Exercise sheet n° 5

Multiple integrales

1. Compute the following double integrals.

$$I_1 = \iint_{[0,3]\times[0,2]} (4 - y^2)\ dxdy, \qquad I_2 = \iint_{[0,3]\times[-2,0]} (x^2 y - 2xy)\ dxdy,$$

$$I_3 = \iint_{[\pi,2\pi]\times[0,\pi]} (\sin x + \cos y)\ dxdy, \quad I_4 = \int_0^{\pi} \left(\int_0^x x \sin y\ dy\right) dx,$$

$$I_5 = \int_0^{\pi} \left(\int_0^{\sin x} y\ dy\right) dx, \qquad I_6 = \int_1^{\ln 8} \left(\int_1^{\ln y} e^{x+y}\ dx\right) dy,$$

$$I_7 = \iiint_{[0,\frac{\pi}{2}]^3} \sin(x + y + z)\ dxdydz.$$

2. Compute the area of each of the following plane regions enclosed by the corresponding curves :

1. $xy = 1,\ x + y = \frac{5}{2}a\ (a > 0)$;
2. $r = a\cos 3\theta$, for $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$, where we are using polar coordinates ;
3. $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$; $x^2 + y^2 \ge a^2$.
   Hint : Use polar coordinates.

3. Compute the volume of the following regions of $\mathbb{R}^3$.

$$\begin{aligned} D_1 &= \{(x, y, z) \in \mathbb{R}^3 /\ z \ge 0,\ x^2 + y^2 \le 1 - z\}, \\ D_2 &= \{(x, y, z) \in \mathbb{R}^3 /\ x^2 + y^2 + z^2 \le 1\}, \\ D_3 &= \{(x, y, z) \in (\mathbb{R}^+)^3 /\ x \le y,\ x + y \le 2,\ z \le x^2 + y^2\}. \end{aligned}$$

4. Compute the double integrals $\iint_D f(x, y)\ dx\ dy$ for the maps and regions given below :

1. $f(x, y) = x^2 y$ and $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$;
2. $f(x, y) = xy$ and $D = \{(x, y) \in \mathbb{R}^2 \mid x \ge 0, y \ge 0, x + y \le 1\}$;
3. $f(x, y) = x^2$ and $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 \le y \le x\}$;
4. $f(x, y) = x\cos(\sqrt{x^2 + y^2})$ and $D = \{(x, y) \in \mathbb{R}^2 \mid x \ge y \ge 0, x^2 + y^2 \le \pi\}$;
5. $f(x, y) = (x^2 - y^2)e^{xy}$ and $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1, x + y \ge 1, x \ge y\}$.
   Hint : Consider the change of variables $(X, Y) = (x + y, x - y)$.

5. Compute the coordinates of the center of mass, or barycenter, of the following surfaces :

1. a homogeneous plate defined by $x^2 \le y \le 1, -1 \le x \le 1$;
2. a horizontal half-disc ;
3. the disc given by the equation
   $$(x - 1)^2 + y^2 \le 1$$
   with surface density $\rho = x|y|$ (given in grams/cm^2 units).

Line integrals

6. Compute the arc length of the arc segments of the following curves :

1. $x = 3t, \quad y = 3t^2, \quad z = 2t^3$, between $O = (0, 0, 0)$ and $A = (3, 3, 2)$
2. $x = e^{-t}\cos t, \quad y = e^{-t}\sin t, \quad z = e^{-t}$, between $t = 0$ and $t = 1$
3. $x^2 + y^2 = z, \quad \frac{y}{x} = \tan z$, between $O = (0, 0, 0)$ and $A = (\sqrt{\frac{\pi}{8}}, \sqrt{\frac{\pi}{24}}, \pi/6)$

7. A material point has a trajectory described by the smooth parameterized curve $x = a\cos t$, $y = a\sin t$, $z = bt$ for $t \in [0, 2\pi]$. Compute the work done by the force $\overrightarrow{F}(x, y, z) = x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}$ on the displacement made by the material point.

8. A material point has a trajectory described by the smooth parameterized curve

$$x = e^t\cos t, \quad y = e^t\sin t \;\text{ et }\; z = e^t,$$

where t indicates the time.

1. What is the distance travelled by the material point between $t = 0$ and $t = 1$? At which time t is the distance travelled equal to $4\sqrt{3}$?
2. Compute the work done by the force $\overrightarrow{F} = z\overrightarrow{k}$ on the displacement made by the material point between $t = 0$ and $t = 1$

9.

1. Let Γ be the parameterized curve in $\mathbb{R}^2$ of equation $y = (x - 1)\ln(x + 1)$, for x going from 0 to 1. Compute
$$I_1 = \int_\Gamma \sqrt{x}\; dy - (\sqrt{x}\ln(x + 1))\; dx.$$
2. Let $C \subset \mathbb{R}^2$ be the circle centered at $(0, 0)$ of radius R, paramaterized counterclockwise. Compute
$$I_2 = \int_C (2x - y)\; dx + (x + y)\; dy.$$
3. Let Γ be a closed smooth parameterized curve in $\mathbb{R}^3$. Compute
$$I_3 = \int_\Gamma yz\; dx + xz\; dy + xy\; dz.$$

10. Compute the work done by the gravitational force $\overrightarrow{F} = \frac{k\overrightarrow{u}_r}{r^2}$, où $r = \sqrt{x^2 + y^2 + z^2}$ on the displacement done by a material point going from $M_1 = (x_1, y_1, z_1)$ to $M_2 = (x_2, y_2, z_2)$.

11. Compute the work done by the force $\overrightarrow{F} = -kr\,\overrightarrow{u}_r$ on the displacement made by a material point travelling along the arc of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ from $M_1 = (a, 0)$ to $M_2 = (0, b)$.

12.

1. Let $\Gamma \subset \mathbb{R}^2$ be a smooth simple curve enclosing a region $\mathcal{D}$ (of type 3) of area A. Use the Green-Riemann theorem to deduce that
$$A = \frac{1}{2}\int_\Gamma x\; dy - y\; dx.$$
2. Compute the area enclosed by the ellipse defined by
$$\begin{cases} x &= a\cos\theta, \\ y &= b\sin\theta, \end{cases} \quad \text{for } 0 \le \theta \le 2\pi.$$

13. Let $D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, \text{ and } x^2 + y^2 \leq 1\}$ and let γ be a (piecewise smooth) simple counterclockwise parameterization of its boundary. Consider the differential form

$$\omega = xy^2 dx + 2xydy.$$

Compute the integral $\int_\gamma \omega$ by means of the Green-Riemann theorem.

14. Use the Green-Riemann theorem to compute the double integral $\int_D xydxdy$, where

$$D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, \text{ and } x + y \leq 2\}.$$