Exercise sheet n° 3

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1. Find all the functions $f(x,y)$ defined on $\mathbb{R}^2$ satisfying that

1. f is C^1 and the partial derivatives of first order vanish everywhere ;
2. f is C^2 and the partial derivatives of second order vanish everywhere ;
3. f is C^3 and

$$\frac{\partial^3 f}{\partial x^2 \partial y} \equiv 0.$$

2. Solve $2\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 0$ by means of the change of variables given by $(u,v) = (x+y, x+2y)$.

3. Solve $2\frac{\partial f}{\partial x} + 3\frac{\partial f}{\partial y} = xy$ by means of the change of variables given by $(u,v) = (x, 3x-2y)$.

4. Solve the following PDE defined on $\mathbb{R}^2 \setminus \{(0,0)\}$: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = a\sqrt{x^2+y^2}$, where $a \in \mathbb{R}$. (Hint : use polar coordinates.)

5. Solve $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - 2\,\frac{\partial^2 f}{\partial x \partial y} = 0$ by means of the change of variables given by $(u,v) = (x-y, x+y)$.

6. Compute the Taylor polynomial of second degree of the following functions :

$$x^2y^2 \text{ at } (1,\ 1), \quad \sin(xy) \text{ at } (0,\ 0), \quad x^3y^2 - 2xy^4 + y^5 \text{ at } (1,\ 2).$$

7. *(bonus)* Consider the map f from $\mathbb{R}^2$ to $\mathbb{R}$ given by

$$f(x,y) = \sqrt{x^2+y^2}.$$

Write down the Taylor polynomial of f of second degree at $a = (3,4)$. Show that the difference between $f(3.1,\ 4.02)$ and $5 + \mathrm{d}f_a(0.1,\ 0.02)$ is less than or equal to 2.10^{-3}.