Exercise sheet n° 4

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1. Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ be the position vector in $\mathbf{R}^3$. Its differential is thus $d\,\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$, and $\|d\,\vec{r}\,\| = \sqrt{(dx)^2 + (dy)^2 + dz^2}$.

1. Recall that $\vec{r} = r\,\vec{u}_r + z\,\vec{k}$, where the first two orthonormal vectors $\vec{u}_r$ and $\vec{u}_\theta$ of the cylindrical coordinate system are defined by
$$\vec{u}_r = \vec{i}\,\cos\theta + \vec{j}\,\sin\theta, \quad \vec{u}_\theta = -\,\vec{i}\,\sin\theta + \vec{j}\,\cos\theta,$$
and the last one is the usual vector $\vec{k}$.
Find the expressions of the differential $d\,\vec{r}$ and the elementary distance $\|d\,\vec{r}\,\|^2$ in cylindrical coordinates.
2. The orthonormal vectors $\vec{u}_r$, $\vec{u}_\theta$ and $\vec{u}_\phi$ of the spherical coordinate system are given by
$$\begin{aligned} \vec{u}_r &= \vec{i}\,\sin\theta\cos\phi + \vec{j}\,\sin\theta\sin\phi + \vec{k}\,\cos\theta, \\ \vec{u}_\phi &= -\,\vec{i}\,\sin\phi + \vec{j}\,\cos\phi, \\ \vec{u}_\theta &= \vec{i}\,\cos\theta\cos\phi + \vec{j}\,\cos\theta\sin\phi - \vec{k}\,\sin\theta. \end{aligned}$$
Define $d\,\vec{r} = d(r\,\vec{u}_r)$. Verify that
$$\begin{aligned} d\,\vec{r} &= dr\,\vec{u}_r + r d\theta\,\vec{u}_\theta + r\sin\theta d\phi\,\vec{u}_\phi, \\ \|d\,\vec{r}\,\|^2 &= (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2. \end{aligned}$$

2. Compute the gradient and the Laplacian of the following functions :
$$\forall (x,y) \in \mathbf{R}^2, \quad f(x,y) = x^2 + y^2,$$
$$\forall (x,y) \in \mathbf{R}^2 \setminus \{(0,0)\}, \quad g(x,y) = \frac{x}{\sqrt{x^2+y^2}}.$$
Repeat the computations with polar coordinates.

3. Compute the divergence and the curl of the following vector fields :
$$\begin{aligned} &\forall (x,y,z) \in \mathbf{R}^3, \quad V(x,y,z) = 2xe^{2z}\sin y\;\vec{i} + x^2e^{2z}\cos y\;\vec{j} + 2x^2e^{2z}\sin y\;\vec{k}, \\ &\forall (r,\theta,z), r > 0, \quad W(r,\theta,z) = r\sin\theta\;\vec{u}_r + r\cos\theta\;\vec{u}_\theta + z\;\vec{k} \end{aligned}$$

4. Let $f : \mathbf{R}^3 \to \mathbf{R}$ and $\vec{v} : \mathbf{R}^3 \to \mathbf{R}^3$ be two maps of class $\mathcal{C}^2$ on $\mathbf{R}^3$. Show that
$$\begin{aligned} \mathrm{div}(f\,\vec{v}) &= f\,\mathrm{div}\,\vec{v} + \mathrm{grad} f \cdot \vec{v}, \\ \mathrm{div}(\mathrm{grad} f) &= \Delta f, \\ \mathrm{rot}(\mathrm{grad} f) &= 0. \end{aligned}$$

5. Given $\alpha \in \mathbf{R}$, define $f_\alpha : \mathbf{R}^3 \setminus \{(0,0,0)\} \to \mathbf{R}$ as
$$\forall (x,y,z) \in \mathbf{R}^3 \setminus \{(0,0,0)\}, \; f_\alpha(x,y,z) = (x^2+y^2+z^2)^\alpha.$$
Find the values of α such that $\Delta f_\alpha = 0$.