MAT332 - Series and integration
Fall term — 2022-2023

Exercise sheet 5: Generalized integrals

1. *Integration by parts.* Find a primitive F of the function

$$\begin{aligned} f : \mathbb{R}_{\geq 0} &\to \mathbb{R}_{\geq 0} \\ t &\mapsto t^2 e^{-t}. \end{aligned}$$

Deduce from the above that the generalized integral $\int_0^{+\infty} f(t)dt$ converges and compute its value.

Solution. Note that $F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by $F(t) = -e^{-t}(t^2 + 2t + 2)$ is a continuous map that is differentiable on $\mathbb{R}_{>0}$ and satisfies that $F'(t) = f(t)$ for all $t > 0$. Moreover, we have that

$$\int_0^{+\infty} f(t)dt = \lim_{A\to+\infty} \Big[F(t)\Big]_0^A = \lim_{A\to+\infty} \big(F(A) - F(0)\big) = \lim_{A\to+\infty} \big(-e^{-A}(A^2+2A+2)+2\big) = 2.$$

2. *A rational fraction.*

(*a*) Given $X > 0$, define

$$I(X) = \int_1^X \frac{dx}{x(x+1)(x+2)}.$$

Compute an explicit expression of $I(X)$.

(*b*) What is its limit when $X \to +\infty$? What can we say about the generalized integral $\int_1^{+\infty} dx/(x(x+1)(x+2))$?

Solution.

(*a*) Since

$$\frac{1}{x(x+1)(x+2)} = \frac{1}{2x} + \frac{1}{2(x+2)} - \frac{1}{x+1}$$

for $x \in \mathbb{R} \setminus \{0, -1, -2\}$, we get that

$$\begin{aligned} I(X) = \int_1^X \frac{dx}{x(x+1)(x+2)} &= \left[\frac{\ln(|x|)}{2} + \frac{\ln(|x+2|)}{2} - \ln(|x+1|)\right]_1^X \\ &= \frac{\ln(X)}{2} + \frac{\ln(X+2)}{2} - \ln(X+1) + \ln(2) - \frac{\ln(3)}{2} = \ln\left(2\sqrt{\frac{X(X+2)}{3(X+1)^2}}\right) \end{aligned}$$

for $X \geq 1$.

(b) Note that

$$\lim_{X\to+\infty} I(X) = \lim_{X\to+\infty} \ln\left(2\sqrt{\frac{X(X+2)}{3(X+1)^2}}\right) = \lim_{X\to+\infty} \ln\left(2\sqrt{\frac{(1+2/X)}{3(1+1/X)^2}}\right) = +\infty,$$

which tells us that $\int_1^{+\infty} dx/(x(x+1)(x+2))$ converges and it is equal to $\ln(2/\sqrt{3})$.

3. *Change of variables.* Let $X \in \mathbb{R}_{\geq 0}$. Compute

$$I(X) = \int_0^X \frac{dt}{\cosh(t)}$$

and find its limit when $X \to +\infty$.

Solution. It is easy to see that

$$\int \frac{dt}{\cosh(t)} = \int \frac{\cosh(t)dt}{1+\sinh^2(t)} = \int \frac{du}{1+u^2} = \arctan(u) + C = \arctan\big(\sinh(t)\big) + C,$$

for any real constant C, where we used that $\cosh^2(t) - \sinh(t) = 1$ for all $t \in \mathbb{R}$, and the change of variables $u = \sinh(t)$ (so $du = \cosh(t)dt$). Hence,

$$I(X) = \int_0^X \frac{dt}{\cosh(t)} = \Big[\arctan\big(\sinh(t)\big)\Big]_0^X = \arctan\big(\sinh(X)\big),$$

for $X \in \mathbb{R}_{\geq 0}$, which tells us that

$$\lim_{X\to+\infty} I(X) = \lim_{X\to+\infty} \arctan\big(\sinh(X)\big) = \frac{\pi}{2}.$$

4. *Convergence and divergence of generalized integrals.* Consider the following integrals :

(a) $\int_1^\infty \ln(x)/(x+e^{-x})dx$,
(b) $\int_0^1 \ln(x)/(x+e^{-x})dx$,
(c) $\int_0^\infty \ln(x)/(x+e^{-x})dx$,
(d) $\int_1^\infty |\sin(x)|/(x^2+1)dx$,
(e) $\int_0^{+\infty} e^{-\sqrt{x}}/\sqrt{x}dx$,
(f) $\int_1^\infty \ln(x)x^{-1}e^{-x}dx$,
(g) $\int_0^\infty (x+2-\sqrt{x^2+4x+1})dx$,
(h) $\int_1^\infty (\sqrt[3]{x^3+1} - \sqrt{x^2+1})dx$,
(i) $\int_1^\infty e^{-\sqrt{x^2-x}}dx$,
(j) $\int_2^\infty \sqrt{x}/\ln^3(x)dx$,
(k) $\int_1^\infty (2+\sin(x)+\sin^2(x))/\sqrt[3]{x^4+x^2}dx$,
(l) $\int_{-\infty}^\infty e^{-x^2}dx$,
(m) $\int_0^1 \ln(x)/(1-x)dx$.

Determine if they converge or diverge.

Solution.

(a) Note first that the integrand is a nonnegative function, so its integral converges if and only if it is bounded above. It is easy to see that

$$\int_1^{+\infty} \frac{\ln(x)}{x+e^{-x}}dx \geq \int_1^{+\infty} \frac{\ln(x)}{2x}dx = \frac{1}{4}\lim_{M\to+\infty}\Big[\ln^2(|x|)\Big]_1^M,$$

where we used that $e^{-x} \leq x$, for all $x \geq 1$. Since the last member goes to $+\infty$ as M goes to $+\infty$, we conclude that the improper integral is divergent.

(b) Note first that the integrand is a nonpositive function. It is easy to see that

$$\begin{aligned}\int_0^1 \frac{|\ln(x)|}{x+e^{-x}}dx &\leq e\int_0^1 |\ln(x)|dx = -e\int_0^1 \ln(x)dx = -e\lim_{\epsilon\to0+}\Big[x\big(\ln(|x|)-1\big)\Big]_\epsilon^1\\ &= e\lim_{\epsilon\to0+}\Big(1-\epsilon(\ln(\epsilon)-1)\Big),\end{aligned}$$

where we used that $e^{-x}+x \geq e^{-1}$, for all $x \in [0,1]$. Since the last member goes to e as ϵ goes to 0, we conclude that the improper integral is convergent.

(c) Since

$$\int_0^{\infty} \frac{\ln(x)}{x+e^{-x}}dx = \int_0^1 \frac{\ln(x)}{x+e^{-x}}dx + \int_1^{+\infty} \frac{\ln(x)}{x+e^{-x}}dx,$$

the first integral of the second member converges whereas the second one diverges, we conclude that $\int_0^\infty \ln(x)/(x+e^{-x})dx$ diverges.

(d) We will prove that the required integral is absolutely convergent. In order to do so, it suffices to show that the integral of the absolute value of the integrand is bounded above. Since

$$\begin{aligned}\int_1^{\infty} \frac{|\sin(x)|}{x^2+1}dx &\leq \int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{M\to+\infty}\Big[\arctan(x)\Big]_1^M\\ &= \lim_{M\to+\infty}\arctan(M) - \frac{\pi}{4} = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4},\end{aligned}$$

we conclude that the improper integral is convergent.

(e) Note first that the integrand is a nonnegative function, so its integral converges if and only if it is bounded above. Since

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{M\to+\infty}\Big[-2e^{-\sqrt{x}}\Big]_1^M = \lim_{M\to+\infty}\big(2e^{-1}-2e^{-\sqrt{M}}\big) = 2e^{-1},$$

we conclude that the improper integral is convergent.

(f) Recall that

$$\lim_{x\to+\infty}\frac{\ln(x)}{x} = 0.$$

As a consequence, there is $C > 1$ such that $\ln(x)/x \leq 1$ for all $x > C$, and in particular

$$\begin{aligned}\int_1^{\infty} \frac{\ln(x)}{x}e^{-x}dx &= \int_1^{C} \frac{\ln(x)}{x}e^{-x}dx + \int_C^{\infty} \frac{\ln(x)}{x}e^{-x}dx\\ &\leq \int_1^{C} \frac{\ln(x)}{x}e^{-x}dx + \int_C^{\infty} e^{-x}dx \leq \int_1^{C} \frac{\ln(x)}{x}e^{-x}dx + \int_C^{\infty} e^{-x}dx\\ &= \int_1^{C} \frac{\ln(x)}{x}e^{-x}dx + \lim_{M\to+\infty}\Big[-e^{-x}\Big]_C^M = \int_1^{C} \frac{\ln(x)}{x}e^{-x}dx + e^{-C}.\end{aligned}$$

We also note that, since $\ln(x)x^{-1}e^{-x}$ is continuous over $[1, C]$, the integral $\int_1^C \ln(x)x^{-1}e^{-x}dx$ exists. We recall that, since $\ln(x)x^{-1}e^{-x}$ is a positive function over $\mathbb{R}_{\geq 1}$, its integral converges if and only if it is bounded above. As a consequence, the given improper integral converges.

(*g*) It is easy to see that

$$\int_0^\infty (x+2-\sqrt{x^2+4x+1})dx$$
$$= \int_0^1 (x+2-\sqrt{x^2+4x+1})dx + \int_1^\infty (x+2-\sqrt{x^2+4x+1})dx.$$

Since $x+2-\sqrt{x^2+4x+1}$ is a continuous function over $[0,1]$, the first integral exists, which implies that the improper integral $\int_0^\infty (x+2-\sqrt{x^2+4x+1})dx$ converges if and only if the improper integral $\int_1^\infty (x+2-\sqrt{x^2+4x+1})dx$ converges. On the other hand,

$$\int_1^\infty (x+2-\sqrt{x^2+4x+1})dx = \int_1^\infty \frac{(x+2)^2-(x^2+4x+1)}{x+2+\sqrt{x^2+4x+1}}dx$$
$$= \int_1^\infty \frac{3}{x+2+\sqrt{x^2+4x+1}}dx \geq \int_1^\infty \frac{1}{2x}dx = \lim_{M\to+\infty}\left[\frac{3\ln(|x|)}{2}\right]_1^M = +\infty,$$

where we have used that

$$x+2+\sqrt{x^2+4x+1} \leq x+2+\sqrt{x^2+4x+4} \leq 2x+4 \leq 6x$$

for all $x \geq 1$. In particular, since the integral $\int_1^\infty (x+2-\sqrt{x^2+4x+1})dx$ is divergent, the integral $\int_0^\infty (x+2-\sqrt{x^2+4x+1})dx$ is divergent.

(*h*) Applying that $a^6 - b^6 = (a-b)(\sum_{k=0}^5 a^k b^{5-k})$ for $a = \sqrt[3]{x^3+1}$ and $b = \sqrt{x^2+1}$, we have that

$$\sqrt[3]{x^3+1}-\sqrt{x^2+1} = \frac{(x^3+1)^2-(x^2+1)^3}{f(x)} = -\frac{3x^4-2x^3+3x^2}{f(x)} = -\frac{x^2(3x^2-2x+3)}{f(x)},$$

where $f(x) = \sum_{k=0}^5 a^k b^{5-k}$ for $a = \sqrt[3]{x^3+1}$ and $b = \sqrt{x^2+1}$. In particular, $\sqrt[3]{x^3+1}-\sqrt{x^2+1} \leq 0$ for all $x \in \mathbb{R}$, since $3x^2-2x+3 = 3(x-1/3)^2+8/3 > 0$ and $f(x) > 0$ for all $x \in \mathbb{R}$. Since

$$\lim_{x\to+\infty} \frac{f(x)}{6x^5} = 1,$$

we have that

$$\lim_{x\to+\infty} \frac{\frac{3x^4-2x^3+3x^2}{f(x)}}{\frac{1}{2x}} = 1,$$

which implies that $\sqrt{x^2+1}-\sqrt[3]{x^3+1} \sim 1/(2x)$ as $x \to +\infty$. Since the integrals $\int_1^C (\sqrt{x^2+1}-\sqrt[3]{x^3+1})dx$ and $\int_1^C 1/x dx$ exist for all $C > 1$, for the integrands are continuous functions, then $\int_1^{+\infty} (\sqrt{x^2+1}-\sqrt[3]{x^3+1})dx$ converges if and only if $\int_1^{+\infty} 1/x dx$ converges. As the latter integral diverges, the same holds for $\int_1^{+\infty} (\sqrt{x^2+1}-\sqrt[3]{x^3+1})dx = -\int_1^{+\infty} (\sqrt[3]{x^3+1}-\sqrt{x^2+1})dx$.

(i) It is clear that

$$\int_1^{+\infty} e^{-\sqrt{x^2-x}}dx = \int_1^2 e^{-\sqrt{x^2-x}}dx + \int_2^{+\infty} e^{-\sqrt{x^2-x}}dx.$$

Since the first integral of the right member converges, for $e^{-\sqrt{x^2-x}}$ is a continuous function defined over a finite and closed interval, the integral in the left member converges if and only if the second integral of the right member converges. Since $e^{-\sqrt{x^2-x}}$ is a positive function, it suffices to show that the integral is bounded above. In this case

$$\int_2^{+\infty} e^{-\sqrt{x^2-x}}dx \leq \int_2^{+\infty} e^{-x/\sqrt{2}}dx = \lim_{M\to+\infty}\Big[-\sqrt{2}e^{-x/\sqrt{2}}\Big]_2^M = \sqrt{2}e^{-\sqrt{2}},$$

where we have used that $x^2 - x \geq x^2/2$, for $x \geq 2$. Since the last integral converges (to 0 when M goes to $+\infty$), we conclude that $\int_1^\infty e^{-\sqrt{x^2-x}}dx$ converges.

(j) Since

$$\lim_{x\to+\infty}\frac{x^\alpha}{\ln(x)} = +\infty$$

for all $\alpha > 0$, we have that

$$\lim_{x\to+\infty}\frac{\sqrt{x}}{\ln^3(x)} = \lim_{x\to+\infty}\left(\frac{\sqrt[6]{x}}{\ln(x)}\right)^3 = +\infty,$$

which in term implies that the integral $\int_2^\infty \sqrt{x}/\ln^3(x)dx$ diverges, by Exercise **5**.

(k) We see that

$$\int_1^\infty \left|\frac{2+\sin(x)+\sin^2(x)}{\sqrt[3]{x^4+x^2}}\right|dx \leq \int_1^\infty \frac{4}{\sqrt[3]{x^4+x^2}}dx \leq \int_1^\infty \frac{4}{\sqrt[3]{x^4}}dx$$
$$= \lim_{M\to+\infty}\Big[-\frac{12}{\sqrt[3]{x}}\Big]_1^M = 12,$$

where we used that $x^4 + x^2 \geq x^4$. Since the integrand is a nonnegative function, its integral converges, for it is bounded above. Hence $\int_1^\infty (2+\sin(x)+\sin^2(x))/\sqrt[3]{x^4+x^2}dx$ is absolutely convergent, so it is convergent.

(l) Since e^{-x^2} is symmetric, *i.e.* $e^{-x^2} = e^{-(-x)^2}$ for all $x \in \mathbb{R}$, which implies that $\int_{-\infty}^\infty e^{-x^2}dx$ converges if and only if $\int_0^\infty e^{-x^2}dx$ converges. Furthermore, since e^{-x^2} is continuous, the integral $\int_0^1 e^{-x^2}dx$ exists, so $\int_0^\infty e^{-x^2}dx$ converges if and only if $\int_1^\infty e^{-x^2}dx$ converges. Moreover, since e^{-x^2} is a nonnegative function, $\int_1^\infty e^{-x^2}dx$ converges if and only if it is bounded. Since $e^{-x^2} \leq e^{-x}$ for all $x \geq 1$, we have that

$$\int_1^\infty e^{-x^2}dx \leq \int_1^\infty e^{-x}dx = \lim_{M\to+\infty}\Big[-e^{-x}\Big]_1^M = e^{-1},$$

so the integral converges.

(m) It is clear that $\ln(x)/(1-x)$ is a continuous function over $]0,1[$. Moreover, the Bernoulli-L'Hospital rule tells us that

$$\lim_{x\to1}\frac{\ln(x)}{1-x} = \lim_{x\to0}\frac{1/x}{-1} = -1,$$

which implies that $\ln(x)/(1-x)$ has a continuous extension to $]0,1]$. In particular, the integral $\int_0^1 \ln(x)/(1-x)dx$ converges if and only if $\int_0^{1/2} \ln(x)/(1-x)dx$ converges. Moreover, since $|\ln(x)/(1-x)| = -\ln(x)/(1-x)$ for $x \in]0,1[$, we see that $\int_0^{1/2} \ln(x)/(1-x)dx$ converges if and only if it is bounded. We also note that

$$\int_0^{1/2} -\frac{\ln(x)}{1-x}dx \leq -2\int_0^{1/2} \ln(x)dx = -2\lim_{\epsilon\to+\infty}\Big[x\ln(x)-x\Big]_\epsilon^{1/2} = 1-\ln(1/2),$$

since $1-x \geq 1/2$ and

$$\lim_{x\to+\infty} x\ln(x) = \lim_{x\to+\infty}\frac{\ln(x)}{1/x} = \lim_{x\to+\infty}\frac{1/x}{-1/x^2} = 0$$

by the Bernoulli-L'Hospital rule. Hence, the integral $\int_0^1 \ln(x)/(1-x)dx$ converges.

5. *Limit and convergence of the integral.*

(a) Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a piecewise continuous function such that $f(t) \to \ell$ when $t \to +\infty$, with $\ell > 0$ or $\ell = +\infty$. Prove that $\int_0^{+\infty} f(t)dt$ diverges.

(b) Give an example of a piecewise continuous function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $g(t)$ does not tend to 0 when $t \to +\infty$ but $\int_0^{+\infty} g(t)dt$ converges.

Solution.

(a) Since $f(t) \to \ell \in \mathbb{R}_{>0} \cup \{+\infty\}$ when $t \to +\infty$, given $\epsilon > 0$, there exists $C > 0$ such that $f(t) > \epsilon$ for all $t > C$. As a consequence,

$$\lim_{M\to+\infty}\int_0^M f(t)dt \geq \lim_{M\to+\infty}\int_C^M f(t)dt \geq \lim_{M\to+\infty}\epsilon(M-C) = +\infty$$

for $M \geq C$.

(b) Consider the map $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by

$$g(t) = \begin{cases} 1, & \text{if } t \in [n-2^{-n}, n+2^{-n}] \text{ for } n \in \mathbb{N}, \\ 0, & \text{if } t \in \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}}[n-2^{-n}, n+2^{-n}]. \end{cases}$$

Then,

$$\int_0^{N+2^{-N}} g(t)dt = \sum_{n=1}^{N}\frac{1}{2^{n-1}} \leq 2$$

for all $N \in \mathbb{N}$. Since the function is nonnegative, $\int_0^{+\infty} g(t)dt$ converges.

6. *Two equivalent descriptions.*

(a) Decide whether the integrals $\int_0^1 t^{-1}e^{-t}dt$ and $\int_1^{+\infty} t^{-1}e^{-t}dt$ converge or diverge.

(b) For any $x > 0$, let $f(x) = \int_x^{+\infty} t^{-1}e^{-t}dt$. Compute its limit at $+\infty$.

(c) Show that

$$f(x) \sim \frac{e^{-x}}{x},$$

when x tends to $+\infty$.

(d) Prove that $f(x) \sim \ln(1/x)$ when $x > 0$ tends to 0.

Solution.

(a) Since the function $t^{-1}e^{-t}$ is positive on $\mathbb{R}_{>0}$, the integral $\int_0^1 t^{-1}e^{-t}dt$ (resp., $\int_1^{+\infty} t^{-1}e^{-t}dt$) converges if it is bounded. Since

$$\int_0^1 t^{-1}e^{-t}dt \geq e^{-1}\int_0^1 t^{-1}dt = e^{-1}\lim_{\epsilon\to 0+}\Big[\ln(x)\Big]_\epsilon^1 = +\infty,$$

the integral $\int_0^1 t^{-1}e^{-t}dt$ diverges. On the other hand,

$$\int_1^{+\infty} t^{-1}e^{-t}dt \leq \int_1^{+\infty} e^{-t}dt = \lim_{M\to+\infty}\Big[-e^{-x}\Big]_1^M = 1,$$

so the integral $\int_1^{+\infty} t^{-1}e^{-t}dt$ converges.

(b) Since the integral $\int_1^{+\infty} t^{-1}e^{-t}dt$ converges, then

$$\lim_{x\to+\infty}\int_x^{+\infty} t^{-1}e^{-t}dt = 0.$$

(c) By the previous item we can use the Bernoulli-L'Hospital rule, which gives

$$\lim_{x\to+\infty}\frac{\int_x^{+\infty} t^{-1}e^{-t}dt}{x^{-1}e^{-x}} = \lim_{x\to+\infty}\frac{x^{-1}e^{-x}}{x^{-2}e^{-x}(x+1)} = \lim_{x\to+\infty}\frac{x}{x+1} = 1,$$

so $f(x) \sim e^{-x}x^{-1}$ as x goes to $+\infty$.

(d) Since the integral $\int_0^{+\infty} e^{-t}t^{-1}dt$ diverges, we can use the Bernoulli-L'Hospital rule, which gives

$$\lim_{x\to 0+}\frac{\int_x^{+\infty} t^{-1}e^{-t}dt}{\ln(1/x)} = \lim_{x\to 0+}\frac{x^{-1}e^{-x}}{1/x} = 1,$$

so $f(x) \sim \ln(x^{-1})$ as x goes to $0+$.

7. *Logarithmic derivative.* Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ be a continuous function. For any $x \in \mathbb{R}_{\geq 0}$, we set $F(x) = \int_0^x f(t)dt$. Prove that the integrals

$$\int_1^{+\infty} f(t)dt \text{ and } \int_1^{+\infty}\frac{f(t)}{F(t)}dt$$

simultaneously converge or diverge.

Solution. Note that $F'(x) = f(x)$ for all $x \in \mathbb{R}_{>0}$. Hence,

$$\lim_{M\to+\infty} \int_1^M \frac{f(t)}{F(t)} dt = \lim_{M\to+\infty} \int_1^M \frac{F'(t)}{F(t)} dt = \lim_{M\to+\infty} \int_1^M \ln\big(F(t)\big)' dt$$
$$= \lim_{M\to+\infty} \Big[\ln\big(F(t)\big)\Big]_1^M = \lim_{M\to+\infty} \ln\left(\frac{F(M)}{F(1)}\right).$$

It is then clear that the previous limit exist if and only if

$$\lim_{M\to+\infty} F(M) = \lim_{M\to+\infty} \int_0^M f(t)dt = \int_1^{+\infty} f(t)dt$$

exists. In consequence, $\int_1^{+\infty} f(t)dt$ converges if and only if $\int_1^{+\infty} f(t)/F(t)dt$ converges.

8. For any integer $n \in \mathbb{N}_0$, we set

$$u_n = \int_{n\pi}^{(n+1)\pi} \frac{\cos^2(x)}{1+x} dx \text{ and } v_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin^2(x)}{1+x} dx.$$

(a) Compute $a_n = u_n + v_n$ and prove that $\sum_{n\in\mathbb{N}_0} a_n$ diverges.

(b) Using integration by parts, prove that we have the inequality

$$|u_n - v_n| \leqslant \frac{1}{2\pi n^2},$$

for all $n \in \mathbb{N}$.

(c) Deduce that $\sum_{n\in\mathbb{N}_0} u_n$ and $\sum_{n\in\mathbb{N}_0} v_n$ are divergent.

(d) Given $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}_{\geq 0}$, set

$$f_\alpha(x) = \frac{\sin^2(x)}{(1+x)^\alpha}.$$

Find the values of α for which the function f_α is integrable on $\mathbb{R}_{\geq 0}$.

Solution.

(a) Note that

$$a_n = \int_{n\pi}^{(n+1)\pi} \frac{\cos^2(x) + \sin^2(x)}{1+x} dx = \int_{n\pi}^{(n+1)\pi} \frac{1}{1+x} dx,$$

for all $n \in \mathbb{N}_0$, so

$$\sum_{n=0}^{N} a_n = \int_0^{(N+1)\pi} \frac{1}{1+x} dx = \ln\big(1 + (N+1)\pi\big)$$

for all $N \in \mathbb{N}_0$, which implies that $\sum_{n\in\mathbb{N}_0} a_n$ diverges.

(b) Note first that

$$u_n - v_n = \int_{n\pi}^{(n+1)\pi} \frac{\cos^2(x) - \sin^2(x)}{1+x} dx = \int_{n\pi}^{(n+1)\pi} \frac{\cos(2x)}{1+x} dx$$
$$= \left[\frac{\sin(2x)}{2(x+1)} \right]_{n\pi}^{(n+1)\pi} + \int_{n\pi}^{(n+1)\pi} \frac{\sin(2x)}{2(1+x)^2} dx = \int_{n\pi}^{(n+1)\pi} \frac{\sin(2x)}{2(1+x)^2} dx,$$

where we did an integration by parts with $u = 1/(1+x)$ and $v' = \cos(2x)$, so $v = \sin(2x)/2$, which implies that

$$|u_n - v_n| \leq \int_{n\pi}^{(n+1)\pi} \left| \frac{\sin(2x)}{2(1+x)^2} \right| dx \leq \frac{\pi}{2n^2\pi^2} = \frac{1}{2\pi n^2},$$

for all $n \in \mathbb{N}_0$, where we used that $1 + x \geq n\pi$ for $x \in [n\pi, (n+1)\pi]$.

(c) If $\sum_{n\in\mathbb{N}_0} u_n$ is convergent, then

$$\sum_{n=0}^{N} v_n = \sum_{n=0}^{N} u_n + \sum_{n=0}^{N} (v_n - u_n)$$

converges as N goes to $+\infty$, since the series $\sum_{n\in\mathbb{N}_0}(v_n - u_n)$ is absolutely convergent. This in turn implies that

$$\sum_{n=0}^{N} a_n = \sum_{n=0}^{N} u_n + v_n = \sum_{n=0}^{N} u_n + \sum_{n=0}^{N} v_n$$

converges as N goes to $+\infty$, which is absurd due to the first item. The same argument shows that if $\sum_{n\in\mathbb{N}_0} v_n$ is convergent, then $\sum_{n\in\mathbb{N}_0} a_n$ is also convergent, which is a contradiction. As a consequence, $\sum_{n\in\mathbb{N}_0} u_n$ and and $\sum_{n\in\mathbb{N}_0} v_n$ are divergent.

(d) Note that, if $\alpha > 1$, then

$$\int_0^{+\infty} f_\alpha(x)dx = \int_0^{+\infty} \frac{\sin^2(x)}{(1+x)^\alpha} dx \leq \int_0^{+\infty} \frac{1}{(1+x)^\alpha} dx$$
$$= \lim_{M\to+\infty} \left[\frac{1}{(1-\alpha)(1+x)^{\alpha-1}} \right]_0^M = \frac{1}{\alpha - 1}.$$

so the integral $\int_0^{+\infty} f_\alpha(x)dx$ converges for $\alpha > 1$.

Moreover, since $f_\alpha(x) \leq f_{\alpha'}(x)$ for all $x \in \mathbb{R}_{\geq 0}$ if $\alpha > \alpha'$, if $\int_0^{+\infty} f_\alpha(x)dx$ converges then $\int_0^{+\infty} f_{\alpha'}(x)dx$ converges, since $(1+x)^{\alpha'} \leq (1+x)^\alpha$ for all $x \in \mathbb{R}_{\geq 0}$ if $\alpha' < \alpha$. On the other hand, we note that $\int_0^{+\infty} f_1(x)dx$ diverges, since

$$\lim_{N\to+\infty} \int_0^{(N+1)\pi} f_1(x)dx = \lim_{N\to+\infty} \sum_{n=0}^{N} a_n$$

diverges. As a consequence, $\int_0^{+\infty} f_\alpha(x)dx$ diverges for all $\alpha \in [0, 1]$.

9. *Convergence and divergence of generalized integrals.* Consider the following integral expressions :

(a) $\int_2^\infty \cos(x)/\ln(x)dx$,

(b) $\int_1^\infty \sin(x-1)/\ln(x)dx$,

(c) $\int_0^1 \sin(1/t)dt$,

(d) $\int_0^1 x^{-1}\cos(2/x)dx$.

Decide whether they are convergent or divergent.

Solution.

(a) Note that

$$\left|\int_2^M \cos(x)dx\right| = \left|\Big[\sin(x)\Big]_2^M\right| \leq 2$$

for all $M \geq 2$. Moreover, since $1/\ln(x)$ is a positive decreasing continuous function whose limit is zero as x goes to $+\infty$, then the Leibniz criterion for integrals tells us that the integral $\int_2^\infty \cos(x)/\ln(x)dx$ converges.

(b) Note that

$$\left|\int_1^M \sin(x-1)dx\right| = \left|\Big[-\cos(x-1)\Big]_1^M\right| \leq 2$$

for all $M \geq 1$. Moreover, since $1/\ln(x)$ is a positive decreasing continuous function whose limit is zero as x goes to $+\infty$, then the Leibniz criterion for integrals tells us that the integral $\int_1^\infty \sin(x-1)/\ln(x)dx$ converges.

(c) Since the function $|\sin(1/t)|$ is continuous, positive and bounded for $t \in]0,1]$, then the integral

$$\int_0^1 |\sin(1/t)|dt = \lim_{\epsilon\to 0+}\int_\epsilon^1 |\sin(1/t)|dt$$

exists, since it is the limit of a bounded function that increases as ϵ decreases to 0. Since the $\int_0^1 \sin(1/t)dt$ is absolutely convergent, it is convergent.

(d) Note first that

$$\lim_{\epsilon\to 0+}\int_\epsilon^1 x^{-1}\cos(2/x)dx = \lim_{\epsilon\to 0+} -\int_1^{1/\epsilon} y^{-1}\cos(2y)dy = -\int_1^{+\infty} y^{-1}\cos(2y)dy.$$

Note that

$$\left|\int_1^M \cos(2y)dy\right| = \left|\left[\frac{\sin(2y)}{2}\right]_1^M\right| \leq 1$$

for all $M \geq 1$. Moreover, since $1/y$ is a positive decreasing continuous function whose limit is zero as y goes to $+\infty$, then the Leibniz criterion for integrals tells us that the integral $\int_1^\infty y^{-1}\cos(2y)dy$ converges, so $\int_0^1 x^{-1}\cos(2/x)dx$ converges.

10. (a) Prove that

$$\sin^2(t) = \frac{\tan^2(t)}{1+\tan^2(t)}$$

for all $t \in [0, \pi/2]$.

(*b*) Let $a > -1$. Using the change of variables $x = \tan(t)$, prove that

$$\int_0^{\frac{\pi}{2}} \frac{dt}{1 + a\sin^2(t)} = \frac{\pi}{2\sqrt{1+a}}.$$

(*c*) Show that

$$\int_0^{\pi} \frac{dt}{1 + a\sin^2(t)} = 2\int_0^{\frac{\pi}{2}} \frac{dt}{1 + a\sin^2(t)}.$$

(*d*) Given $\alpha \in \mathbb{R}_{>0}$, define

$$u_n = \int_0^{\pi} \frac{dt}{1 + (n\pi)^\alpha \sin^2(t)}$$

for $n \in \mathbb{N}_0$. Prove that the series with general term u_n converges if and only if $\alpha > 2$.

(*e*) Let

$$v_n = \int_{n\pi}^{(n+1)\pi} \frac{dt}{1 + t^\alpha \sin^2(t)}$$

for $n \in \mathbb{N}_0$. Study the convergence of the series with general term v_n.

(*f*) Does the integral

$$\int_0^{+\infty} \frac{dt}{1 + t^\alpha \sin^2(t)}$$

converge or diverge ?

Solution.

(*a*) It is clear that

$$\frac{\tan^2(t)}{1 + \tan^2(t)} = \frac{\frac{\sin^2(t)}{\cos^2(t)}}{1 + \frac{\sin^2(t)}{\cos^2(t)}} = \frac{\sin^2(t)}{\cos^2(t) + \sin^2(t)} = \sin^2(t)$$

for all $t \in [0, \pi/2]$.

(*b*) Using the change of variables $x = \tan(t)$, we see that

$$\int_0^{\frac{\pi}{2}} \frac{dt}{1 + a\sin^2(t)} = \int_0^{+\infty} \frac{1}{1 + \frac{ax^2}{1+x^2}} \frac{1}{1+x^2} dx = \int_0^{+\infty} \frac{1}{1 + (a+1)x^2} dx$$

$$= \lim_{M \to +\infty} \left[\frac{\arctan(\sqrt{1+a}\,x)}{\sqrt{1+a}} \right]_0^M = \frac{\pi}{2\sqrt{1+a}},$$

since $dx = dt/\cos^2(t)$, or equivalently

$$dt = \left(1 - \frac{x^2}{1+x^2}\right) dx = \frac{dx}{1+x^2}.$$

(*c*) Note that Show that

$$\int_0^{\pi} \frac{dt}{1+a\sin^2(t)} = \int_0^{\frac{\pi}{2}} \frac{dt}{1+a\sin^2(t)} + \int_{\frac{\pi}{2}}^{\pi} \frac{dt}{1+a\sin^2(t)}$$

and using the change of variables $s = \pi - t$ we have

$$\int_{\frac{\pi}{2}}^{\pi} \frac{dt}{1+a\sin^2(t)} = \int_0^{\frac{\pi}{2}} \frac{ds}{1+a\sin^2(s)},$$

which gives

$$\int_0^{\pi} \frac{dt}{1+a\sin^2(t)} = 2\int_0^{\frac{\pi}{2}} \frac{dt}{1+a\sin^2(t)}.$$

(*d*) Note that

$$u_n = \int_0^{\pi} \frac{dt}{1+(n\pi)^\alpha \sin^2(t)} = 2\int_0^{\frac{\pi}{2}} \frac{dt}{1+(n\pi)^\alpha \sin^2(t)} = \frac{\pi}{\sqrt{1+(n\pi)^\alpha}},$$

for $n \in \mathbb{N}_0$, where we used the two previous items. Since $u_n \sim \pi/n^{\alpha/2}$ as n goes to $+\infty$ and the series $\sum_{n\in\mathbb{N}} 1/n^s$ converges if and only if $s > 1$, we see that $\sum_{n\in\mathbb{N}_0} u_n$ converges if and only if $\alpha > 2$.

(*e*) Note that

$$u_{n+1} = \int_{n\pi}^{(n+1)\pi} \frac{dt}{1+((n+1)\pi)^\alpha \sin^2(t)} \leq v_n \leq \int_{n\pi}^{(n+1)\pi} \frac{dt}{1+(n\pi)^\alpha \sin^2(t)} = u_n$$

for $n \in \mathbb{N}_0$, since $n\pi \leq t \leq (n+1)\pi$ and

$$\int_{n\pi}^{(n+1)\pi} \frac{dt}{1+(n\pi)^\alpha \sin^2(t)} = \int_0^{\pi} \frac{dt}{1+(n\pi)^\alpha \sin^2(t)}.$$

Since $u_{n+1} \leq v_n \leq u_n$ for all $n \in \mathbb{N}_0$,

$$\sum_{n=1}^{N+1} u_n \leq \sum_{n=0}^{N} v_n \leq \sum_{n=0}^{N} u_n$$

so the series $\sum_{n\in\mathbb{N}_0} v_n$ converges precisely when $\sum_{n\in\mathbb{N}_0} u_n$ converges, *i.e.* $\alpha > 2$.

(*f*) Recall that the integral

$$\int_0^{+\infty} \frac{dt}{1+t^\alpha \sin^2(t)}$$

converges if and only if it is bounded, since the integrand is a positive function. Moreover, since the series

$$\int_0^{(N+1)\pi} \frac{dt}{1+t^\alpha \sin^2(t)} = \sum_{n=0}^{N} v_n$$

converges if and only if $\alpha > 2$, the integral also converges if and only if $\alpha > 2$.

11. Given $n \in \mathbb{N}_0$, define

$$I_n = \int_0^{+\infty} \frac{dx}{(1+x^2)^{n+1}}.$$

(*a*) Prove that I_n converges for all $n \in \mathbb{N}_0$.

(*b*) Prove that

$$I_n - I_{n+1} = \int_0^{+\infty} x \frac{x}{(1+x^2)^{n+2}} dx$$

and use an integration by parts to show that

$$I_{n+1} = \frac{2n+1}{2(n+1)} I_n.$$

(*c*) Set $u_n = \sqrt{n} I_n$. Prove that

$$\ln(u_{n+1}) - \ln(u_n) = \ln\left(1 + \frac{1}{2n}\right) - \frac{1}{2}\ln\left(1 + \frac{1}{n}\right).$$

(*d*) Using a Taylor polynomial of order 2 at 0 of $\ln(1+x)$, prove that the series with general term $\ln(u_{n+1}) - \ln(u_n)$ converges.

(*e*) Deduce the existence of a constant $C > 0$ such that $I_n \sim \frac{C}{\sqrt{n}}$ when n tends to $+\infty$.

Solution.

(*a*) Note that

$$\frac{1}{(1+x^2)^{n+1}} \le \frac{1}{1+x^2}$$

for all $n \in \mathbb{N}_0$. Since the previous function is positive and the integral $\int_0^{+\infty} dx/(1+x^2)$ converges,

$$I_n = \int_0^{+\infty} \frac{dx}{(1+x^2)^{n+1}} \le \int_0^{+\infty} \frac{dx}{1+x^2} = I_0$$

implies that I_n converges.

(*b*) Note that

$$I_n - I_{n+1} = \int_0^{+\infty} \frac{dx}{(1+x^2)^{n+1}} - \int_0^{+\infty} \frac{dx}{(1+x^2)^{n+2}} = \int_0^{+\infty} \frac{(1+x^2)-1}{(1+x^2)^{n+2}} dx$$
$$= \int_0^{+\infty} x \frac{x}{(1+x^2)^{n+2}} dx.$$

By an integration by parts for the functions $u = x$ and $v' = x/((1+x^2)^{n+2})$, so $v = (1+x^2)^{-n-1}/(2(n+1))$, we get

$$\int_0^{+\infty} x \frac{x}{(1+x^2)^{n+2}} dx = \lim_{M \to +\infty} \left[-\frac{x}{2(n+1)(1+x^2)^{n+1}} \right]_0^M + \frac{1}{2(n+1)} \int_0^{+\infty} \frac{1}{(1+x^2)^{n+2}} dx,$$

which is tantamount to $I_{n+1} - I_n = I_{n+1}/(2(n+1))$, *i.e.*

$$I_{n+1} = \frac{2n+1}{2(n+1)} I_n,$$

for all $n \in \mathbb{N}_0$.

(c) It is clear that

$$\begin{aligned}\ln(u_{n+1}) - \ln(u_n) = \ln\left(\frac{u_{n+1}}{u_n}\right) = \ln\left(\frac{\sqrt{n+1}I_{n+1}}{\sqrt{n}I_n}\right) \\ = \ln\left(\frac{1+1/(2n)}{\sqrt{1+1/n}}\right) = \ln\left(1+\frac{1}{2n}\right) - \frac{1}{2}\ln\left(1+\frac{1}{n}\right).\end{aligned}$$

(d) The Bernoulli-L'Hospital rule tells us that

$$\lim_{x\to 0} \frac{\ln(1+x/2) - \ln(1+x)/2}{x^2/4} = \lim_{x\to 0} \frac{\frac{x}{2(2+x)(1+x)}}{x/2} = 1.$$

Applying the previous identity for $x = 1/n$ and $n \in \mathbb{N}$ we get that $\ln(u_{n+1}) - \ln(u_n) \sim 1/(2n)^2$ for $n \to +\infty$, so the series with general term $\ln(u_{n+1}) - \ln(u_n)$ converges.

(e) Since

$$\sum_{n=1}^{N} \ln(u_{n+1}) - \ln(u_n) = \ln(u_{N+1}) - \ln(u_1)$$

converges as N goes to $+\infty$, then $\ln(u_{N+1})$ has a limit $\ln(C)$ as N goes to $+\infty$. Moreover, $C > 0$ since $I_n > 0$ for all $n \in \mathbb{N}_0$. This implies that $I_N \sim \frac{C}{\sqrt{N}}$ when N tends to $+\infty$.