MAT332 - Series and integration

Fall term — 2023-2024

Exercise sheet 4: Riemann integrals

1. *Riemann sums.*

(a) Prove that the sequence $(u_n)_{n\in\mathbb{N}_0}$ whose general term is

$$u_n = \sum_{k=1}^{n} \frac{n+k}{n^2+k^2}$$

is a sequence of Riemann sums which converges and compute its limit.

(b) Compute the limit when n tends to $+\infty$ of $\sum_{k=n+1}^{2n} 1/k$.

(c) For which real number α is the sequence $(v_n)_{n\in\mathbb{N}_0}$ with general term

$$v_n = \frac{1}{n^2}\sum_{k=1}^{n} k^\alpha \sin(k/n)$$

a sequence of Riemann sums ? What is its limit ? What about the other values of α in $]-1,+\infty[$?

(d) Using Riemann sums, prove the equivalences

$$\sum_{k=1}^{n} k^\alpha \sim \frac{1}{\alpha+1} n^{\alpha+1} \text{ for } \alpha > 0, \text{ and } \sum_{k=n+1}^{2n} \ln(k) \sim n\ln(n)$$

when n tends to $+\infty$.

2. *Primitives.* Consider the following integrals :

(a) $\int_a^b t^n dt$, with $n \in \mathbb{N}_0$,

(b) $\int_a^b P(t)dt$, with P a polynomial of degree d,

(c) $\int_a^b e^{\alpha t} dt$, with $\alpha \in \mathbb{C}$,

(d) $\int_a^b \sqrt{t}dt$,

(e) $\int_a^b 1/\sqrt{t}dt$,

(f) $\int_a^b t^{1/3}dt$,

(g) $\int_a^b 1/(1+t^2)dt$.

For each of them find $[a,b]$ such that the function is Riemann integrable on $[a,b]$ and compute the integral on the interval.

3. *Primitives of rational functions.* Compute the following primitives

(a) $\int x^3/(x^2+1)dx$,

(b) $\int dx/(x(1+x)^2)$,

(c) $\int dx/(4x^2-3x+2)$

(d) $\int x^2/(x^4-1)dx$,

(e) $\int dx/(49-4x^2)$,

(f) $\int (5x-12)/(x(x-4))dx$,

(g) $\int (x-1)/(x^2+x+1)dx$.

4. *Change of variables.* Give primitives for the functions $\sqrt{x^2+1}$, $\sqrt{x^2-1}$ and $\sqrt{1-x^2}$ using the change of variables $x=\sinh(u)$, $x=\cosh(u)$ or $x=\sin(u)$.

5. *Primitives of trigonometric functions.* Compute the following primitives

(a) $\int \sin^3(x)dx$,

(b) $\int \sin(x)/(2+\cos^2(x))dx$,

(c) $\int dx/(1+\sin^2(x))$,

(d) $\int \sin(x/2)\cos(x/3)dx$,

(e) $\int \sin^2(x)\cos^3(x)dx$,

(f) $\int \sin^2(x)\cos^4(x)dx$.

6. *More primitives.* Compute the primitives

$$\int x^3 \ln(x)dx \text{ and } \int e^{-x}\cos(x)dx.$$

7. *Integration and derivatives.* Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Prove that the function g defined on $\mathbb{R}$ by $g(x)=\int_{2x}^{x^2} f(t)dt$ is differentiable and compute its derivative.

8. *A special case.*

(a) Let $f:[a,b]\to\mathbb{R}$ be a continuous function. Give a necessary and sufficient condition on f such that $|\int_a^b f(x)dx| = \int_a^b |f(x)|dx$.

(b) Same question for $f:[a,b]\to\mathbb{C}$.

9. *Primitive of exponentials.*

(a) Prove that a primitive of the function defined on $\mathbb{R}$ given by $x\mapsto P(x)e^{ax}$, where $P\in\mathbb{R}[X]$ is a polynomial and $a\in\mathbb{R}$ is of the form $x\mapsto Q(x)e^{ax}+C$ where $Q\in\mathbb{R}[X]$ is a polynomial and $C\in\mathbb{R}$ is a constant.

(b) Prove that a primitive of the function defined on $\mathbb{R}$ given by $x\mapsto P(x)\cos(\alpha x)$, where $P\in\mathbb{R}[X]$ is a polynomial and $\alpha\in\mathbb{R}$ is a real function of the form $x\mapsto Q_1(x)\cos(\alpha x)+Q_2(x)\sin(\alpha x)+C$ where $Q_1,Q_2\in\mathbb{R}[X]$ are polynomials and $C\in\mathbb{R}$ is a constant.