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MAT332

Fall 2021

Final examination - June 2023

Unjustified answers will be automatically excluded.

The grading is only approximate.

4, 5pt

1. Determine if following series are convergent or divergent :

(*a*) $\sum_{n=2}^{+\infty} \frac{(-2)^n \cos(n)}{3^n+n}$, (*b*) $\sum_{n=2}^{+\infty} \frac{e^{n+1}}{n!}$, (*c*) $\sum_{n=2}^{+\infty} \left(1-\frac{2}{n}\right)^{n^2}$.

Solution.

(*a*) Let $u_n = 2^n/(3^n+n)$ and $v_n = (-1)^n \cos(n)$ for $n \geq 2$. Note that $u_{n+1} \leq u_n$ for all integers $n \geq 2$, since the latter is equivalent to $n \leq 3^n + 1$ for $n \geq 2$, which is trivially proved by induction. Let $w_n = e^{in(1+\pi)}$ for $n \geq 2$. Hence, $v_n = \mathrm{Re}(w_n)$ for all integers $n \geq 2$, which implies that

$$\left|\sum_{n=2}^{N+2} v_n\right| = \left|\sum_{n=2}^{N+2} \mathrm{Re}(w_n)\right| = \left|\mathrm{Re}\left(\sum_{n=2}^{N+2} w_n\right)\right| = \left|\mathrm{Re}\left(e^{i2(1+\pi)}\frac{1-e^{i(N+1)(1+\pi)}}{1-e^{i(1+\pi)}}\right)\right|$$
$$\leq \left|e^{i2(1+\pi)}\frac{1-e^{i(N+1)(1+\pi)}}{1-e^{i(1+\pi)}}\right| \leq \frac{2}{|1-e^{i(1+\pi)}|}$$

for all nonnegative integers N, where we used the sum of the geometric series of ratio $e^{i(1+\pi)}$ in the third equality and that $|\mathrm{Re}(z)| \leq |z|$ for all $z \in \mathbb{C}$ in the first inequality. The Leibniz's criterion tells us that the series $\sum_{n=2}^{+\infty} u_n v_n$ is convergent.

(*b*) Let $u_n = e^{n+1}/n!$ for $n \geq 2$. Since $u_n > 0$ for all integers $n \geq 2$ and

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \frac{e}{n+1} = 0 < 1,$$

the ratio test tells us that the series $\sum_{n=2}^{+\infty} u_n$ is (absolutely) convergent.

(*c*) Recall that

$$\lim_{x\to0}(1+x)^{1/x} = e > 1$$

Let

$$u_n = \left(1-\frac{2}{n}\right)^{n^2}$$

for $n \geq 2$. Since $u_n > 0$ for all integers $n \geq 2$ and

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty}\left(1-\frac{2}{n}\right)^n = \left(\lim_{n\to+\infty}\left(1-\frac{2}{n}\right)^{-n/2}\right)^{-2} = e^{-2} < 1,$$

the ratio test tells us that the series $\sum_{n=2}^{+\infty} u_n$ is (absolutely) convergent.

3pt

2. Let $\alpha \in \mathbb{R}_{>0}$. Consider the sequence $(u_{\alpha,n})_{n\in\mathbb{N}_{\geq 2}}$ given by

$$u_{\alpha,n} = \frac{(-1)^n}{n^\alpha + (-1)^n}$$

for all integers $n \geq 2$.

(a) Determine the set

$$\mathsf{AC} = \left\{ \alpha \in \mathbb{R}_{>0} : \sum_{n=2}^{+\infty} u_{\alpha,n} \text{ is absolutely convergent} \right\}.$$

(b) Determine the set

$$\mathsf{C} = \left\{ \alpha \in \mathbb{R}_{>0} : \sum_{n=2}^{+\infty} u_{\alpha,n} \text{ is convergent} \right\}.$$

Solution.

(a) Note first that

$$|u_{\alpha,n}| = \frac{1}{n^\alpha + (-1)^n} \tag{1}$$

if $n \geq 2$, since in that case $n^\alpha \geq 2^\alpha = e^{\alpha \ln(2)} > e^0 = 1$. Moreover, the series

$$\sum_{n=2}^{+\infty} \frac{1}{n^\alpha + (-1)^n} \tag{2}$$

is convergent if and only if it is bounded, since all their terms are nonnegative. Note now that

$$\lim_{n\to+\infty} \frac{\frac{1}{n^\alpha+(-1)^n}}{\frac{1}{n^\alpha}} = 1$$

for all $\alpha > 0$. Hence, for $\alpha > 1$, since $\sum_{n=1}^{+\infty} n^{-\alpha}$ is convergent, we conclude that (2) is convergent, whereas, for $\alpha \leq 1$, since $\sum_{n=1}^{+\infty} n^{-\alpha}$ is divergent, we conclude that (2) is divergent. As a consequence,

$$\mathsf{AC} = \mathbb{R}_{>1}.$$

(b) Recall that $\mathsf{AC} \subseteq \mathsf{C}$, so $\mathbb{R}_{>1} \subseteq \mathsf{C}$. Moreover, we claim that $\alpha = 1$ also belongs to C. Indeed, doing a block summation we see that

$$\sum_{n=2}^{+\infty} \frac{(-1)^n}{n+(-1)^n} = \sum_{k=1}^{+\infty} \left[\frac{1}{(2k)+1} - \frac{1}{(2k+1)-1} \right] = -\sum_{k=1}^{+\infty} \frac{1}{2k(2k+1)}$$

is convergent, since

$$\lim_{k\to+\infty} \frac{\frac{1}{2k(2k+1)}}{\frac{1}{4k^2}} = 1,$$

and the series $\sum_{k=1}^{+\infty}(4k)^{-2}$ is convergent. Hence, $\mathbb{R}_{\geq 1} \subseteq \mathsf{C}$.

Let us assume that $\alpha < 1$. Doing a block summation we see that

$$\sum_{n=2}^{+\infty} \frac{(-1)^n}{n^\alpha+(-1)^n} = \sum_{k=1}^{+\infty}\left[\frac{1}{(2k)^\alpha+1} - \frac{1}{(2k+1)^\alpha-1}\right] = \sum_{k=1}^{+\infty} \frac{(2k+1)^\alpha-(2k)^\alpha-2}{(2k)^\alpha(2k+1)^\alpha}.$$

Let

$$v_{\alpha,k} = \frac{(2k+1)^\alpha-(2k)^\alpha-2}{(2k)^\alpha(2k+1)^\alpha}$$

for all positive integers k. Given $x \in \mathbb{R}_{>1}$, recall that $(x+1)^\alpha - x^\alpha = \alpha y^{\alpha-1}$ for all some $y \in [x, x+1]$, by the Mean Value Theorem. This implies that

$$\lim_{k\to+\infty} (2k+1)^\alpha-(2k)^\alpha = \lim_{k\to+\infty} \alpha(2k+\theta_k)^{\alpha-1} = 0,$$

where we used the Mean Value Theorem to write $(2k+1)^\alpha-(2k)^\alpha = \alpha(2k+\theta_k)^{\alpha-1}$ for some $\theta_k \in [0,1]$. This in turn implies that there exists $k_\alpha \in \mathbb{N}$ such that $v_{\alpha,k} \leq 0$ for all integers $k \geq k_\alpha$. Moreover, this also tells us that

$$\lim_{k\to+\infty} \frac{v_{\alpha,k}}{\frac{2}{(2k)^{2\alpha}}} = \lim_{k\to+\infty} \frac{((2k+1)^\alpha-(2k)^\alpha)/2-1}{((2k+1)/(2k))^\alpha} = 1.$$

Since the series $\sum_{k=1}^{+\infty}(2k)^{-\beta}$ is convergent if and only if $\beta > 1$, we conclude that

$$\sum_{n=2}^{+\infty} \frac{(-1)^n}{n^\alpha+(-1)^n}$$

is convergent if and only if $2\alpha > 1$, *i.e.* $\alpha > 1/2$. In consequence,

$$\mathsf{AC} = \mathbb{R}_{>1/2}.$$

3,5pt

3. Determine if the following integrals are convergent or divergent :

(a) $\int_2^{+\infty} \frac{1}{x^{2/3}(x-2)^{2/3}} dx,$ (b) $\int_1^{+\infty} \sqrt{x}e^{-x} dx.$

Solution.

(*a*) Note that the integrand is a continuous function over $\mathbb{R}_{>2}$. Since

$$\lim_{x\to+\infty} \frac{\frac{1}{x^{2/3}(x-2)^{2/3}}}{\frac{1}{x^{4/3}}} = 1,$$

and $\int_3^{+\infty} x^{-4/3}dx$ converges,

$$\int_3^{+\infty} \frac{dx}{x^{2/3}(x-2)^{2/3}}$$

converges as well. Moreover, since

$$\lim_{x\to 2+} \frac{\frac{1}{x^{2/3}(x-2)^{2/3}}}{\frac{1}{2^{2/3}(x-2)^{2/3}}} = 1,$$

and $\int_2^3 (x-2)^{-2/3}dx$ converges, for

$$\int_2^3 (x-2)^{-2/3}dx = \lim_{\epsilon\to 0+}\left[3(x-2)^{1/3}\right]_{2+\epsilon}^3 = 3,$$

then

$$\int_2^3 \frac{dx}{x^{2/3}(x-2)^{2/3}}$$

converges as well. As a consequence,

$$\int_2^{+\infty} \frac{1}{x^{2/3}(x-2)^{2/3}}dx$$

converges.

(*b*) Note that the integrand is a continuous and nonnegative function over $\mathbb{R}_{\geq 1}$. Then, the integral we are interested in converges if and only if it is bounded. Note first that $\sqrt{x}e^{-x} \leq xe^{-x}$ for all $x \geq 1$. Moreover, since

$$\int_1^{+\infty} xe^{-x}dx = \lim_{x\to+\infty}\left[-\frac{1+x}{e^x}\right]_1^M = \frac{2}{e},$$

we conclude that

$$\int_1^{+\infty} \sqrt{x}e^{-x}dx \leq \int_1^{+\infty} xe^{-x}dx = \frac{2}{e},$$

so the required integral is bounded and thus convergent.

9pt

4. Consider the functional expression given by

$$f(x) = \frac{\ln\left(|1-x^3|\right)}{x^3}.$$

(a) Determine the maximal domain of definition of f within $\mathbb{R}_{\geq 0}$.

(b) Prove that the integral

$$\int_0^x f(t)dt \tag{3}$$

converges for all $x \in [0,1]$. Define thus the function $F : [0,1] \to \mathbb{R}$ whose value at $x \in [0,1]$ is given by the integral (3).

(c) Let $x \in]0,1[$. Using integration by parts, express the value of $F(x)$ in terms of an integral between 0 and 1 of a rational fraction to be determined.

(d) Let $g : [0,1[\to \mathbb{R}$ be the function given by $g(x) = 3/(1-x^3)$ for $x \in [0,1[$. Determine the partial fraction decomposition of $g(x)$.

(e) Compute a primitive of the function g.

(f) Obtain an explicit expression of $F(x)$ for $x \in [0,1[$ in terms of elementary functions.

(g) Prove that

$$F(1) = -\frac{3}{4}\ln(3) - \frac{\pi}{4\sqrt{3}}.$$

Solution.

(a) It is clear that the maximal domain of definition within $\mathbb{R}_{\geq 0}$ of the functional expression f is $\mathbb{R}_{>0} \setminus \{1\}$. We also note that the function $f : \mathbb{R}_{>0} \setminus \{1\} \to \mathbb{R}$ it defines is continuous, since it is the quotient of continuous functions with nonzero denominator.

(b) We note first that

$$\lim_{x\to 0+} \frac{\ln(1-x^3)}{x^3} = \lim_{x\to 0+} \frac{\frac{-3x^2}{1-x^3}}{3x^2} = -\lim_{x\to 0+} \frac{1}{1-x^3} = -1,$$

where the second identity follows from the Bernoulli-L'Hospital rule. This implies that the function $\bar{f} : \mathbb{R}_{\geq 0} \setminus \{1\} \to \mathbb{R}$ given by $\bar{f}(x) = f(x)$ for $x \in \mathbb{R}_{>0} \setminus \{1\}$ and $\bar{f}(0) = -1$ is continuous. As a consequence, the definite integral

$$\int_0^x \bar{f}(t)dt = \int_0^x f(t)dt = F(x)$$

exists for all $x \in [0,1[$. Note that $F(0) = 0$ by definition of the integral. It remains to consider the case $x \in \mathbb{R}_{\geq 1}$.

We will finally show that

$$\int_0^1 \frac{\ln\left(|1-t^3|\right)}{t^3} dt = \int_0^{1/2} \frac{\ln\left(|1-t^3|\right)}{t^3} dt + \int_{1/2}^1 \frac{\ln\left(|1-t^3|\right)}{t^3} dt \tag{4}$$

converges. Since the integrand is a nonnegative function, it suffices to show that the integral is bounded. Moreover, by the argument in the previous paragraph, the integral (4) converges if and only if the last integral in (4) converges. Moreover since the integrand has a continuous at every point of Note that

$$\frac{\ln\left(|1-t^3|\right)}{t^3} = \frac{\ln\left(|1-t|\right)}{t^3} + \frac{\ln(t^2+t+1)}{t^3},$$

where we used that $1-t^3 = (1-t)(t^2+t+1)$. This implies that the last integral in (4) converges if and only if

$$\int_{1/2}^1 \frac{\ln\left(|1-t|\right)}{t^3} dt \tag{5}$$

converges. Applying the change of variables $u = 1-t$ to the previous expression, we get that

$$\begin{aligned}\int_{1/2}^1 \left|\frac{\ln(1-t)}{t^3}\right| dt &= \int_0^{1/2} \frac{\left|\ln(u)\right|}{(1-u)^3} du \leq 8 \int_0^{1/2} \left|\ln(u)\right| du \\ &= -8 \lim_{\epsilon\to 0+} \Big[u\big(\ln(u)-1\big)\Big]_{\epsilon}^{1/2} = 4\big(\ln(2)+1\big),\end{aligned}$$

where we used that $(1-u)^3 \geq 1/8$ for all $u \in [0,1/2]$. Hence, the integral (5) converges, which in turn implies that the integral (4) converges

(c) We have that

$$\begin{aligned}F(x) = \int_0^x \frac{\ln\left(|1-t^3|\right)}{t^3} dt &= \lim_{\epsilon\to 0+}\left[-\frac{\ln\left(|1-t^3|\right)}{2t^2}\right]_{\epsilon}^{x} - \frac{3}{2}\int_0^x \frac{1}{1-t^3} dt \\ &= -\frac{\ln\left(|1-x^3|\right)}{2x^2} - \frac{3}{2}\int_0^x \frac{1}{1-t^3} dt,\end{aligned}$$

where we used integration by parts in the second equality with $u = \ln(|1-t^3|)$ and $v' = t^{-3}$ (and $v = -t^{-2}/2$), and that

$$\lim_{\epsilon\to 0+} \frac{\ln(1-\epsilon^3)}{2\epsilon^2} = \lim_{\epsilon\to 0+} \frac{\frac{-3\epsilon^2}{1-\epsilon^3}}{4\epsilon} = \lim_{\epsilon\to 0+} \frac{-3\epsilon}{4(1-\epsilon^3)} = 0,$$

where the second identity follows from the Bernoulli-L'Hospital rule.

(d) It is easy to check that

$$g(x) = \frac{3}{1-x^3} = \frac{x+2}{x^2+x+1} + \frac{1}{1-x},$$

for all $x \in [0,1[$.

(*e*) Using the previous item we see that a primitive of g is given by

$$\begin{aligned}
G(x) &= \int g(x)dx = \int \frac{x+2}{x^2+x+1}dx + \int \frac{dx}{1-x} \\
&= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx + \frac{3}{2}\int \frac{1}{x^2+x+1}dx + \int \frac{dx}{1-x} \\
&= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}dx + \frac{3}{2}\int \frac{1}{(x+1/2)^2+3/4}dx + \int \frac{dx}{1-x} \\
&= \frac{\ln(x^2+x+1)}{2} + \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) - \ln\left(|1-x|\right)
\end{aligned}$$

for all $x \in [0,1[$. In particular,

$$G(0) = \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}}.$$

(*f*) By the three previous items we have that

$$\begin{aligned}
F(x) &= -\frac{\ln(1-x^3)}{2x^2} - \frac{G(x)-G(0)}{2} \\
&= -\frac{\ln(1-x^3)}{2x^2} - \frac{\ln(x^2+x+1)}{4} - \frac{\sqrt{3}}{2}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + \frac{\ln(1-x)}{2} \\
&\quad + \frac{\pi}{4\sqrt{3}} \\
&= -\frac{(1-x^2)\ln(1-x)}{2x^2} - \frac{(2+x^2)\ln(x^2+x+1)}{4x^2} - \frac{\sqrt{3}}{2}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) \\
&\quad + \frac{\pi}{4\sqrt{3}}
\end{aligned}$$

for all $x \in [0,1[$, where we used that $1-x^3 = (1-x)(x^2+x+1)$ in the last identity.

(*g*) Recall that

$$\frac{\sqrt{3}}{2}\arctan\left(\frac{3}{\sqrt{3}}\right) = \frac{\pi}{2\sqrt{3}}$$

and that

$$\begin{aligned}
\lim_{x\to 1-}\frac{(1-x^2)\ln(1-x)}{2x^2} &= \lim_{x\to 1-}\frac{1+x}{2x^2}\frac{\ln(1-x)}{(1-x)^{-1}} = \lim_{x\to 1-}\frac{\ln(1-x)}{(1-x)^{-1}} \\
&= -\lim_{x\to 1-}\frac{(1-x)^{-1}}{(1-x)^{-2}} = -\lim_{x\to 1-}(1-x) = 0,
\end{aligned}$$

where the third identity follows from the Bernoulli-L'Hospital rule. Using the two previous identities and the previous item we get that

$$\lim_{x\to 1-} F(x) = -\frac{3}{4}\ln(3) - \frac{\pi}{4\sqrt{3}}.$$

Since F is continuous at every point of $\mathbb{R}_{\geq 0}$, by the definition of generalized integral, we conclude that

$$F(1) = -\frac{3}{4}\ln(3) - \frac{\pi}{4\sqrt{3}}.$$