MAT332 - SERIES AND INTEGRATION
Fall term — 2022-2023

Exercise sheet 4: Riemann integrals

1. *Riemann sums.*

(a) Prove that the sequence $(u_n)_{n\in\mathbb{N}_0}$ whose general term is

$$u_n = \sum_{k=1}^{n} \frac{n+k}{n^2+k^2}$$

is a sequence of Riemann sums which converges and compute its limit.

(b) Compute the limit when n tends to $+\infty$ of $\sum_{k=n+1}^{2n} 1/k$.

(c) For which real number α is the sequence $(v_n)_{n\in\mathbb{N}_0}$ with general term

$$v_n = \frac{1}{n^2}\sum_{k=1}^{n} k^{\alpha}\sin(k/n)$$

a sequence of Riemann sums ? What is its limit ? What about the other values of α in $]-1,+\infty[$?

(d) Using Riemann sums, prove the equivalences

$$\sum_{k=1}^{n} k^{\alpha} \sim \frac{1}{\alpha+1}n^{\alpha+1} \text{ for } \alpha > 0, \text{ and } \sum_{k=n+1}^{2n} \ln(k) \sim n\ln(n)$$

when n tends to $+\infty$.

Solution.

(a) It is clear that

$$S_n = \sum_{k=1}^{n} \frac{n+k}{n^2+k^2} = \sum_{k=1}^{n} \frac{1}{n}\frac{1+k/n}{1+(k/n)^2}$$

coincides with a Riemann sum associated with the map $f : [0,1] \to \mathbb{R}$ given by $f(x) = (1+x)/(1+x^2)$ for the partition $\{x_j = j/n : j \in [\![0,n]\!]\}$ of the interval $[0,1]$. In consequence,

$$\lim_{n\to+\infty} S_n = \int_0^1 \frac{1+x}{1+x^2}dx = \left[\arctan(x) + \frac{\ln(1+x^2)}{2}\right]_0^1 = \frac{\pi+\ln(4)}{4}.$$

(b) It is clear that

$$S_n = \sum_{k=n+1}^{2n} \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{k+n} = \sum_{k=1}^{n} \frac{1}{n}\frac{1}{1+k/n}$$

is a Riemann sum of the map $f : [0,1] \to \mathbb{R}$ given by $f(x) = 1/x$ for the partition $\{x_j = 1 + j/n : j \in \{0,\dots,n\}\}$ of the interval $[1,2]$. In consequence,

$$\lim_{n\to+\infty} S_n = \int_1^2 \frac{1}{x}dx = \left[\ln(x)\right]_1^2 = \ln(2).$$

(*c*) It is clear that

$$v_n = \frac{1}{n^2}\sum_{k=1}^{n} k^\alpha \sin(k/n) = n^{\alpha-1}\sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^\alpha \sin(k/n)$$

is the product of $n^{\alpha-1}$ by a Riemann sum of the map $f : [0,1] \to \mathbb{R}$ given by $f(x) = x^\alpha \sin(x)$ for the partition $\{x_j = j/n : j \in \{0,\dots,n\}\}$ of the interval $[0,1]$. In consequence,

$$\lim_{n\to+\infty}\frac{v_n}{n^{\alpha-1}} = \int_0^1 x^\alpha \sin(x)dx.$$

If $\alpha > 1$, we thus conclude that v_n goes to $+\infty$ as n goes to $+\infty$. If $\alpha = 1$, then

$$\lim_{n\to+\infty} v_n = \int_0^1 x\sin(x)dx = \Big[\sin(x) - x\cos(x)\Big]_1^2 = \sin(1) - \cos(1).$$

Finally, if $-1 \geq \alpha < 1$, then the integral $\int_0^1 x^\alpha \sin(x)dx$ is finite and we thus conclude that v_n goes to 0 as n goes to $+\infty$.

(*d*) Assume $\alpha > 0$. Note that

$$S_n = \sum_{k=1}^{n}\frac{1}{n}\left(\frac{k}{n}\right)^\alpha$$

is a Riemann sum of the map $f : [0,1] \to \mathbb{R}$ given by $f(x) = x^\alpha$ for the partition $\{x_j = j/n : j \in \{0,\dots,n\}\}$ of the interval $[0,1]$. In consequence,

$$\lim_{n\to+\infty} S_n = \int_0^1 x^\alpha dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 = \frac{1}{\alpha+1},$$

which immediately implies that $\sum_{k=1}^{n} k^\alpha \sim n^{\alpha+1}/(\alpha+1)$ as n goes to $+\infty$.

On the other hand, note that

$$S_n = \sum_{j=1}^{n}\frac{1}{n}\ln\left(1+\frac{j}{n}\right) = \frac{\left(\sum_{j=1}^{n}\ln(j+n)\right) - n\ln(n)}{n}$$

is a Riemann sum of the map $f : [0,1] \to \mathbb{R}$ given by $f(x) = \ln(1+x)$ for the partition $\{x_j = j/n : j \in \{0,\dots,n\}\}$ of the interval $[0,1]$. In consequence,

$$\lim_{n\to+\infty} S_n = \int_0^1 \ln(1+x)dx = \Big[(1+x)\ln(1+x) - x\Big]_0^1 = \ln(4) - 1.$$

This implies that

$$\lim_{n\to+\infty}\frac{\sum_{k=n+1}^{2n}\ln(k)}{n\ln(n)} - 1 = \lim_{n\to+\infty}\frac{S_n}{\ln(n)} = 0,$$

which tells us that $\sum_{k=n+1}^{2n}\ln(k) \sim n\ln(n)$ as n goes to $+\infty$.

2. *Primitives.* Consider the following integrals :

(a) $\int_a^b t^n dt$, with $n \in \mathbb{N}_0$,

(b) $\int_a^b P(t)dt$, with P a polynomial of degree d,

(c) $\int_a^b e^{\alpha t} dt$, with $\alpha \in \mathbb{C}$,

(d) $\int_a^b \sqrt{t}dt$,

(e) $\int_a^b 1/\sqrt{t}dt$,

(f) $\int_a^b t^{1/3}dt$,

(g) $\int_a^b 1/(1+t^2)dt$.

For each of them find $[a,b]$ such that the function is Riemann integrable on $[a,b]$ and compute the integral on the interval.

Solution.

(a) For any interval $[a,b] \subseteq \mathbb{R}$ we have that

$$\int_a^b t^n dt = \left[\frac{t^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1}-a^{n+1}}{n+1},$$

for $n \in \mathbb{N}$.

(b) Assume that $P = \sum_{i=0}^d a_i t^i$, with $a_d \neq 0$. For any interval $[a,b] \subseteq \mathbb{R}$ we have that

$$\int_a^b P(t)dt = \sum_{i=0}^d a_i \int_a^b t^i dt = \sum_{i=0}^d a_i \left[\frac{t^{i+1}}{i+1}\right]_a^b = \sum_{i=0}^d a_i \frac{b^{i+1}-a^{i+1}}{i+1}.$$

(c) If $\alpha = 0$, $e^{\alpha t} = 1$ and this case is already included in the first item for $n = 0$. Assume that $\alpha \neq 0$. For any interval $[a,b] \subseteq \mathbb{R}$ we have that

$$\int_a^b e^{\alpha t} dt = \left[\frac{e^{\alpha t}}{\alpha}\right]_a^b = \frac{e^{\alpha b}-e^{\alpha a}}{\alpha}.$$

(d) For any interval $[a,b] \subseteq \mathbb{R}_{\geq 0}$ we have that

$$\int_a^b \sqrt{t}dt = \left[\frac{2t^{3/2}}{3}\right]_a^b = 2\frac{b^{3/2}-a^{3/2}}{3}.$$

(e) For any interval $[a,b] \subseteq \mathbb{R}_{>0}$ we have that

$$\int_a^b \frac{dt}{\sqrt{t}} = \left[2\sqrt{t}\right]_a^b = 2(\sqrt{b}-\sqrt{a}).$$

(f) For any interval $[a,b] \subseteq \mathbb{R}$ we have that

$$\int_a^b \sqrt[3]{t}dt = \left[3\frac{t^{4/3}}{4}\right]_a^b = 3\frac{b^{4/3}-a^{4/3}}{4}.$$

(g) For any interval $[a,b] \subseteq \mathbb{R}$ we have that

$$\int_a^b \frac{1}{1+t^2}dt = \left[\arctan(t)\right]_a^b = \arctan(b) - \arctan(a).$$

3. *Primitives of rational functions.* Compute the following primitives

(a) $\int x^3/(x^2+1)dx$,
(b) $\int dx/(x(1+x)^2)$,
(c) $\int dx/(4x^2-3x+2)$
(d) $\int x^2/(x^4-1)dx$,
(e) $\int dx/(49-4x^2)$,
(f) $\int (5x-12)/(x(x-4))dx$,
(g) $\int (x-1)/(x^2+x+1)dx$.

Solution. In the following, $C \in \mathbb{R}$ will denote a general real constant.

(a) It is clear that

$$\int \frac{x^3 dx}{x^2+1} = \int x dx - \frac{x dx}{x^2+1} = \frac{x^2}{2} - \frac{\ln(x^2+1)}{2} + C,$$

whose domain of definition is $\mathbb{R}$.

(b) It is clear that

$$\int \frac{dx}{x(1+x)^2} = \int \left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\right)dx = \ln(|x|) - \ln(|x+1|) + \frac{1}{(x+1)} + C,$$

whose domain of definition is $\mathbb{R} \setminus \{0, -1\}$.

(c) It is clear that

$$\int \frac{dx}{4x^2-3x+2} = \frac{1}{4}\int \frac{dx}{(x-\frac{3}{8})^2 + \frac{23}{64}} = \frac{2\arctan\left(\frac{8x-3}{\sqrt{23}}\right)}{\sqrt{23}} + C,$$

whose domain of definition is $\mathbb{R}$.

(d) It is clear that

$$\begin{aligned}\int \frac{x^2 dx}{x^4-1} &= \int \left(\frac{1}{2(x^2+1)} - \frac{1}{4(x+1)} + \frac{1}{4(x-1)}\right)dx \\ &= \frac{1}{4}\Big(\ln(|x-1|) - \ln(|x+1|) + 2\arctan(x)\Big) + C,\end{aligned}$$

whose domain of definition is $\mathbb{R} \setminus \{\pm 1\}$.

(e) It is clear that

$$\int \frac{dx}{49-4x^2} = \frac{1}{28}\int \left(\frac{1}{7+2x} + \frac{1}{7-2x}\right)dx = \frac{1}{28}\ln\left(\left|\frac{2x+7}{2x-7}\right|\right) + C,$$

whose domain of definition is $\mathbb{R} \setminus \{\pm 7/2\}$.

(f) It is clear that

$$\int \frac{(5x-12)dx}{x(x-4)} = \int \left(\frac{3}{x} + \frac{2}{x-4}\right)dx = 3\ln(|x|) + 2\ln(|x-4|) + C,$$

whose domain of definition is $\mathbb{R} \setminus \{0, 4\}$.

(g) It is clear that

$$\begin{aligned}\int \frac{(x-1)dx}{x^2+x+1} &= \int \left(\frac{2x+1}{2(x^2+x+1)} - \frac{3}{2(x^2+x+1)}\right)dx \\ &= \int \left(\frac{2x+1}{2(x^2+x+1)} - \frac{6}{(2x+1)^2+3}\right)dx \\ &= \frac{1}{2}\left(\ln(x^2+x+1) - 2\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\right) + C,\end{aligned}$$

whose domain of definition is $\mathbb{R}$.

4. *Change of variables.* Give primitives for the functions $\sqrt{x^2+1}$, $\sqrt{x^2-1}$ and $\sqrt{1-x^2}$ using the change of variables $x = \sinh(u)$, $x = \cosh(u)$ or $x = \sin(u)$.

Solution. We just give the corresponding integrals and let the reader verify that they are correct :

$$\int \sqrt{1+x^2}dx = \frac{1}{2}\Big(x\sqrt{1+x^2} + \operatorname{argsinh}(x)\Big) + C,$$
$$\int \sqrt{x^2-1}dx = \frac{1}{2}\Big(x\sqrt{x^2-1} - \ln\big(x+\sqrt{x^2-1}\big)\Big) + C,$$
$$\int \sqrt{1-x^2}dx = \frac{1}{2}\Big(x\sqrt{1-x^2} + \arcsin(x)\Big) + C,$$

where $C \in \mathbb{R}$ is a real general constant.

5. *Primitives of fonctions of trigonametric functions.* Compute the following primitives

(a) $\int \sin^3(x)dx$,	(d) $\int \sin(x/2)\cos(x/3)dx$,
(b) $\int \sin(x)/(2+\cos^2(x))dx$,	(e) $\int \sin^2(x)\cos^3(x)dx$,
(c) $\int dx/(1+\sin^2(x))$,	(f) $\int \sin^2(x)\cos^4(x)dx$.

Solution. Assume that m is an odd positive integer of the form $m = 2m'+1$ with $m' \in \mathbb{N}_0$. Then,

$$\int \cos^m(x)\sin^n(x)dx = \int \cos^{2m'}(x)\sin^n(x)\cos(x)dx = \int \big(1-\sin^2(x)\big)^{m'}\sin^n(x)\cos(x)dx$$
$$= \sum_{k=0}^{m'}(-1)^k\binom{m'}{k}\int \sin^{n+2k}(x)\cos(x)dx = \sum_{k=0}^{m'}(-1)^k\binom{m'}{k}\frac{\sin^{n+2k+1}(x)}{n+2k+1}. \tag{1}$$

In a similar way, if n is an odd positive integer of the form $n = 2n'+1$ with $n' \in \mathbb{N}$, we have

$$\int \cos^m(x)\sin^n(x)dx = \int \cos^m(x)\sin^{2n'}(x)\sin(x)dx = \int \big(1-\cos^2(x)\big)^{n'}\cos^m(x)\sin(x)dx$$
$$= \sum_{k=0}^{n'}(-1)^k\binom{n'}{k}\int \cos^{m+2k}(x)\sin(x)dx = \sum_{k=0}^{n'}(-1)^{k+1}\binom{n'}{k}\frac{\cos^{m+2k+1}(x)}{m+2k+1}. \tag{2}$$

In the following, $C \in \mathbb{R}$ will denote a general real constant.

(a) Using (2) we see that

$$\int \sin^3(x)dx = \frac{1}{3}\big(\cos^3(x) - 3\cos(x)\big) + C,$$

whose domain of definition is $\mathbb{R}$.

(b) Using the change of variables $y = \cos(x)/\sqrt{2}$, so $dy = -\sin(x)dx/\sqrt{2}$, we see that

$$\int \frac{\sin(x)}{2+\cos^2(x)}dx = -\frac{1}{\sqrt{2}}\int \frac{dy}{1+y^2} = -\frac{\arctan(\cos(x)/\sqrt{2})}{\sqrt{2}} + C,$$

whose domain of definition is $\mathbb{R}$.

(c) Using the change of variables $y = \tan(x)$, then $\sin^2(x) = y^2/(1+y^2)$ and $dx = dy/(1+y^2)$. This tells us that

$$\int \frac{1}{1+\sin^2(x)}dx = \int \frac{1}{1+2y^2}dy = \frac{1}{\sqrt{2}}\arctan(\sqrt{2}y) + C$$
$$= \frac{1}{\sqrt{2}}\arctan\big(\sqrt{2}\tan(x)\big) + C,$$

whose domain of definition is $]-\pi/2, \pi/2[$.

(d) It is direct to see that

$$\int \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{3}\right)dx = \int \left(\sin\left(\frac{x}{6}\right)+\sin\left(\frac{5x}{6}\right)\right)\frac{dx}{2} = -\frac{3}{5}\left(5\cos\left(\frac{x}{6}\right)+\cos\left(\frac{5x}{6}\right)\right)+C,$$

whose domain of definition is $\mathbb{R}$, where we have used the identity

$$\sin(\alpha)\cos(\beta) = \big(\sin(\alpha-\beta)+\sin(\alpha+\beta)\big)/2.$$

(e) Using (1) we see immediately that

$$\int \sin^2(x)\cos^3(x)dx = \frac{1}{15}\big(5\sin^3(x) - 3\sin^5(x)\big) + C,$$

whose domain of definition is $\mathbb{R}$.

(f) We note first that

$$\int \cos^2(x)dx = \int \frac{\cos(2x)+1}{2}dx = \frac{\sin(2x)}{4} + \frac{x}{2} + C = \frac{\sin(x)\cos(x)+x}{2} + C, \quad (3)$$

where we used that $2\cos^2(x) = \cos(2x)+1$ and $\sin(2x) = 2\sin(x)\cos(x)$ for all $x \in \mathbb{R}$. Moreover, we also have that

$$\int \cos^m(x)dx = \frac{\sin(x)\cos^{m-1}(x)}{m} + \frac{m-1}{m}\int \cos^{m-2}(x)dx,$$

for all integers $m \geq 2$. This follows immediately from an integration by parts, by taking $u(x) = \cos^{m-1}(x)$ and $v'(x) = \cos(x)$ (so $v(x) = \sin(x)$) in the integral in the first member. Hence, by applying this expression to $m = 4$ as well as (3) we get that

$$\int \cos^4(x)dx = \frac{\sin(x)\cos^3(x)}{4} + \frac{3}{4}\int \cos^2(x)dx$$
$$= \frac{2\sin(x)\cos^3(x) + 3\sin(x)\cos(x) + 3x}{8} + C,$$

Analogously, if $m = 6$ we get that

$$\int \cos^6(x)dx = \frac{\sin(x)\cos^5(x)}{6} + \frac{5}{6}\int \cos^4(x)dx$$
$$= \frac{8\sin(x)\cos^5(x) + 10\sin(x)\cos^3(x) + 15\sin(x)\cos(x) + 15x}{48} + C,$$

Finally, we conclude that

$$\begin{aligned}\int \sin^2(x)\cos^4(x)dx &= \int \left(\cos^4(x)-\cos^6(x)\right)dx \\ &= \frac{2\sin(x)\cos^3(x)+3\sin(x)\cos(x)+3x}{8} \\ &\quad -\frac{8\sin(x)\cos^5(x)+10\sin(x)\cos^3(x)+15\sin(x)\cos(x)+15x}{48} \\ &\quad +C \\ &= \frac{-8\sin(x)\cos^5(x)+2\sin(x)\cos^3(x)+3\sin(x)\cos(x)+3x}{48}+C.\end{aligned}$$

The domain of definition is clearly $\mathbb{R}$.

6. *More primitives.* Compute the primitives

$$\int x^3\ln(x)dx \text{ and } \int e^{-x}\cos(x)dx.$$

Solution. By doing an integration by parts with $u=\ln(x)$ et $v'=x^n$, *i.e.* $v=x^{n+1}/(n+1)$, we get that, if $n\neq -1$,

$$\int x^n\ln(x)dx = \frac{x^{n+1}\big((n+1)\ln(x)-1\big)}{(n+1)^2}+C,$$

with domain of definition $\mathbb{R}_{>0}$. On the other hand, using the change of variables $u=\ln(x)$ we see that

$$\int \frac{\ln(x)}{x}dx = \frac{\ln^2(x)}{2}+C,$$

with domain of definition $\mathbb{R}_{>0}$.

Finally,

$$\begin{aligned}\int e^{-x}\cos(x)dx &= \mathrm{Re}\left(\int e^{x(-1+i)}dx\right) \\ &= \mathrm{Re}\left(\frac{e^{(-1+i)x}}{-1+i}\right)+C = \frac{e^{-x}}{2}\big(\sin(x)-\cos(x)\big)+C,\end{aligned}$$

with domain of definition $\mathbb{R}$.

7. *Integration and derivatives.* Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Prove that the function g defined on $\mathbb{R}$ by $g(x)=\int_{2x}^{x^2} f(t)dt$ is differentiable and compute its derivative.

Solution. Let $F:\mathbb{R}\to\mathbb{R}$ be a primitive of f, which exists since f is continuous. For instance, one could take

$$F(x) = \int_0^x f(t)dt,$$

for $x \in \mathbb{R}$. It is clear that $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = F(x^2) - F(2x)$ for $x \in \mathbb{R}$, by Barrow's theorem. Since F is differentiable, g is also differentiable, being the sum of compositions of differentiable functions. Using the chain rule we see that $g'(x) = 2xf(x^2) - 2f(2x)$ for $x \in \mathbb{R}$.

8. *A special case.*

(a) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Give a necessary and sufficient condition on f such that $|\int_a^b f(x)dx| = \int_a^b |f(x)|dx$.

(b) Same question for $f : [a,b] \to \mathbb{C}$.

Solution.

(a) We have $|\int_a^b f(x)dx| = \int_a^b |f(x)|dx$ if and only if $f(x) \geq 0$ for all $x \in [a,b]$ or $f(x) \leq 0$ for all $x \in [a,b]$. Indeed, it is clear that this condition is sufficient, since, if $f(x) \geq 0$ for all $x \in [a,b]$, then

$$\left|\int_a^b f(x)dx\right| = \int_a^b f(x)dx = \int_a^b \left|f(x)\right|dx,$$

where we used that the integral of a nonnegative function is nonnegative. The case for nonpositive functions is analogous.

To prove the converse, recall first that if the integral of a continuous function $g : [a,b] \to \mathbb{R}_{\geq 0}$ is zero, then $g(x) = 0$ for all $x \in [a,b]$. Indeed, assume that there exist c in $[a,b]$ such that $f(c) > 0$. Let $m = f(c)/2 > 0$. Since g is continuous, there exists $\delta > 0$ such that $f(x) \geq m > 0$ for all $x \in [c-\delta, c+\delta]$. Then,

$$0 = \int_a^b g(x)dx = \int_a^{c-\delta} g(x)dx + \int_{c-\delta}^{c+\delta} g(x)dx + \int_c^b g(x)dx \geq \int_{c-\delta}^{c+\delta} g(x)dx \geq 2\delta m > 0,$$

which is absurd.

Recall now that given any real (continuous) function $f : [a,b] \to \mathbb{R}$, there exist (continuous) functions $f_{\pm} : [a,b] \to \mathbb{R}_{\geq 0}$ such that $f(x) = f_+(x) - f_-(x)$ for all $x \in [a,b]$, and $f_{\pm}(x) = 0$ if $\pm f(x) \leq 0$. Indeed, define $f_+(x) = \max(f(x), 0)$ and $f_-(x) = -\in(f(x), 0)$ for $x \in [a,b]$. Let $A_{\pm} = \int_a^b f_{\pm}(x)dx$. Note that $A_{\pm} \geq 0$, since the integral of a nonnegative function is nonnegative. Then,

$$\left|\int_a^b f(x)dx\right| = \left|\int_a^b \big(f_+(x) - f_-(x)\big)dx\right| = \left|\int_a^b f_+(x)dx - \int_a^b f_-(x)dx\right| = |A_+ - A_-|$$

and

$$\int_a^b \left|f(x)\right|dx = \int_a^b f\big(_+(x) + f_-(x)\big)dx = \int_a^b f_+(x)dx + \int_a^b f_-(x)dx = A_+ + A_-,$$

since $|f(x)| = f_x(x) + f_-(x)$ for all $x \in [a,b]$. The only possible solution to $|A_+ - A_-| = A_+ + A_-$ with $A_{\pm} \geq 0$ is $A_+ = 0$ or $A_- = 0$. By the comment in the previous paragraph, this implies $f_+(x) = 0$ for all $x \in [a,b]$ or $f_-(x) = 0$ for all $x \in [a,b]$, respectively, which is tantamount to $f(x) \geq 0$ for all $x \in [a,b]$ or $f(x) \leq 0$ for all $x \in [a,b]$.

(*b*) Recall that given any (continuous) function $f : [a, b] \to \mathbb{C}$, there exist (continuous) functions $u, v : [a, b] \to \mathbb{R}$ such that $f(x) = u(x) + iv(x)$ for all $x \in [a, b]$. Moreover,

9. *Primitive of exponentials.*

(*a*) Prove that a primitive of the function defined on $\mathbb{R}$ given by $x \mapsto P(x)e^{ax}$, where $P \in \mathbb{R}[X]$ is a polynomial and $a \in \mathbb{R}$ is of the form $x \mapsto Q(x)e^{ax} + C$ where $Q \in \mathbb{R}[X]$ is a polynomial and $C \in \mathbb{R}$ is a constant.

(*b*) Prove that a primitive of the function defined on $\mathbb{R}$ given by $x \mapsto P(x)\cos(\alpha x)$, where $P \in \mathbb{R}[X]$ is a polynomial and $\alpha \in \mathbb{R}$ is a real function of the form $x \mapsto Q_1(x)\cos(\alpha x) + Q_2(x)\sin(\alpha x) + C$ where $Q_1, Q_2 \in \mathbb{R}[X]$ are polynomials and $C \in \mathbb{R}$ is a constant.

Solution.

(*a*) We can prove the result by induction on the degree of P. If $\deg(P) \leq 0$, then $P = c \in \mathbb{R}$ and we set $Q(x) = c/a$. Assume the statement holds for every polynomial P of degree strictly less than $d \in \mathbb{N}$. We will prove it for the case P has degree d. Since $(Q(x)e^{ax} + C)' = (Q'(x) + aQ(x))e^{ax}$, it suffices to show that there exists a polynomial $Q \in \mathbb{R}[X]$ such that $Q'(x) + aQ(x) = P(x)$. Let $P = cx^d + \bar{P}$, with $c \neq 0$ and $\bar{P}$ of degree strictly less than d. Set $R = cx^d/a$. It is clear that $R' + aR - P$ is a polynomial of degree strictly less than d. By the inductive assumption, there exists a polynomial T such that $T' + aT = R' + aR - P$. Hence $Q = R - T$ satisfies the required property.

(*b*) The same argument as the one given in the previous item applies *mutatis mutandi* in this case.