L2 - UE MAT334

Exercise sheet n° 2

Functions of several variables

1. Determine the maximal domain of definition inside $\mathbb{R}^2$ of the following functions.

$$f_1(x,y) = \frac{1}{x} + \frac{1}{y}, \qquad f_2(x,y) = \frac{1}{x+y}, \qquad f_3(x,y) = \frac{1}{|x|+|y|}, \qquad f_4(x,y) = \ln(\frac{1}{xy}).$$

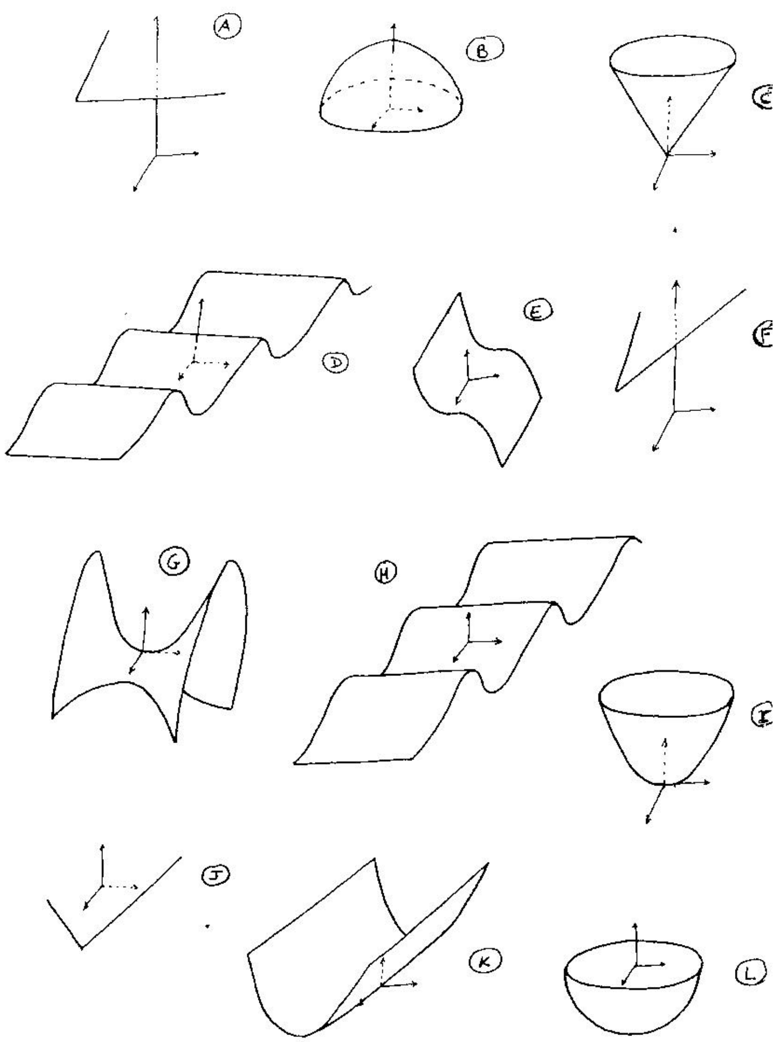
2. Match each of the following expressions with the corresponding surface.

$$f_1(x,y) = x^2, \qquad f_2(x,y) = \frac{1}{6}(5 - x + 2y), \qquad f_3(x,y) = y^2 - x^2, \qquad f_4(x,y) = y,$$

$$f_5(x,y) = y^2, \qquad f_6(x,y) = -y^3, \qquad f_7(x,y) = -\sin x, \qquad f_8(x,y) = 1 - (x^2 + y^2),$$

$$f_9(x,y) = 5, \qquad f_{10}(x,y) = x^2 + y^2, \qquad f_{11}(x,y) = \sin x, \qquad f_{12}(x,y) = x^2 + y^2 - 1,$$

$$f_{13}(x,y) = \cos x, \qquad f_{14}(x,y) = \sqrt{x^2+y^2}, \qquad f_{15}(x,y) = 3 - x - y.$$



3. Describe the level sets of the following collection of functions defined on $\mathbb{R}^2$.

$$f_1(x,y) = (x-2)^2, \quad f_2(x,y) = x^2 - y^2, \quad f_3(x,y) = x^2 + x - y, \quad f_4(x,y) = x^2 + y^2 - 2x + 4y + 5.$$

4. Express the following functions in polar coordinates and deduce the corresponding level sets.

$$f_1(x,y) = \sqrt{x^2+y^2}, \qquad f_2(x,y) = \frac{1}{x^2+y^2}, \qquad f_3(x,y) = \frac{y}{x}, \qquad f_4(x,y) = \arctan\frac{y}{x}.$$

5. Determine the maximal domain of definition in $\mathbb{R}^2$ of the following functions and compute the partial derivatives up to second order.

$$f_1(x,y) = 4x^4y^2 - 3x^2y^3 + xy - y + 1, \qquad f_2(x,y) = \frac{x-y}{x+y}, \qquad f_3(x,y) = x^2 + xy^2 - 5y^4,$$

$$f_4(x,y) = \sin(x^2y), \qquad f_5(x,y) = \exp(xy)\sin x, \qquad f_6(x,y) = \ln(\sqrt{x^2+y^2}),$$

$$f_7(x,y) = \frac{x^2-y^2}{x^2+y^2}, \qquad f_8(x,y) = \frac{xy}{x^2+y^2}, \qquad f_9(x,y) = \frac{1}{x^2-xy+y^2+1}.$$

6. Compute the directional derivative along d of the following functions :

$$f(x,y) = xe^{x+y},\ d = (1,2), \quad \text{and} \quad g(x,y) = \frac{x-y}{x+y},\ d = (3,-1).$$

7. Compute the differential of each of the following functions f, and write down explicitly $df(p) : \mathbb{R}^2 \to \mathbb{R}$.

1. $f_1 = e^x + y^2$, $p_1 = (0,0)$,
2. $f_2 = \sin(x)\cos(y)$, $p_2 = (\pi, \pi/2)$,
3. $f_3 = \ln(2x-3y)$, $p_3 = (1,-1)$,
4. $f_4 = x^2y^3$, $p_4 = (1,1)$.

8. Determine if the following differential forms on $\mathbb{R}^2$ are *exact* (we recall that a differential form $\omega = g_1dx + g_2dy$ is exact if there is a function f such that $\omega = df$). If they are, compute a function f satisfying that $\omega = df$.

1. $\omega_1 = xdx + ydy$,
2. $\omega_2 = xdx + xdy$,
3. $\omega_3 = ydx + xdy$,
4. $\omega_1 = y^2dx + ydy$.

9. Show that $\omega = \frac{1}{(x+y)^2}\left(2yz\,\mathrm{dx} - 2xz\,\mathrm{dy} + (x^2-y^2)\,\mathrm{dz}\right)$ is exact and compute a function f such that that $\omega = df$.

10. *(bonus)* Let f be defined as

$$\begin{cases} f(x,y) = (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0), \\ f(0,0) = 0. \end{cases}$$

Show that f is continuous on $\mathbb{R}^2$ and that the partial derivatives of first order exist at every point of $\mathbb{R}^2$. Are they continuous on $\mathbb{R}^2$?

11. *(bonus)* Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^3y}{x^2+y^2}, \text{ if } (x,y) \neq (0,0),$$

and $f(0,0) = 0$. Show that f is of class C^1 on $\mathbb{R}^2$. Compute $\mathrm{d}f_{(0,0)}$, and $\mathrm{d}f_{(1,1)}$.

Vector maps

12. (Moving orthonormal frame). Let $\vec{u}_r : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ be the vector field defined on $\mathbb{R}^2 \setminus \{0\}$ given by sending every point $P = (x, y) \in \mathbb{R}^2 \setminus \{0\}$ to

$$\vec{u}_r(P) = \text{ the } \textit{unitary} \text{ vector of the same direction as } \overrightarrow{OP}.$$

Let $\vec{u}_\theta$ be the vector field on $\mathbb{R}^2 \setminus \{0\}$ given by sending P in $\mathbb{R}^2 \setminus \{0\}$ to

$$\vec{u}_\theta(P) = \text{ the counterclockwise rotation of } \vec{u}_r(P) \text{ through an angle } \frac{\pi}{2}.$$

Find $\vec{u}_r(P)$ and $\vec{u}_\theta(P)$ explicitly using Cartesian coordinates (x, y). Do the same for the polar coordinates (r, θ) of the point P.

13. Compute the Jacobian matrix $J_{\vec{u}}(x, y)$ of each of the following vector fields $\vec{u} : \mathbb{R}^n \to \mathbb{R}^n$.

1. $\vec{u}_1(x, y) = (x^2 + xy^2, \sin(x + y))$,
2. $\vec{u}_2(x, y, z) = (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0)$.

14. Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ and $\vec{v} : \mathbb{R}^n \to \mathbb{R}^n$ be two vector fields on $\mathbb{R}^n$.

1. Let $g : \mathbb{R}^n \to \mathbb{R}$ be the map defined by $g(P) = \vec{F}(P) \cdot \vec{v}(P)$. Show that we have that

$$\frac{\partial g}{\partial \vec{h}}(P) = \frac{\partial \vec{F}}{\partial \vec{h}}(P) \cdot \vec{v}(P) + \vec{F}(P) \cdot \frac{\partial \vec{v}}{\partial \vec{h}}(P)$$

   for all $\vec{h} \in \mathbb{R}^n$.

2. Deduce that, if $||\vec{F}||$ is constant, then

$$\frac{\partial \vec{F}}{\partial \vec{h}} \cdot \vec{F} = 0,$$

   for all P and all $\vec{h}$.