Exercise sheet n° 8

Fetch your electronic calculators !

Application : extrema of a function of several variables

1. Compute the local maxima and minima of the following functions defined on $\mathbb{R}^2$.

$$f_1(x,y) = 4 - 2x^2 - y^2 \qquad f_2(x,y) = x^2 + y^2 - 1 \qquad f_3(x,y) = x^2 - 2x + y^2 - 1$$

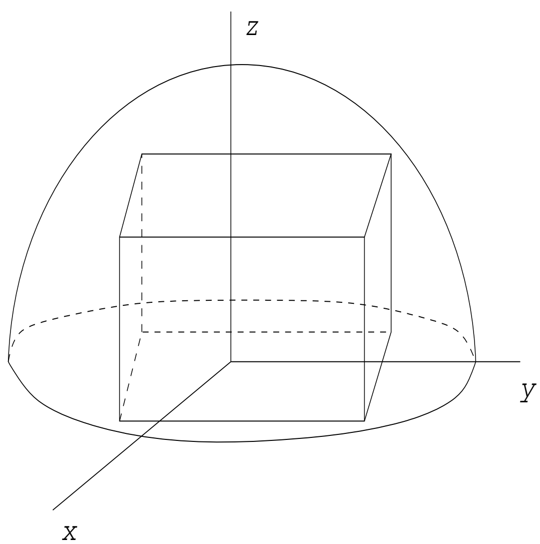
$$f_4(x,y) = -x^2 + 2xy - 2y^2 - 4 \qquad f_5(x,y) = x - y^2 - x^3 \qquad f_6(x,y) = 3x + 12y - x^3 - y^3$$

$$f_7(x,y) = (x-y)^2 + x^3 + y^3 \qquad f_8(x,y) = (x-y)^2 + x^2 + y^2 \qquad f_9(x,y) = -2(x-y)^2 + x^4 + y^4$$

2. Determine if the following functions have an extremum (maximum or mimimum) at $(0,0)$:

$$f_1(x,y) = x^2, \quad f_2(x,y) = xy, \quad f_3(x,y) = x^2 - 4y^2, \quad f_4(x,y) = x^2 + xy + y^2,$$
$$f_5(x,y) = x^2 - xy + y^2, \quad f_6(x,y) = -2(x-y)^2 + x^4 + y^4$$
$$f_7(x,y,z) = x^2 + 2y^2 + 3z^2, \quad f_8(x,y,z) = x^4 + y^2 + z^2 - 4x - 2y - 2z + 4,$$
$$f_9(x,y,z) = 3(x^2 + 2y^2 + z^2) - 2(x+y+z)^4, \quad f_{10}(x,y,z) = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$$

3. What are the lengths of the edges of the rectangular parallelepiped of largest volume that is inscribed in the semisphere of radius a ?



4. Anne and Paul have invested $20,000e$ in the creation and development of a new product. Its unitary production cost is $2e$. The marketing consultant suggests that they should invest Ae in advertising. The estimated sales volume is given by

$$2000 + 4\sqrt{A} - 20p$$

units of product at unitary price p.

1. Express the net gain of Anne and Paul as a function A and p.
2. Find the values of the price p and the total invest A in paid advertising that give the maximal net gain ?

Application : the method of least squares

5. For this kind of problem, is it possible to have several values of a variable z (different or not) associated to one value of the independent variable t ?

For example, is it possible to to have 2 different values of z for each value of t ? Or 4 equal values of z for a unique value of t ?

6.[Qualitative analysis] We study the weights of a population of rats divided into different groups. The rats of each group follow a different diet. We measure the variable `weight`, which we try to explain in terms of the (independent) variables `dosage` and `regime` given below :

$$\begin{aligned}
\textbf{weight} &= [1.2941979;\ 1.9777068;\ 0.784177;\ 1.1130249;\ 0.7522560;\ 1.8111625] \\
\textbf{dosage} &= [0.4665600;\ 0.8916927;\ 0.580997;\ 0.4992316;\ 0.3973825;\ 0.9306090] \\
\textbf{regime} &= [\text{proteins},\ \text{proteins},\ \text{cereal},\ \text{cereal},\ \text{mixture},\ \text{cereal}].
\end{aligned}$$

1. Compute the least square regression using the independent variable `regime`.
2. Compute the matrix of regressors for the parameter estimation given by the method of least squares.

7.[Nonlinear transformations]

1. Consider the following values

   $$t_i = i, \quad z_i = t_i^3 + t_i^2,$$

   for $i = -3, -2, ..., 2, 3$. Compute the straight line given by the method of least squares for these data. Plot the given points and the computed straight line.
2. Write down the system of linear equations of the method of least squares that gives a polynomial of degree less than or equal to 3 fitting the data $(t_i, e^{-2t_i})_{i=0,\dots,20}$ for $t_i = 0.2i$.
3. Describe the method of least squares that uses the fitting expression $f(t) = ate^{-bt}$ for some parameters a and b to be determined, in terms of the data (t_i, z_i).

8. Consider the model $y = X\beta + \xi$, where X is a matrix of size $n \times k$ (with $k \leq n$), $y \in I\!R^n$, $\xi \in I\!R^n$ is noise, and $\beta \in I\!R^k$ is the unknown parameter of the linear regression. We assume that the matrix X has full rank, *i.e.* $\text{rank}(X) = k$.

1. Verify that the matrix of size $k \times k$ defined as $H = X(X^tX)^{-1}X^t$ is symmetric and *idempotent*, *i.e.* $H^2 = H$.
2. Show that the eigenvalues λ_i, $i = 1, ..., k$ of an idempotent matrix of size $k \times k$ are included in $\{0, 1\}$.

3. Verify that $Hu = u$ for all $u \in \mathcal{X}$, where $\mathcal{X} = \{z \in \mathbb{R}^n : \; z = X\beta, \; \beta \in \mathbb{R}^k\}$, and $Hu = 0$ for all $u \perp \mathcal{X}$. Equivalently, H is an *orthogonal projection* (or *projection* for short) onto the subspace $\mathcal{X}$ of $\mathbb{R}^n$.

   Let $\widehat{\beta} = (X^t X)^{-1} X^t y$ be the least square estimator for β and set $\widehat{y} = X\widehat{\beta} = X(X^t X)^{-1} X^t y = Hy$ (the *estimated value*).

4. Show that the *residue* $\widehat{\xi} = y - \widehat{y}$ is orthogonal to $\widehat{y}$. Let $K = I - H = I - X(X^t X)^{-1} X^t$. Verify that K is a projection, and identify the subspace target of $\mathbb{R}^n$.