Exercise sheet n° 1

Basic geometry of $\mathbb{R}^n$ for $n = 2$ or 3

We suppose that the space $\mathbb{R}^n$ is provided with the usual Euclidean scalar product.

1. Plot the following sets in $\mathbb{R}^2$.

- $D_1 = \{(x, y) | \, 0 \leq x + y \leq 1, \, y > 0\}$
- $D_2 = \{(x, y) | \, x + y + 1 \geq 0, \, x \leq 0, \, y \leq 0\}$
- $D_3 = \{(x, y) | \, |x + y| \leq 1, \, |x - y| \leq 1\}$
- $D_4 = \{(x, y) | \, x^2 + y^2 \leq 4\}$
- $D_5 = \{(x, y) | \, x^2 + y^2 > 1\}$
- $D_6 = \{(x, y) | \, x^2 + y^2 - 2x - 4y \leq 4\}$
- $D_7 = \{(x, y) | \, x^2 + y^2 \leq 1, x > 0, y \geq 0\}$
- $D_8 = \{(x, y) | \, x^2 + y^2 \leq 4, x + y \geq 0\}$
- $D_9 = \{(x, y) | \, x^2 + y^2 \leq 1, x + y \geq 1\}$
- $D_{10} = \{(x, y) | \, x^2 + y^2 \leq 1 \text{ or } x + y \geq 1\}$

2. Consider four points A, B, C, D in the space $\mathbb{R}^n$. Compute the following expression

$$\overrightarrow{AB} \cdot \overrightarrow{CD} + \overrightarrow{AC} \cdot \overrightarrow{DB} + \overrightarrow{AD} \cdot \overrightarrow{BC}.$$

3. Consider four points A, B, C, D in the space $\mathbb{R}^3$. Compute

$$\overrightarrow{AB} \times \overrightarrow{AC} - \overrightarrow{BC} \times \overrightarrow{BA}$$

and

$$\overrightarrow{CA} \times \overrightarrow{CB} - \overrightarrow{DA} \times \overrightarrow{DB} - \overrightarrow{DB} \times \overrightarrow{DC} - \overrightarrow{DC} \times \overrightarrow{DA}.$$

4. Let $\vec{u}, \vec{v}$ et $\vec{w}$ be three vectors in the space $\mathbb{R}^3$. We recall the following identity for the **double vector product**

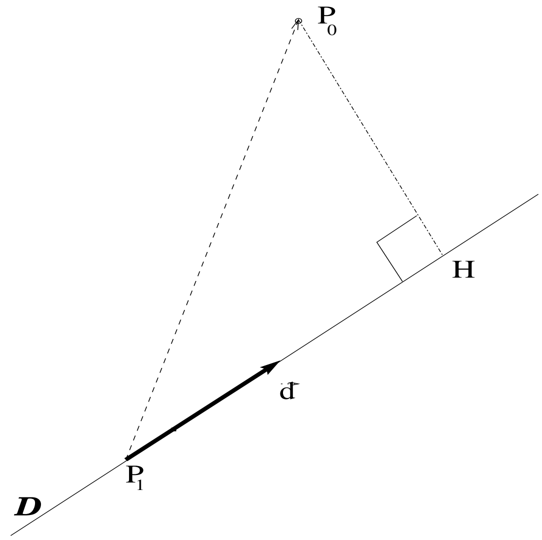
$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w}) \, \vec{v} - (\vec{u} \cdot \vec{v}) \, \vec{w} \quad \text{(note that the order is important!)}$$

Compute

$$\vec{u} \times (\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}).$$

Reminder

We recall that the **orthogonal projection** of a point P_0 onto the line D is the point H on D such that the distance $d(P_0, H)$ is minimal with respect to all the points of the line. The value of this minimum is called the **distance** from P_0 to the line D (see the figure below).



5. Let D be the line in $\mathbb{R}^2$ given by $y = 2x - 1$ and let A be the point $(1, 2)$.

1. Compute the square of the distance between the points A and $M(x, 2x-1)$ of the line, as a function of x. Show that this function has a minimum and deduce the distance $d(A, D)$ from A to the line D. Compute the coordinates of the projection H of A onto D.
2. Give a direction vector of D. Let B be point $(-1, 1)$. Compute the distance from B to the line D.

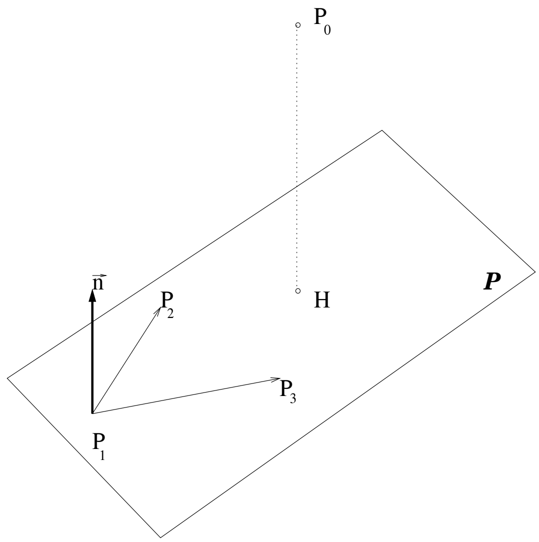
6. Let $P_1(3, 1, -2)$ and $P_2(-1, 2, 4)$ be two points in $\mathbb{R}^3$.

1. Find the equation of the line D containing P_1 and P_2 and a direction vector $\vec{v}$ of norm 1 of this line.
2. Compute the distance from $P_0(1, 3, -1)$ to the line D.
3. Determine the coordinates (x, y, z) of the orthogonal projection H of P_0 onto the line D.

7. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be the the planes of equations

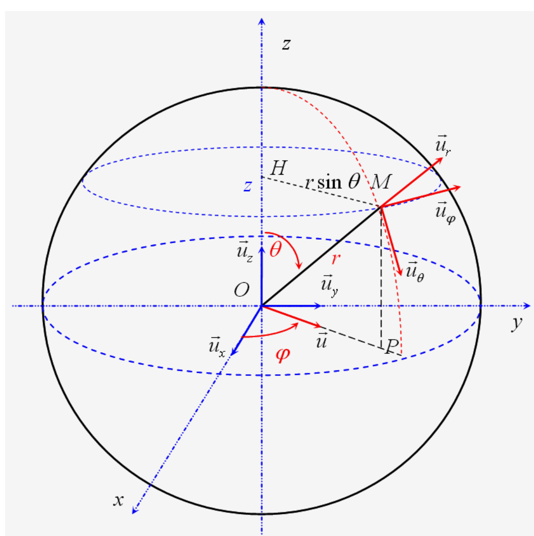
$$\mathcal{P}_1 : \quad 2x + 3y + z - 4 = 0;$$
$$\mathcal{P}_2 : \quad 3x - y - 3z - 2 = 0.$$

1. Show that the two planes are orthogonal.
2. Compute the distance from the origin O to the plane $\mathcal{P}_1$, to the plane $\mathcal{P}_2$, and to the line $\mathcal{D} = \mathcal{P}_1 \cap \mathcal{P}_2$.
3. Consider the sphere $\mathcal{S}$ given by the equation $x^2 + y^2 + z^2 - 2x - 2y - 2z + 2 = 0$. Is the intersection between the plane $\mathcal{P}_1$ and the sphere $\mathcal{S}$ nonempty ? And the intersection between the line $\mathcal{D}$ and the sphere $\mathcal{S}$?



Polar, cylindrical and spherical coordinates

8. Using polar coordinates, what does each of the two equations $r = constant$ and $\theta = constant$ represent, respectively ?
9. Consider cylindrical coordinates. What does each of the three equations $r = constant$, $\theta = constant$ and $z = constant$ represent, respectively ? And what does the set given by the intersection of the conditions $r = constant$ and $\theta = constant$ represent ? Answer the same question for the intersection between $r = constant$ and $z = constant$?
10. Consider now spherical coordinates. What does each of the three equations $r = constant$, $\theta = constant$ and $\phi = constant$ represent, respectively ? And what does the set given by the intersection of the conditions $r = constant$ and $\theta = constant$ represent ? Answer the same question for the intersection between $r = constant$ and $\phi = constant$?



11. Let C be the circle of equation $(x-1)^2 + y^2 = 1$. Give the equation determining C in polar coordinates.
12. Let $\mathcal{C}$ be the cone in $\mathbb{R}^3$ defined as $z = \sqrt{x^2 + y^2}$ in Cartesian coordinates. Give the equation of $\mathcal{C}$ in cylindrical coordinates.

Homework

13. Let $\mathcal{D}_1$ be the line containing the point M_1 and of direction vector $\vec{u}_1$, and let $\mathcal{D}_2$ be the line containing the point M_2 and of direction vector $\vec{u}_2$. The **distance** $d(\mathcal{D}_1, \mathcal{D}_2)$ **between** $\mathcal{D}_1$ **and** $\mathcal{D}_2$ is defined as the minimum of the distances $d(M, M')$ between any two points $M \in \mathcal{D}_2$ and $M' \in \mathcal{D}_2$. We assume that $\vec{u}_1$ et $\vec{u}_2$ are linearly independent.

1. Let $\mathcal{P}$ be the plane containing the point M_1 and parallel to $\vec{u}_1$ and $\vec{u}_2$. Give the equation of $\mathcal{P}$, and show that for every point M' of the line $\mathcal{D}_2$ we have that $d(\mathcal{P}, M') = d(\mathcal{P}, M_2)$. Deduce that $\mathcal{D}_2$ is parallel to the plane $\mathcal{P}$.
2. Compute the volume of the parallelepiped of sides $\vec{u}_1$, $\vec{u}_2$ and $\overrightarrow{M_1M_2}$. Deduce the distance from the point M_2 to the plane $\mathcal{P}$, and the distance between the lines $\mathcal{D}_1$ and $\mathcal{D}_2$.

14. A sailor is planning a journey from Brest to New-York by travelling along an arc of a circle over the Atlantic Ocean. Compute the length of this arc Brest-New-York, if the Earth radius is $R = 6400$km and the geographic coordinates of the cities are

Brest : latitude $48^o50'N$, longitude 0^o

New-York : latitude $40^o40'N$, longitude $73^o50'W$.

Hint : Let $\vec{a}_1$ and $\vec{a}_2$ be two vectors of the same norm. Show that the angle θ_{12} between $\vec{a}_1$ and $\vec{a}_2$ satisfies the identity

$$\cos\theta_{12} = \sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) + \cos\theta_1 \cos\theta_2,$$

where θ_1, θ_2, ϕ_1 and ϕ_2 are the angles describing the corresponding vectors $\vec{a}_1$ and $\vec{a}_2$ in spherical coordinates.

The velocity and the acceleration

15. Consider a material point described by the smooth parameterized curve given by

$$\begin{aligned} x(t) &= \sin t, \\ y(t) &= 1 - \cos^2 t, \end{aligned}$$

where $t \in \mathbb{R}$ indicates the time.

1. Plot the trajectory of the point and show that the motion is in fact periodic. Determine the (smallest strictly positive) period.
2. Compute the velocity and determine the values of t where it vanishes. Compute also the values of t where the speed (*i.e.* the norm of the velocity) is maximal.