MAT332
Fall 2021

Midterm examination

Unjustified answers will be automatically excluded.

The grading is only approximate.

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2
3
4
5

3pt **1.** Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers such that $\sum_{n\in\mathbb{N}} z_n$ is absolutely convergent. Prove that $\sum_{n\in\mathbb{N}} z_n$ is then convergent.

Solution. Consider the sequence $\{s_N\}_{N\in\mathbb{N}}$ of partial sums $s_N = \sum_{n=1}^{N} z_n$ for $N \in \mathbb{N}$. Since $\sum_{n\in\mathbb{N}} z_n$ converges absolutely, the sequence $\{S_N\}_{N\in\mathbb{N}}$ of partial sums given by $S_N = \sum_{n=1}^{N} |z_n|$ for $N \in \mathbb{N}$ is a Cauchy sequence. Given $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\sum_{n=N}^{M-1} |z_n| = |S_M - S_N| \leq \epsilon$$

for all $M > N \geq N_0$. Using the triangle inequality we see that

$$|s_M - s_N| = \left|\sum_{n=N}^{M-1} z_n\right| \leq \sum_{n=N}^{M-1} |z_n| = |S_M - S_N| \leq \epsilon$$

for all $M > N \geq N_0$. This tells us that $\{s_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence of complex numbers. Since any Cauchy sequence of complex numbers is convergent, then $\{s_N\}_{N\in\mathbb{N}}$ is convergent, *i.e.* $\sum_{n\in\mathbb{N}} z_n$ converges.

2pt **2.** Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ given by

$$a_{2k-1} = \frac{2^{k-1}}{3^{k-1}} \text{ and } a_{2k} = \frac{2^{k-1}}{3^k}$$

for $k \in \mathbb{N}$. Determine if $\sum_{n\in\mathbb{N}} a_n$ is convergent or divergent.

Solution. We will show that $a_n^{1/n}$ converges to $\sqrt{2/3}$ as n goes to $+\infty$. Indeed, note that

$$\lim_{k\to+\infty} a_{2k-1}^{1/(2k-1)} = \lim_{k\to+\infty} \frac{2^{\frac{k-1}{2k-1}}}{3^{\frac{k-1}{2k-1}}} = \lim_{k\to+\infty} \frac{2^{\frac{1-1/k}{2-1/k}}}{3^{\frac{1-1/k}{2-1/k}}} = \sqrt{\frac{2}{3}}$$

and

$$\lim_{k\to+\infty} a_{2k}^{1/(2k)} = \lim_{k\to+\infty} \frac{2^{\frac{k-1}{2k}}}{3^{\frac{k}{2k}}} = \lim_{k\to+\infty} \frac{2^{\frac{1-1/k}{2}}}{3^{\frac{1}{2}}} = \sqrt{\frac{2}{3}}.$$

Since the subsequences $\{a_{2k-1}^{1/(2k-1)}\}_{k\in\mathbb{N}}$ and $\{a_{2k}^{1/(2k)}\}_{k\in\mathbb{N}}$ of $\{a_n^{1/n}\}_{n\in\mathbb{N}}$ converge to $\sqrt{2/3}$ as n goes to $+\infty$, then

$$\lim_{k\to+\infty} a_n^{1/n} = \sqrt{\frac{2}{3}}.$$

The root test tells us then that $\sum_{n\in\mathbb{N}} a_n$ is convergent.

5pt **3.** Consider the sequence $\{v_n\}_{n\in\mathbb{N}}$ given by

$$v_n = \sin\left(\pi\frac{n^3+1}{n^2+1}\right)$$

for $n\in\mathbb{N}$.

(*a*) Show that

$$v_n = \sin\left(\pi\frac{n^3+1}{n^2+1}\right) = (-1)^{n+1}\sin\left(\pi\frac{n-1}{n^2+1}\right)$$

for $n\in\mathbb{N}$.

(*b*) Show that the map $f:\mathbb{R}\to\mathbb{R}$ given by

$$f(x) = \sin\left(\pi\frac{x-1}{x^2+1}\right)$$

is strictly decreasing for all $x>3$.

(*c*) Determine if $\sum_{n\in\mathbb{N}} v_n$ is convergent

(*d*) Determine if $\sum_{n\in\mathbb{N}} v_n$ is absolutely convergent.

Solution.

(*a*) Note that

$$\begin{aligned}\sin\left(\pi\frac{n^3+1}{n^2+1}\right) &= \sin\left(\pi\frac{n(n^2+1)-(n-1)}{n^2+1}\right) = \sin\left(\pi n-\pi\frac{n-1}{n^2+1}\right)\\ &= (-1)^n\sin\left(-\pi\frac{n-1}{n^2+1}\right) = (-1)^{n+1}\sin\left(\pi\frac{n-1}{n^2+1}\right),\end{aligned}$$

where we used that $\sin(n\pi+x) = (-1)^n\sin(x)$ and $\sin(-x) = -\sin(x)$ for $x\in\mathbb{R}$.

(*b*) It is clear that

$$f'(x) = \cos\left(\pi\frac{x-1}{x^2+1}\right)\pi\frac{x^2-1-(x-1)2x}{(x^2+1)^2} = \cos\left(\pi\frac{x-1}{x^2+1}\right)\pi\underbrace{\frac{-x^2+2x+1}{(x^2+1)^2}}_{r(x)},$$

for all $x \in \mathbb{R}$. Since $-x^2+2x+1 = -(x-1+\sqrt{2})(x-1-\sqrt{2})$, it is clear that $r(x) < 0$ for $x > 1+\sqrt{2}$. Moreover, since $2(x-1) < x^2+1$ is tantamount to $0 < x^2-2x+3$, which clearly holds since $x^2-2x+3 = (x-1)^2+2 > 0$, we have that

$$0 < \pi\frac{x-1}{x^2+1} < \frac{\pi}{2} \tag{1}$$

for all $x > 1$, which implies that $f'(x) < 0$ for all $x > 1+\sqrt{2}$, and in particular for $x \geq 3$. As a consequence, f is strictly decreasing for $x \geq 3$.

(*c*) It is clear that

$$\lim_{n\to+\infty} \sin\left(\pi\frac{n-1}{n^2+1}\right) = 0,$$

since the sinus function is continuous and its argument converges to 0 as n goes to $+\infty$. Moreover, (1) tells us that

$$\sin\left(\pi\frac{n-1}{n^2+1}\right) \geq 0$$

for all integers $n > 1$, and it clearly vanishes for $n = 1$. Then, the sequence $\{b_n\}_{n\in\mathbb{N}}$ given by

$$b_n = \sin\left(\pi\frac{n-1}{n^2+1}\right)$$

is nonnegative, decreasing and has limit zero. Now, since the sequence $\{s_n\}_{n\in\mathbb{N}}$ of partial sums of $\{a_n\}_{n\in\mathbb{N}}$ given by $a_n = (-1)^{n+1}$ is bounded, because $|s_n| = |(1+(-1)^{n+1})/2| \leq 1$ for all $n \in \mathbb{N}$, the Leibniz criterion tells us that the series

$$\sum_{n\in\mathbb{N}} v_n = \sum_{n\in\mathbb{N}} a_n b_n = \sum_{n\in\mathbb{N}} (-1)^{n+1} \sin\left(\pi\frac{n-1}{n^2+1}\right)$$

converges.

(*d*) Since

$$\lim_{n\to+\infty} \frac{\sin\left(\pi\frac{n-1}{n^2+1}\right)}{1/(\pi n)} = \lim_{n\to+\infty} \frac{\sin\left(\pi\frac{1-1/n}{n+1/n}\right)}{1/(\pi n)} = 1,$$

and the series $\sum_{n\in\mathbb{N}} 1/n$ is divergent, the series $\sum_{n\in\mathbb{N}} b_n = \sum_{n\in\mathbb{N}} |v_n|$ is not convergent. Hence, $\sum_{n\in\mathbb{N}} v_n$ is not absolutely convergent.

7pt **4.** Given $\alpha \in \mathbb{R}$, consider the sequence $\{w_{n,\alpha}\}_{n\in\mathbb{N}}$ given by

$$w_{n,\alpha} = \left(n.\sin\left(\frac{1}{n}\right)\right)^{n^\alpha}$$

for $n \in \mathbb{N}$.

(*a*) Prove that, given $\beta \geq \alpha$, $\sum_{n\in\mathbb{N}} w_{n,\beta}$ converges if $\sum_{n\in\mathbb{N}} w_{n,\alpha}$ converges.
Hint : use that $\sin(x) < x$ for all $x > 0$.

(*b*) Prove that

$$\lim_{x\to 0+} \frac{6\ln\left(\frac{\sin(x)}{x}\right) + x^2}{x^{2+\epsilon}} = 0$$

for all $\varepsilon \in [0, 1]$.
Hint : prove first that

$$\lim_{x\to 0+} \frac{-\sin(x) + x\cos(x)}{x^\varepsilon \sin(x)} = 0.$$

(*c*) Using the previous item for $\varepsilon = 0$, prove that $\sum_{n\in\mathbb{N}} w_{n,\alpha}$ diverges if $\alpha \leq 2$.

(*d*) Given ε such that $0 < \varepsilon \leq 1$, show that

$$w_{n,\alpha} \sim e^{-\frac{n^\varepsilon}{6}}$$

as n goes to $+\infty$.

(*e*) Prove that $\sum_{n\in\mathbb{N}} w_{n,\alpha}$ converges for all $\alpha > 2$.
Hint : use that

$$\lim_{n\to+\infty} \frac{e^{-n^\varepsilon/6}}{1/n^2} = 0$$

for all $\varepsilon > 0$.

Solution.

(*a*) Since $0 < \sin(x) < x$ for all $0 < x < \pi$,

$$0 < n.\sin\left(\frac{1}{n}\right) < n.\frac{1}{n} = 1 \tag{2}$$

for all $n \in \mathbb{N}$, because in this case $0 < 1/n < \pi$. Now, $\alpha \geq \beta$ tells us that $n^\alpha \leq n^\beta$, which, together with (2), implies that

$$0 < w_{n,\beta} = \left(n\sin\left(\frac{1}{n}\right)\right)^{n^\beta} \leq \left(n\sin\left(\frac{1}{n}\right)\right)^{n^\alpha} = w_{n,\alpha}$$

for all $n \in \mathbb{N}$. Since $\sum_{n\in\mathbb{N}} w_{n,\alpha}$ converges, the comparison test tells us that $\sum_{n\in\mathbb{N}} w_{n,\beta}$ also converges.

(*b*) Given $\varepsilon \in [0,1]$, note first that

$$\begin{aligned}\lim_{x\to 0+} \frac{-\sin(x)+x\cos(x)}{x^{\varepsilon}\sin(x)} &= \lim_{x\to 0+} \frac{-x\sin(x)}{x^{\varepsilon}\cos(x)+\varepsilon x^{\varepsilon-1}\sin(x)} \\ &= \lim_{x\to 0+} \frac{-x^{1-\varepsilon}\sin(x)}{\cos(x)+\varepsilon\frac{\sin(x)}{x}} = 0,\end{aligned} \tag{3}$$

where we have applied the Bernoulli-L'Hospital rule in the first equality and we used that $\sin(x)/x$ converges to 1 when x goes to 0 in the last equality. Now,

$$\begin{aligned}\lim_{x\to 0+} \frac{6\ln\left(\frac{\sin(x)}{x}\right)+x^2}{x^{2+\varepsilon}} &= \lim_{x\to 0+} \frac{6\frac{x}{\sin(x)}\frac{x\cos(x)-\sin(x)}{x^2}+2x}{(2+\varepsilon)x^{1+\varepsilon}} \\ &= \frac{2}{2+\varepsilon}\lim_{x\to 0+} \frac{3\big(x\cos(x)-\sin(x)\big)+x^2\sin(x)}{x^{2+\varepsilon}\sin(x)} \\ &= \frac{2}{2+\varepsilon}\lim_{x\to 0+} \frac{-x\sin(x)+x^2\cos(x)}{(2+\varepsilon)x^{1+\varepsilon}\sin(x)+x^{2+\varepsilon}\cos(x)} \\ &= \frac{2}{2+\varepsilon}\lim_{x\to 0+} \frac{-\sin(x)+x\cos(x)}{(2+\varepsilon)x^{\varepsilon}\sin(x)+x^{1+\varepsilon}\cos(x)} \\ &= \frac{2}{2+\varepsilon}\lim_{x\to 0+} \frac{-\sin(x)+x\cos(x)}{x^{\varepsilon}\sin(x)}\frac{1}{(2+\varepsilon)+\frac{x}{\sin(x)}\cos(x)} \\ &= \frac{2}{2+\varepsilon}0.1 = 0,\end{aligned}$$

where we have applied the Bernoulli-L'Hospital rule in the first and third equalities, we used (3) and that $\sin(x)/x$ converges to 1 when x goes to 0 in the last equality. In particular, if $\varepsilon = 0$ we get that

$$\lim_{x\to 0+} \frac{\ln\left(\frac{\sin(x)}{x}\right)}{x^2} = -\frac{1}{6}. \tag{4}$$

(*c*) Using (4) we have that

$$\lim_{x\to 0+} e^{\frac{\ln\left(\frac{\sin(x)}{x}\right)}{x^2}} = e^{\lim_{x\to 0+}\frac{\ln\left(\frac{\sin(x)}{x}\right)}{x^2}} = e^{-1/6} > 0,$$

since the exponential function is continuous. In particular, replacing x by $1/n$ in the previous identity, we get

$$\lim_{n\to+\infty} w_{n,2} = \lim_{n\to+\infty} e^{n^2\ln\left(n.\sin\left(\frac{1}{n}\right)\right)} = e^{-1/6} > 0.$$

Since the general term $w_{n,2}$ does not tend to zero when n goes to $+\infty$, we conclude that $\sum_{n\in\mathbb{N}} w_{n,2}$ diverges. Moreover, by the first item, $\sum_{n\in\mathbb{N}} w_{n,\alpha}$ diverges if $\alpha \leq 2$.

(*d*) Using the second item we have that

$$\lim_{x\to 0+} e^{\frac{6\ln\left(\frac{\sin(x)}{x}\right)+x^2}{6x^{2+\varepsilon}}} = e^{\frac{1}{6}\lim_{x\to 0+}\frac{6\ln\left(\frac{\sin(x)}{x}\right)+x^2}{x^{2+\varepsilon}}} = e^0 = 1,$$

since the exponential function is continuous. In particular, replacing x by $1/n$ in the previous identity, we get

$$1 = \lim_{n\to+\infty} e^{\frac{6n^{2+\varepsilon}\ln\left(n.\sin\left(\frac{1}{n}\right)\right)+n^{\varepsilon}}{6}} = \lim_{n\to+\infty} \frac{w_{n,2+\varepsilon}}{e^{-\frac{n^{\varepsilon}}{6}}},$$

which tells us that $w_{n,\alpha} \sim e^{-\frac{n^{\varepsilon}}{6}}$ as n goes to $+\infty$.

(*e*) We note that

$$\lim_{n\to+\infty} \frac{e^{-\frac{n^{\varepsilon}}{6}}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{n^2}{e^{\frac{n^{\varepsilon}}{6}}} = 0,$$

for all $\varepsilon > 0$. Then, $w_{n,2+\varepsilon} = o(1/n^2)$ as n goes to $+\infty$. Since $\sum_{n\in\mathbb{N}} 1/n^2$ converges, $\sum_{n\in\mathbb{N}} w_{n,2+\varepsilon}$ converges for all ε such that $0 < \varepsilon < 1$. By the first item, we conclude that $\sum_{n\in\mathbb{N}} w_{n,\alpha}$ converges for all $\alpha > 2$.

3pt **5.** Compute the limit of

$$S_n = \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2-k^2}}$$

as n goes to $+\infty$.

Solution. It is clear that

$$S_n = \sum_{k=0}^{n-1} \frac{1}{\sqrt{n^2-k^2}} = \sum_{k=0}^{n-1} \frac{1}{n} \frac{1}{\sqrt{1-(k/n)^2}}$$

is a Riemann sum of the map $f : [0,1] \to \mathbb{R}$ given by $f(x) = 1/\sqrt{1-x^2}$ for the partition $\{x_j = j/n : j \in \{0,\dots,n\}\}$ of the interval $[0,1]$. In consequence,

$$\lim_{n\to+\infty} S_n = \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = \Big[\arcsin(x)\Big]_0^1 = \frac{\pi}{2}.$$