Exercise sheet n° 6

Matrix calculus

1. Compute the products AB and BA in case they are defined.

1. $A = (1,\ 2,\ -1,\ 3)$, $B = (-1,\ 0,\ 2,\ 1)^t$
2. $A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -2 \end{pmatrix}$
3. $A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

   (**Hint :** Use the product by blocks in this example).

2. Answer the following yes/no questions.
(a) Every diagonal matrix is symmetric.
(b) For every matrix A and every scalar number c, $(cA)^t = cA^t$.
(c) Every tridiagonal matrix is symmetric.
(d) Given any pair of matrices A and B, $(AB)^t = B^tA^t$.

3. Complete the following sentences so they become true statements.
(a) The rank of the matrix $\begin{pmatrix} 0 & -1 & 2 \\ 0 & 3 & 4 \end{pmatrix}$ is ______.
(b) If A is a matrix of 3×7, then its rank is greater than or equal to ______ and less than or equal to ______.
(c) The rank of a nonzero matrix of size 3×3 and all equal entries is ______.
(d) If A is a matrix of size 4×8, then its nullity is greater then or equal to ______.
(e) If A is a nonzero matrix of size 4×3 each of whose columns contains the same constant entry, then the rank of A is ______.
(f) An example of matrix of rank 2 and nullity 1 is ______.

4. Let $u = (1, 2, 3)^t$ and $v = (a, b, c)^t$ be two column vectors. Consider the matrix given by $P = uv^t$. What is the rank of P ? Compute P^2, P^k, $k = 3, 4, \ldots.$ What is the image of P ? And its kernel ?

5. Compute the matrix representation of the following linear maps relative to the given bases.

1. $f : \begin{cases} \mathbb{R}^3 & \rightarrow & \mathbb{R}^3 \\ (x, y, z) & \mapsto & (x + 2y + 3z, 2y - z, x + z) \end{cases}$,
2. the symmetry of the plane $\mathbb{R}^2$ given by the mirror reflection through the axis defined by e_1, where (e_1, e_2) is any basis of $\mathbb{R}^2$,

3. the orthogonal projection of $\mathbb{R}^2$ onto the axis defined by a nonzero vector e_1,
4. $f : \left\{ \begin{array}{ccc} \mathbb{R}_2[X] & \to & \mathbb{R}_2[X] \\ P & \mapsto & 2(X+1)P - (X^2-1)P' \end{array} \right.$.

6. Consider the plane $\mathbb{R}^2$ endowed of the canonical basis $B = (e_1, e_2)$. Let f be the linear map given by $f : \left\{ \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (u,v) & \mapsto & (2u, -v) \end{array} \right.$.

1. Compute the matrix representation A of f relative to the basis B.
2. Show that the vectors $e'_1 = (3,1)$ and $e'_2 = (5,2)$ give a new basis B' of $\mathbb{R}^2$.
3. Compute the matrix representation A' of f relative to the basis B' by explicitly calculating $f(e'_1)$ and $f(e'_2)$.
4. Compute the transition matrices P and Q between the bases B and B'.
5. Reobtain A' by means of the change of basis formula.
6. Compute the matrices of f^5 relatives to both bases
(**Hint** : Note that $A = PA'P^{-1}$ implies $A^n = PA'^n P^{-1}$).

7.Consider the space $\mathbb{R}^3$ endowed of the canonical basis B.

1. Show that $u_1 = (2,-1,-2)$, $u_2 = (1,0,-1)$ and $u_3 = (-2,1,3)$ form a basis B' of $\mathbb{R}^3$.
2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map whose matrix relative to the basis B is $A = \begin{pmatrix} 9 & -6 & 10 \\ -5 & 2 & -5 \\ -12 & 6 & -13 \end{pmatrix}$. Compute the transition matrices between the two bases.
3. Compute the matrix of f relative to the basis B'.

8. Consider four square matrices A, B, C and D of size $n \times n$.

1. Show that tr(AB)=tr(BA).
2. Deduce that the equalities $AC + DB = I_n$ and $CA + BD = 0_n$ cannot hold simultaneously.
3. Let u be any linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. Deduce that the trace of the matrix of u relative to any basis B is independent of the chosen basis.

9. Express the quadratic form

$$Q(x,y,z) = 2x^2 + y^2 + z^2 + 2xy + 4yz - 6xz$$

as a matrix product en $v^t A v$, where $v = (x,y,z)^t$ and
(i) A is a lower triangular matrix ;
(ii) A is a symmetric matrix.

Systems of linear equations and determinants

10. Solve the following systems of linear equations by using the Gauss or the Gauss-Jordan method.

$$(a)\quad \begin{aligned} x+y+z&=0\\ x+2y+3z&=2\\ x+3y+4z&=3 \end{aligned} \qquad (b)\quad \begin{aligned} x+3y-z &= 9\\ 3x+9y-3z &= 27\\ -2x+y-5z &= 10 \end{aligned}$$

$$(c)\quad \begin{aligned} x+y&=1\\ 2x+y&=2\\ 3x+2y&=5; \end{aligned} \qquad (d)\quad \begin{aligned} x_3+x_4&=0\\ -2x_1-4x_2+x_3&=0\\ -x_3+x_4&=0; \end{aligned} \qquad (e)\quad \begin{aligned} x_1+x_2+3x_3&=2\\ 2x_1+5x_2+9x_3&=1\\ x_1+2x_2+4x_3&=1; \end{aligned}$$

11. In this exercise all matrices will be square.

1. Given any matrix M, let M' be the matrix obtained from M by the row operation $L_1 \to 2L_1 + L_2$. Does the equality $\det(M) = \det(M')$ hold ?
2. Assume that M and M' are two square matrices of the same size and that there exists $x \in \mathbb{R}^n \setminus \{0\}$ such that $Mx = M'x$. Is it true that $\det(M) = \det(M')$? And $\det(M - M') = 0$?
3. Assume now that $n = 3$. Is it true that $\det_{\mathcal{B}}(v_1, v_2, v_3) = \det_{\mathcal{B}}(v_2, v_3, v_1)$?
4. Let v be an element of an n-dimensional vector space E. If $\det(v_1 + v, v_2, ..., v_n) = \det(v_1, ..., v_n)$, is it true that $v \in vect(v_2, .., v_n)$?

12. Compute the determinants of the following matrices.

$$A = \begin{pmatrix} 1 & 1 & -1\\ 2 & 3 & -4\\ 4 & 1 & -4 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} 4 & 1 & 2\\ -1 & 1 & -1\\ -2 & -1 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 1 & -2\\ 0 & 2 & -2\\ 1 & 0 & 0 \end{pmatrix}.$$

$$a = \begin{vmatrix} 1 & 2 & 1 & 3\\ 4 & 0 & 3 & 1\\ -1 & 2 & -3 & 0\\ 1 & 6 & -1 & -1 \end{vmatrix}, \qquad b = \begin{vmatrix} 1 & \cos x & \cos 2x\\ 1 & \cos y & \cos 2y\\ 1 & \cos z & \cos 2z \end{vmatrix} \quad \forall (x,y,z) \in \mathbb{R}^3,$$

13. We recall the following identity :

$$\det(u, v, w) = u \cdot (v \wedge w)[= v \cdot (w \wedge u) = w \cdot (u \wedge v)].$$

1. Give a geometrical interpretation of the quantity $\|v\|\,\|w\|\,|\sin\theta|$, where θ is the angle between v and w.
2. Deduce that $|\det(u, v, w)| = \mathrm{Vol}(P)$, where P is the parallelepiped generated by u, v and w.
3. Under what conditions has this parallelepiped volume 0 ?

14. Let $A \in \mathcal{M}_n(\mathbb{R})$ such that $\mathrm{rank}(A) = 1$.

1. We denote by $(C_1, \ldots, C_n)$ the columns of A. Show that $\exists C \in I\!R^n, \exists(\lambda_1, \ldots, \lambda_n) \in I\!R^n$ tq
   $\forall i \in \{1, \cdots, n\}, C_i = \lambda_i C$.
2. By making use of an appropriate change of basis, show that $\det(I_n + A) = 1 + \mathrm{tr}(A)$.

15. Let $A = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix} \in \mathcal{M}_{2n}(I\!R)$.

1. Compute $\det A$.
2. Let $S \in M_{2n}(I\!R)$ such that $S^T A S = A$. Show that S is invertible and that S^T and S^{-1} are similar.