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MAT332

Fall 2022

Final examination - January 2023

Unjustified answers will be automatically excluded.

The grading is only approximate.

5pt

1. Determine if the following series are convergent or divergent :

(*a*) $\sum_{n=1}^{+\infty} \frac{\sin(n)}{n^2+\sin^2(n)}$, (*b*) $\sum_{n=1}^{+\infty} \sqrt{\frac{2+n}{2+5^n}}$, (*c*) $\sum_{n=1}^{+\infty} \frac{1}{4+(-1)^n n^{2/3}}$.

Solution.

(*a*) The series $\sum_{n=1}^{+\infty} \sin(n)/(n^2+\sin^2(n))$ is absolutely convergent, so it is convergent. To prove this, note first that

$$\left|\frac{\sin(n)}{n^2+\sin^2(n)}\right| \leq \frac{1}{n^2+\sin^2(n)} \leq \frac{1}{n^2}$$

for all $n \in \mathbb{N}$, where we used that $n^2+\sin^2(n) \geq n^2$. In consequence, the series

$$\sum_{n=1}^{+\infty}\left|\frac{\sin(n)}{n^2+\sin^2(n)}\right|$$

is convergent, since it has the upper bound given by $\sum_{n=1}^{+\infty} 1/n^2$, which is a convergent series.

(*b*) The series $\sum_{n=1}^{+\infty} \sqrt{(2+n)/(2+5^n)}$ is convergent. To prove this, note first that

$$\lim_{n\to+\infty} \frac{\sqrt{\frac{2+n}{2+5^n}}}{\left(\sqrt{\frac{2}{5}}\right)^n} = \lim_{n\to+\infty} \frac{\sqrt{\frac{2+n}{2+5^n}}}{\sqrt{\frac{2^n}{5^n}}} = \lim_{n\to+\infty} \sqrt{\frac{(2+n)5^n}{2^n(2+5^n)}} = \lim_{n\to+\infty} \sqrt{\frac{(2+n)/2^n}{1+2/5^n}} = 0.$$

In consequence, $\sqrt{(2+n)/(2+5^n)} = o((\sqrt{2/5})^n)$ as $n \to +\infty$. Since $\sqrt{2/5} < 1$, the geometric series $\sum_{n=1}^{+\infty} (\sqrt{2/5})^n$ converges, which implies that the required series is also convergent.

(*c*) The series $\sum_{n=1}^{+\infty} 1/(4+(-1)^n n^{2/3})$ is convergent. Since

$$\sum_{n=1}^{+\infty} \frac{1}{4+(-1)^n n^{2/3}}$$

converges if and only if

$$\sum_{n=n_0}^{+\infty} \frac{1}{4+(-1)^n n^{2/3}}$$

converges for some $n_0 \in \mathbb{N}$, we will focus on the latter, for some $n_0 \in \mathbb{N}$ to be determined later.

Consider the map $f : \mathbb{R} \setminus \{-1/4\} \to \mathbb{R}$ given by $f(x) = x/(1+4x)$ for all $x \in \mathbb{R} \setminus \{-1/4\}$. Then, $f'(x) = 1/(1+4x)^2$ and $f''(x) = -8/(1+4x)^3$ for all $x \in \mathbb{R} \setminus \{-1/4\}$. Note in particular that $f(0) = 0$ and $f'(0) = 1$. Moreover, let $u_n = (-1)^n/n^{2/3}$ for all $n \in \mathbb{N}$. We see that $|u_n|$ is decreasing as a function of n, and that $u_8 = 1/4$, so $u_n \in \,]1/4, 1/4[\, \subseteq \mathbb{R} \setminus \{-1/4\}$ for all integers $n \geq 9$. Moreover, a simple computation shows that $u_n \in \mathbb{R} \setminus \{-1/4\}$ for all positive integers $n < 9$. Furthermore, we note that

$$f(u_n) = \frac{\frac{(-1)^n}{n^{2/3}}}{1 + \frac{4(-1)^n}{n^{2/3}}} = \frac{1}{4 + (-1)^n n^{2/3}}$$

for all integers $n \in \mathbb{N}$. By the mean value theorem we have that

$$|f(x) - x| = |f(x) - f(0) - f'(0)x| \leq \sup_{y \in I_x} |f''(y)| \frac{|x|^2}{2} = \sup_{y \in I_x} \frac{4x^2}{|1+4y|^3} \tag{1}$$

for all $x \in \,]1/4, 1/4[\,$, where I_x is the interval with limits 0 and x.

Since $(u_n)_{n \in \mathbb{N}}$ converges to zero as n goes to $+\infty$, let $n_0 \in \mathbb{N}$ satisfy that $|u_n| < 1/8$ for all $n \geq n_0$. Note that $|1+4y| > 1/2$ for all $y \in \mathbb{R}$ such that $|y| < 1/8$, since the latter is tantamount to $-1/8 < y < 1/8$, which implies $1/2 < 1+4y < 3/2$, giving the result. In consequence, $|1+4u_n| > 1/2$ for all $n \geq n_0$, which together with (1) implies that

$$|f(u_n) - u_n| \leq 32 u_n^2 = \frac{32}{n^{4/3}} \tag{2}$$

for all $n \geq n_0$. Since the series $32 \sum_{n=n_0}^{+\infty} 1/n^{4/3}$ converges, the series $\sum_{n=n_0}^{+\infty} (f(u_n) - u_n)$ is absoultely convergent, and in particular convergent. On the other hand, since the series $\sum_{n=n_0}^{+\infty} u_n$ is convergent, by a direct application of the Leibniz criterion (since the partial sums $\sum_{n=n_0}^{N} (-1)^n \in \{-1, 0, 1\}$ are bounded for all integers $N \geq n_0$ and $(1/n^{2/3})_{n \in \mathbb{N}}$ is a decreasing sequence converging to zero), the series

$$\sum_{n=n_0}^{+\infty} f(u_n) = \sum_{n=n_0}^{+\infty} (f(u_n) - u_n) + \sum_{n=n_0}^{+\infty} u_n$$

is also convergent, as was to be shown.

2,5pt

2. Compute the value of the integral

$$\int_1^2 \frac{dx}{x^2(3-x)}.$$

Solution. We note first that

$$\frac{1}{x^2(3-x)} = \frac{1}{3x^2} + \frac{1}{9x} + \frac{1}{9(3-x)}$$

for all $x \in \mathbb{R} \setminus \{0,3\}$. Then,

$$\int \frac{dx}{x^2(3-x)} = \int \frac{dx}{3x^2} + \int \frac{dx}{9x} + \int \frac{dx}{9(3-x)} = \int \frac{dx}{3x^2} + \int \frac{dx}{9x} - \int \frac{dy}{9y}$$
$$= -\frac{1}{3x} + \frac{\ln\left(|x|\right)}{9} - \frac{\ln\left(|y|\right)}{9} + C = -\frac{1}{3x} + \frac{\ln\left(|x|\right)}{9} - \frac{\ln\left(|3-x|\right)}{9} + C.$$

where we used the change of variables $y = 3-x$ (so $dy = -dx$) in the third integral of the second member. As a consequence,

$$\int_1^2 \frac{dx}{x^2(3-x)} = \left[-\frac{1}{3x} + \frac{\ln\left(|x|\right)}{9} - \frac{\ln\left(|3-x|\right)}{9} + C \right]_1^2 = \frac{1}{6} + \frac{\ln(4)}{9}.$$

2,5pt

3. Consider the following integrals

(*a*) $\int_0^1 \frac{\sqrt{1-x}}{\ln(x)} dx$, (*b*) $\int_0^{+\infty} \frac{\sin(x)}{e^{x^2}} dx$.

Determine if they are divergent or convergent.

Solution.

(*a*) We claim that the required integral is convergent. Let $f :]0,1[\to \mathbb{R}$ be the map given by $f(x) = -\sqrt{1-x}/\ln(x)$ for $x \in]0,1[$. Note that f is continuous and positive on its domain. Since

$$\int_0^1 f(x)dx = -\int_0^1 \frac{\sqrt{1-x}}{\ln(x)} dx,$$

$\int_0^1 \bar{f}(x)dx$ converges if and only if the required integral converges. We will thus work with f from now on. Since

$$\lim_{x\to 0+} f(x) = -\lim_{x\to 0+} \frac{\sqrt{1-x}}{\ln(x)} = 0,$$

the function f admits a continuous extension $\bar{f} : [0,1[\to \mathbb{R}$ such that $\bar{f}(0) = 0$ (and $\bar{f}(x) = f(0)$ for $x \in]0,1[$, by definition). Further, note that

$$\lim_{x\to 1-} \frac{\bar{f}(x)}{\frac{1}{\sqrt{1-x}}} = \lim_{x\to 1-} -\frac{1-x}{\ln(x)} = \lim_{x\to 1-} \frac{1}{1/x} = 1,$$

where we used the Bernoulli-L'Hospital rule for the second equality. Hence,

$\int_0^1 \bar{f}(x)dx$ converges if and only if the integral $\int_0^1 1/\sqrt{1-x}dx$ converges. Moreover, the latter integral converges, since

$$\int_0^1 \frac{1}{\sqrt{1-x}}dx = \lim_{\ell\to 1-}\Big[-2\sqrt{1-x}\Big]_0^\ell = \lim_{\ell\to 1-} -2\sqrt{1-\ell}+2 = 2,$$

so the required integral converges as well.

(b) We claim that the integral $\int_0^{+\infty} \sin(x)/e^{x^2}dx$ is absolutely convergent, so it is convergent. To prove this, note first that

$$\left|\frac{\sin(x)}{e^{x^2}}\right| \le \frac{1}{e^{x^2}} \le \frac{1}{e^x} = e^{-x}$$

for all $x \in \mathbb{R}_{\ge 1}$, where the second inequality follows from the fact that in this $x^2 \ge x$ so $e^{x^2} \ge e^x$, as the exponential function is strictly increasing. Since the integral $\int_1^{+\infty} e^{-x} = e^{-1}$ is convergent, we conclude that the integral

$$\int_1^{+\infty}\left|\frac{\sin(x)}{e^{x^2}}\right|dx$$

is also convergent. Moreover, since

$$\int_0^1\left|\frac{\sin(x)}{e^{x^2}}\right|dx$$

is the integral of a continuous function over a bounded and closed interval, it exists. As a consequence,

$$\int_0^{+\infty}\left|\frac{\sin(x)}{e^{x^2}}\right|dx = \int_0^1\left|\frac{\sin(x)}{e^{x^2}}\right|dx + \int_1^{+\infty}\left|\frac{\sin(x)}{e^{x^2}}\right|dx$$

also exists, as was to be shown.

3pt

4. Given $\alpha \in \mathbb{R}$, consider the integral

$$I_\alpha = \int_1^{+\infty}\frac{dx}{x(1+x^\alpha)}.$$

(a) Determine the set $\mathsf{C} = \{\alpha \in \mathbb{R} : I_\alpha \text{ converges}\}$.

(b) Compute the value of I_α for every $\alpha \in \mathsf{C}$.
Hint : use the change of variables $y = x^\alpha$.

Solution.

(a) We claim that $\mathsf{C} = \mathbb{R}_{>0}$. Indeed, note first that the integrand in the definition of

I_α is a continuous function on $\mathbb{R}_{\geq 1}$. Moreover, if $\alpha > 0$, we note that

$$\lim_{x\to+\infty} \frac{\frac{1}{x(1+x^\alpha)}}{\frac{1}{x^{1+\alpha}}} = \lim_{x\to+\infty} \frac{x^\alpha}{1+x^\alpha} = \lim_{x\to+\infty} \frac{1}{1+1/x^\alpha} = 1.$$

As a consequence, for $\alpha > 0$, I_α is convergent if and only if $\int_1^{+\infty} dx/x^{1+\alpha}$ converges, which in turn implies that I_α is convergent for all $\alpha > 0$. We further note that $I_0 = 2^{-1}\int_1^{+\infty} dx/x$, which is divergent. Finally, if $\alpha < 0$, we note that

$$\lim_{x\to+\infty} \frac{\frac{1}{x(1+x^\alpha)}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{1}{1+x^\alpha} = 1.$$

In consequence, for $\alpha < 0$, I_α is convergent if and only if $\int_1^{+\infty} dx/x$ converges. Since the latter is a divergent integral, I_α is divergent for all $\alpha < 0$.

(*b*) Assume $\alpha \in C$, *i.e.* $\alpha > 0$. Consider the change of variables $y = x^\alpha$. Then, $dy = \alpha x^{\alpha-1}dx$, which in turn implies that $dy = \alpha x^\alpha dx/x = \alpha y dx/x$, *i.e.* $dy/y = \alpha dx/x$. Then,

$$\begin{aligned}\int \frac{dx}{x(1+x^\alpha)} = \int \frac{dy}{\alpha y(1+y)} &= \frac{1}{\alpha}\int \left(\frac{1}{y} - \frac{1}{1+y}\right)dy \\ &= \frac{\ln(|y|)}{\alpha} - \frac{\ln(|1+y|)}{\alpha} + C = \frac{1}{\alpha}\ln\left(\frac{|y|}{|1+y|}\right) + C \\ &= \frac{1}{\alpha}\ln\left(\frac{|x^\alpha|}{|1+x^\alpha|}\right) + C.\end{aligned}$$

As a consequence,

$$\begin{aligned}\int_1^{+\infty} \frac{dx}{x(1+x^\alpha)} &= \lim_{A\to+\infty}\left[\frac{1}{\alpha}\ln\left(\frac{|x^\alpha|}{|1+x^\alpha|}\right)\right]_1^A = \lim_{A\to+\infty}\frac{1}{\alpha}\ln\left(\frac{A^\alpha}{1+A^\alpha}\right) - \frac{1}{\alpha}\ln\left(\frac{1}{2}\right) \\ &= \lim_{A\to+\infty}\frac{1}{\alpha}\ln\left(\frac{1}{1+1/A^\alpha}\right) - \frac{1}{\alpha}\ln\left(\frac{1}{2}\right) = -\frac{1}{\alpha}\ln\left(\frac{1}{2}\right) = \frac{\ln(2)}{\alpha}.\end{aligned}$$

7pt

5. Given $n \in \mathbb{N}_0$, let

$$u_n = \frac{\pi}{4} - \sum_{i=0}^{n} \frac{(-1)^i}{2i+1}. \tag{3}$$

(*a*) Show that

$$u_n = (-1)^{n+1}\int_0^1 \frac{t^{2n+2}}{1+t^2}dt.$$

Hint : Use that

$$\frac{\pi}{4} = \int_0^1 \frac{dt}{1+t^2} \text{ and } \frac{1}{2i+1} = \int_0^1 t^{2i}dt$$

for $i \in \mathbb{N}_0$.

(b) Show that, given $N \in \mathbb{N}_0$, there exists $m_N \in \mathbb{N}_0$ with $m_N > N$ such that

$$\sum_{n=0}^{N} u_n = -\int_0^1 \frac{t^2 + (-t^2)^{m_N}}{(1+t^2)^2} dt.$$

(c) Show that

$$\lim_{m\to+\infty} \int_0^1 \frac{(-t^2)^m}{(1+t^2)^2} dt = 0.$$

(d) Using the previous items, prove that the series $\sum_{n=0}^{+\infty} u_n$ converges, and that its sum is equal to

$$-\int_0^1 \frac{t^2}{(1+t^2)^2} dt.$$

(e) Show that

$$\int_0^1 \frac{t^2}{(1+t^2)^2} dt = \int_0^{\pi/4} \sin^2(s) ds$$

and determine the numeric value of the sum $\sum_{n=0}^{+\infty} u_n$.
Hint : Use the change of variables $t = \tan(s)$.

(f) Using the previous items, show that the series

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}$$

converges and that its sum is $\pi/4$.

Solution.

(a) Using the hint we get that

$$\begin{aligned} u_n &= \frac{\pi}{4} - \sum_{i=0}^{n} \frac{(-1)^i}{2i+1} = \int_0^1 \frac{1}{1+t^2} dt - \sum_{i=0}^{n} (-1)^i \int_0^1 t^{2i} dt \\ &= \int_0^1 \left(\frac{1}{1+t^2} - \sum_{i=0}^{n} (-t^2)^i \right) dt = \int_0^1 \left(\frac{1}{1+t^2} - \frac{1-(-t^2)^{n+1}}{1+t^2} \right) dt \\ &= \int_0^1 \frac{(-t^2)^{n+1}}{1+t^2} dt = (-1)^{n+1} \int_0^1 \frac{t^{2n+2}}{1+t^2} dt, \end{aligned}$$

where we used the geometric sum $\sum_{i=0}^{n} q^i = (1-q^{n+1})/(1-q)$, which is valid for all $q \in \mathbb{R} \setminus \{1\}$, for the particular case $q = -t^2$. Note that $t \in [0,1]$ implies that $-t^2 \neq 1$.

(*b*) Using the previous item we see that

$$\sum_{n=0}^{N} u_n = \sum_{n=0}^{N} (-1)^{n+1} \int_0^1 \frac{t^{2n+2}}{1+t^2} dt = \int_0^1 \sum_{n=0}^{N} \frac{(-t^2)^{n+1}}{1+t^2} dt$$
$$= \int_0^1 \frac{(-t^2)-(-t^2)^{N+2}}{(1+t^2)^2} dt = -\int_0^1 \frac{t^2+(-t^2)^{N+2}}{(1+t^2)^2} dt,$$

where we used in the third equality the expression of the geometric sum recalled in the previous item. Hence, we obtained the required expression for $\sum_{n=0}^{N} u_n$ with $m_N = N+2 > N$.

(*c*) Note that

$$\left| \frac{(-t^2)^m}{(1+t^2)^2} \right| \leq |-t^2|^m = t^{2m}$$

for all $t \in \mathbb{R}$, since $(1+t^2)^2 \geq 1$, which implies that

$$0 \leq \left| \int_0^1 \frac{(-t^2)^m}{(1+t^2)^2} dt \right| \leq \int_0^1 \left| \frac{(-t^2)^m}{(1+t^2)^2} \right| dt$$
$$\leq \int_0^1 t^{2m} dt = \frac{1}{2m+1},$$

for all $m \in \mathbb{N}_0$. Using the sandwich theorem we conclude that

$$\lim_{m \to +\infty} \int_0^1 \frac{(-t^2)^m}{(1+t^2)^2} dt = 0.$$

(*d*) The second item tells us that

$$0 \leq \left| \sum_{n=0}^{N} u_n + \int_0^1 \frac{t^2}{(1+t^2)^2} dt \right| = \left| \int_0^1 \frac{(-t^2)^{N+2}}{(1+t^2)^2} dt \right|$$

for all $N \in \mathbb{N}_0$. Since the latter term converges to zero as N goes to $+\infty$, by the previous item, we conclude that

$$\sum_{n=0}^{+\infty} u_n = \lim_{N \to +\infty} \sum_{n=0}^{N} u_n = -\int_0^1 \frac{t^2}{(1+t^2)^2},$$

as was to be shown.

(*e*) Using the change of variables $t = \tan(s)$, so $dt = ds/\cos^2(s)$, we see that

$$\int \frac{t^2}{(1+t^2)^2} dt = \int \frac{\tan(s)^2}{(1+\tan^2(s))^2 \cos^2(s)} ds = \int \sin^2(s) ds$$
$$= \int \frac{1-\cos(2s)}{2} dt = \frac{2s - \sin(2s)}{4} + C.$$

Hence, since the tangent function restricted to $[0,\pi/4]$ is a strictly increasing map, whocse image is precisely $[0,1]$, we further conclude that

$$\int_0^1 \frac{t^2}{(1+t^2)^2}dt = \int_0^{\pi/4} \sin^2(s)ds = \left[\frac{2s-\sin(2s)}{4}\right]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{4}.$$

In consequence,

$$\sum_{n=0}^{+\infty} u_n = -\int_0^1 \frac{t^2}{(1+t^2)^2}dt = \frac{1}{4} - \frac{\pi}{8}.$$

(*f*) Since the series $\sum_{n=0}^{+\infty} u_n$ converges, its general term converges to zero, *i.e.*

$$0 = \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{\pi}{4} - \sum_{i=0}^{n} \frac{(-1)^i}{2i+1} = \frac{\pi}{4} - \lim_{n\to+\infty} \sum_{i=0}^{n} \frac{(-1)^i}{2i+1} = \frac{\pi}{4} - \sum_{i=0}^{+\infty} \frac{(-1)^i}{2i+1},$$

which implies that

$$\frac{\pi}{4} = \sum_{i=0}^{+\infty} \frac{(-1)^i}{2i+1},$$

as was to be shown.