C}}(i,j)$ has exactly one element if $j \succeq i$, and $\mathsf{Hom}_{\mathscr{C}}(i,j)$ is empty otherwise.*

(ii) *Prove that if I satisfies the Moore-Smith condition, i.e. given $i,j \in I$ there exists $k \in I$ such that $k \succeq i$ and $k \succeq j$, then $\mathscr{C}$ is filtered.*

We will call $\mathscr{C}$ the ***category associated with the poset I***.

4.46. Let $\mathscr{C}$ be a category. A morphism $f \in \mathsf{Hom}_{\mathscr{C}}(X,Y)$ is called an **isomorphism** if there exists a morphism $g \in \mathsf{Hom}_{\mathscr{C}}(Y,X)$ such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$. In this case, we also say that X and Y are **isomorphic** objects, and we will write it $X \cong Y$. The set of isomorphisms from X to Y is denoted by $\mathsf{Iso}_{\mathscr{C}}(X,Y)$, and we define $\mathsf{Aut}_{\mathscr{C}}(X,Y)$ as the group $\mathsf{Iso}_{\mathscr{C}}(X,X)$ under the composition $\circ_{\mathscr{C}}$, which is called the group of **automorphisms** of X.

4.47. Let $f \in \mathsf{Hom}_{\mathscr{C}}(X,Y)$. We say that f is a **monomorphism** (resp., **epimorphism**) if given morphisms $g,g' \in \mathsf{Hom}_{\mathscr{C}}(Y,Z)$ (resp., $g,g' \in \mathsf{Hom}_{\mathscr{C}}(Z,X)$) such that $f \circ g = f \circ g'$ (resp., $g \circ f = g' \circ f$) then $g = g'$. It is clear that an isomorphism is also a monomorphism and an epimorphism.

4.48. Exercise. *Prove that there exists at most one morphism $g \in \mathsf{Hom}_{\mathscr{C}}(Y,X)$ satisfying the previous condition. This morphism, if it exists, is called the* ***inverse*** *of f.*

4.49. Exercise. *Prove that a morphism in* Set *or in* Grp *is an isomorphism if and only if it is bijective. Show that this does not hold in* Top.

Functors

4.50. Definition. *Given two categories $\mathscr{C}$ and $\mathscr{D}$, a* ***(covariant) functor*** *F from $\mathscr{C}$ to $\mathscr{D}$, usually written as $F : \mathscr{C} \to \mathscr{D}$, is a pair of maps (F_0, F_1) of the form $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ and $F_1 : \mathscr{C}_1 \to \mathscr{D}_1$ such that $t_{\mathscr{D}} \circ F_1 = F_0 \circ t_{\mathscr{C}}$, $s_{\mathscr{D}} \circ F_1 = F_0 \circ s_{\mathscr{C}}$, $F_1 \circ i_{\mathscr{C}} = i_{\mathscr{D}}$ and $F_1 \circ (\circ_{\mathscr{C}}) = (\circ_{\mathscr{D}}) \circ F$. A* ***contravariant functor*** *from $\mathscr{C}$ to $\mathscr{D}$ is just a (covariant) functor from $\mathscr{C}^{\mathrm{op}}$ to $\mathscr{D}$.*

4.51. Remark. *More intuitively, a functor $F : \mathscr{C} \to \mathscr{D}$ is given by a map $F_0 : \mathscr{C}_0 \to \mathscr{D}_0$ together with maps*

$$F_{X,Y} = F_1|_{\mathrm{Hom}_{\mathscr{C}}(X,Y)} : \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F_0(X), F_0(Y))$$

for all pairs $(X,Y) \in \mathscr{C}_0^2$ satisfying that

$$F_{X,X}(\mathrm{id}_X) = \mathrm{id}_{F_0(X)} \text{ and } F_{X,Z}(f \circ g) = F_{Y,Z}(f) \circ F_{X,Y}(g),$$

for all $X, Y, Z \in \mathscr{C}_0$ and for all morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(Y,Z)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$. By abuse of notation, and if it is clear from the context, it is usual to omit the subscripts and to denote both F_0 and F_1 simply by F.

4.52. Exercise. *Let $\mathscr{C}$ be a category, and consider $F = (F_0, F_1) : \mathscr{C} \to \mathscr{C}$ given as follows. For every object $X \in \mathscr{C}_0$, set $F(X) = X$, and for every morphism $f : X \to Y$ in $\mathscr{C}_1$ between the objects X and Y, define $F_1(f) = f$. Prove that this defines a functor, called the* ***identity functor****.*

4.53. Exercise. *Let $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ be a functor between two categories $\mathscr{C}$ and $\mathscr{D}$, and let $f \in \mathscr{C}_1$ be an isomorphism. Prove that $F_1(f)$ is an isomorphism.*

4.54. Exercise. *Let $\mathscr{C}$ be a category, and consider $F = (F_0, F_1) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathsf{Set}$ given as follows. For every pair of objects $(X,Y) \in \mathscr{C}_0^2$, set $F_0(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$, and for every pair of morphisms (f,g), with $f : X' \to X$ and $g : Y \to Y'$ in $\mathscr{C}_1$, define $F_1(f,g) : \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}}(X',Y')$ as the map sending $h \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ to $g \circ h \circ f$. Prove that this defines a functor, called the* ***Hom functor****, that we denote simply by $\mathrm{Hom}_{\mathscr{C}}(-,-)$.*

4.55. Exercise. *Let A be a ring. Prove that $(F_0, F_1) : \mathrm{Mod}_A \times {}_A\mathrm{Mod} \to {}_{\mathbb{Z}}\mathrm{Mod}$, given by $F_0(M,N) = M \otimes_A N$ and $F_1(f,g) = f \otimes_A g$ is a functor, called the* ***tensor product functor****, that we denote simply by $(-) \otimes_A (-)$.*

4.56. A functor $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ is an **equivalence** if $F_{X,Y} : \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F_0(X), F_0(Y))$ is bijective for all objects X and Y in $\mathscr{C}_0$, and for every object Z in $\mathscr{D}$, there is an object X in $\mathscr{C}_0$ such that $F_0(X) \cong Z$. If a functor satisfies the first part of the definition it is called **fully faithful**, whereas a functor satisfies the second part it is called **dense**.

4.57. Given functors $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ and $G = (G_0, G_1) : \mathscr{D} \to \mathscr{E}$, it is easy to see that $(G_0 \circ F_0, G_1 \circ F_1)$ is a functor from $\mathscr{C}$ to $\mathscr{E}$, called the **composition functor**, and it is denoted by $G \circ F$.

Natural transformations

4.58. Given two functors $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ and $G = (G_0, G_1) : \mathscr{C} \to \mathscr{D}$, a **natural transformation** from F to G is a map $h : \mathscr{C}_0 \to \mathscr{D}_1$ such that $s_{\mathscr{D}} \circ h = F_0$, $t_{\mathscr{D}} \circ h = G_0$, and for every $f \in \mathscr{C}_1$, $G_1(f) \circ h(s_{\mathscr{C}}(f)) = h(t_{\mathscr{C}}(f)) \circ F_1(f)$. Equivalently, a natural transformation from F to G is a class of morphisms $\{h_X = h(X) \in \mathrm{Hom}_{\mathscr{D}}(F_0(X), G_0(X)) : X \in \mathscr{C}_0\}$ such that $G_1(f) \circ h_X = h_Y \circ F_1(f)$, for every morphism $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, which can be represented graphically as the commutation of the following diagram

$$\begin{tikzcd}
F_0(X) \arrow[r, "h_X"] \arrow[d, "F_1(f)"] & G_0(X) \arrow[d, "G_1(f)"] \\
F_0(Y) \arrow[r, "h_Y"] & G_0(Y)
\end{tikzcd}$$

A **natural isomorphism** is a natural transformation h from F to G such that $h(X)$ is an isomorphism in $\mathscr{D}$, for all $X \in \mathscr{C}_0$. We say in this case that F and G are **naturally isomorphic**.

4.59. Exercise. *Let $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ be a functor between two categories $\mathscr{C}$ and $\mathscr{D}$. Prove that F is an equivalence if and only if there exists a functor $G = (G_0, G_1) : \mathscr{D} \to \mathscr{C}$ such that $F \circ G$ is naturally isomorphic to the identity functor of $\mathscr{D}$ and $G \circ F$ is naturally isomorphic to the identity functor of $\mathscr{C}$. The functor G is called a **quasi-inverse**.*

4.60. Exercise. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, such that $\mathscr{C}$ is small. Define $\mathsf{Fun}(\mathscr{C}, \mathscr{D})_0$ be the class formed by all functors from $\mathscr{C}$ to $\mathscr{D}$. Given functors $F, G : \mathscr{C} \to \mathscr{D}$, define $\mathsf{Hom}_{\mathsf{Fun}(\mathscr{C},\mathscr{D})}(F, G)$ to be the set of all natural transformations from F to G. Given a functor $F : \mathscr{C} \to \mathscr{D}$, define the natural transformation ${}_F$ from F to F by ${}_F(X) = \mathrm{id}_{F(X)}$ for all $X \in \mathscr{C}_0$. Moreover, given functors $F, G, H : \mathscr{C} \to \mathscr{D}$, if h is natural transformation from F to G and k is natural transformation from G to H, define the natural transformation $k \circ h$ from F to H as $(k \circ h)(X) = k(X) \circ h(X)$ for all $X \in \mathscr{C}_0$. Prove that the previous data defines a category $\mathsf{Fun}(\mathscr{C}, \mathscr{D})$.*

4.61. Exercise. *Let $\mathscr{C}$ be a category. Given an object $X \in \mathscr{C}_0$, let $\mathsf{Yo}^{\mathrm{op}}(X) = \mathsf{Hom}_{\mathscr{C}}(X, -)$ be the functor in $\mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ and $\mathsf{Yo}(X) = \mathsf{Hom}_{\mathscr{C}}(-, X)$ be the functor in $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$. If $f : X' \to X$ is a morphism in $\mathscr{C}$, let $\mathsf{Yo}^{\mathrm{op}}(f)$ be the natural transformation*

$$\mathsf{Hom}_{\mathscr{C}}(X, -) \to \mathsf{Hom}_{\mathscr{C}}(X', -)$$

given by the family of maps

$$\mathsf{Yo}^{\mathrm{op}}(f)(Y) : \mathsf{Hom}_{\mathscr{C}}(X, Y) \to \mathsf{Hom}_{\mathscr{C}}(X', Y)$$

sending $g \in \mathsf{Hom}_{\mathscr{C}}(X, Y)$ to $g \circ f$, for all $Y \in \mathscr{C}_0$. Analogously, let $\mathsf{Yo}(f)$ be the natural transformation

$$\mathsf{Hom}_{\mathscr{C}}(-, X') \to \mathsf{Hom}_{\mathscr{C}}(-, X)$$

given by the family of maps

$$\mathsf{Yo}(f)(Y) : \mathsf{Hom}_{\mathscr{C}}(Y, X) \to \mathsf{Hom}_{\mathscr{C}}(Y, X')$$

*sending $g \in \mathsf{Hom}_{\mathscr{C}}(Y, X)$ to $f \circ g$, for all $Y \in \mathscr{C}_0$. Prove that the previous constructions induce fully faithful functors $\mathsf{Yo}^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ and $\mathsf{Yo} : \mathscr{C} \to \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$. They are called the **Yoneda** functors.*

4.62. Given a functor $F : \mathscr{C} \to \mathsf{Set}$, we say it is **representable** if there exists an object X in $\mathscr{C}$ and a natural isomorphism $F \cong \mathsf{Hom}_{\mathscr{C}}(X, -)$, and we call X a **representative** of F. Note that previous exercise tells us that if X and X' are two representatives of F, then $X \cong X'$ in $\mathscr{C}$. Moreover, if the functor is $F : \mathscr{C}^{\mathrm{op}} \to \mathsf{Set}$ is representable, we will usually express this by means of $F \cong \mathsf{Hom}_{\mathscr{C}}(-, X)$, to which the previous comment also apply.

4.63. Given a category $\mathscr{C}$, a subset $S \subseteq \mathscr{C}_0$ is called a **set of isomorphic classes of objects** of $\mathscr{C}$ if for every $Y \in \mathscr{C}_0$ there is $X \in S$ such that $X \cong Y$, and given $X, X' \in S$, $X \cong S'$ implies that $S = S'$.

4.64. Exercise. *Let $\mathscr{C}$ be a category with a set S of isomorphic classes of objects.*

(i) Assume that $S' \subseteq \mathscr{C}_0$ is another set of isomorphic classes of objects of $\mathscr{C}$. Prove that S and S' are in bijection.

(ii) and let $F = (F_0, F_1) : \mathscr{C} \to \mathscr{D}$ be an equivalence between two categories $\mathscr{C}$ and $\mathscr{D}$. Prove that the set $\{F(X) : X \in S\} \subseteq \mathscr{D}_0$ is a set of isomorphic classes of objects of $\mathscr{D}$.

Various universal constructions

4.65. Given a family of objects $\{X_i\}_{i \in I}$ indexed by a set in a category $\mathscr{C}$, we define its **product** as the object $P \in \mathscr{C}_0$ together with the family of morphisms $p_i : P \to X_i$ satisfying that, given any other object $P' \in \mathscr{C}_0$ together with the family of morphisms $p'_i : P' \to X_i$, there exists a unique morphism $f : P' \to P$ such that $p_i \circ f = p'_i$, for all $i \in I$. It is easy to see that, if the product exists, it is unique up to unique isomorphism. We denote the product object P typically by $\prod_{i \in I} X_i$.

4.66. Dually, given a family of objects $\{X_i\}_{i\in I}$ in a category $\mathscr{C}$ indexed by a set I, we define its **coproduct** as the object $C \in \mathscr{C}_0$ together with the family of morphisms $u_i : X_i \to C$ satisfying that, given any other object $C' \in \mathscr{C}_0$ together with the family of morphisms $u'_i : X_i \to C'$, there exists a unique morphism $f : C \to C'$ such that $f \circ u_i = u'_i$, for all $i \in I$. Again, it is easy to see that, if the product exists, it is unique up to unique isomorphism. The careful reader would have also noticed that a coproduct in $\mathscr{C}$ is exactly the same as a product in the opposite category $\mathscr{C}^{\mathrm{op}}$ of $\mathscr{C}$ (see Exercise 4.42). We denote the product object C typically by $\coprod_{i\in I} X_i$.

4.67. An object I in a category $\mathscr{C}$ is called **initial** (resp., **final**) if given any other object $X \in \mathscr{C}_0$ there exists a unique morphism $I \to X$ (resp., $X \to F$). It is easy to see that the initial (resp., final) object is unique up to unique isomorphism. The careful reader can verify that a final object in $\mathscr{C}$ is exactly the same as an initial object in the opposite category $\mathscr{C}^{\mathrm{op}}$ of $\mathscr{C}$. Moreover, the initial (resp., final) object is the same as the coproduct (resp., product) over the empty family of objects.

4.68. Exercise. *Let $\{X_i\}_{i\in I}$ be a family of objects in a category $\mathscr{C}$ indexed by a set I. Prove that $(P, (p_i)_{i\in I})$ is a product (resp., $(C, (u_i)_{i\in I})$ is a coproduct) if and only if the natural transformation*

$$\mathsf{Hom}_{\mathscr{C}}(-, P) \longrightarrow \prod_{i\in I} \mathsf{Hom}_{\mathscr{C}}(-, X_i) \qquad \left(\textit{resp., } \mathsf{Hom}_{\mathscr{C}}(C, -) \longrightarrow \prod_{i\in I} \mathsf{Hom}_{\mathscr{C}}(X_i, -) \right)$$

sending $f : X \to P$ to $(p_i \circ f)_{i\in I}$ (resp., $f : C \to X$ to $(f \circ u_i)_{i\in I}$), for $X \in \mathscr{C}_0$ is an isomorphism. In the language of representable functors (see paragraph 4.62), a product (resp., coproduct) of $\{X_i\}_{i\in I}$ is a representative of the contravariant (resp., covariant) functor

$$\prod_{i\in I} \mathsf{Hom}_{\mathscr{C}} \qquad \left(\textit{resp., } \prod_{i\in I} \mathsf{Hom}_{\mathscr{C}}(X_i, -) \right).$$

4.69. Exercise. *Let $\mathscr{C}$ be a category and let X be an object in $\mathscr{C}$. Assume that both the coproduct $X \coprod X$ and the product $X \prod X$ exist. We denote the associated morphisms of the coproduct by $u_1, u_2 : X \to X \coprod X$, and those of the product by $p_1, p_2 : X \prod X \to X$.*

*(i) Prove that there exists a unique morphism $\nabla_X : X \coprod X \to X$, called the **codiagonal**, satisfying that $\nabla_X \circ u_i = \mathrm{id}_X$ for $i = 1, 2$.*

*(ii) Prove that there exists a unique morphism $\Delta_X : X \to X \prod X$, called the **diagonal**, satisfying that $p_i \circ \Delta_X = \mathrm{id}_X$ for $i = 1, 2$.*

4.70. Exercise. *(i) Let Set be the category of all sets. Prove that products exist and they coincide with the usual cartesian products. Analogously, prove that coproducts exist and they are given by disjointed unions. Finally, prove that the empty set is an initial object and that the singleton set is a final object.*

(ii) Let Grp be the category of all groups. Prove that products exist and they coincide with the usual cartesian products. Moreover, prove that the initial and the final objects are given by the trivial group $\{1\}$.

(iii) Let ${}_A\mathsf{Mod}$ be the category of all modules over a ring A. Prove that products exist and they coincide with the usual cartesian products. Analogously, prove that coproducts exist and they are given by sirect sums of modules. Moreover, prove that the initial and the final objects are given by the trivial module $0 = \{\mathbf{0}\}$.

(iv) Let Top be the category of all topological spaces. Prove that products exist and they coincide with the usual cartesian products, endowed with the usual (Tychonov) topology. Analogously, prove that coproducts exist and they are given by disjointed unions, endowed with the disjoint union topology. Finally, prove that the empty set is an initial object and that the singleton set is a final object.

4.71. Exercise. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor with $\mathscr{C}$ small. A **cone** (resp., **cocone**) of F is an object $Y \in \mathscr{D}$ and a class of morphisms $p_X : Y \to F(X)$ (resp., $p_X : F(X) \to Y$) for all $X \in \mathscr{C}_0$ such that $F(f) \circ p_X = p_{X'}$ ($p_{X'} \circ F(f) = p_X$) for all morphism $f \in \mathsf{Hom}_{\mathscr{C}}(X, X')$. A **morphism of cones** (resp., **morphism of cococones**) from (Y, p) to (Z, q) is a morphism $g : Y \to Z$ in $\mathscr{D}$ such that $q_X \circ g = p_X$ (resp., $g \circ p_X = q_X$) for all $X \in \mathscr{C}_0$.*

Prove that composition and units of $\mathscr{D}$ make of the class of all cones (resp., cocones) of F together with their morphisms a category, that we denote by $\mathsf{Cone}(F)$ (resp., $\mathsf{Cocone}(F)$).

4.72. If it exists, the final (resp., initial) object of the category $\mathsf{Cone}(F)$ (resp., $\mathsf{Cocone}(F)$) defined in the previous exercise is called the **limit** (resp., **colimit**) of F. It is denoted by

$$\lim_{\mathscr{C}} F \qquad \left(\text{resp., } \operatorname{colim}_{\mathscr{C}} F\right).$$

4.73. Exercise. *Let $\mathscr{D}$ be the category Set of sets, let $\mathscr{C}$ be a small category and let $F : \mathscr{C} \to \mathscr{D}$ be a functor.*

(i) Prove that $\lim_{\mathscr{C}} F$ exists and it is given by

$$\left\{(m_X)_{X\in\mathscr{C}_0} \in \prod_{X\in\mathscr{C}_0} F(X) : F(f)(m_X) = m_Y, \text{ for all } f \in \mathsf{Hom}_{\mathscr{C}}(X,Y)\right\} \subseteq \prod_{X\in\mathscr{C}_0} F(X)$$

together with the structure maps $\{p_X : \lim_{\mathscr{C}} F \to F(X)\}_{X\in\mathscr{C}_0}$, where p_X is given as the composition of the inclusion into $\prod_{X\in\mathscr{C}_0} F(X)$ and the canonical projection onto $F(X)$.

(ii) Prove that $\operatorname{colim}_{\mathscr{C}} F$ exists and it is given as the quotient of the disjoint union $\sqcup_{X\in\mathscr{C}_0} F(X)$ by the equivalence relation generated by $F(f)(m_X) \sim m_Y$ for all $f \in \mathsf{Hom}_{\mathscr{C}}(X,Y)$, $m_X \in F(X)$ and $X \in \mathscr{C}_0$, where $F(f)(m_X) \in F(Y)$, together with the structure maps $\{u_X : F(X) \to \operatorname{colim}_{\mathscr{C}} F\}_{X\in\mathscr{C}_0}$, where u_X is given as the composition of the inclusion into $\sqcup_{X\in\mathscr{C}_0} F(X)$ and the canonical projection onto the quotient.

4.74. Exercise. *Let $\mathscr{D}$ be the category ${}_A\mathsf{Mod}$ for some ring A, let $\mathscr{C}$ be a small category and let $F : \mathscr{C} \to \mathscr{D}$ be a functor.*

(i) Let $\mathscr{O} : {}_A\mathsf{Mod} \to \mathsf{Set}$ be the forgetful functor, i.e. the one sending a module to its underlying set. Prove that the object $\lim_{\mathscr{C}}(\mathscr{O} \circ F)$ described in the first item of the previous exercise endowed with the canonical module structure over A is the limit $\lim_{\mathscr{C}} F$.

(ii) Prove that $\operatorname{colim}_{\mathscr{C}} F$ exists and it is given as the quotient of $\oplus_{X\in\mathscr{C}_0} F(X)$ by the submodule generated by the subset

$$\left\{F(f)(m_X) - m_Y : \text{ for all } f \in \mathsf{Hom}_{\mathscr{C}}(X,Y), m_X \in F(X), X \in \mathscr{C}_0\right\},$$

where $F(f)(m_X) \in F(Y)$, together with the structure maps $\{u_X : F(X) \to \operatorname{colim}_{\mathscr{C}} F\}_{X\in\mathscr{C}_0}$, where u_X is given as the composition of the inclusion into $\oplus_{X\in\mathscr{C}_0} F(X)$ and the canonical projection onto the quotient.

4.75. Exercise. *Let $\mathscr{C}$ be a small category and let $\mathscr{D}$ be a complete (resp., cocomplete) category.*

(i) Let F, G be two functors from $\mathscr{C}$ to $\mathscr{D}$ and let $t : F \to G$ be a natural transformation. Prove that there exists a unique morphism $\hat{t} : \lim_{\mathscr{C}} F \to \lim_{\mathscr{C}} G$ in $\mathscr{D}$ such that $q_X \circ \hat{t} = t_X \circ p_X$ (resp., $v_X \circ t_X = \hat{t} \circ u_X$) for all $X \in \mathscr{C}_0$, where $p_X : \lim_{\mathscr{C}} F \to F(X)$ and $q_X : \lim_{\mathscr{C}} G \to G(X)$ (resp., $u_X : F(X) \to \operatorname{colim}_{\mathscr{C}} F$ and $v_X : G(X) \to \lim_{\mathscr{C}} G$) are the structure morphisms for all $X \in \mathscr{C}_0$.

(ii) Prove that the assignement $\mathsf{Fun}(\mathscr{C},\mathscr{D}) \to \mathscr{D}$ sending the functor F to its limit (resp., colimit) $\lim_{\mathscr{C}}$ (resp., $\operatorname{colim}_{\mathscr{C}}$) and a natural transformation $t : F \to G$ to the morphism $\hat{t} : \lim_{\mathscr{C}} F \to \lim_{\mathscr{C}} G$ (resp., $\hat{t} : \operatorname{colim}_{\mathscr{C}} F \to \operatorname{colim}_{\mathscr{C}} G$) in the previous item is a functor. It is denoted $\lim_{\mathscr{C}}$ (resp., $\operatorname{colim}_{\mathscr{C}}$).

4.76. Exercise. *Let $\mathscr{C}$ be a category, $\mathscr{D}$ be a complete and cocomplete category and let $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ be the category of functors. Let $\mathscr{E}$ be a small category and $F : \mathscr{E} \to \mathsf{Fun}(\mathscr{C},\mathscr{D})$ be a functor. The aim of this exercise is to prove that the limit and the colimit of F exist, so $\mathsf{Fun}(\mathscr{C},\mathscr{D})$ is complete and cocomplete.*

(i) Given an object Y of $\mathscr{C}$, let $\mathsf{ev}_Y(F)$ denote the assignement $\mathscr{E} \to \mathscr{D}$ sending an object X in $\mathscr{E}$ to $F(X)(Y)$ and a morphism $f : X \to X'$ in $\mathscr{E}$ to $F(f)(Y)$. Prove that $\mathsf{ev}_Y(F)$ is a functor from $\mathscr{E}$ to $\mathscr{D}$.

(ii) Given a morphism $g : Y' \to Y$ in $\mathscr{C}$, let $\mathsf{ev}_g(F)$ denote the family of morphisms

$$\{\mathsf{ev}_g(F) : \mathsf{ev}_{Y'}(F)(X) \to \mathsf{ev}_Y(F)(X)\}_{X\in\mathscr{E}_0}$$

given by $\mathsf{ev}_g(F) = F(X)(g)$, for all $X \in \mathscr{E}_0$. Prove that $\mathsf{ev}_g(F)$ is a natural transformation from $\mathsf{ev}_{Y'}(F)$ to $\mathsf{ev}_Y(F)$.

(iii) *Prove that the assignment* $\mathsf{ev}(F) : \mathscr{C} \to \mathsf{Fun}(\mathscr{E}, \mathscr{D})$ *sending an object* Y *of* $\mathscr{C}$ *to* $\mathsf{ev}_Y(F)$ *and a morphism* $g : Y' \to Y$ *in* $\mathscr{C}$ *to* $\mathsf{ev}_Y(F)$ *is a functor.*

(iv) *Prove that there are natural isomorphisms*

$$\lim_{\mathscr{E}} F \cong \lim_{\mathscr{E}} \circ \mathsf{ev}(F) \text{ and } \operatorname{colim}_{\mathscr{E}} F \cong \operatorname{colim}_{\mathscr{E}} \circ \mathsf{ev}(F)$$

in $\mathsf{Fun}(\mathscr{C}, \mathsf{Set})$, *where the right occurrences of* $\lim_{\mathscr{E}}$ *and* $\operatorname{colim}_{\mathscr{E}}$ *denote the functors* $\mathsf{Fun}(\mathscr{E}, \mathscr{D}) \to \mathscr{D}$ *given by taking limit and colimit, respectively (see Exercise 4.75).*

4.77. A category is said to be **complete** (resp., **cocomplete**) if it has all limits (resp., colimits) of all functors whose domain is a small category. A functor $G : \mathscr{C} \to \mathscr{D}$ is said to **preserve limits** (resp., **preserve colimits**)if given any limit (resp., colimit) (Y, p) of a functor $F : \mathscr{E} \to \mathscr{C}$, $(G(Y), G(p))$ is a limit (resp., colimit) of the functor $G \circ F$.

4.78. Exercise (*cf.* Exercise 4.68)**.** *Let* $F : \mathscr{E} \to \mathscr{C}$, *where* $\mathscr{E}$ *is a small category. Recall the Yoneda functors* $\mathsf{Yo} : \mathscr{C} \to \mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$ *and* $\mathsf{Yo}^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathscr{C}, \mathsf{Set})$ *defined in Exercise 4.61. Prove that a limit (resp., colimit) of* F *can be equivalently defined as a representative of the functor* $\lim_{\mathscr{E}} \mathsf{Yo} \circ F$ *in* $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Set})$ *(resp.,* $\lim_{\mathscr{E}} \mathsf{Yo}^{\mathrm{op}} \circ F$ *in* $\mathsf{Fun}(\mathscr{C}, \mathsf{Set})$*, where we denote also by* F *the functor* $\mathscr{E}^{\mathrm{op}} \to \mathscr{C}^{\mathrm{op}}$ *induced by* $F : \mathscr{E} \to \mathscr{C}$*).*

4.79. Exercise. *Let* $\{X_i\}_{i \in I}$ *be a family of objects in a category* $\mathscr{D}$ *indexed by a set* I*.*

(i) *Prove that there exists a unique category* $\mathscr{C}$ *with set of objects* I *such that* $\mathsf{Hom}_{\mathscr{C}}(i, j)$ *is* $\{\mathsf{id}_i\}$ *if* $i = j$ *and empty if* $i \neq j$*.*

(ii) *Prove that there is a unique functor* $F : \mathscr{C} \to \mathscr{D}$ *sending* $i \in I$ *to* X_i*, for all* $i \in I$*.*

(iii) *Prove that, if it exists, the limit (resp., colimit) of* F *is precisely the product (resp., coproduct) of* $\{X_i\}_{i \in I}$*.*

4.80. Exercise. *Let* $f : X \to Y$ *be a morphism in a category* $\mathscr{D}$*.*

(i) *Let* $\mathscr{C}$ *be category associated with the partially ordered set* $I = \{a, b, c\}$ *endowed with the partial order* $a \succ c$ *and* $b \succ c$*, and* $F : \mathscr{C} \to \mathscr{D}$ *be the unique functor sending* a, b *to* X *and* c *to* Y*, and* $F(\mathsf{Hom}_{\mathscr{C}}(a, c)) = F(\mathsf{Hom}_{\mathscr{C}}(b, c)) = \{f\}$*. Prove that* f *is a monomorphism if and only if the limit of* F *exists and is the object* X *together with the morphisms* $p_a = p_b = \mathsf{id}_X$ *and* $p_c = f$*.*

(ii) *Let* $\mathscr{C}$ *be category associated with the partially ordered set* $I = \{a, b, c\}$ *endowed with the partial order* $a \succ c$ *and* $a \succ b$*, and* $F : \mathscr{C} \to \mathscr{D}$ *be the unique functor sending* a *to* X *and* b, c *to* Y*, and* $F(\mathsf{Hom}_{\mathscr{C}}(a, b)) = F(\mathsf{Hom}_{\mathscr{C}}(a, c)) = \{f\}$*. Prove that* f *is an epimorphism if and only if the colimit of* F *exists and it is the object* X *together with the morphisms* $p_a = f$ *and* $p_b = p_c = \mathsf{id}_Y$*.*

4.81. A limit (resp., colimit) of a functor $F : \mathscr{C} \to \mathscr{D}$ is said to be **filtered** if the category $\mathscr{C}$ is filtered. We usually denote this situation by

$$\lim_{\leftarrow \mathscr{C}} F \qquad \left(\text{resp., } \operatorname{colim}_{\rightarrow \mathscr{C}} F \right).$$

4.82. Exercise. *Let* $\mathscr{D}$ *be the category* ${}_A\mathsf{Mod}$ *for some ring* A*, let* $\mathscr{C}$ *be a filtered small category and let* $F : \mathscr{C} \to \mathscr{D}$ *be a functor.*

(i) *Let* $C' = \sqcup_{X \in \mathscr{C}_0} F(X)$ *and* $\sim$ *the equivalence relation on* L' *given by* $m_X \sim m_{X'}$ *for* $m_X \in F(X)$ *and* $m_{X'} \in F(X')$ *if there exists* $X'' \in \mathscr{C}_0$ *and morphisms* $f : X \to X''$ *and* $g : X' \to X''$ *in* $\mathscr{C}$ *such that* $F(f)(m_X) = F(f)(m_{X'})$*. Define* $C = C/\sim$ *and for* $X \in \mathscr{C}_0$ *set* $u_X : F(X) \to C$ *as the composition of the canonical inclusion* $F(X) \to C'$ *and the canonical projection* $C' \to C$*. Prove that* C *has a natural structure of* A*-module such that* u_X *is a morphism of* A*-modules for all* $X \in \mathscr{C}_0$*. Prove moreover that* C *together with the family of morphisms* $(u_X)_{X \in \mathscr{C}_0}$ *is a filtered colimit of* F*.*

(ii) *From the previous description, deduce in particular that, given* $m \in \operatorname{colim}_{\rightarrow \mathscr{C}} F$*, there exists* $X \in \mathscr{C}_0$ *and* $m_X \in F(X)$ *such that* m *is the image of* m_X *under the structure morphism* $F(X) \to \operatorname{colim}_{\rightarrow \mathscr{C}} F$*.*

(iii) *From the description of the filtered colimit in the previous item, deduce in particular that, for every* $X \in \mathscr{C}_0$*, the kernel of the structure morphism* $F(X) \to \operatorname{colim}_{\rightarrow \mathscr{C}} F$ *is the union of the kernels of the maps* $F(f) : F(X) \to F(Y)$ *for all morphisms* $f : X \to Y$ *in* $\mathscr{C}$*.*

Adjoint pairs of functors

4.83. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be two functors. We say that (F, G) is an **adjoint pair** of functors if there exists an natural isomorphism between the functors $\mathsf{Hom}_{\mathscr{D}}(F(-), -), \mathsf{Hom}_{\mathscr{C}}(-, G(-)) : \mathscr{C}^{\mathsf{op}} \times \mathscr{D} \to \mathsf{Set}$. We also say that F is a **left adjoint** and that G is a **right adjoint**.

4.84. Exercise. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, and $F, F' : \mathscr{C} \to \mathscr{D}$ and $G, G' : \mathscr{D} \to \mathscr{C}$ be functors.*

(i) Prove that if (F, G) and (F, G') are adjoint pairs, there is a natural isomorphism between G and G'.

(ii) Prove that if (F, G) and (F', G) are adjoint pairs, there is a natural isomorphism between F and F'.

4.85. Exercise. *Let A and B be two rings and N be a $A \otimes_{\mathbb{Z}} B^{\mathsf{op}}$-module. We recall that, given an A-module M and a B-module M', $\mathsf{Hom}_A(N, M)$ has a natural structure of B-module via $(b \cdot f)(n) = f(n \cdot b)$, for all $b \in B$, $n \in N$ and $f \in \mathsf{Hom}_A(N, M)$, and $N \otimes_B M'$ has a natural structure of A-module via $a \cdot (n \otimes_B m') = (a \cdot n) \otimes_B m'$, for all $a \in A$, $n \in N$ and $m' \in M'$.*

(i) Prove that the functor $N \otimes_B (-) : {}_B\mathsf{Mod} \to {}_{\mathbb{Z}}\mathsf{Mod}$ in Exercise 4.55 factors canonically through the forgetful functor ${}_A\mathsf{Mod} \to {}_{\mathbb{Z}}\mathsf{Mod}$. We will denote the resulting functor also by $N \otimes_B (-) : {}_B\mathsf{Mod} \to {}_A\mathsf{Mod}$.

(ii) Prove that the functor $\mathsf{Hom}_A(N, -) : {}_A\mathsf{Mod} \to {}_{\mathbb{Z}}\mathsf{Mod}$ in Exercise 4.54 factors canonically through the forgetful functor ${}_B\mathsf{Mod} \to {}_{\mathbb{Z}}\mathsf{Mod}$. We denote the resulting functor by $\mathsf{Hom}_A(N, -) : {}_A\mathsf{Mod} \to {}_B\mathsf{Mod}$.

(iii) Prove that the functor $N \otimes_B (-) : {}_B\mathsf{Mod} \to {}_A\mathsf{Mod}$ is left adjoint to $\mathsf{Hom}_A(N, -) : {}_A\mathsf{Mod} \to {}_B\mathsf{Mod}$.

4.86. Theorem. *Let $\mathscr{C}$ and $\mathscr{D}$ be two categories, and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be two functors such that (F, G) is an adjoint pair. Assume that $\mathscr{C}$ and $\mathscr{D}$ have limits and colimits. Prove that F preserves colimits and G preserves limits. In particular, F preserves coproducts and epimorphisms, and G preserves products and monomorphisms (see Exercises 4.79 and 4.80).*

Proof. We will prove that G preserves limits and leave the case of colimits to the reader. Let $H : \mathscr{E} \to \mathscr{D}$ be a functor, where $\mathscr{E}$ is a small category, and let $(L, u_X : L \to H(X))_{X \in \mathscr{E}_0}$ be a limit of H. Consider the functor $G \circ H$. By Exercise 4.78, its limit is exactly a representative of the $\lim_{\mathscr{E}} \mathsf{Yo} \circ G \circ H$. Using Exercise 4.76, the fact that G is right adjoint to F as well as that L is a representative of the $\lim Y \circ H$ we get that

$$\begin{aligned}
\Big(\lim_{\mathscr{E}} \mathsf{Yo} \circ G \circ H\Big)(X) &\cong \lim_{\mathscr{E}} \mathsf{ev}_X \circ \mathsf{Yo} \circ G \circ H = \lim_{\mathscr{E}} \mathsf{Hom}_{\mathscr{C}}(X, (G \circ H)(-)) \\
&\cong \lim_{\mathscr{E}} \mathsf{Hom}_{\mathscr{D}}(F(X), H(-)) = \lim_{\mathscr{E}} \mathsf{ev}_{F(X)} \circ \mathsf{Yo} \circ H \\
&\cong \Big(\lim_{\mathscr{E}} \mathsf{Yo} \circ H\Big)(F(X)) \cong \mathsf{Yo}(L)(F(X)) = \mathsf{Hom}_{\mathscr{D}}(F(X), L) \cong \mathsf{Hom}_{\mathscr{C}}(X, G(L)),
\end{aligned}$$

for every object X in $\mathscr{C}$. Moreover, it is easy to see that the previous isomorphisms are natural in X, which implies that $G(L)$ is a representative of $\lim_{\mathscr{E}} \mathsf{Yo} \circ G \circ H$, *i.e.* $G(L)$ is its limit by Exercise 4.78. □

4.87. Exercise. *Let $\mathscr{C}$ be a small filtered category and $\mathscr{D}$ be a complete (resp., cocomplete) category.*

(i) Consider the assignment $(F_0, F_1) : \mathscr{D} \to \mathsf{Fun}(\mathscr{C}, \mathscr{D})$ sending $X \in \mathscr{D}_0$ to the functor $F(X)$ satisfying that $F(X)(Y) = X$ and $F(X)(f) = \mathsf{id}_X$ for all $Y \in \mathscr{C}_0$ and $f \in \mathscr{C}_1$. Prove that this defines a functor, called the ***constant functor****.*

(ii) Prove that the functor $\lim_{\mathscr{C}} : \mathsf{Fun}(\mathscr{C}, \mathscr{D}) \to \mathscr{D}$ (resp., $\mathsf{colim}_{\mathscr{C}} : \mathsf{Fun}(\mathscr{C}, \mathscr{D}) \to \mathscr{D}$) defined in Exercise 4.75 is right (resp., left) adjoint to the constant functor defined in the previous item.

4.88. Exercise. *Let $p \in \mathbb{Z}$ be a prime number.*

(i) Let $\mathscr{C}$ be the category associated with $\mathbb{N}$ endowed with the usual order. Recall that given $n \in \mathbb{N}$, there exists a canonical monomorphism of abelian groups $\mathbb{Z}/p^n.\mathbb{Z} \to \mathbb{Z}/p^{n+1}.\mathbb{Z}$ given by multiplication by p. Prove that the previous morphisms induce a functor $F : \mathscr{C} \to {}_{\mathbb{Z}}\mathsf{Mod}$ sending $n \in \mathbb{N}$ to $\mathbb{Z}/p^n.\mathbb{Z}$. The colimit of this functor is called the ***Prüfer p-group****. It is usually denoted by $\mathbb{Z}_{p^\infty}$.*

(ii) *Given $n \in \mathbb{N}$, define $u_n : \mathbb{Z}/p^n.\mathbb{Z} \to \mathbb{Z}[1/p]/\mathbb{Z}$ as the unique morphism of abelian groups sending $\bar{1} \in \mathbb{Z}/p^n.\mathbb{Z}$ to the class of $1/p^n$ in $\mathbb{Z}[1/p]/\mathbb{Z}$. Prove that these morphisms induce an isomorphism of abelian groups $\mathbb{Z}_{p^\infty} \cong \mathbb{Z}[1/p]/\mathbb{Z}$.*

(iii) *Let $\mathscr{C}'$ be the category associated with $\mathbb{N}$ endowed with the reversed order. Recall that given $n \in \mathbb{N}$, there exists a canonical epimorphism of abelian groups $\mathbb{Z}/p^{n+1}.\mathbb{Z} \to \mathbb{Z}/p^n.\mathbb{Z}$ induced by the identity. Prove that the previous morphisms induce a functor $F' : \mathscr{C}' \to {}_{\mathbb{Z}}\mathsf{Mod}$ sending $n \in \mathbb{N}$ to $\mathbb{Z}/p^n.\mathbb{Z}$. The limit of this functor is called the* ***group of p-adic integers****. It is usually denoted by $\mathbb{Z}_p$.*

(iv) *Prove that there is an isomorphism of abelian groups $\mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}_p$.*

Additive categories

4.89. If an object is both initial and final coincide we say that the category has **zero object**, which is usually denoted by $0_{\mathscr{C}}$. If we fix a zero object $0_{\mathscr{C}}$, given two objects X and Y in $\mathscr{C}$ we call the unique map $X \to 0_{\mathscr{C}} \to Y$ the **zero morphism**, and we denote it simply by 0. The careful reader will verify that the definition of zero morphism is independent of the choice of zero object.

4.90. Exercise. *Let $\{X_i\}_{i\in I}$ be a family of objects in a category $\mathscr{C}$ indexed by a set I. Assume that $\mathscr{C}$ has a zero objects, and that both its product $(\prod_{i\in I} X_i, (p_i)_{i\in I})$ and its coproduct $(\coprod_{i\in I} X_i, (u_i)_{i\in I})$ exist in $\mathscr{C}$. Given $i, j \in I$, define the morphism $\delta_{i,j} : X_i \to X_j$ as the zero morphism if $i \neq j$ and as the identity of X_i if $i = j$.*

(i) *Prove that the property of the product and the coproduct induce morphisms $\varepsilon_i : X_i \to \prod_{i\in I} X_i$ and $\pi_i : \coprod_{i\in I} X_i \to X_i$ for all $i \in I$ such that $\pi_i \circ u_j = \delta_{i,j} = p_i \circ \varepsilon_j$ for all $i, j \in I$.*

(ii) *Deduce that there exists a unique morphism $\iota : \coprod_{i\in I} X_i \to \prod_{i\in I} X_i$ such that $\pi_i \circ \iota \circ u_j = \delta_{i,j}$ for all $i, j \in I$.*

4.91. Let $\mathscr{C}$ be a category with zero object. Given a morphism $f : X \to Y$ in $\mathscr{C}$, we define its **kernel** is an object K in $\mathscr{C}$ and a morphism $i : K \to X$ such that $f \circ i = 0$, and given any other object K' and morphism $i' : K' \to X$ satisfying that $f \circ i' = 0$, there exits a unique morphism $j : K' \to K$ such that $i' = i \circ j$. Dually, a **cokernel** is an object C in $\mathscr{C}$ and a morphism $p : Y \to C$ such that $p \circ f = 0$, and given any other object C' and morphism $p' : Y \to C'$ satisfying that $p' \circ f = 0$, there exits a unique morphism $q : C \to C'$ such that $p' = q \circ p$. It is clear that if a kernel or cokernel exists, it is unique up to unique isomorphism. We usually denote the kernel of f by $\mathsf{Ker}(f)$ and its cokernel by $\mathsf{Coker}(f)$. Moreover, we see that a cokernel of a morphism in $\mathscr{C}$ is precisely the kernel of the same morphism regarded in $\mathscr{C}^{\mathrm{op}}$.

4.92. Exercise. *Prove that a morphism whose kernel (resp., cokernel) is the zero object is a monomorphism (resp., epimorphism).*

4.93. Exercise. *Let $\mathscr{C}$ be a category with zero object. Assume that $\mathscr{C}$ has all the kernels and cokernels of all its morphisms. Given a morphism $f : X \to Y$ in $\mathscr{C}$, let $i : \mathsf{Ker}(f) \to X$ be its kernel and $p : Y \to \mathsf{Coker}(f)$ its cokernel. Let $q : X \to \mathsf{Coker}(i)$ be the cokernel of i and $j : \mathsf{Ker}(p) \to Y$ be the kernel of p. We call $\mathsf{Ker}(p)$ the* ***(abelian) image*** *of f, and we denote it $\mathsf{Img}(f)$. Prove that there exists a unique morphism $\bar{f} : \mathsf{Coker}(i) \to \mathsf{Ker}(p)$ such that $f = j \circ \bar{f} \circ q$.*

4.94. We say that a category $\mathscr{C}$ is **semiadditive** if it has a zero object, finite products and finite coproducts exist and the map ι in Exercise 4.90 is an isomorphism for every finite family of objects.

4.95. Exercise. *Let $\mathscr{C}$ be an semiadditive category. We recall that in this case finite coproducts and finite products are canonically isomorphic via the morphism ι in Exercise 4.90. Let X and Y be two objects in $\mathscr{C}$.*

(i) *Let $u_i : X \to X \oplus X$ and $v_i : Y \to Y \oplus Y$ for $i = 1, 2$ be the morphisms of the corresponding coproducts. Given $f, g \in \mathsf{Hom}_{\mathscr{C}}(X, Y)$, prove that there exists a unique morphism $f \coprod g : X \coprod X \to Y \coprod Y$ satisfying that $(f \coprod g) \circ u_1 = f \circ u_1$ and $(f \coprod g) \circ u_2 = g \circ u_2$.*

(ii) *Given $f, g \in \mathsf{Hom}_{\mathscr{C}}(X, Y)$, where define $f + g$ as the composition*

$$X \xrightarrow{\Delta_X} X \prod X \xrightarrow{\iota^{-1}} X \coprod X \xrightarrow{f \coprod Y} Y \coprod Y \xrightarrow{\nabla_Y} Y.$$

Prove that this defines a structure of abelian monoid with unit on $\mathsf{Hom}_{\mathscr{C}}(X, Y)$, where the unit is the zero morphism.

Prove that the composition $\circ_{X,Y,Y} : \mathsf{Hom}_{\mathscr{C}}(Y, Z) \times \mathsf{Hom}_{\mathscr{C}}(X, Y) \to \mathsf{Hom}_{\mathscr{C}}(X, Z)$ of the category $\mathscr{C}$ is bilinear for the previous structures of abelian monoid with unit.

4.96. We say that a category $\mathscr{C}$ is **additive** if it is semiadditive and the structure of abelian monoid with unit in the previous exercise is in fact an abelian group. A functor $F : \mathscr{C} \to \mathscr{D}$ between additive categories is called **additive** if it preserves finite (co)products.

4.97. Exercise. *Prove that the category of modules over a ring is additive.*

4.98. Exercise. *Let $F : \mathscr{C} \to \mathscr{D}$ be a functor, where $\mathscr{C}$ and $\mathscr{D}$ are additive categories. Prove that the following conditions are equivalent:*

(Ad.1) *F is additive;*

(Ad.2) *the induced map $\mathsf{Hom}_{\mathscr{C}}(X, Y) \to \mathsf{Hom}_{\mathscr{C}}(F(X), F(Y))$ is a morphism of abelian groups, for all objects X, Y in $\mathscr{C}$.*

Note in particular that $F(0_{\mathscr{C}}) = 0_{\mathscr{D}}$.

Abelian categories

4.99. Let $\mathscr{C}$ be an additive category having all kernels and cokernels of all of its morphisms. We say that $\mathscr{C}$ is **abelian** if, given any morphism f in $\mathscr{C}$, the morphism $\bar{f}$ in Exercise 4.93 is an isomorphism. If $\mathscr{C}$ is an abelian category we can define the notion of exact sequence of morphisms, in precisely the same way as we did for complexes of modules in paragraph 1.31. Namely, given an integer $n \geq 3$, a **(finite) sequence** of objects in a category $\mathscr{C}$ is the data

$$M^1 \xrightarrow{f^1} M^2 \xrightarrow{f^2} M^3 \xrightarrow{f^3} \dots \xrightarrow{f^{n-2}} M^{n-1} \xrightarrow{f^{n-1}} M^n,$$

where M^i) is an object in $\mathscr{C}$ for $i \in [\![1, n]\!]$ and f^i is a morphism in $\mathscr{C}$ for $i \in [\![1, n-1]\!]$. The number n is called the **length** of the sequence. If $\mathscr{C}$ is abelian, we say that the finite sequence is **exact** at the position $i \in [\![2, n-1]\!]$ if $\mathsf{Img}(f^i) = \mathsf{Ker}(f^{i-1})$. As in the case of complexes of A-modules, the sequence is said to be **exact** if it is exact at the position i for every $i \in [\![2, n-1]\!]$. A **short exact sequence** in an abelian category $\mathscr{C}$ is an exact finite sequence of length 5 with $M^1 = M^5 = 0$.

4.100. Exercise. *Let A be a ring. Prove that ${}_A\,\mathsf{Mod}$ is an abelian category.*

4.101. Exercise. *Let $\mathscr{C}$ be an abelian category. Define $\mathsf{Gr}(\mathscr{C})_0$ as the class of all sequences $X = \{X_n\}_{n \in \mathbb{Z}}$ with $X_n \in \mathscr{C}_0$ for all $n \in \mathbb{Z}$. Given two sequences $X = \{X_n\}_{n \in \mathbb{Z}}$ and $Y = \{Y_n\}_{n \in \mathbb{Z}}$ of objects in $\mathscr{C}$, a morphism from X to Y is a sequence $f = \{f_n\}_{n \in \mathbb{Z}}$, where $f_n \in \mathsf{Hom}_{mathscrC}(X_n, Y_n)$ for all $n \in \mathbb{Z}$. Given another morphism $g = \{g_n\}_{n \in \mathbb{Z}}$ from Y to $Z = \{Z_n\}_{n \in \mathbb{Z}}$ The composition $g \circ f$ is given by the sequence $\{g_n \circ f_n\}_{n \in \mathbb{Z}}$. The identity id_X is the sequence $\{\mathsf{id}_{X_n}\}_{n \in \mathbb{Z}}$.*

(i) *Prove that $\mathsf{Gr}(\mathscr{C})$ is an abelian category.*

(ii) *Let $\mathscr{C} = {}_A\,\mathsf{Mod}$. Prove that $\mathsf{Gr}(\mathscr{C})$ is equivalent to ${}_A\,\mathsf{GMod}$.*

4.102. Exercise. *Let $\mathscr{C}$ be an abelian category. Define $\mathsf{Ch}(\mathscr{C})$ as the category whose objects (X, d) are given by sequences $\{X_n\}_{n \in \mathbb{Z}}$ with $X_n \in \mathscr{C}_0$ together with morphisms $d_n : X_n \to X_{n-1}$ such that $d_n \circ d_{n-1} = 0$ for all $n \in \mathbb{Z}$. Given two objects (X, d) and (X', d'), a morphism from (X, d) to (X', d') is a sequence $f = \{f_n\}_{n \in \mathbb{Z}}$ with $f_n \in \mathsf{Hom}_{\mathscr{C}}(X_n, Y_n)$ such that $d'_n \circ f_n = f_{n-1} \circ d_n$ for all $n \in \mathbb{Z}$. The composition and the units are the same as in Exercise 4.101.*

(i) *Prove that* $\mathsf{Ch}(\mathscr{C})$ *is an abelian category.*

(ii) *Let* $\mathscr{C} = {}_A\mathsf{Mod}$. *Prove that* $\mathsf{Ch}(\mathscr{C})$ *is equivalent to* ${}_A\mathsf{DGMod}$.

(iii) *Given and object* (X, d) *in* $\mathsf{Ch}(\mathscr{C})$, *denote by* $H(X, d)$ *the object in* $\mathsf{Gr}(\mathscr{C})$ *whose n-th component is* $\mathsf{Ker}(d_n)/\mathsf{Img}(d_{n+1})$, *for* $n \in \mathbb{Z}$. *If* f *is a morphism from* (X, d) *to* (X', d') *in* $\mathsf{Ch}(\mathscr{C})$, *define* $H(f)$ *as the morphism in* $\mathsf{Gr}(\mathscr{C})$ *whose n-th component is the morphism*

$$\mathsf{Ker}(d_n)/\mathsf{Img}(d_{n+1}) \to \mathsf{Ker}(d'_n)/\mathsf{Img}(d'_{n+1})$$

induced by f_n, *for* $n \in \mathbb{Z}$. *Prove that* $H : \mathsf{Ch}(\mathscr{C}) \to \mathsf{Gr}(\mathscr{C})$ *is a functor, that coincides with (1) under the identifications in the second item of this and the previous exercises.*

4.103. Let $\mathscr{C}$ and $\mathscr{D}$ be two abelian categories and let $F : \mathscr{C} \to \mathscr{D}$ be an additive functor. We say that F is **right** (resp., **left**) **exact** if, given a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathscr{C}$, the sequence $0 \to F(M') \to F(M) \to F(M'')$ (resp., $F(M') \to F(M) \to F(M'') \to 0$) in $\mathscr{D}$ is exact. The functor F is called **exact** if it is left and right exact.

4.104. Exercise. *Let* $\mathscr{C}$ *and* $\mathscr{D}$ *be two abelian categories, and* $F : \mathscr{C} \to \mathscr{D}$ *and* $G : \mathscr{D} \to \mathscr{C}$ *be functors such that* (F, G) *is an adjoint pair.*

(i) *Prove that* F *is right exact and* G *is left exact.*

(ii) *Prove that if* F *preserves monomorphisms, then* G *preserves injective objects. Analogously, if* G *preserves epimorphisms, then* F *preserves projective objects.*

4.105. Exercise (cont. to Exercise 4.102)**.** *Let* $F : \mathscr{C} \to \mathscr{D}$ *be an additive functor between abelian categories. Consider the assignement* $\mathsf{Gr}(\mathscr{C}) \to \mathsf{Gr}(\mathscr{D})$ *(resp.,* $\mathsf{Ch}(\mathscr{C}) \to \mathsf{Ch}(\mathscr{D})$*) that sends the object* X *(resp.,* (X, d)*) given by the objects* $\{X_n\}_{n\in\mathbb{Z}}$ *with* $X_n \in \mathscr{C}_0$ *(resp., and the morphisms* $d_n : X_n \to X_{n-1}$*) for all* $n \in \mathbb{Z}$ *to the object, which we will denote by* $F(X)$ *(resp.,* $(F(X), F(d))$*), given by the objects* $\{F(X_n)\}_{n\in\mathbb{Z}}$ *(resp., and the morphisms* $F(d_n) : F(X_n) \to F(X_{n-1})$*) for all* $n \in \mathbb{Z}$. *Moreover, given two objects* X *(resp.,* (X, d)*) and* X' *(resp.,* (X', d')*), and a morphism* f *from* X *(resp.,* (X, d)*) to* X' *(resp.,* (X', d')*) given by a sequence* $f = \{f_n\}_{n\in\mathbb{Z}}$ *with* $f_n \in \mathsf{Hom}_{\mathscr{C}}(X_n, Y_n)$ *for all* $n \in \mathbb{Z}$, *define the morphism* $F(f)$ *from* $F(X)$ *(resp.,* $(F(X), F(d))$*) to* $F(X')$ *(resp.,* $(F(X'), F(d'))$*) by the sequence* $\{F(f)_n\}_{n\in\mathbb{Z}}$ *with* $F(f)_n = F(f_n)$ *for all* $n \in \mathbb{Z}$.

(i) *Prove that the previous assignements define functors* $\mathsf{Gr}(\mathscr{C}) \to \mathsf{Gr}(\mathscr{D})$ *and* $\mathsf{Ch}(\mathscr{C}) \to \mathsf{Ch}(\mathscr{D})$, *which we will also denote by* F.

(ii) *Assume that* F *is exact. Prove that* $H \circ F = F \circ H$, *for all* $n \in \mathbb{Z}$, *where* H *is the homology functor defined in Exercise 4.102.*

4.106. Exercise. *Let* $\mathscr{D}$ *be the category* ${}_A\mathsf{Mod}$ *for some ring* A, *let* $\mathscr{C}$ *be a filtered small category and let* $F : \mathscr{C} \to \mathscr{D}$ *be a functor.*

(i) *Using Theorem 4.86 and Exercise 4.87, deduce that the functor* $\mathsf{colim}_{\to\mathscr{C}} : \mathsf{Fun}(\mathscr{C}, \mathscr{D}) \to \mathscr{D}$ *defined in Exercise 4.75 is right exact.*

(ii) *Using Exercise 4.82, prove that the functor* $\mathsf{colim}_{\to\mathscr{C}} : \mathsf{Fun}(\mathscr{C}, \mathscr{D}) \to \mathscr{D}$ *in the previous item preserves monomorphisms, so it is exact.*

(iii) *Prove that there exist equivalences of categories*

$$\phi : \mathsf{Gr}\left(\mathsf{Fun}(\mathscr{C}, \mathscr{D})\right) \to \mathsf{Fun}\left(\mathscr{C}, \mathsf{Gr}(\mathscr{D})\right),$$

and

$$\Phi : \mathsf{Ch}\left(\mathsf{Fun}(\mathscr{C}, \mathscr{D})\right) \to \mathsf{Fun}\left(\mathscr{C}, \mathsf{Ch}(\mathscr{D})\right)$$

(see Exercises 4.101 and 4.102).

(iv) *Using Exercise 4.105 and the previous items, show that there is a natural isomorphism*

$$H \circ \underset{\to\mathscr{C}}{\mathsf{colim}} \circ \Phi \cong \underset{\to\mathscr{C}}{\mathsf{colim}} \circ \phi \circ H.$$

The previous identity means precisely that taking homology commutes with filtered colimits.

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