Vertex algebras and 2-monoidal categories

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Abstract

In this article we address Problem 5.12 in [8]. More precisely, we prove that the singular tensor product introduced by R. Borcherds in the previous reference is part of a 2-monoidal category structure in a certain category of functors. We also complete some missing points in the previously mentioned article, most notably in the definitions of singular tensor products and of vertex algebras themselves, which are however verified in all the examples appearing in that reference. To prove our results it will be extremely useful, if not essential, to frame our objects within the language of bicategories. We also introduce a slightly more general notion of (quantum) vertex algebra than the one in [8], that we call categorical (quantum) vertex algebra, enjoying all the properties mentioned by Borcherds in that article and having as particular example the definition presented by that author.

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1 Introduction

Vertex algebras were introduced by R. Borcherds in his seminal article [6], where he provided the axiomatic framework to deal with vertex operators, which arose in the study of dual resonance models in string theory in the 1970s but also independently around the same time in the representation theory of certain infinite dimensional Lie algebras (see [13, 16]). Vertex algebras have become a rather pervasive object of study in mathematics, mainly in representation theory, finite group theory, combinatorics, algebraic geometry and number theory.

In the article [8] (see also [7]), Borcherds proposed a new formulation for the mainstay of vertex algebras, namely the use of a new tensor product, called **singular tensor product**, defined in a rather convoluted category of functors (*cf.* [3]). The main purpose of this new tensor product is, in Borcherds' own words, "to make the theory of vertex algebras trivial", which should presumably be interpreted in the spirit of the structuralist approach, typically attributed to A. Grothendieck, that claims that "it is better to have a good category with bad objects than a bad category with good objects".

The article [8] is however rather scarce in details –or even statements– concerning the singular tensor product, since its objective is presumably of programmatic nature. As the reader can verify from our exposition below, there are even some missing structural hypotheses in the definition of singular tensor product of [8]. However, as the reader might also expect, these missing hypotheses hold in all the examples considered in that article, which shows that the ideas laid in [8] are right. On the other hand, we want to remark that the article [8] has not received much attention. Indeed, even though several authors studied (and generalized) some constructions on bicharacters of bialgebras and their application to vertex algebras following [8] (see *e.g.* [2, 24]), as far as we know the categorical constructions in [8] and the singular tensor product have not been studied in general. However, in the interesting preprint [10] the author compares the definition of vertex algebra in [8] for the specific choice recalled in Thm. 5.9, and the classical definition of vertex algebra.

The question posed in [8] we mainly concern us with is his Problem 5.12, where he states the need for a theory of categories with two tensor products, that applies to the singular tensor product, together with a canonical tensor product of the objects he considers. The main goal of this article is to show that the theory of 2-monoidal categories introduced in [1] (see also [9, 25]) fits this situation, verifying all the properties mentioned by Borcherds, in particular the exchange law as well as the fact that classical tensor product of vertex algebras should naturally be vertex algebras (see the penultimate paragraph of p. 64 in [8]). More precisely, we remark that **vertex algebras**, introduced in [8], Def. 3.12, are indeed particular examples of the **categorical vertex algebras** we introduce (see Definitions 5.3 and 5.8). Moreover, the latter satisfy all the categorical properties Borcherds mentioned in Section 3 of [8] (see Theorems 4.5, 4.8 and 4.10). To be fair, since the latter article does not contain many details on which properties these tensor products should satisfy (others than the ones we mentioned), we cannot be sure if our proposal fits exactly what Borcherds had in mind, but we believe that our results should be the closest possible.

The structure of the article is as follows. In Section 2 we recall the basic terminology on (braided) monoidal categories, (co)algebras and (co)modules over them. In Subsection 2.3 we present some elementary constructions from bicategory theory. In particular, we introduce a notion of module over a homomorphism of bicategories and present an elementary but useful result (see Proposition 2.2), both of which we were unable to find in the literature, although we are convinced they should be well known. Even though most, if not all, of the contents of the section are well known, the main reason for recalling them briefly is to establish the notation we shall later use for the proof of our results. In Section 3 we lay the basic objects we will study and prove the basic results we will need to define singular tensor products.

Finally, in Section 4 we prove the main results of this work, namely that a certain category $\mathscr{C}$ of modules over a commutative algebra has a natural structure of symmetric 2-monoidal category (see Theorems 4.5, 4.8 and 4.10). To deal with the rather convoluted algebraic properties of the objects we are presented with, it will prove essential to make use of the constructions of bicategories recalled in Subsection 2.3. Not only do they provide a very natural framework to work in, but also they will be doing some heavy lifting. After setting a general categorical framework, we introduce in Subsection 5.1 the notion of categorical (quantum) vertex algebra, which verifies all the categorical properties Borcherds mentioned in Section 3 of [8] for his (quantum) vertex algebras. Finally, in Subsection 5.2 we present the notion of (quantum) vertex algebra introduced in [8], Def. 3.12, as a special case of categorical (quantum) vertex algebra. We remark that the version of quantum vertex algebra in [8] is different from others in the literature (*cf.* [12]).

I thank Carina Boyallian for pointing out the interesting reference [10].

2 Preliminaries

In this section we briefly review the well-known notions and results of (braided) monoidal categories as well as the basic algebraic structures within them we will use. The main purpose is to lay the notation we will use in the sequel. In Subsection 2.3, we recall some basic results coming from bicategory theory. In the last part of this section we briefly recall the not so well-known notion of 2-monoidal (or duoidal) category.

2.1 Notation

We will denote by $\mathbb{N}_0$ (resp., $\mathbb{N}$) the set of nonnegative (resp., positive) integers $\{0, 1, 2, \dots\}$ (resp., $\{1, 2, \dots\}$), and given $n', n'' \in \mathbb{N}_0$ we denote by $[\![n', n'']\!]$ the set $\{n \in \mathbb{N}_0 : n' \leq n \leq n''\}$. Given $n \in \mathbb{N}$, let $\mathbb{S}_n$ be the group of bijections of $[\![1, n]\!]$.

To alleviate the burden of notation, and as it is widely done in the literature, we will often commit the abuse of omitting the extra structure of an algebraic object if this does not cause any confusion (*e.g.* we will denote a monoidal category $\mathscr{C}$ simply by the underlying class of objects and morphisms). The reason is typically that the notation we follow for the extra structure is obtained by simply decorating fixed symbols with the object one mentioned (*e.g.* for the case of a monoidal category $\mathscr{C}$, $\otimes_{\mathscr{C}}$ will denote the tensor product and $\mathbf{I}_{\mathscr{C}}$ the unit).

2.2 Review of basic facts on monoidal categories

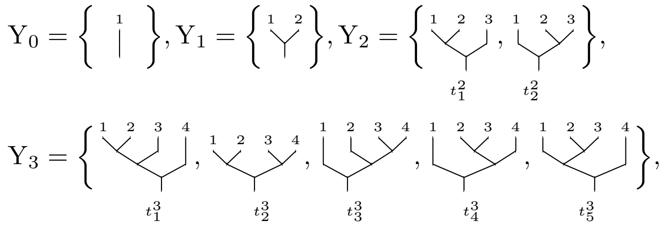
We assume the reader is familiar with the basic definitions of categories (see [4]), as well as the particular case of (**braided**) **monoidal categories** (see [14], Ch. XI and XIII). We will only recall some basic facts about them, which also allow us to set the notation. Moreover, to avoid set-theoretic issues –and as it is implicitly assumed in the literature– we fix an infinite universe $\mathscr{U}$ such that the classes of objects and morphisms of all of our categories are subsets of the universe (*i.e.* we deal with $\mathscr{U}$-categories), except for the category $\mathbf{Cat}$ of all ($\mathscr{U}$-)categories. We just recall that a small category in this situation simply means that the class of objects is an element of the universe.

We remark that, if $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\mathscr{C}})$ is a monoidal category, we will denote the **associativity**, **left unit** and **right unit constraints** by

$$\begin{gathered} a^{\mathscr{C}}(X, Y, Z) : (X \otimes_{\mathscr{C}} Y) \otimes_{\mathscr{C}} Z \longrightarrow X \otimes_{\mathscr{C}} (Y \otimes_{\mathscr{C}} Z), \\ \ell^{\mathscr{C}}(X) : \mathbf{I}_{\mathscr{C}} \otimes_{\mathscr{C}} X \longrightarrow X \text{ and } r^{\mathscr{C}}(X) : X \otimes_{\mathscr{C}} \mathbf{I}_{\mathscr{C}} \longrightarrow X, \end{gathered}$$

respectively, for objects X, Y and Z of $\mathscr{C}$. We will sometimes omit the arguments in the previous natural transformations if they are clear from the context. Moreover, in some expressions, specially to alleviate the notation in the composition of several morphisms of algebraic objects such as (co)algebras and (co)modules, we will sometimes omit the associativity constraint, if it is clear from the context. We shall typically denote the monoidal category simply by its underlying category $\mathscr{C}$, since we will use the previous notation to denote the rest of its structure, unless otherwise stated. Analogously, a braided monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\mathscr{C}}, \tau^{\mathscr{C}})$ will simply be denoted by $\mathscr{C}$. We say that the unit $\mathbf{I}_{\mathscr{C}}$ of the monoidal category $\mathscr{C}$ is **strict** if $\ell(X)^{\mathscr{C}} = \mathrm{id}_X = r(X)^{\mathscr{C}}$, for all objects X of $\mathscr{C}$, so in particular $\mathbf{I}_{\mathscr{C}} \otimes_{\mathscr{C}} X = X = X \otimes_{\mathscr{C}} \mathbf{I}_{\mathscr{C}}$.

To keep track of the different parenthesizations of tensor products we will use the following typical bookkeeping device. Given $n \in \mathbb{N}_0$, denote by Y_n the set of all **rooted planar binary trees** with $(n+1)$ leaves, which we we typically label with the elements of $[\![1, n+1]\!]$ from left to right (see [17], Appendix C.1). For example,



(2.1)

where we have numbered the elements of Y_2 and Y_3 for future reference.

We will denote the set of vertices of $t \in \mathrm{Y}_n$ by $\mathrm{Ver}(t)$ and the set of leaves by $\mathrm{Lvs}(t)$. We recall that a tree $t \in \mathrm{Y}_n$ for $n \geq 1$ has n vertices, $n-1$ **edges** –*i.e.* edges connecting two vertices–, and $n+2$ **half-edges**, –*i.e.* edges incident on at most one vertex–, $n+1$ of which are leaves and one extra half-edge, called the **root**. The tree $t \in \mathrm{Y}_0$ is supposed to be formed of just the root, which is simultaneously considered as a leaf. We also recall that given rooted planar binary trees $t \in \mathrm{Y}_n$ and $t' \in \mathrm{Y}_{n'}$, **grafting** the root of t' to the i-th leaf of t forms a new rooted planar binary tree $t'' \in \mathrm{Y}_{n+n'}$. We will say that $t \in \mathrm{Y}_n$ is **thin** if given any vertex $v \in \mathrm{Ver}(t)$ there is a leaf i adjacent on v.

Given a monoidal category $\mathscr{C}$ and $t \in \mathrm{Y}_n$, denote by $t(\otimes_{\mathscr{C}}) : \mathscr{C}^{n+1} \to \mathscr{C}$ the functor defined recursively as follows. If $t \in \mathrm{Y}_0$, $t(\otimes_{\mathscr{C}})$ is the identity functor of $\mathscr{C}$, and for $n \geq 1$ if $t \in \mathrm{Y}_n$ is obtained from trees $t_\ell \in \mathrm{Y}_{n'}$ and $t_r \in \mathrm{Y}_{n''}$ with $n' + n'' + 1 = n$ by grafting the root of t_ℓ (resp., t_r) to the left (resp., right) leaf of $t_0 \in \mathrm{Y}_1$, then $t(\otimes_{\mathscr{C}}) = \otimes_{\mathscr{C}} \circ (t_\ell(\otimes_{\mathscr{C}}), t_r(\otimes_{\mathscr{C}}))$. Given $t, t' \in \mathrm{Y}_n$, we denote by $\mathcal{a}^{\mathscr{C}}_{t \to t'} : t(\otimes_{\mathscr{C}}) \to t'(\otimes_{\mathscr{C}})$ the natural isomorphism obtained from using the associativity constraint of $\mathscr{C}$, which is unique by Mac Lane's coherence theorem (see [14], Thm. XI.5.3).

The previous construction can be extended to the case of a monoidal category $\mathscr{C}$ with a symmetric braiding $\tau^{\mathscr{C}}$. For $n \in \mathbb{N}$, if $\sigma \in \mathbb{S}_{n+1}$, we denote by $\mathcal{c}(\sigma) : \mathscr{C}^{n+1} \to \mathscr{C}^{n+1}$ the auto-functor sending $(X_1, \dots, X_{n+1})$ to $(X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(n+1)})$, for all $(n+1)$-tuples of objects or morphisms of $\mathscr{C}$. Set $\mathrm{Y}^{\mathrm{sym}}_n = \mathrm{Y}_n \times \mathbb{S}_{n+1}$. Then, given $T = (t, \sigma), T' = (t', \sigma') \in \mathrm{Y}^{\mathrm{sym}}_n$, we denote by $\mathcal{ac}^{\mathscr{C}}_{T \to T'} : t(\otimes_{\mathscr{C}}) \circ \mathcal{c}(\sigma) \to t'(\otimes_{\mathscr{C}}) \circ \mathcal{c}(\sigma')$ the natural isomorphism obtained from using the associativity constraint $\mathcal{a}^{\mathscr{C}}$ of $\mathscr{C}$ and the symmetric braiding $\tau^{\mathscr{C}}$, which is unique by Mac Lane's coherence theorem (see [18], Thm. XI.1.1). If $T = (t, \mathrm{id}_{[\![1,n+1]\!]}), T' = (t, \sigma) \in \mathrm{Y}^{\mathrm{sym}}_n$, we will denote the morphism $\mathcal{ac}^{\mathscr{C}}_{T \to T'}$ simply by $\tau^{\mathscr{C}}(\sigma, t)$.

We recall that a monoidal category $\mathscr{C}$ is said to be **semicocartesian** if its unit $\mathbf{I}_{\mathscr{C}}$ is the initial object of the underlying category of $\mathscr{C}$ (*cf.* [11]). In this case, given $t \in Y_n$, an $(n+1)$-tuple $\hat{X} = (X_1, \dots, X_{n+1})$ of objects of $\mathscr{C}$ and $i \in [\![1, n+1]\!]$, let $\hat{X}_i$ be the $(n+1)$-tuple whose j-th component is X_i if $j = i$ and $\mathbf{I}_{\mathscr{C}}$ else, and $\hat{f}_i$ be the $(n+1)$-tuple of morphisms of $\mathscr{C}$ whose j-th component is id_{X_i} if $j = i$ and $i_{X_j} : \mathbf{I}_{\mathscr{C}} \to X_j$ else. We define $\iota_{i,t}(\hat{X}) : X_i \to t(\otimes_{\mathscr{C}})(\hat{X})$ as the composition of the unique isomorphism $X_i \to t(\otimes_{\mathscr{C}})(\hat{X}_i)$ given by using the left and right unit constraints of $\mathscr{C}$ and $t(\otimes_{\mathscr{C}})(\hat{f}_i)$. It is clear that $\iota_{i,t}(\hat{X})$ is natural in $\hat{X}$, and,

$$\mathcal{a}^{\mathscr{C}}_{t \to t'}(\hat{X}) \circ \iota_{i,t}(\hat{X}) = \iota_{i,t'}(\hat{X}), \tag{2.2}$$

for any $t, t' \in \mathrm{Y}_n$, by Mac Lane's coherence theorem for monoidal categories, which also tells us the following result. Given $t' \in \mathrm{Y}_m$ and $t \in \mathrm{Y}_n$, as well as tuples $\hat{Y} = (Y_1, \dots, Y_{m+1})$ and $\hat{X} = (X_1, \dots, X_{n+1})$ of objects of $\mathscr{F}$, such that $Y_j = t(\otimes_{\mathscr{C}})(\hat{X})$ for some $j \in [\![1, m+1]\!]$, consider $t'' \in \mathrm{Y}_{n+m}$ obtained by grafting the root of t on the j-th leaf of t', and $\hat{Z} = (Z_1, \dots, Z_{n+m+1})$ given by $Z_i = Y_i$ if $i \in [\![1, j-1]\!]$, $Z_i = X_{i-j+1}$ if $i \in [\![j, j+n]\!]$, and $Z_i = Y_{i-n}$ if $i \in [\![j+n+1, m+n+1]\!]$. Then,

$$\iota_{i,t''}(\hat{Z}) = \begin{cases} \iota_{i,t'}(\hat{Y}), & \text{if } i \in [\![1, j-1]\!], \\ \iota_{j,t'}(\hat{Y}) \circ \iota_{i,t}(\hat{X}), & \text{if } i \in [\![j, j+n]\!], \\ \iota_{i-n,t'}(\hat{Y}), & \text{if } i \in [\![j+n+1, n+m]\!]. \end{cases} \tag{2.3}$$

If $t \in \mathrm{Y}_1$, we will denote $\iota_{i,t}(\hat{X})$ simply by $\iota_i(\hat{X})$.

We remark that the notion of **monoidal** (or **tensor**) **functor** in [14], Def. XI.4.1, is nowadays called **strong** monoidal, specially due to the pervasiveness of **lax** and

oplax monoidal functors (see [1], Section 3.1.1). We will denote the **structure morphisms** of a(n) (op)lax monoidal functor $F : \mathscr{C} \to \mathscr{D}$ between monoidal categories by $\varphi^F(X,Y) : F(X) \otimes_{\mathscr{D}} F(Y) \to F(X \otimes_{\mathscr{C}} Y)$ and $\upsilon^F : \mathbf{I}_{\mathscr{D}} \to F(\mathbf{I}_{\mathscr{C}})$ (resp., $\varphi^F(X,Y) : F(X \otimes_{\mathscr{C}} Y) \to F(X) \otimes_{\mathscr{D}} F(Y)$ and $\upsilon^F : F(\mathbf{I}_{\mathscr{C}}) \to \mathbf{I}_{\mathscr{D}}$). The morphisms $\varphi^F(X,Y)$ can be assembled into a family of natural morphisms

$$\varphi^{F,t}(\hat{X}) : t(\otimes_{\mathscr{D}})\big(F(X_1),\dots,F(X_n)\big) \longrightarrow F\big(t(\otimes_{\mathscr{C}})(\hat{X})\big),$$
$$\Big(\text{resp., } \varphi^{F,t}(\hat{X}) : F\big(t(\otimes_{\mathscr{C}})(\hat{X})\big) \longrightarrow t(\otimes_{\mathscr{D}})\big(F(X_1),\dots,F(X_n)\big)\Big)$$

for all $n \geq 2$, $t \in \mathrm{Y}_{n-1}$ and $\hat{X} = (X_1,\dots,X_n) \in \mathscr{C}^n$. Then, the definition of (op)lax monoidal functor tells us that

$$\begin{aligned} &\varphi^{F,t'}(\hat{X}) \circ a^{\mathscr{D}}_{t\to t'}\big(F(X_1),\dots,F(X_n)\big) = F\big(a^{\mathscr{C}}_{t\to t'}(\hat{X})\big) \circ \varphi^{F,t}(\hat{X}) \\ &\Big(\text{resp., } a^{\mathscr{D}}_{t\to t'}\big(F(X_1),\dots,F(X_n)\big) \circ \varphi^{F,t}(\hat{X}) = \varphi^{F,t'}(\hat{X}) \circ F\big(a^{\mathscr{C}}_{t\to t'}(\hat{X})\big)\Big), \end{aligned} \tag{2.4}$$

for all $n \geq 2$, $t, t' \in \mathrm{Y}_{n-1}$ and $\hat{X} = (X_1,\dots,X_n) \in \mathscr{C}^n$.

2.3 Some constructions from bicategory theory

We will also assume that the reader is familiar with the basic notions of bicategories (see the short but very clear introduction [15], whose notation we shall follow, or the more comprehensive [4]). For the reader's convenience, we provide some basic notions in the rather restricted setting we will be interested in. We also provide some fundamental results we will extensively use. They will allow us to effectively organize the several algebraic structures appearing from Section 3 onward, and they will also do some heavy lifting for us.

We recall first that the category $\mathbf{Cat}$ of all categories has a canonical structure of 2-category, and let $\mathscr{F}$ be a small category, which is considered as a 2-category where the morphism space $\mathrm{Mor}_{\mathscr{F}}(I,J)$ is considered as a discrete category for every pair of objects I and J of $\mathscr{F}$. In particular, $\mathscr{F}$ and $\mathbf{Cat}$ are bicategories.

We recall that a (**unit preserving**) **homomorphism of bicategories** $\mathscr{f} : \mathscr{F} \to \mathbf{Cat}$ consists of a map $I \mapsto \mathscr{f}(I)$ sending every object I of $\mathscr{F}$ to a category $\mathscr{f}(I)$, a map $f \mapsto \mathscr{f}(f)$ sending every morphism $f : I \to J$ in $\mathscr{F}$ to a functor $\mathscr{f}(f) : \mathscr{f}(I) \to \mathscr{f}(J)$, and natural isomorphisms $\phi^{\mathscr{f}}(g,f) : \mathscr{f}(g) \circ \mathscr{f}(f) \to \mathscr{f}(g \circ f)$ of functors for every pair of composable morphisms f and g in $\mathscr{F}$, satisfying that $\mathscr{f}(\mathrm{id}_I)$ is the identity functor of $\mathscr{f}(I)$ for every object I of $\mathscr{F}$ and

$$\phi^{\mathscr{f}}(h, g\circ f)(-) \circ \mathscr{f}(h)\big(\phi^{\mathscr{f}}(g,f)(-)\big) = \phi^{\mathscr{f}}(h\circ g, f)(-) \circ \phi^{\mathscr{f}}(h,g)\big(\mathscr{f}(f)(-)\big), \tag{2.5}$$

for every triple of composable morphisms f, g and h in $\mathscr{F}$. It is easy to see that any functor $\mathscr{f} : \mathscr{F} \to \mathbf{Cat}$ of the underlying categories gives a **strict** homomorphism of bicategories, *i.e.* such that $\phi^{\mathscr{f}}(g,f)$ is the identity.

Let $\mathscr{f}, \mathscr{g} : \mathscr{F} \to \mathbf{Cat}$ be two homomorphisms of bicategories. Then, there exists a natural category $\mathbf{Trans_{st}}(\mathscr{f},\mathscr{g})$, which we recall in elementary terms. The objects of the category $\mathbf{Trans_{st}}(\mathscr{f},\mathscr{g})$, called (**unit preserving**) **strong transformations**, are given by functors $\zeta(I) : \mathscr{f}(I) \to \mathscr{g}(I)$ for all objects I of $\mathscr{F}$, and natural isomorphisms of functors $\zeta(f) : \mathscr{g}(f) \circ \zeta(I) \to \zeta(J) \circ \mathscr{f}(f)$ for all morphisms $f : I \to J$ in $\mathscr{F}$ such that $\zeta(\mathrm{id}_I)$ is the identity natural transformation and

$$\zeta(g\circ f)(-)\circ\phi^{\mathscr{g}}(g,f)\big(\zeta(I)(-)\big) = \zeta(K)\big(\phi^{\mathscr{f}}(g,f)(-)\big)\circ\zeta(g)\big(\mathscr{f}(f)(-)\big)\circ\mathscr{g}(g)\big(\zeta(f)(-)\big), \tag{2.6}$$

for all morphisms $f : I \to J$ and $g : J \to K$ in $\mathscr{F}$. Given two objects ζ and ζ' of $\mathbf{Trans_{st}}(\mathscr{f},\mathscr{g})$, a morphism $\Gamma : \zeta \to \zeta'$ in $\mathbf{Trans_{st}}(\mathscr{f},\mathscr{g})$, called a **modification**, is given by natural transformations $\Gamma(I) : \zeta(I) \to \zeta'(I)$ for all objects I of $\mathscr{F}$ such that

$$\zeta'(f)(-) \circ \mathcal{g}(f)\big(\Gamma(I)(-)\big) = \Gamma(J)\big(\mathcal{f}(f)(-)\big) \circ \zeta(f)(-) \tag{2.7}$$

for all morphisms $f : I \to J$ in $\mathscr{F}$. The composition of modifications $\Gamma'' : \zeta'' \to \zeta$ and $\Gamma' : \zeta \to \zeta'$ is the modification given by the natural transformation $\Gamma(I)(-) = \Gamma'(I)(-) \circ \Gamma(I)(-)$, for every object I of $\mathscr{F}$.

We also recall that a strong transformation $\zeta : \mathcal{f} \to \mathcal{f}'$ of homomorphisms of bicategories is said to be **strict** if $\zeta(f)$ is the identity for all morphisms f in $\mathscr{F}$. Given a homomorphisms of bicategories $\mathcal{f} : \mathscr{F} \to \mathbf{Cat}$ define the **identity strong transformation** $\mathrm{id}_{\mathcal{f}} : \mathcal{f} \to \mathcal{f}$ as the unique strict natural transformation such that $\mathrm{id}_{\mathcal{f}}(I) : \mathcal{f}(I) \to \mathcal{f}(I)$ is the identity functor for all objects I of $\mathscr{F}$. Moreover, given homomorphisms of bicategories $\mathcal{f}'', \mathcal{f}, \mathcal{f}' : \mathscr{F} \to \mathbf{Cat}$ and strong transformations $\zeta'' : \mathcal{f}'' \to \mathcal{f}$ and $\zeta' : \mathcal{f} \to \mathcal{f}'$, define the **composed** strong transformation $\zeta' \circ \zeta''$ to be the strong transformation $\bar{\zeta}$ such that $\bar{\zeta}(I) = \zeta(I) \circ \zeta'(I)$ for all objects I of $\mathscr{F}$, and $\bar{\zeta}(f)(-) = \zeta(J)(\zeta'(f)(-)) \circ \zeta(f)(\zeta'(I)(-))$ for all morphisms $f : I \to J$ in $\mathscr{F}$. It is straightforward to check that $\zeta' \circ \zeta''$ indeed defines a strong transformation.

The category $\mathbf{Trans_{st}}(\mathcal{f}, \mathcal{g})$ is just the space of morphisms from $\mathcal{f}$ to $\mathcal{g}$ in the 2-category of homomorphisms from $\mathscr{F}$ to $\mathbf{Cat}$ (see the 2-category $[\mathscr{F}, \mathbf{Cat}]$ in [15], Section 2.0). We will only need a very small fragment of this well-known 2-category structure, which is contained in the next result.

Lemma 2.1. *Let $\mathcal{f}, \mathcal{f}', \mathcal{g} : \mathscr{F} \to \mathbf{Cat}$ be homomorphisms of bicategories. A strong transformation $\zeta' : \mathcal{f} \to \mathcal{f}'$ induces a functor*

$$\mathbf{Trans_{st}}(\zeta', \mathcal{g}) : \mathbf{Trans_{st}}(\mathcal{f}', \mathcal{g}) \longrightarrow \mathbf{Trans_{st}}(\mathcal{f}, \mathcal{g}) \tag{2.8}$$

satisfying that $\mathbf{Trans_{st}}(\mathrm{id}_{\mathcal{f}}, \mathcal{g})$ is the identity functor, and

$$\mathbf{Trans_{st}}(\zeta' \circ \zeta'', \mathcal{g}) = \mathbf{Trans_{st}}(\zeta'', \mathcal{g}) \circ \mathbf{Trans_{st}}(\zeta', \mathcal{g}),$$

for all strong transformations $\zeta' : \mathcal{f} \to \mathcal{f}'$ and $\zeta'' : \mathcal{f}'' \to \mathcal{f}$, and all homomorphisms of bicategories $\mathcal{f}'' : \mathscr{F} \to \mathbf{Cat}$.

Proof. Given an object ζ of $\mathbf{Trans_{st}}(\mathcal{f}', \mathcal{g})$, let $\mathbf{Trans_{st}}(\zeta', \mathcal{g})(\zeta)$ be the object $\zeta \circ \zeta'$ of $\mathbf{Trans_{st}}(\mathcal{f}, \mathcal{g})$. Moreover, given a modification $\Gamma : \zeta_1 \to \zeta_2$ for objects ζ_1 and ζ_2 of $\mathbf{Trans_{st}}(\mathcal{f}', \mathcal{g})$, let $\mathbf{Trans_{st}}(\zeta', \mathcal{g})(\Gamma)$ be the modification $\bar{\Gamma}$ given by $\bar{\Gamma}(I)(-) = \Gamma(I)(\zeta'(I)(-))$ for all objects I of $\mathscr{F}$. This gives a well-defined functor (2.8), due to the strong hypothesis on ζ'. The last part of the statement follows immediately from the definitions. □

We will also make use of the previous result where the strong transformation $\zeta' : \mathcal{g} \to \mathcal{g}'$ appears in the second argument.

We have not been to able to find the following definition in the literature, but we believe it should be well known. Given a homomorphism of bicategories $\mathcal{g} : \mathscr{F} \to \mathbf{Cat}$, we define a **$\mathcal{g}$-module** M as a collection of objects $M_I \in \mathcal{g}(I)$ indexed by all objects I in $\mathscr{F}$ and a family of morphisms $M_f : \mathcal{g}(f)(M_I) \to M_J$ in $\mathcal{g}(J)$ indexed by all morphisms $f : I \to J$ in $\mathscr{F}$ such that M_{id_I} is the identity of M_I for all I in $\mathscr{F}$, and

$$M_g \circ \mathcal{g}(g)(M_f) = M_{g \circ f} \circ \phi^{\mathcal{g}}(g, f)(M_I), \tag{2.9}$$

for all morphisms $f : I \to J$ and $g : J \to K$ in $\mathscr{F}$. A **morphism of $\mathcal{g}$-modules** from M' to M is a collection of morphisms $F_I : M'_I \to M_I$ for all objects I of $\mathscr{F}$, such that

$$F_J \circ M'_f = M_f \circ \mathcal{g}(f)(F_I), \tag{2.10}$$

for all morphisms $f : I \to J$ in $\mathscr{F}$. Then, $\mathcal{g}$-modules and their morphisms, together with the usual composition, form a category that we will denote by $\mathcal{g}$-$\mathbf{Mod}$.

We will make intensive use of the next result.

Proposition 2.2. *Let $\mathcal{f}, \mathcal{f}', \mathcal{g} : \mathscr{F} \to \mathbf{Cat}$ be homomorphisms of bicategories. Suppose further that $\mathcal{g}(I)$ is cocomplete for all objects I of $\mathscr{F}$, and $\mathcal{g}(f)$ preserves colimits for all morphisms $f : I \to J$ in $\mathscr{F}$. Taking colimits induces a functor*

$$\operatorname{colim}_{\mathcal{f},\mathcal{g}} : \mathbf{Trans_{st}}(\mathcal{f}, \mathcal{g}) \longrightarrow \mathcal{g}\text{-}\mathbf{Mod}. \tag{2.11}$$

Moreover, a strong transformation $\zeta' : \mathcal{f} \to \mathcal{f}'$ induces a natural transformation

$$Z' : \operatorname{colim}_{\mathcal{f},\mathcal{g}} \circ \mathbf{Trans_{st}}(\zeta', \mathcal{g}) \longrightarrow \operatorname{colim}_{\mathcal{f}',\mathcal{g}},$$

which is an isomorphism if $\zeta'(I)$ is an equivalence for all objects I of $\mathscr{F}$.

Proof. Given an object ζ of $\mathbf{Trans_{st}}(\mathcal{f}, \mathcal{g})$, we define the $\mathcal{g}$-module $\operatorname{colim}_{\mathcal{f},\mathcal{g}} \zeta$ such that $(\operatorname{colim}_{\mathcal{f},\mathcal{g}} \zeta)_I$ is the colimit of the functor $\zeta(I)$ for I in $\mathscr{F}$, and, for every morphism $f : I \to J$ in $\mathscr{F}$, $(\operatorname{colim}_{\mathcal{f},\mathcal{g}} \zeta)_f : \mathcal{g}(f)((\operatorname{colim}_{\mathcal{f},\mathcal{g}} \zeta)_I) \to (\operatorname{colim}_{\mathcal{f},\mathcal{g}} \zeta)_J$ is the composition of

$$\mathcal{g}(f)\big(\operatorname{colim} \zeta(I)\big) \xrightarrow{\sim} \operatorname{colim}\big(\mathcal{g}(f) \circ \zeta(I)\big) \xrightarrow{\sim} \operatorname{colim}\big(\zeta(J) \circ \mathcal{f}(f)\big) \longrightarrow \operatorname{colim} \zeta(J), \tag{2.12}$$

where the first map follows from the fact that $\mathcal{g}(f)$ preserves colimits, the second map is induced by $\zeta(f)$ and the last one follows from the general property of colimits. A straightforward computation using (2.6) and the universal property of colimits tells us that $\operatorname{colim}_{\mathcal{f},\mathcal{g}} \zeta$ is indeed a $\mathcal{g}$-module. If $\Gamma : \zeta \to \zeta''$ is a modification, for ζ and ζ'' in $\mathbf{Trans_{st}}(\mathcal{f}, \mathcal{g})$, $\Gamma(I)$ induces a morphism $\operatorname{colim} \zeta(I) \to \operatorname{colim} \zeta''(I)$ for every object I of $\mathscr{F}$, defining a morphism of $\mathcal{g}$-modules $\operatorname{colim}_{\mathcal{f},\mathcal{g}} \zeta \to \operatorname{colim}_{\mathcal{f},\mathcal{g}} \zeta''$, by (2.7) and the universal property of colimits. It is straightforward to check that this definition is indeed a functor, since taking colimits is functorial.

Finally, given an object ζ of $\mathbf{Trans_{st}}(\mathcal{f}, \mathcal{g})$ and an object I of $\mathscr{F}$, define the morphism

$$Z'(\zeta)_I : \operatorname{colim}\big(\zeta(I) \circ \zeta'(I)\big) \longrightarrow \operatorname{colim} \zeta(I) \tag{2.13}$$

by the general property of colimits. Using this and the definition of $\zeta \circ \zeta'(f)$ we see that the previous construction gives a morphism of $\mathcal{g}$-modules $Z'(\zeta) : \operatorname{colim}_{\mathcal{f},\mathcal{g}} \circ \mathbf{Trans_{st}}(\zeta', \mathcal{g})(\zeta) \to \operatorname{colim}_{\mathcal{f}',\mathcal{g}}(\zeta)$. The naturality in ζ follows from (2.7) and the universal property of colimits, so Z' is a natural transformation. Finally, it is clear that if $\zeta'(I)$ is an equivalence for all objects I of $\mathscr{F}$, then (2.13) is also an isomorphism, which implies that $Z'(\zeta)$ is an isomorphism of $\mathcal{g}$-modules. □

We will usually omit the subscripts of the functor (2.11), if they are clear from the context, so we will simply write colim.

2.4 Review of basic facts on (co)algebras and (co)modules

We also assume that the reader is familiar with the basic notions of (**co**)**algebras** (also called (**co**)**monoids**) and their (**co**)**modules** in monoidal categories, as well as **bialgebras** (or **bimonoids**) in braided monoidal categories (see [21], Ch. 1, [19, 20, 22, 23] or the nice reference [1]). All (co)algebras in this article are assumed to be (co)unitary, their morphisms respect the (co)unit, and all (co)modules over them are assumed to be compatible with the (co)unit, as well as their morphisms. If A (resp., C) is an algebra (resp., a coalgebra), we will denote its (co)product by $\mu_A : A \otimes_{\mathscr{C}} A \to A$ (resp., $\Delta_C : C \to C \otimes_{\mathscr{C}} C$) and its (co)unit by $\eta_A : \mathbf{I}_{\mathscr{C}} \to A$ (resp., $\epsilon_C : C \to \mathbf{I}_{\mathscr{C}}$). If A is an algebra in a monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\mathscr{C}})$, we denote by ${}_A\mathbf{Mod}(\mathscr{C})$ the category of left A-modules in $\mathscr{C}$ and morphisms of A-modules. Each A-module is typically written as a pair (M, ρ_M), with structure morphism $\rho_M : A \otimes_{\mathscr{C}} M \to M$, or just by M. Right A-modules are defined analogously, but we will consider left modules unless otherwise stated.

An algebra A (resp., a coalgebra C) in a braided monoidal category $\mathscr{C}$ is said to be **(co)commutative** if $\mu_A \circ \tau^{\mathscr{C}}(A,A) = \mu_A$ (resp., $\tau^{\mathscr{C}}(C,C) \circ \Delta_C = \Delta_C$). We denote by $\mathbf{Alg}(\mathscr{C})$ (resp., $\mathbf{coAlg}(\mathscr{C})$) the subcategory of $\mathscr{C}$ formed by all (co)algebras and morphisms of (co)algebras. Given an algebra A (resp., a coalgebra C) in a monoidal category $\mathscr{C}$ and $t \in \mathrm{Y}_n$, denote by $\hat{A}$ (resp., $\hat{C}$) the $(n+1)$-tuple whose each entry is A (resp., C). We will denote by $\mu_A^t : t(\otimes_{\mathscr{C}})(\hat{A}) \to A$ (resp., $\Delta_C^t : C \to t(\otimes_{\mathscr{C}})(\hat{C})$) the morphism defined recursively as $\mu_A^t = \mathrm{id}_A$ (resp., $\Delta_C^t = \mathrm{id}_C$) if $n = 0$, and, for $n \geq 1$, if t is obtained from trees $t_\ell \in \mathrm{Y}_{n'}$ and $t_r \in \mathrm{Y}_{n''}$ with $n' + n'' + 1 = n$ by grafting the root of t_ℓ (resp., t_r) to the left (resp., right) leaf of $t_0 \in \mathrm{Y}_1$, then

$$\mu_A^t = \mu_A \circ (\mu_A^{t_\ell} \otimes_{\mathscr{C}} \mu_A^{t_r}) \left(\text{resp., } \Delta_C^t = (\Delta_C^{t_\ell} \otimes_{\mathscr{C}} \Delta_C^{t_r}) \circ \Delta_C\right),$$

We also set $\mu_A^{[0]} = \eta_A$ (resp., $\Delta_C^{[0]} = \epsilon_C$). By making use of the (co)associativity of μ_A (resp., Δ_C), given $n \in \mathbb{N}$, we also write $\mu_A^{[n]}$ (resp., $\Delta_C^{[n]}$) instead of μ_A^t (resp., Δ_C^t) for $t \in \mathrm{Y}_{n-1}$.

We remark that, given two algebras A and A' in $\mathscr{C}$ an A-A'-bimodule is a pair (M, ρ), with structure morphism $\rho : A \otimes_{\mathscr{C}} M \otimes_{\mathscr{C}} A' \to M$ satisfying the usual associativity and unit conditions. It can be equivalently defined as the datum of a left A-module structure $\rho_\ell : A \otimes_{\mathscr{C}} M \to M$ and a right A'-module structure $\rho_r : M \otimes_{\mathscr{C}} A' \to M$ such that $\rho_r \circ (\rho_\ell \otimes_{\mathscr{C}} \mathrm{id}_{A'}) = \rho_\ell \circ (\mathrm{id}_A \otimes_{\mathscr{C}} \rho_r)$. It is a simple verification that, if $\mathscr{C}$ is (resp., finitely) cocomplete, *i.e.* $\mathscr{C}$ has all (resp., finite) colimits, and the tensor product $\otimes_{\mathscr{C}}$ commutes with (resp., finite) colimits on both sides, then ${}_A\mathbf{Mod}(\mathscr{C})$ is cocomplete and the inclusion functor ${}_A\mathbf{Mod}(\mathscr{C}) \to \mathscr{C}$ preserves (resp., finite) colimits. The analogous result holds for right modules and bimodules as well.

If $\mathscr{C}$ has finite colimits, given a right A-module $(M, \rho_{M,r})$ and a left A-module $(N, \rho_{N,\ell})$ one further defines $M \otimes_A N$ as the coequalizer in $\mathscr{C}$ of the pair of maps $\rho_{M,r} \otimes_{\mathscr{C}} \mathrm{id}_N, \mathrm{id}_M \otimes_{\mathscr{C}} \rho_{N,\ell} : M \otimes_{\mathscr{C}} A \otimes_{\mathscr{C}} N \to M \otimes_{\mathscr{C}} N$. Assume further that the tensor product of $\mathscr{C}$ commutes with finite colimits on each side and suppose that M has an A'-A-bimodule structure for a left A'-action given by $\rho_{M,\ell}$. Then, $M \otimes_A N$ has a natural structure of left A'-module induced by $\rho_{M,\ell} \otimes_{\mathscr{C}} \mathrm{id}_N$. Analogously, given a morphism $g : N \to N'$ of left A-modules, $\mathrm{id}_M \otimes_{\mathscr{C}} g$ naturally induces a morphism of left A'-modules $\mathrm{id}_M \otimes_A g : M \otimes_A N \to M \otimes_A N'$. The previous construction gives a functor

$$M \otimes_A (-) : {}_A\mathbf{Mod}(\mathscr{C}) \longrightarrow {}_{A'}\mathbf{Mod}(\mathscr{C}).$$

Note that, if $M = A$ has the A-A-bimodule structure given by $\mu_A^{[3]}$ and N is a left A-module, then $\rho_{N,\ell} : A \otimes_{\mathscr{C}} N \to N$ is precisely a coequalizer of the pair $\rho_{A,r} \otimes_{\mathscr{C}} \mathrm{id}_N$ and $\mathrm{id}_A \otimes_{\mathscr{C}} \rho_{N,\ell}$. This allows us to set $A \otimes_A N = N$, where the left A-module structures also coincide. Analogously, we can (and will) set the identity $N' \otimes_A A = N'$ of right A-modules, for every right A-module N'.

If $f : A \to A'$ is a morphism of algebras in $\mathscr{C}$, one defines a functor

$$\mathrm{Res}_f : {}_{A'}\mathbf{Mod}(\mathscr{C}) \longrightarrow {}_A\mathbf{Mod}(\mathscr{C}) \tag{2.14}$$

that sends an A'-module (M, ρ) to the A-module $(M, \rho \circ (f \otimes_{\mathscr{C}} \mathrm{id}_M))$ and is the identity on morphisms. If $\mathscr{C}$ has finite colimits and the tensor product commutes with them on both sides, then a standard computation shows that Res_f has the left adjoint

$$\mathrm{Ind}_f = A' \otimes_A (-) : {}_A\mathbf{Mod}(\mathscr{C}) \longrightarrow {}_{A'}\mathbf{Mod}(\mathscr{C}), \tag{2.15}$$

where A' has the A'-A-bimodule structure morphism $\mu_{A'}^{[3]} \circ (\mathrm{id}_{A'} \otimes_{\mathscr{C}} \mathrm{id}_{A'} \otimes_{\mathscr{C}} f)$.

For the following result, consider $\mathbf{Alg}(\mathscr{C})$ as a 2-category where every morphism space is a discrete category, and $\mathbf{Cat}$ has the usual structure of 2-category. The proof is straightforward.

Fact 2.3. *Assume that $\mathscr{C}$ is a monoidal category. Consider the map*

$$\mathrm{R} : \mathbf{Alg}(\mathscr{C})^{\mathrm{op}} \longrightarrow \mathbf{Cat} \tag{2.16}$$

sending an algebra A to ${}_A\mathbf{Mod}(\mathscr{C})$ and a morphism $f : A' \to A$ of algebras to the functor Res_f given in (2.14). Then, (2.16) is a functor, so a strict homomorphism of bicategories.

Assume moreover that $\mathscr{C}$ is finitely cocomplete and the tensor product commutes with colimits on both sides. Consider the map

$$\mathrm{I} : \mathbf{Alg}(\mathscr{C}) \longrightarrow \mathbf{Cat} \tag{2.17}$$

sending an algebra A to ${}_A\mathbf{Mod}(\mathscr{C})$, a morphism $f : A' \to A$ of algebras to Ind_f, and, given a pair of morphisms $f : A' \to A$ and $g : A \to A''$ of algebras, consider the unique natural isomorphism $\phi^{\mathrm{I}}(g, f) : \mathrm{Ind}_g \circ \mathrm{Ind}_f \to \mathrm{Ind}_{g\circ f}$ of functors coming from the fact that both are left adjoints of $\mathrm{Res}_f \circ \mathrm{Res}_g = \mathrm{Res}_{g\circ f}$. Then, (2.17) is a homomorphism of bicategories.

Assume that A is a commutative algebra in a symmetric monoidal category $\mathscr{C}$. As usual, in this case one identifies left A-module structures $\rho_\ell : A \otimes_{\mathscr{C}} M \to M$ and right A-modules structures $\rho_r : M \otimes_{\mathscr{C}} A \to M$ on an object M of $\mathscr{C}$ by means of $\rho_r = \rho_\ell \circ \tau^{\mathscr{C}}(M, A)$, and we write either ρ_ℓ or ρ_r simply by ρ. If $\mathscr{C}$ has finite colimits and the tensor product commutes with them on both sides, we recall that the category ${}_A\mathbf{Mod}(\mathscr{C})$ endowed with the tensor product $\otimes_A$, the associativity constraint induced by that of $\mathscr{C}$, the unit given by A with the regular structure given by the product of A and the symmetric braiding induced by that of $\mathscr{C}$ is also a symmetric monoidal category with a strict unit, denoted by ${}_A\mathbf{Mods}(\mathscr{C})$. [1]

Lemma 2.4. *Assume $\mathscr{C}$ is a finitely cocomplete symmetric monoidal category and the tensor product commutes with finite colimits on both sides. Let $f : A \to A'$ be an morphism of commutative algebras in $\mathscr{C}$. If M and N are A-modules, there is a canonical natural isomorphism*

$$\mathrm{Ind}_f(M \otimes_A N) \longrightarrow \mathrm{Ind}_f(M) \otimes_{A'} \mathrm{Ind}_f(N) \tag{2.18}$$

of A'-modules. Moreover, if f is an epimorphism of commutative algebras in $\mathscr{C}$, given A'-modules M' and N', the canonical morphism

$$\mathrm{Res}_f(M') \otimes_A \mathrm{Res}_f(N') \longrightarrow \mathrm{Res}_f(M' \otimes_{A'} N') \tag{2.19}$$

of A-modules is an isomorphism.

Proof. The first isomorphism is an immediate consequence of the associativity constraint and the symmetric braiding of ${}_A\mathbf{Mods}(\mathscr{C})$. Let us prove the second. Since (2.19) factors through

$$\begin{aligned}\mathrm{Res}_f(M') \otimes_A \mathrm{Res}_f(N') &\simeq \mathrm{Res}_f(M' \otimes_{A'} A') \otimes_A \mathrm{Res}_f(A' \otimes_{A'} N') \\ &\simeq \mathrm{Res}_f\left(M' \otimes_{A'} (A' \otimes_A A') \otimes_{A'} N'\right),\end{aligned}$$

it suffices to show that the product $\mu_{A'}$ of A' induces an isomorphism $A' \otimes_A A' \to A'$. The last identity follows from the fact that a morphism $f : A \to A'$ of commutative algebras is an epimorphism if and only if

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_{A'}} \\ A' & \xrightarrow{\mathrm{id}_{A'}} & A' \end{array}$$

[1]Note that we defined $A \otimes_A M = M = M \otimes_A A$ three paragraphs before, so the unit of ${}_A\mathbf{Mods}(\mathscr{C})$ is strict regardless of the unit of $\mathscr{C}$ being strict or not.

is a push-out, and that a push-out of a pair of morphisms $f : A \to A'$ and $g : A \to A''$ of commutative algebras in $\mathscr{C}$ is the commutative algebra $\bar{A} = \mathrm{Res}_f(A') \otimes_A \mathrm{Res}_g(A'')$, together with the maps $A' \to \bar{A}$ and $A'' \to \bar{A}$ given by $\mathrm{id}_{A'} \otimes_A g$ and $f \otimes_A \mathrm{id}_{A''}$, respectively, and the usual product given as the composition of the canonical projection

$$(A' \otimes_A A'') \otimes_{\mathscr{C}} (A' \otimes_A A'') \longrightarrow (A' \otimes_A A'') \otimes_A (A' \otimes_A A''),$$

the associativity constraint in $\mathscr{A} = {}_A\mathbf{Mods}(\mathscr{C})$ with $\mathrm{id}_{A'} \otimes_A \tau^{\mathscr{A}}(A'', A') \otimes_A \mathrm{id}_{A''}$ and $\bar{\mu}_{A'} \otimes_A \bar{\mu}_{A''}$, where $\bar{\mu}_X : X \otimes_A X \to X$ is the morphism induced by the product $\mu_X : X \otimes_{\mathscr{C}} X \to X$ of $X \in \{A', A''\}$. □

We recall that, if $\mathscr{C}$ is a monoidal category with a braiding $\tau^{\mathscr{C}}$ and B is a bialgebra in $\mathscr{C}$, then the tensor product $M \otimes_{\mathscr{C}} N$ of two B-modules (M, ρ_M) and (N, ρ_N) has a natural structure of B-module, called **diagonal**, given by

$$\big(\rho_M \otimes_{\mathscr{C}} \rho_N\big) \circ \big(\mathrm{id}_B \otimes_{\mathscr{C}} \tau^{\mathscr{C}}(B, M) \otimes_{\mathscr{C}} \mathrm{id}_N\big) \circ \big(\Delta_B \otimes_{\mathscr{C}} \mathrm{id}_{M \otimes_{\mathscr{C}} N}\big), \tag{2.20}$$

where Δ_B denotes the coproduct of B, and $\mathbf{I}_{\mathscr{C}}$ has the structure of B-module given by the composition of $r^{\mathscr{C}}(B)$ and the counit ϵ_B of B. With this structure ${}_B\mathbf{Mod}(\mathscr{C})$ is a monoidal category and the inclusion inside of $\mathscr{C}$ is strong monoidal (see [19], Lemma 1.1). We denote by ${}_B\mathbf{Modt}(\mathscr{C})$ the category ${}_B\mathbf{Mod}(\mathscr{C})$ endowed with the previous monoidal structure. Recall that a **B-module algebra** is an algebra in the monoidal category ${}_B\mathbf{Modt}(\mathscr{C})$.

On the other hand, recall that, if $F : \mathscr{C} \to \mathscr{C}'$ is a(n) (op)lax monoidal functor, then F sends $\mathbf{Alg}(\mathscr{C})$ (resp., $\mathbf{coAlg}(\mathscr{C})$) to $\mathbf{Alg}(\mathscr{C}')$ (resp., $\mathbf{coAlg}(\mathscr{C}')$), and given an algebra A it sends the subcategory ${}_A\mathbf{Mod}(\mathscr{C})$ of $\mathscr{C}$ to ${}_{F(A)}\mathbf{Mod}(\mathscr{C}')$. If $\mathscr{C}$ and $\mathscr{C}'$ are braided, B is a bialgebra in $\mathscr{C}$ and F is braided strong monoidal, then it restricts to a braided strong monoidal functor ${}_B\mathbf{Modt}(\mathscr{C}) \to {}_{F(B)}\mathbf{Modt}(\mathscr{C}')$.

If the braiding $\tau^{\mathscr{C}}$ of $\mathscr{C}$ is further assumed to be symmetric and B is cocommutative, then $\tau^{\mathscr{C}}(M, N) : M \otimes_{\mathscr{C}} N \to N \otimes_{\mathscr{C}} M$ is a morphism of B-modules for all B-modules M and N, which implies that ${}_B\mathbf{Modt}(\mathscr{C})$ is a symmetric monoidal category and the inclusion inside of $\mathscr{C}$ is a braided strong monoidal functor (see [21], Section 1.8 for the case of vector spaces, but the general case is analogous). We will always assume that ${}_B\mathbf{Modt}(\mathscr{C})$ is endowed with the previous symmetric braiding in this situation. We remark that, if $f : B \to B'$ is a morphism of (resp., cocommutative) bialgebras in $\mathscr{C}$, then (2.14) is a (resp., braided) lax monoidal functor.

Remark 2.5. *Given a cocommutative bialgebra B in a braided monoidal category $\mathscr{C}$ and B-modules M and N in $\mathscr{C}$, the braiding $\tau^{\mathscr{C}}(M, N)$* ***is not*** *in general a morphism of B-modules (take for instance the category $\mathscr{C}$ of modules over the Fomin-Kirillov algebra on 3 generators). In the same spirit, given two algebras A and A' in a braided monoidal category $\mathscr{C}$, even though their tensor product $A \otimes_{\mathscr{C}} A'$ is naturally an algebra (see the proof of Lemma 2.4), the braidings $\tau^{\mathscr{C}}(A, A')$ and $\tau^{\mathscr{C}}(A', A)$* ***are not*** *in general morphisms of algebras. Moreover, the tensor product of bialgebras (resp., commutative algebras) in a braided monoidal category* ***is not*** *in general a bialgebra (resp., commutative algebra).*

2.5 Duoidal (or 2-monoidal) categories

A 2-monoidal category (or duoidal category) is a category $\mathscr{C}$ endowed with two monoidal structures $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes})$ and $(\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes})$, satisfying some assumptions. We denote the associativity, and the left and right unit constraints of $(\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes})$ by $a^{\boxtimes}(X, Y, Z)$, $\ell^{\boxtimes}(X)$ and $r^{\boxtimes}(X)$, respectively, and those of the monoidal structure $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes})$ by $a^{\otimes}(X, Y, Z)$, $\ell^{\otimes}(X)$ and $r^{\otimes}(X)$, respectively.

Before providing the definition, it will be useful to establish some terminology to be able to deal later with the parenthesizations. First, for $n \in \mathbb{N}_0$, we define

$$\mathrm{Y}_n^{\mathrm{lab}} = \big\{ \mathfrak{t} = (t, \lambda) : t \in \mathrm{Y}_n, \lambda : \mathrm{Ver}(t) \to \{\bullet, \blacksquare\} \big\}.$$

Given $\mathfrak{t} = (t, \lambda) \in \mathrm{Y}_n^{\mathrm{lab}}$, denote by $\mathfrak{t}(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}}) : \mathscr{C}^{n+1} \to \mathscr{C}$ the functor defined recursively as follows. If $\mathfrak{t} \in \mathrm{Y}_0^{\mathrm{lab}}$, $\mathfrak{t}(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}})$ is the identity functor of $\mathscr{C}$. Let $n \geq 1$ and $\mathfrak{t} = (t, \lambda) \in \mathrm{Y}_n^{\mathrm{lab}}$ be constructed from $\mathfrak{t}_\ell = (t_\ell, \lambda_\ell) \in \mathrm{Y}_{n'}$ and $\mathfrak{t}_r = (t_r, \lambda_r) \in \mathrm{Y}_{n''}$ with $n' + n'' + 1 = n$ as follows. First, t is obtained by grafting the root of t_ℓ (resp., t_r) to the left (resp., right) leaf of $t_0 \in \mathrm{Y}_1$, whose unique vertex is denoted by v. Moreover, we assume that λ coincides with λ_ℓ (resp., λ_r) on the vertices of t_ℓ (resp., t_r). Then, we set

$$\mathfrak{t}(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}}) = \begin{cases} \otimes_{\mathscr{C}} \circ \big(\mathfrak{t}_\ell(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}}), \mathfrak{t}_r(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}})\big), & \text{if } \lambda(v) = \bullet, \\ \boxtimes_{\mathscr{C}} \circ \big(\mathfrak{t}_\ell(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}}), \mathfrak{t}_r(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}})\big), & \text{if } \lambda(v) = \blacksquare. \end{cases}$$

We recall now the not so well-known definition of 2-monoidal category (see [1], Def. 6.1.1).

Definition 2.6. *A* ***2-monoidal category*** *(also called* ***duoidal category*** *in [9, 25]) is a tuple $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_\otimes, \boxtimes_{\mathscr{C}}, \mathbf{I}_\boxtimes)$, where $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_\otimes)$ and $(\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_\boxtimes)$ are monoidal categories, together with a natural morphism*

$$\mathrm{sh}(A,B,C,D) : (A \otimes_{\mathscr{C}} B) \boxtimes_{\mathscr{C}} (C \otimes_{\mathscr{C}} D) \longrightarrow (A \boxtimes_{\mathscr{C}} C) \otimes_{\mathscr{C}} (B \boxtimes_{\mathscr{C}} D) \tag{2.21}$$

in $\mathscr{C}$ and three morphisms

$$\mu_\boxtimes : \mathbf{I}_\otimes \boxtimes_{\mathscr{C}} \mathbf{I}_\otimes \longrightarrow \mathbf{I}_\otimes, \quad \Delta_\otimes : \mathbf{I}_\boxtimes \longrightarrow \mathbf{I}_\boxtimes \otimes_{\mathscr{C}} \mathbf{I}_\boxtimes, \quad \textit{and} \quad \nu : \mathbf{I}_\boxtimes \longrightarrow \mathbf{I}_\otimes, \tag{2.22}$$

in $\mathscr{C}$ such that the following conditions hold.

(i) The triple $(\mathbf{I}_\otimes, \mu_\boxtimes, \nu)$ is an algebra in the monoidal category $(\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_\boxtimes)$, and the triple $(\mathbf{I}_\boxtimes, \Delta_\otimes, \nu)$ is a coalgebra in the monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_\otimes)$.

(ii) For any pair of objects X and Y in $\mathscr{C}$, the diagrams

$$\begin{array}{ccc} \mathbf{I}_\boxtimes \boxtimes_{\mathscr{C}} (X \otimes_{\mathscr{C}} Y) & \xrightarrow{\Delta_\otimes \boxtimes_{\mathscr{C}} \mathrm{id}_{X \otimes_{\mathscr{C}} Y}} & (\mathbf{I}_\boxtimes \otimes_{\mathscr{C}} \mathbf{I}_\boxtimes) \boxtimes_{\mathscr{C}} (X \otimes_{\mathscr{C}} Y) \\ \wr \big\downarrow \ell^\boxtimes(X \otimes_{\mathscr{C}} Y) & & \big\downarrow \mathrm{sh}(\mathbf{I}_\boxtimes, \mathbf{I}_\boxtimes, X, Y) \\ X \otimes_{\mathscr{C}} Y & \xrightarrow[(\ell^\boxtimes(X) \otimes_{\mathscr{C}} \ell^\boxtimes(Y))^{-1}]{\sim} & (\mathbf{I}_\boxtimes \boxtimes_{\mathscr{C}} X) \otimes_{\mathscr{C}} (\mathbf{I}_\boxtimes \boxtimes_{\mathscr{C}} Y) \end{array}$$

$$\begin{array}{ccc} (X \otimes_{\mathscr{C}} Y) \boxtimes_{\mathscr{C}} \mathbf{I}_\boxtimes & \xrightarrow{\mathrm{id}_{X \otimes_{\mathscr{C}} Y} \boxtimes_{\mathscr{C}} \Delta_\otimes} & (X \otimes_{\mathscr{C}} Y) \boxtimes_{\mathscr{C}} (\mathbf{I}_\boxtimes \otimes_{\mathscr{C}} \mathbf{I}_\boxtimes) \\ \wr \big\downarrow r^\boxtimes(X \otimes_{\mathscr{C}} Y) & & \big\downarrow \mathrm{sh}(X, Y, \mathbf{I}_\boxtimes, \mathbf{I}_\boxtimes) \\ X \otimes_{\mathscr{C}} Y & \xrightarrow[(r^\boxtimes(X) \otimes_{\mathscr{C}} r^\boxtimes(Y))^{-1}]{\sim} & (X \boxtimes_{\mathscr{C}} \mathbf{I}_\boxtimes) \otimes_{\mathscr{C}} (Y \boxtimes_{\mathscr{C}} \mathbf{I}_\boxtimes) \end{array}$$

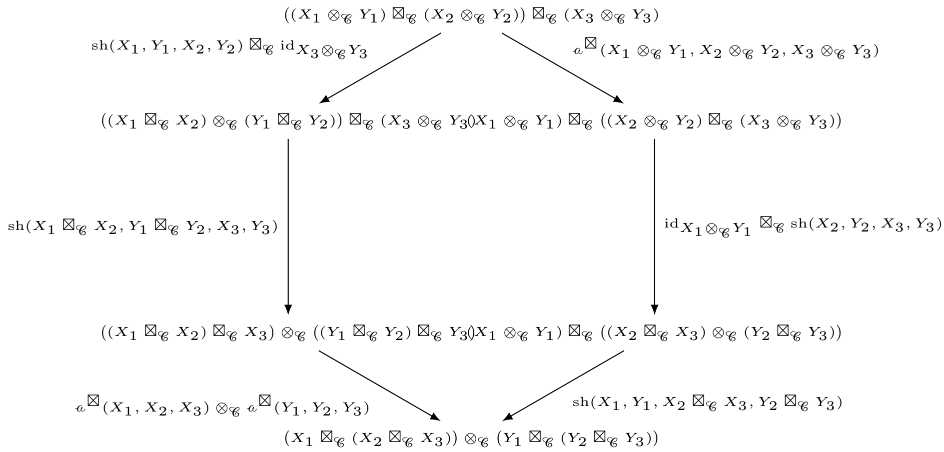
and

$$\begin{array}{ccc} (\mathbf{I}_\otimes \otimes_{\mathscr{C}} X) \boxtimes_{\mathscr{C}} (\mathbf{I}_\otimes \otimes_{\mathscr{C}} Y) & \xrightarrow[\sim]{\ell^\otimes(X) \boxtimes_{\mathscr{C}} \ell^\otimes(Y)} & X \boxtimes_{\mathscr{C}} Y \\ \big\downarrow \mathrm{sh}(\mathbf{I}_\otimes, X, \mathbf{I}_\otimes, Y) & & \wr \big\downarrow \ell^\otimes(X \boxtimes_{\mathscr{C}} Y)^{-1} \\ (\mathbf{I}_\otimes \boxtimes_{\mathscr{C}} \mathbf{I}_\otimes) \otimes_{\mathscr{C}} (X \boxtimes_{\mathscr{C}} Y) & \xrightarrow{\mu_\boxtimes \otimes_{\mathscr{C}} \mathrm{id}_{X \boxtimes_{\mathscr{C}} Y}} & \mathbf{I}_\otimes \otimes_{\mathscr{C}} (X \boxtimes_{\mathscr{C}} Y) \end{array}$$

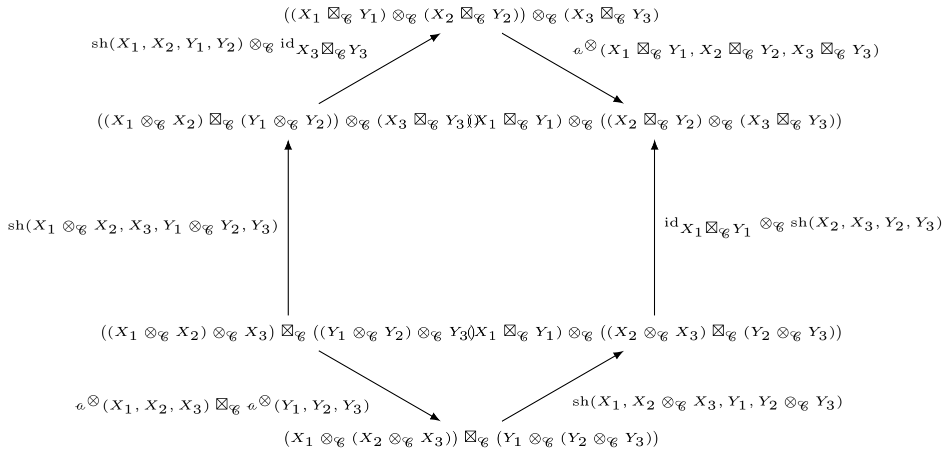
$$\begin{array}{ccc} (X \otimes_{\mathscr{C}} \mathbf{I}_\otimes) \boxtimes_{\mathscr{C}} (Y \otimes_{\mathscr{C}} \mathbf{I}_\otimes) & \xrightarrow[\sim]{r^\otimes(X) \boxtimes_{\mathscr{C}} r^\otimes(Y)} & X \boxtimes_{\mathscr{C}} Y \\ \big\downarrow \mathrm{sh}(X, \mathbf{I}_\otimes, Y, \mathbf{I}_\otimes) & & \wr \big\downarrow r^\otimes(X \boxtimes_{\mathscr{C}} Y)^{-1} \\ (X \boxtimes_{\mathscr{C}} Y) \otimes_{\mathscr{C}} (\mathbf{I}_\otimes \boxtimes_{\mathscr{C}} \mathbf{I}_\otimes) & \xrightarrow{\mathrm{id}_{X \boxtimes_{\mathscr{C}} Y} \otimes_{\mathscr{C}} \mu_\boxtimes} & (X \boxtimes_{\mathscr{C}} Y) \otimes_{\mathscr{C}} \mathbf{I}_\otimes \end{array}$$

commute.

(iii) Given $X_1, Y_1, X_2, Y_2, X_3, Y_3$ objects in $\mathscr{C}$, the following diagrams



and



commute.

*Moreover, a 2-monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes}, \mathrm{sh})$ is said to be **$\otimes_{\mathscr{C}}$-braided** if the monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes})$ is provided with a braiding $\tau^{\otimes}$ such that $(\mathbf{I}_{\boxtimes}, \Delta_{\otimes}, \nu)$ is a cocommutative coalgebra in $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes}, \tau^{\otimes})$, and for any objects A, B, C and D in $\mathscr{C}$ the diagram*

$$\begin{array}{ccc}
(A \otimes_{\mathscr{C}} B) \boxtimes_{\mathscr{C}} (C \otimes_{\mathscr{C}} D) & \xrightarrow{\mathrm{sh}(A,B,C,D)} & (A \boxtimes_{\mathscr{C}} C) \otimes_{\mathscr{C}} (B \boxtimes_{\mathscr{C}} D) \\
{\scriptstyle \tau^{\otimes}(A,B) \boxtimes_{\mathscr{C}} \tau^{\otimes}(C,D)} \downarrow & & \downarrow {\scriptstyle \tau^{\otimes}(A \boxtimes_{\mathscr{C}} C, B \boxtimes_{\mathscr{C}} D)} \\
(B \otimes_{\mathscr{C}} A) \boxtimes_{\mathscr{C}} (D \otimes_{\mathscr{C}} C) & \xrightarrow{\mathrm{sh}(B,A,D,C)} & (B \boxtimes_{\mathscr{C}} D) \otimes_{\mathscr{C}} (A \boxtimes_{\mathscr{C}} C)
\end{array}$$

*commutes. The analogous definition of **$\boxtimes_{\mathscr{C}}$-braided** 2-monoidal category is clear. A 2-monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes}, \mathrm{sh})$ is called **braided** if it is $\otimes_{\mathscr{C}}$-braided and $\boxtimes_{\mathscr{C}}$-braided. The analogous definitions for symmetric braidings are immediate.*

Example 2.7. *Any braided (resp., symmetric) monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\mathscr{C}}, \tau^{\mathscr{C}})$ can be regarded as a braided (resp., symmetric) 2-monoidal category where both braided (resp., symmetric) monoidal structures coincide, the unit $\mathbf{I}_{\mathscr{C}}$ has the obvious algebra and coalgebra structures, and $\mathrm{sh}(A, B, C, D)$ is given by $\mathrm{id}_A \otimes_{\mathscr{C}} \tau^{\mathscr{C}}(B, C) \otimes_{\mathscr{C}} \mathrm{id}_D$ (see [1], Prop. 6.10).*

Remark 2.8. *Given two structures of monoidal categories* $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes})$ *and* $(\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes})$ *on* $\mathscr{C}$ *with structure morphisms* (2.21) *and* (2.22)*, they define a structure of* 2*-monoidal category if either of the following equivalent conditions holds:*

(a) the functors $\otimes_{\mathscr{C}} : (\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes}) \times (\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes}) \to (\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes})$ *and* $\mathbf{I}_{\otimes} : \mathscr{e} \to (\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes})$ *are lax monoidal,*

(b) the functors $\boxtimes_{\mathscr{C}} : (\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes}) \times (\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes}) \to (\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes})$ *and* $\mathbf{I}_{\boxtimes} : \mathscr{e} \to (\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes})$ *are oplax monoidal,*

where $\mathscr{e}$ *indicates the monoidal category with one object whose space of endomorphisms is the identity. The proof just follows from writing down the definitions (see* [1]*, Prop. 6.4). We may equivalently rephrase (b) (resp., (a)) as saying that a* 2*-monoidal category is a pseudomonoid in the monoidal* 2*-category* $\mathscr{opl}(\mathrm{Cat})$ *(resp.,* $\mathscr{l}(\mathrm{Cat})$*) whose* 0*-cells are monoidal categories, whose* 1*-cells are oplax (resp., lax) monoidal functors, and whose* 2*-cells are monoidal natural transformations (see* [1]*, Prop. 6.73).*

We recall the following result, the proof of which is just a direct consequence of the definitions.

Proposition 2.9. *Let* $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes}, \mathrm{sh})$ *be a* 2*-monoidal category, and let* A *and* A' *(resp.,* C *and* C'*) be two (co)algebras in* $(\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes})$ *(resp.,* $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes})$*). Then,* $A \otimes_{\mathscr{C}} A'$ *(resp.,* $C \boxtimes_{\mathscr{C}} C'$*) is a (co)algebra in* $(\mathscr{C}, \boxtimes_{\mathscr{C}}, \mathbf{I}_{\boxtimes})$ *(resp.,* $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\otimes})$*) for the (co)product*

$$(\mu_A \otimes_{\mathscr{C}} \mu_{A'}) \circ \mathrm{sh}(A, A', A, A') \left(\textit{resp.}, \mathrm{sh}(C, C, C', C') \circ (\Delta_C \boxtimes_{\mathscr{C}} \Delta_{C'})\right),$$

and the (co)unit $(\eta_A \otimes_{\mathscr{C}} \eta_{A'}) \circ \Delta_{\otimes}$ *(resp.,* $\mu_{\boxtimes} \circ (\epsilon_C \boxtimes_{\mathscr{C}} \epsilon_{C'})$*).*

If $\mathscr{C}$ *is also* $\boxtimes_{\mathscr{C}}$*-symmetric (resp.,* $\otimes_{\mathscr{C}}$*-symmetric) and* A *and* A' *(resp.,* C *and* C'*) are (co)commutative, then* $A \otimes_{\mathscr{C}} A'$ *(resp.,* $C \boxtimes_{\mathscr{C}} C'$*) is (co)commutative.*

3 Some preparations: categories of functors

3.1 Induced monoidal structures on categories of functors

For the rest of the article we assume that $\mathscr{F}$ is a small category. Given a category $\mathscr{C}$, denote by $\mathbf{Fun}(\mathscr{F}, \mathscr{C})$ the category whose objects are functors from $\mathscr{F}$ to $\mathscr{C}$ and whose morphisms are natural transformations. We recall that $\mathbf{Fun}(\mathscr{F}, \mathscr{C})$ is (co)complete if $\mathscr{C}$ is so, and (co)limits are computed pointwise.

Moreover, if $\mathscr{C}$ is endowed with a (resp., braided, symmetric) monoidal structure with tensor product $\otimes_{\mathscr{C}}$ and unit $\mathbf{I}_{\mathscr{C}}$, then $\mathbf{Fun}(\mathscr{F}, \mathscr{C})$ is also a (resp., braided, symmetric) monoidal category with tensor product $(F \otimes G)(I) = F(I) \otimes_{\mathscr{C}} G(I)$ and $(F \otimes G)(f) = F(f) \otimes_{\mathscr{C}} G(f)$ for any object I and any morphism f in $\mathscr{F}$, unit given by the constant functor of value $\mathbf{I}_{\mathscr{C}}$ (so in particular it sends any morphism to the identity of $\mathbf{I}_{\mathscr{C}}$), and the associativity, left unit and right unit constraints (resp., as well as braiding,) induced by that of $\mathscr{C}$. We will call this (resp., braided) monoidal structure on $\mathbf{Fun}(\mathscr{F}, \mathscr{C})$ **induced** and we denote it by $\mathbf{Funi}(\mathscr{F}, \mathscr{C})$.

3.2 Algebras and coalgebras in categories of functors

For the rest of the article we assume that $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\mathscr{C}})$ is a monoidal category that is cocomplete and such that the tensor product commutes with colimits on each side. The next result follows directly from the definitions.

Fact 3.1. *A functor* $F \in \mathbf{Fun}(\mathscr{F}, \mathscr{C})$ *is a (co)algebra for the induced monoidal structure if and only if* F *factors through the canonical inclusion of* $\mathbf{Alg}(\mathscr{C})$ *(resp.,* $\mathbf{coAlg}(\mathscr{C})$*) inside of* $\mathscr{C}$*. If* $\mathscr{C}$ *is also braided, the analogous result holds for (co)commutative (co)algebras and bialgebras.*

To reduce notation we will denote the subcategory of $\mathbf{Funi}(\mathscr{F},\mathscr{C})$ formed by all its (co)algebras simply by $\mathbf{Alg}(\mathscr{F},\mathscr{C})$ (resp., $\mathbf{coAlg}(\mathscr{F},\mathscr{C})$) .

Assume that $(\mathscr{F},\square,\mathbf{I}_{\mathscr{F}})$ is a monoidal category. We say that an object A in $\mathbf{Alg}(\mathscr{F},\mathscr{C})$ (resp., $\mathbf{coAlg}(\mathscr{F},\mathscr{C})$) is **oplax** if the underlying functor $A:\mathscr{F}\to\mathscr{C}$ is endowed with an oplax monoidal structure such that the structure morphisms $A(I\square J)\to A(I)\otimes_{\mathscr{C}}A(J)$ and $A(\mathbf{I}_{\mathscr{F}})\to\mathbf{I}_{\mathscr{C}}$ are morphisms of (co)algebras in $\mathscr{C}$ for all objects I, J in $\mathscr{C}$, where $A(I)\otimes_{\mathscr{C}}A(J)$ has the usual structure of (co)algebra of the tensor product. The definition of **lax** algebras and coalgebras is analogous. Assume further that $\mathscr{C}$ is endowed with a braiding. A bialgebra B in $\mathbf{Funi}(\mathscr{F},\mathscr{C})$ is said to be **oplax** (resp., **lax**) if the underlying functor $B:\mathscr{F}\to\mathscr{C}$ is endowed with an oplax (resp., a lax) monoidal structure such that the underlying algebra and coalgebra structures of B are oplax (resp., lax). On the other hand, we further say that (co)algebra (or bialgebra) A in $\mathbf{Funi}(\mathscr{F},\mathscr{C})$ **braided** if the underlying functor $A:\mathscr{F}\to\mathscr{C}$ is braided monoidal.

Lemma 3.2. *Assume that $\mathscr{C}$ is endowed with a symmetric braiding $\tau^{\mathscr{C}}$ and that $\mathscr{F}$ is a semicocartesian symmetric monoidal category with tensor product $\square$, unit $\mathbf{I}_{\mathscr{F}}$ and symmetric braiding $\tau^{\mathscr{F}}$. Let A (resp., C) be a (co)commutative (co)algebra in $\mathbf{Funi}(\mathscr{F},\mathscr{C})$ (resp., $\mathbf{Funi}(\mathscr{F}^{\mathrm{op}},\mathscr{C})$). Let $\upsilon^A:\mathbf{I}_{\mathscr{C}}\to A(\mathbf{I}_{\mathscr{F}})$ (resp., $\upsilon^C:C(\mathbf{I}_{\mathscr{F}})\to\mathbf{I}_{\mathscr{C}}$) be the (co)unit of $A(\mathbf{I}_{\mathscr{F}})$ (resp., $C(\mathbf{I}_{\mathscr{F}})$) and given objects I and J in $\mathscr{F}$, let $\varphi^A(I,J):A(I)\otimes_{\mathscr{C}}A(J)\to A(I\square J)$ (resp., $\varphi^C(I,J):C(I\square J)\to C(I)\otimes_{\mathscr{C}}C(J)$) be the composition of (resp., the comultiplication of $C(I\square J)$ and)*

$$A\big(\iota_1(I,J)\big)\otimes_{\mathscr{C}}A\big(\iota_2(I,J)\big):A(I)\otimes_{\mathscr{C}}A(J)\longrightarrow A(I\square J)\otimes_{\mathscr{C}}A(I\square J)$$
$$\Big(\textit{resp., } C\big(\iota_1(I,J)\big)\otimes_{\mathscr{C}}C\big(\iota_2(I,J)\big):C(I\square J)\otimes_{\mathscr{C}}C(I\square J)\longrightarrow C(I)\otimes_{\mathscr{C}}C(J)\Big)$$

and the multiplication of $A(I\square J)$, where $\iota_1(I,J):I\to I\square J$ and $\iota_2(I,J):J\to I\square J$ are the morphisms explained in the penultimate paragraph of Subsection 2.2. Using the previous morphisms, A (resp., C) becomes a braided (op)lax (co)algebra.

Conversely, if A (resp., C) is a braided (op)lax (co)algebra, then the structure morphisms are given as before.

Proof. We will prove the statement for algebras since that of coalgebras follows from duality. We first remark that A being an algebra in $\mathbf{Funi}(\mathscr{F},\mathscr{C})$ together with Fact 3.1 immediately implies that $\varphi^A(I,J)$ is natural in I and J. We also note that υ^A is trivially a morphism of algebras in $\mathscr{C}$. Furthermore, since $A(I)$ is a commutative algebra for all objects I in $\mathscr{F}$, the product map $A(I)\otimes_{\mathscr{C}}A(I)\to A(I)$ is a morphism of algebras in $\mathscr{C}$, which implies that $\varphi^A(I,J)$ is a morphism of algebras for all objects I and J in $\mathscr{F}$. It remains to prove that the previous structure morphisms make A a braided lax monoidal functor.

We now prove the compatibility with the units of $\mathscr{F}$ and $\mathscr{C}$ (see [14], XI.4, (4.2) and (4.3)). Indeed, note first that

$$\begin{aligned}
&A\big(\ell^{\mathscr{F}}(I)\big)\circ\varphi^A(\mathbf{I}_{\mathscr{F}},I)\circ(\upsilon^A\otimes_{\mathscr{C}}\mathrm{id}_{A(I)})\\
&=A\big(\ell^{\mathscr{F}}(I)\big)\circ\mu_{A(\mathbf{I}_{\mathscr{F}}\square I)}\circ\Big(A\big(\iota_1(\mathbf{I}_{\mathscr{F}},I)\big)\otimes_{\mathscr{C}}A\big(\iota_2(\mathbf{I}_{\mathscr{F}},I)\big)\Big)\circ(\upsilon^A\otimes_{\mathscr{C}}\mathrm{id}_{A(I)})\\
&=\mu_{A(I)}\circ\Big(A\big(\ell^{\mathscr{F}}(I)\big)\otimes_{\mathscr{C}}A\big(\ell^{\mathscr{F}}(I)\big)\Big)\circ\Big(A\big(\iota_1(\mathbf{I}_{\mathscr{F}},I)\big)\otimes_{\mathscr{C}}A\big(\iota_2(\mathbf{I}_{\mathscr{F}},I)\big)\Big)\circ(\upsilon^A\otimes_{\mathscr{C}}\mathrm{id}_{A(I)})\\
&=\mu_{A(I)}\circ\Big(\Big(A\big(\ell^{\mathscr{F}}(I)\circ\iota_1(\mathbf{I}_{\mathscr{F}},I)\big)\circ\upsilon^A\Big)\otimes_{\mathscr{C}}A\big(\ell^{\mathscr{F}}(I)\circ\iota_2(\mathbf{I}_{\mathscr{F}},I)\big)\Big)\\
&=\mu_{A(I)}\circ\Big(\Big(A\big(\ell^{\mathscr{F}}(I)\circ\iota_1(\mathbf{I}_{\mathscr{F}},I)\big)\circ\upsilon^A\Big)\otimes_{\mathscr{C}}\mathrm{id}_{A(I)}\Big)\\
&=\mu_{A(I)}\circ(\eta_{A(I)}\otimes_{\mathscr{C}}\mathrm{id}_{A(I)})=\ell^{\mathscr{C}}\big(A(I)\big),
\end{aligned}$$

where we have used that $A(\ell^{\mathscr{F}}(I))$ is a morphism of algebras in $\mathscr{C}$ in the second equality, that $\ell^{\mathscr{F}}(I)\circ\iota_2(\mathbf{I}_{\mathscr{F}},I)=\mathrm{id}_I$ by definition of $\iota_2(\mathbf{I}_{\mathscr{F}},I)$ in the fourth equality, and that $\upsilon^A = \eta_{A(\mathbf{I}_{\mathscr{F}})}$ is the unit of $A(\mathbf{I}_{\mathscr{F}})$ and that $A(\ell^{\mathscr{F}}(I) \circ \iota_1(\mathbf{I}_{\mathscr{F}}, I))$ is a morphism of algebras in $\mathscr{C}$ in the penultimate equality. This shows that A is compatible with the unit on the left. The case for the unit on the right is analogous.

Let us next prove that the structure morphism φ^A is compatible with the tensor products (see [14], XI.4, (4.1)). Indeed, the reader can verify that

$$
\begin{aligned}
&\varphi^A(I, J\Box K) \circ \big(\mathrm{id}_{A(I)} \otimes_{\mathscr{C}} \varphi^A(J,K)\big) \circ a^{\mathscr{C}}\big(A(I), A(J), A(K)\big) \\
&= \mu_{A(I\Box(J\Box K))} \circ \Big(A\big(\iota_1(I, J\Box K)\big) \otimes_{\mathscr{C}} A\big(\iota_2(I, J\Box K)\big)\Big) \circ (\mathrm{id}_{A(I)} \otimes_{\mathscr{C}} \mu_{A(I,J\Box K)}) \\
&\quad \circ \Big(\mathrm{id}_{A(I)} \otimes_{\mathscr{C}} A\big(\iota_1(J,K)\big) \otimes_{\mathscr{C}} A\big(\iota_2(J,K)\big)\Big) \circ a^{\mathscr{C}}\big(A(I), A(J), A(K)\big) \\
&= \mu_{A(I\Box(J\Box K))} \circ (\mathrm{id}_{A(I\Box(J\Box K))} \otimes_{\mathscr{C}} \mu_{A(I\Box(J\Box K))}) \circ \Big(A\big(\iota_1(I, J\Box K)\big) \\
&\quad \otimes_{\mathscr{C}} \Big(A\big(\iota_2(I, J\Box K) \circ \iota_1(J,K)\big) \otimes_{\mathscr{C}} A\big(\iota_2(I, J\Box K) \circ \iota_2(J,K)\big)\Big)\Big) \\
&\quad \circ a^{\mathscr{C}}\big(A(I), A(J), A(K)\big) \\
&= \mu_{A(I\Box(J\Box K))} \circ (\mathrm{id}_{A(I\Box(J\Box K))} \otimes_{\mathscr{C}} \mu_{A(I\Box(J\Box K))}) \\
&\quad \circ a^{\mathscr{C}}\Big(A\big(I\Box(J\Box K)\big), A\big(I\Box(J\Box K)\big), A\big(I\Box(J\Box K)\big)\Big) \\
&\quad \circ \Big(\Big(A\big(\iota_1(I, J\Box K)\big) \otimes_{\mathscr{C}} A\big(\iota_2(I, J\Box K) \circ \iota_1(J,K)\big)\Big) \otimes_{\mathscr{C}} A\big(\iota_2(I, J\Box K) \circ \iota_2(J,K)\big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&A\big(a^{\mathscr{F}}(I,J,K)\big) \circ \varphi^A(I\Box J, K) \circ \big(\varphi^A(I,J) \otimes_{\mathscr{C}} \mathrm{id}_{A(K)}\big) \\
&= A\big(a^{\mathscr{F}}(I,J,K)\big) \circ \mu_{A((I\Box J)\Box K)} \circ \Big(A\big(\iota_1(I\Box J, K)\big) \otimes_{\mathscr{C}} A\big(\iota_2(I\Box J, K)\big)\Big) \\
&\quad \circ (\mu_{A(I\Box J)} \otimes_{\mathscr{C}} \mathrm{id}_{A(K)}) \circ \Big(\Big(A\big(\iota_1(I,J)\big) \otimes_{\mathscr{C}} A\big(\iota_2(I,J)\big)\Big) \otimes_{\mathscr{C}} \mathrm{id}_{A(K)}\Big) \\
&= \mu_{A(I\Box(J\Box K))} \circ \Big(A\big(a^{\mathscr{F}}(I,J,K) \circ \iota_1(I\Box J, K)\big) \otimes_{\mathscr{C}} A\big(a^{\mathscr{F}}(I,J,K) \circ \iota_2(I\Box J, K)\big)\Big) \\
&\quad \circ \big(\mu_{A(I\Box J)} \otimes_{\mathscr{C}} \mathrm{id}_{A(K)}\big) \circ \Big(\Big(A\big(\iota_1(I,J)\big) \otimes_{\mathscr{C}} A\big(\iota_2(I,J)\big)\Big) \otimes_{\mathscr{C}} \mathrm{id}_{A(K)}\Big) \\
&= \mu_{A(I\Box(J\Box K))} \circ (\mu_{A(I\Box(J\Box K))} \otimes_{\mathscr{C}} \mathrm{id}_{A(I\Box(J\Box K))}) \\
&\quad \circ \Big(\Big(A\big(a^{\mathscr{F}}(I,J,K) \circ \iota_1(I\Box J, K) \circ \iota_1(I,J)\big) \\
&\qquad \otimes_{\mathscr{C}} A\big(a^{\mathscr{F}}(I,J,K) \circ \iota_1(I\Box J, K) \circ \iota_2(I,J)\big)\Big) \otimes_{\mathscr{C}} A\big(a^{\mathscr{F}}(I,J,K) \circ \iota_2(I\Box J, K)\big)\Big)
\end{aligned}
$$

coincide, since $\mu_{A(I\Box(J\Box K))}$ is associative and we have used (2.2) and (2.3).

Finally, A is braided (see [14], XIII.3.6, (3.12)), since

$$
\begin{aligned}
&A\big(\tau^{\mathscr{F}}(I,J)\big) \circ \varphi^A(I,J) = A\big(\tau^{\mathscr{F}}(I,J)\big) \circ \mu_{A(I\Box J)} \circ \Big(A\big(\iota_1(I,J)\big) \otimes_{\mathscr{C}} A\big(\iota_2(I,J)\big)\Big) \\
&= \mu_{A(J\Box I)} \circ \Big(A\big(\tau^{\mathscr{F}}(I,J) \circ \iota_1(I,J)\big) \otimes_{\mathscr{C}} A\big(\tau^{\mathscr{F}}(I,J) \circ \iota_2(I,J)\big)\Big) \\
&= \mu_{A(J\Box I)} \circ \Big(A\big(\iota_2(J,I)\big) \otimes_{\mathscr{C}} A\big(\iota_1(J,I)\big)\Big) \\
&= \mu_{A(J\Box I)} \circ \tau^{\mathscr{C}}\big(A(J\Box I), A(J\Box I)\big) \circ \Big(A\big(\iota_2(J,I)\big) \otimes_{\mathscr{C}} A\big(\iota_1(J,I)\big)\Big) \\
&= \mu_{A(J\Box I)} \circ \Big(A\big(\iota_1(J,I)\big) \otimes_{\mathscr{C}} A\big(\iota_2(J,I)\big)\Big) \circ \tau^{\mathscr{C}}\big(A(I), A(J)\big) \\
&= \varphi^A(J,I) \circ \tau^{\mathscr{C}}\big(A(I), A(J)\big).
\end{aligned}
$$

The first part of the lemma is thus proved.

We will now prove the last part of the lemma. We first note that, given an algebra $\bar{A}$ in a monoidal category $\mathscr{C}$, there is a unique morphism of algebras $\mathbf{I}_{\mathscr{F}} \to \bar{A}$.

This implies in particular that there is a unique morphism $\upsilon^A : \mathbf{I}_{\mathscr{C}} \to A(\mathbf{I}_{\mathscr{F}})$, given by the unit of $A(\mathbf{I}_{\mathscr{F}})$. We shall finally prove that the structure morphism $\varphi^A(I, J)$ is uniquely determined, for all objects I and J in $\mathscr{F}$. We first note that the proof of Lemma 2.4 tells us that the finite (co)product of (co)commutative (co)algebras in a symmetric monoidal category $\mathscr{C}$ is given by their tensor product. This immediately implies that a morphism of commutative algebras $\varphi : A' \otimes_{\mathscr{C}} A'' \to \bar{A}$ in $\mathscr{C}$ is exactly given by $\mu_{\bar{A}} \circ (\varphi_{A'} \otimes_{\mathscr{C}} \varphi_{A''})$, where $\varphi_{A'} : A' \to \bar{A}$ (resp., $\varphi_{A''} : A'' \to \bar{A}$) is $\varphi \circ (\mathrm{id}_{A'} \otimes_{\mathscr{C}} \eta_{A''}) \circ \boldsymbol{r}^{\mathscr{C}}(A')^{-1}$ (resp., $\varphi \circ (\eta_{A'} \otimes_{\mathscr{C}} \mathrm{id}_{A''}) \circ \boldsymbol{\ell}^{\mathscr{C}}(A'')^{-1}$). Applying the previous result for $A' = A(I)$, $A'' = A(J)$, $\bar{A} = A(I \square J)$, and $\varphi = \varphi^A(I, J)$, we see that is suffices to show that the morphisms $\varphi^A(I, J) \circ (\mathrm{id}_{A(I)} \otimes_{\mathscr{C}} \eta_{A(J)}) \circ \boldsymbol{r}^{\mathscr{C}}(A(I))^{-1}$ and $\varphi^A(I, J) \circ (\eta_{A(I)} \otimes_{\mathscr{C}} \mathrm{id}_{A(J)}) \circ \boldsymbol{\ell}^{\mathscr{C}}(A(J))^{-1}$ are uniquely determined. Finally, by definition of monoidal functor we see that

$$\begin{aligned}
\varphi^A(I, J) \circ (\mathrm{id}_{A(I)} \otimes_{\mathscr{C}} \eta_{A(J)}) &= \varphi^A(I, J) \circ \big(\mathrm{id}_{A(I)} \otimes_{\mathscr{C}} A(i_J)\big) \circ \big(\mathrm{id}_{A(I)} \otimes_{\mathscr{C}} \upsilon^A\big) \\
&= A(\mathrm{id}_I \square i_J) \circ \varphi^A(I, \mathbf{I}_{\mathscr{F}}) \circ \big(\mathrm{id}_{A(I)} \otimes_{\mathscr{C}} \upsilon^A\big) \\
&= A(\mathrm{id}_I \square i_J) \circ A\big(\boldsymbol{r}^{\mathscr{F}}(I)^{-1}\big) \circ \boldsymbol{r}^{\mathscr{C}}\big(A(I)\big) \\
&= A\big(\iota_1(I, J)\big) \circ \boldsymbol{r}^{\mathscr{C}}\big(A(I)\big),
\end{aligned}$$

where $i_J : \mathbf{I}_{\mathscr{F}} \to J$ is the unique morphism in $\mathscr{F}$ introduced in the penultimate paragraph of Subsection 2.2, and the analogous identity holds for $\varphi^A(I, J) \circ (\eta_{A(I)} \otimes_{\mathscr{C}} \mathrm{id}_{A(J)})$. This concludes the proof of the lemma. □

Corollary 3.3. *Assume the same hypotheses as in Lemma 3.2. Let B be a cocommutative bialgebra in $\mathbf{Funi}(\mathscr{F}, \mathscr{C})$. Then the structure morphisms given in the previous lemma turn B into an oplax bialgebra.*

Proof. This follows from the previous lemma, together with the fact that both υ^B and $\varphi^B(I, J)$ are morphisms of coalgebras, since the unit and the product of a bialgebra are so. □

We also collect the following result, which is a direct corollary of Fact 3.1 and the elementary fact that (op)lax monoidal functors of monoidal categories send (co)algebras to (co)algebras.

Fact 3.4. *Let $F : \mathscr{F} \to \mathscr{F}'$ be a functor of small categories and $G : \mathscr{C}' \to \mathscr{C}$ be a(n) (op)lax monoidal functor between monoidal categories. Then, the induced functor $(F^*, G_*) : \mathbf{Funi}(\mathscr{F}', \mathscr{C}') \to \mathbf{Funi}(\mathscr{F}, \mathscr{C})$ sending M to $G \circ M \circ F$ sends a (co)algebra of $\mathbf{Funi}(\mathscr{F}', \mathscr{C}')$ to a (co)algebra of $\mathbf{Funi}(\mathscr{F}, \mathscr{C})$. If G is a braided (op)lax monoidal functor between braided monoidal categories, (F^*, G_*) sends (co)commutative (co)algebras to (co)commutative (co)algebras.*

3.3 External modules over algebras in functor categories

Let $A \in \mathbf{Funi}(\mathscr{F}^{\mathrm{op}}, \mathscr{C})$ be an algebra. Consider the strict homomorphism

$$\hbar : \mathscr{F} \longrightarrow \mathbf{Cat} \tag{3.1}$$

of bicategories given as the composition of the functor $A : \mathscr{F} \to \mathbf{Alg}(\mathscr{C})^{\mathrm{op}}$ and the functor $\mathrm{R} : \mathbf{Alg}(\mathscr{C})^{\mathrm{op}} \to \mathbf{Cat}$ introduced in (2.16).

Definition 3.5. *Let $A \in \mathbf{Funi}(\mathscr{F}^{\mathrm{op}}, \mathscr{C})$ be an algebra, and let $\hbar$ be the strict homomorphism of bicategories (3.1). An $\hbar$-module will be called an **external module** over A, and a morphism of $\hbar$-modules $F : M \to N$ will be called a **morphism of external modules**. We will denote by ${}_A\mathbf{ExMod}(\mathscr{F}, \mathscr{C})$ the category of external modules over A.*

Lemma 3.6. *Let $A \in \mathbf{Funi}(\mathscr{F}^{\text{op}}, \mathscr{C})$ be an algebra. Then, the notion of external module M given in Definition 3.5 coincides with that of module in [8], Def. 3.6. More precisely, an external module is equivalently described as a functor $M \in \mathbf{Fun}(\mathscr{F}, \mathscr{C})$ such that $M(I)$ is an $A(I)$-module via a morphism $\rho_I : A(I) \otimes_{\mathscr{C}} M(I) \to M(I)$, for all I in $\mathscr{F}$, satisfying that*

$$M(f) \circ \rho_I \circ \big(A(f) \otimes_{\mathscr{C}} \mathrm{id}_{M(I)}\big) = \rho_J \circ \big(\mathrm{id}_{A(J)} \otimes_{\mathscr{C}} M(f)\big), \tag{3.2}$$

for all morphisms $f : I \to J$ in $\mathscr{F}$. Moreover, a morphism of external modules $F : M \to N$ is the same as a natural transformation F between the underlying functors such that $F(I) : M(I) \to N(I)$ is a morphism of $A(I)$-modules for all I in $\mathscr{F}$.

Proof. The result follows directly from noting that (3.2) is a translation of (2.9). Indeed, a functor $M \in \mathbf{Fun}(\mathscr{F}, \mathscr{C})$ is an external module over A if and only if $M(I) \in {}_{A(I)}\mathbf{Mod}(\mathscr{C})$ for all I in $\mathscr{F}$, and given a morphism $f : I \to J$ in $\mathscr{F}$, $M(f)$ induces a morphism $\mathrm{Res}_{A(f)}(M(I)) \to M(J)$ of $A(J)$-modules in $\mathscr{C}$. Analogously, given two external modules M and N over A, a natural transformation $F : M \to N$ between the underlying functors of M, N from $\mathscr{F}$ to $\mathscr{C}$ is a morphism of external modules if and only if $F(I) : M(I) \to N(I)$ is a morphism of $A(I)$-modules in $\mathscr{C}$, for all objects I in $\mathscr{F}$. In consequence, the condition on the morphisms of external modules is just a translation of (2.10). □

3.4 The monoidal structure on the category of external modules

The next result is implicit in [8], so we present it for clarity.

Lemma 3.7. *Assume that $\mathscr{C}$ is endowed with a braiding $\tau^{\mathscr{C}}$. Let $B \in \mathbf{Funi}(\mathscr{F}^{\text{op}}, \mathscr{C})$ be a bialgebra, and $M', M'' \in {}_B\mathbf{ExMod}(\mathscr{F}, \mathscr{C})$ be two external modules over B, with actions ρ'_I and ρ''_I for all I in $\mathscr{F}$, respectively. Given I in $\mathscr{F}$, define the morphism $\rho_I : B(I) \otimes_{\mathscr{C}} M'(I) \otimes_{\mathscr{C}} M''(I) \to M'(I) \otimes_{\mathscr{C}} M''(I)$ by (2.20). Then, the functor $M = M' \otimes M''$ together with the previous morphisms ρ_I is a external module over B. Analogously, the constant functor with value $\mathbf{I}_{\mathscr{C}}$ has a structure of external module over B via $\sigma_I : B(I) \otimes_{\mathscr{C}} \mathbf{I}_{\mathscr{C}} \to \mathbf{I}_{\mathscr{C}}$ given by the $\epsilon_{B(I)} \circ \boldsymbol{r}^{\mathscr{C}}(B(I))$, where $\epsilon_{B(I)}$ is the counit of $B(I)$. Moreover, the subcategory ${}_B\mathbf{ExMod}(\mathscr{F}, \mathscr{C})$ of $\mathbf{Funi}(\mathscr{F}, \mathscr{C})$ formed by all external modules over B is a monoidal subcategory.*

Proof. The fact that $M'(I) \otimes_{\mathscr{C}} M''(I)$ is a module over $B(I)$ via ρ_I is a general result recalled in Subsection 2.4. We need to prove it verifies (3.2). This follows from

$$\begin{aligned}
&M(f) \circ \rho_I \circ \big(B(f) \otimes_{\mathscr{C}} \mathrm{id}_{M(I)}\big) \\
&= \Big(\big(M'(f) \circ \rho'_I\big) \otimes_{\mathscr{C}} \big(M''(f) \circ \rho''_I\big)\Big) \circ \big(\mathrm{id}_{B(I)} \otimes_{\mathscr{C}} \tau^{\mathscr{C}}(B(I), M(I)) \otimes_{\mathscr{C}} \mathrm{id}_{M''(I)}\big) \\
&\quad \circ \Big(\big(\Delta(I) \circ B(f)\big) \otimes_{\mathscr{C}} \mathrm{id}_{M'(I) \otimes_{\mathscr{C}} M''(I)}\Big) \\
&= \Big(\big(M'(f) \circ \rho'_I\big) \otimes_{\mathscr{C}} \big(M''(f) \circ \rho''_I\big)\Big) \circ \big(\mathrm{id}_{B(I)} \otimes_{\mathscr{C}} \tau^{\mathscr{C}}(B(I), M(I)) \otimes_{\mathscr{C}} \mathrm{id}_{M''(I)}\big) \\
&\quad \circ \Big(\big(B(f) \otimes_{\mathscr{C}} B(f)\big) \otimes_{\mathscr{C}} \mathrm{id}_{M'(I) \otimes_{\mathscr{C}} M''(I)}\Big) \circ \Big(\Delta_{B(J)} \otimes_{\mathscr{C}} \mathrm{id}_{M'(I) \otimes_{\mathscr{C}} M''(I)}\Big) \\
&= \Big(\big(M'(f) \circ \rho'_I\big) \otimes_{\mathscr{C}} \big(M''(f) \circ \rho''_I\big)\Big) \circ \Big(\big(B(f) \otimes_{\mathscr{C}} \mathrm{id}_{M'(I)} \otimes_{\mathscr{C}} B(f) \otimes_{\mathscr{C}} \mathrm{id}_{M''(I)}\Big) \\
&\quad \circ \big(\mathrm{id}_{B(J)} \otimes_{\mathscr{C}} \tau^{\mathscr{C}}(B(J), M(I)) \otimes_{\mathscr{C}} \mathrm{id}_{M''(I)}\big) \circ \Big(\Delta_{B(J)} \otimes_{\mathscr{C}} \mathrm{id}_{M'(I) \otimes_{\mathscr{C}} M''(I)}\Big) \\
&= \big(\rho'_J \otimes_{\mathscr{C}} \rho''_J\big) \circ \big(\mathrm{id}_{B(J)} \otimes_{\mathscr{C}} M'(f) \otimes_{\mathscr{C}} \mathrm{id}_{B(J)} \otimes_{\mathscr{C}} M''(f)\big) \\
&\quad \circ \big(\mathrm{id}_{B(J)} \otimes_{\mathscr{C}} \tau^{\mathscr{C}}(B(J), M(I)) \otimes_{\mathscr{C}} \mathrm{id}_{M''(I)}\big) \circ \Big(\Delta_{B(J)} \otimes_{\mathscr{C}} \mathrm{id}_{M'(I) \otimes_{\mathscr{C}} M''(I)}\Big) \\
&= \big(\rho'_J \otimes_{\mathscr{C}} \rho''_J\big) \circ \big(\mathrm{id}_{B(J)} \otimes_{\mathscr{C}} \tau^{\mathscr{C}}(B(J), M'(J)) \otimes_{\mathscr{C}} \mathrm{id}_{M''(J)}\big) \\
&\quad \circ \big(\Delta_{B(J)} \otimes_{\mathscr{C}} M'(f) \otimes_{\mathscr{C}} M''(f)\big) = \rho_J \circ \big(\mathrm{id}_{B(J)} \otimes_{\mathscr{C}} M(f)\big)
\end{aligned}$$

for all morphisms $f : I \to J$ in $\mathscr{F}$, where we have used that $B(f) : B(J) \to B(I)$ is a morphism of coalgebras in the second equality, the naturality of the braiding in the third and fifth equalities, and (3.2) in the fourth equality. Hence $M' \otimes M''$ together with the previous morphisms ρ_I is an external module over B.

Analogously, to show that the constant functor with value $\mathbf{I}_{\mathscr{C}}$ has a structure of external module over B with the given morphisms σ_I it suffices to prove that it verifies (3.2). Since $M(f)$ is the identity in this case, (3.2) is tantamount to $\epsilon_{B(I)} \circ B(f) = \epsilon_{B(J)}$, which follows from the fact that B is a coalgebra and Fact 3.1.

It remains to show that the tensor product functor of $\mathbf{Funi}(\mathscr{F}, \mathscr{C})$ also preserves morphisms of modules over B, since the associativity, left unit and right unit morphisms are immediately seen to be morphisms of modules over B, because this is the case for the category of modules over the bialgebra $B(I)$ in $\mathscr{C}$. Since the tensor product of morphisms in $\mathbf{Funi}(\mathscr{F}, \mathscr{C})$ is computed pointwise, as stated in the first paragraph of this subsection, the result follows immediately. The lemma is thus proved. □

Lemma 3.8. *Assume the same hypotheses as in Lemma 3.7. Suppose further that the braiding of $\mathscr{C}$ is symmetric and B is cocommutative. Then, the symmetric braiding of the monoidal category $\mathbf{Funi}(\mathscr{F}, \mathscr{C})$ restricts to a symmetric braiding on its monoidal subcategory ${}_B\mathbf{ExMod}(\mathscr{F}, \mathscr{C})$.*

Proof. It suffices to prove that $\tau^{\mathscr{C}}(M(I), N(I)) : M(I) \otimes_{\mathscr{C}} N(I) \to N(I) \otimes_{\mathscr{C}} M(I)$ is a morphism of $B(I)$-modules in $\mathscr{C}$ for all I in $\mathscr{F}$, with respect to the diagonal structure given by (2.20), provided $B(I)$ is cocommutative. This is a standard result in symmetric monoidal categories recalled in Subsection 2.4. □

We will denote by ${}_B\mathbf{ExModt}(\mathscr{F}, \mathscr{C})$ the (resp., braided) monoidal category ${}_B\mathbf{ExMod}(\mathscr{F}, \mathscr{C})$ with the tensor product and unit in Lemma 3.7 (resp., and the braiding in Lemma 3.8).

Remark 3.9. *Note that, by the comments in Remark 2.5, Lemma 3.8 does not necessary hold if the braiding is not symmetric.*

We can describe algebras in the monoidal category ${}_B\mathbf{ExModt}(\mathscr{F}, \mathscr{C})$.

Lemma 3.10. *Assume the same hypotheses as in Lemma 3.7. Let $B \in \mathbf{Funi}(\mathscr{F}^{\mathrm{op}}, \mathscr{C})$ be a bialgebra. Then, $A \in \mathbf{Fun}(\mathscr{F}, \mathscr{C})$ is an algebra in ${}_B\mathbf{ExModt}(\mathscr{F}, \mathscr{C})$ if and only if $A(I)$ is an algebra in ${}_{B(I)}\mathbf{Modt}(\mathscr{C})$ for all I in $\mathscr{F}$, and given a morphism $f : I \to J$ in $\mathscr{F}$, $A(f)$ induces a morphism $\mathrm{Res}_{B(f)}(A(I)) \to A(J)$ of $B(J)$-module algebras in $\mathscr{C}$.*

Proof. Note first that, since (2.14) is a lax monoidal functor, it sends the $B(I)$-module algebra $A(I)$ to the $B(J)$-module algebra $\mathrm{Res}_{B(f)}(A(I))$. The result now directly follows from the definitions. □

Let $\mathscr{C}$ be a braided monoidal category and $(\mathscr{F}, \Box, \mathbf{I}_{\mathscr{F}})$ a monoidal category. Let B be an oplax bialgebra in the braided monoidal category $\mathbf{Funi}(\mathscr{F}^{\mathrm{op}}, \mathscr{C})$ and A an algebra in the monoidal category ${}_B\mathbf{ExModt}(\mathscr{F}, \mathscr{C})$. We say that A is **lax** if the underlying algebra $A \in \mathbf{Alg}(\mathscr{F}, \mathscr{C})$ is lax monoidal and the structure morphism $\varphi^A(I, J) : \mathrm{Res}_{\varphi^B(I,J)}(A(I) \otimes_{\mathscr{C}} A(J)) \to A(I \Box J)$ of A is a morphism of $B(I \Box J)$-module algebras for all objects I, J in $\mathscr{C}$, where $\varphi^B(I, J) : B(I \Box J) \to B(I) \otimes_{\mathscr{C}} B(J)$ is the morphism of bialgebras given by the structure morphism of B. We further say that A is **braided** if the underlying functor $A : \mathscr{F} \to \mathscr{C}$ is braided monoidal.

3.5 Modules over algebras in the category of external modules

Assume that $\mathscr{C}$ has a symmetric braiding. Let $B \in \mathbf{Funi}(\mathscr{F}^{\mathrm{op}}, \mathscr{C})$ be a cocommutative bialgebra and let A be an algebra in ${}_B\mathbf{ExModt}(\mathscr{F}, \mathscr{C})$. Consider the homomorphism

$$\underset{\sim}{g} : \mathscr{F} \longrightarrow \mathbf{Cat} \tag{3.3}$$

of bicategories defined as follows. Given an object I of $\mathscr{F}$, define the category $\mathcal{g}(I) = {}_{A(I)}\mathbf{Mod}({}_{B(I)}\mathbf{Modt}(\mathscr{C}))$, and if $f : I \to J$ is a morphism in $\mathscr{F}$, let $\mathcal{g}(f)$ be the functor $\mathrm{Ind}_{A(f)} \circ \mathrm{Res}_{B(f)}$, where $A(f)$ denotes the morphism $\mathrm{Res}_{B(f)}(A(I)) \to A(J)$ of $A(J)$-module algebras in $\mathscr{C}$. The natural isomorphism

$$\phi^{\mathcal{g}}(g, f) : \mathcal{g}(g) \circ \mathcal{g}(f) \longrightarrow \mathcal{g}(g \circ f) \tag{3.4}$$

for morphisms $f : I \to J$ and $g : J \to K$ in $\mathscr{F}$ is given by $\phi^{\mathrm{I}}(A(g), A(f))$ in Fact 2.3.

We can finally give the following characterization of modules over algebras in the monoidal category ${}_B\mathbf{ExModt}(\mathscr{F}, \mathscr{C})$.

Lemma 3.11. *Assume that the monoidal category $\mathscr{C}$ has a symmetric braiding. Let $B \in \mathbf{Funi}(\mathscr{F}^{\mathrm{op}}, \mathscr{C})$ be a cocommutative bialgebra and $A \in \mathbf{Alg}({}_B\mathbf{ExModt}(\mathscr{F}, \mathscr{C}))$. Then, there is a canonical equivalence between the category ${}_A\mathbf{Mod}({}_B\mathbf{ExModt}(\mathscr{F}, \mathscr{C}))$ of modules over A and the category $\mathcal{g}$-$\mathbf{Mod}$ of $\mathcal{g}$-modules.*

Proof. By unraveling the definition of module over A, we see that $M \in \mathbf{Fun}(\mathscr{F}, \mathscr{C})$ is a module over A if and only if $M(I) \in {}_{A(I)}\mathbf{Mod}({}_{B(I)}\mathbf{Modt}(\mathscr{C}))$ for all I in $\mathscr{F}$, and given $f : I \to J$ in $\mathscr{F}$, $M(f)$ induces a morphism $\mathrm{Res}_{B(f)}(M(I)) \to \mathrm{Res}_{A(f)}(M(J))$ of $\mathrm{Res}_{B(f)}(A(I))$-modules in ${}_{B(J)}\mathbf{Modt}(\mathscr{C})$, where $A(f)$ denotes the morphism $\mathrm{Res}_{B(f)}(A(I)) \to A(J)$ of $B(J)$-module algebras. By using the adjunction between the restriction and the extension of scalars recalled in (2.15), we may equivalently rephrase the morphism $\mathrm{Res}_{B(f)}(M(I)) \to \mathrm{Res}_{A(f)}(M(J))$ by a morphism $\mathrm{Ind}_{A(f)}(\mathrm{Res}_{B(f)}(M(I))) \to M(J)$ of $A(J)$-modules in ${}_{B(J)}\mathbf{Modt}(\mathscr{C})$. This proves the first part of the statement.

Analogously, given $M, N \in {}_A\mathbf{Mod}({}_B\mathbf{ExModt}(\mathscr{F}, \mathscr{C}))$, a natural transformation $F : M \to N$ between the underlying functors from $\mathscr{F}$ to $\mathscr{C}$ is a morphism of A-modules if and only if $F(I) : M(I) \to N(I)$ is a morphism of $A(I)$-modules in ${}_{B(I)}\mathbf{Modt}(\mathscr{C})$, for all objects I in $\mathscr{F}$, proving the second part of the lemma. ◻

4 The main result(s)

In this section we prove the main result of this article, namely that the category of $\mathcal{g}$-modules has a natural structure of symmetric 2-monoidal category.

4.1 The setting

For the rest of the article, we will consider the following elements, some of which have already been mentioned:

(VA.1) a small semicocartesian monoidal category $(\mathscr{F}, \square, \mathbf{I}_{\mathscr{F}})$;

(VA.2) a symmetric monoidal category $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_{\mathscr{C}}, \tau^{\mathscr{C}})$ that is cocomplete and such that the tensor product $\otimes_{\mathscr{C}}$ commutes with colimits on each side;

(VA.3) an oplax cocommutative bialgebra B in the symmetric monoidal category $\mathbf{Funi}(\mathscr{F}^{\mathrm{op}}, \mathscr{C})$;

(VA.4) a lax commutative algebra $A : \mathscr{F} \to \mathscr{C}$ in the symmetric monoidal category $\mathscr{A} = {}_B\mathbf{ExModt}(\mathscr{F}, \mathscr{C})$ (see Lemma 3.8);

(VA.5) the symmetric monoidal category $\mathcal{C} = {}_A\mathbf{Mods}(\mathscr{A})$ of A-modules in $\mathscr{A}$ with tensor product $\otimes_A$, strict unit A and braiding induced by that of $\mathscr{A}$.

Note that the monoidal structures of the categories appearing in (VA.3)-(VA.5) only depend on the symmetric monoidal structure of the category $\mathscr{C}$ (and A in the last case).

Neither the semicocartesian condition on $\mathscr{F}$ nor the oplax and lax conditions on B and A, respectively, appear in [8]. The last two hypotheses **are even necessary to properly define singular tensor products**, to be introduced in the next section.

We will also eventually assume:

(VA.6) the semicocartesian monoidal category $\mathscr{F}$ has a symmetric braiding $\tau^{\mathscr{F}}$,

(VA.7) the bialgebra B and the algebra A are braided;

(VA.8) the structure morphisms of the underlying monoidal functors of the bialgebra B and the algebra A are those of Lemma 3.2.

Notice that, under the assumption (VA.6), the oplax condition on B and the lax condition on A stated in (VA.3) and (VA.4), respectively, are superfluous by Lemma 3.2 and Corollary 3.3. Moreover, the same results tell us that (VA.6) implies (VA.7) and (VA.8).

4.2 Singular tensor products

Recall that we will use the notation explained in the last paragraph of Section 2. **For the rest of this section we assume the hypotheses (VA.1)-(VA.5).**

4.2.1 The definition and first properties

We explain here the singular tensor product on the category ${}_A\mathbf{Mod}(\mathscr{A})$, following the spirit but mainly completing the programmatic article [8]. Let $n \geq 2$ be an integer and $t \in \mathrm{Y}_{n-1}$ be a rooted planar binary tree with n leaves. We will first consider the functor

$$\mathscr{f}_t : \mathscr{F} \longrightarrow \mathbf{Cat} \tag{4.1}$$

defined as follows. Given an object I of $\mathscr{F}$, define the small category $\mathscr{f}_t(I)$ whose objects are pairs $(\hat{I}, \mu_{\hat{I}})$, where $\hat{I} = (I_1, \dots, I_n) \in \mathscr{F}^n$ is an n-tuple of objects and $\mu_{\hat{I}} : t(\square)(\hat{I}) \to I$ is a morphism in $\mathscr{F}$, and whose morphisms $(\hat{I}, \mu_{\hat{I}}) \to (\hat{I}', \mu_{\hat{I}'})$, for $\hat{I} = (I_1, \dots, I_n)$ and $\hat{I}' = (I'_1, \dots, I'_n)$, are given by n-tuples $(f_1, \dots, f_n)$ of morphisms in $\mathscr{F}$ with $f_i : I_i \to I'_i$ for $i \in [\![1, n]\!]$ such that $\mu_{\hat{I}'} \circ t(\square)(f_1, \dots, f_n) = \mu_{\hat{I}}$. The composition is the obvious one. If $f : I \to J$ is a morphism in $\mathscr{F}$, let $\mathscr{f}_t(f) : \mathscr{f}_t(I) \to \mathscr{f}_t(J)$ be the functor sending $(\hat{I}, \mu_{\hat{I}})$ to $(\hat{I}, f \circ \mu_{\hat{I}})$, and that is the identity on the morphisms. If we consider $\mathscr{F}$ as a 2-category where every morphism space $\mathrm{Hom}_{\mathscr{F}}(I, J)$ is a discrete category, then the functor $\mathscr{f}_t$ is thus a strict homomorphism of bicategories.

Recall the homomorphism of bicategories $\mathscr{g}$ defined in (3.3), and the category $\mathbf{Trans_{st}}(\mathscr{f}_t, \mathscr{g})$ introduced in Subsection 2.3. Given an integer $n \geq 2$ and $t \in \mathrm{Y}_{n-1}$, we will construct a functor

$$\mathscr{t}_t : \mathscr{C}^n \longrightarrow \mathbf{Trans_{st}}(\mathscr{f}_t, \mathscr{g}) \tag{4.2}$$

as follows.

At the level of the objects, given an n-tuple $\hat{M} = (M_1, \dots, M_n)$ of objects in $\mathscr{C} = {}_A\mathbf{Mod}(\mathscr{A})$, we will start by defining an object in $\mathbf{Trans_{st}}(\mathscr{f}_t, \mathscr{g})$ by a functor

$$\mathscr{t}_t(\hat{M}, I) : \mathscr{f}_t(I) \longrightarrow \mathscr{g}(I) = {}_{A(I)}\mathbf{Mod}\big({}_{B(I)}\mathbf{Modt}(\mathscr{C})\big) \tag{4.3}$$

for every object I of $\mathscr{F}$, and a natural isomorphism

$$\mathscr{t}_t(\hat{M}, f) : \mathscr{g}(f) \circ \mathscr{t}_t(\hat{M}, I) \longrightarrow \mathscr{t}_t(\hat{M}, J) \circ \mathscr{f}_t(f) \tag{4.4}$$

for every morphism $f : I \to J$ in $\mathscr{F}$. These data will satisfy the conditions stated in Subsection 2.3. Let $(\hat{I}, \mu_{\hat{I}})$ be an object in $\mathscr{f}_t(I)$. To simplify, we will denote $\hat{M}(\hat{I}) =$

$(M_1(I_1), \ldots, M_n(I_n))$, $B(\hat{I}) = (B(I_1), \ldots, B(I_n))$, $A(\hat{I}) = (A(I_1), \ldots, A(I_n))$. In particular, we will write $t(\otimes_{\mathscr{C}})(\hat{M}(\hat{I})) = t(\otimes_{\mathscr{C}})(M_1(I_1), \ldots, M_n(I_n))$.

We will first start with the definition of (4.3). Note that $t(\otimes_{\mathscr{C}})(M(\hat{I}))$ is a module over the bialgebra $t(\otimes_{\mathscr{C}})(B(\hat{I}))$ in $\mathscr{C}$. Since B is an oplax bialgebra, the structure map $\varphi^{B,t}(\hat{I}) : B\big(t(\Box)(\hat{I})\big) \longrightarrow t(\otimes_{\mathscr{C}})\big(B(\hat{I})\big)$ of B is a morphism of bialgebras. Hence, by means of $\varphi^{B,t}(\hat{I}) \circ B(\mu_{\hat{I}})$, we get that $\mathrm{Res}_{\varphi^{B,t}(\hat{I})\circ B(\mu_{\hat{I}})}(t(\otimes_{\mathscr{C}})(\hat{M}(\hat{I})))$ is a module over the bialgebra $B(I)$ in $\mathscr{C}$. Since the restriction functor is lax monoidal, the previous module is also a module over the algebra

$$\mathrm{Res}_{\varphi^{B,t}(\hat{I})\circ B(\mu_{\hat{I}})}\Big(t(\otimes_{\mathscr{C}})\big(A(\hat{I})\big)\Big)$$

in ${}_{B(I)}\mathbf{Modt}(\mathscr{C})$. Moreover, since A is a lax algebra, the structure morphism

$$\varphi^{A,t}(\hat{I}) : \mathrm{Res}_{\varphi^{B,t}(\hat{I})}\Big(t(\otimes_{\mathscr{C}})\big(A(\hat{I})\big)\Big) \longrightarrow A\big(t(\Box)(\hat{I})\big)$$

of A is a morphism of $B(t(\Box)(\hat{I}))$-module algebras, which implies that the composition

$$\mathrm{Res}_{\varphi^{B,t}(\hat{I})\circ B(\mu_{\hat{I}})}\Big(t(\otimes_{\mathscr{C}})\big(A(\hat{I})\big)\Big) \overset{\varphi^{A,t}(\hat{I})}{\longrightarrow} \mathrm{Res}_{B(\mu_{\hat{I}})}\Big(A\big(t(\Box)(\hat{I})\big)\Big) \overset{A(\mu_{\hat{I}})}{\longrightarrow} A(I)$$

is a morphism of $B(I)$-module algebras. Hence, we can consider the $A(I)$-module

$$\mathcal{t}_t(\hat{M}, I)(\hat{I}, \mu_{\hat{I}}) = \mathrm{Ind}_{A(\mu_{\hat{I}})\circ\varphi^{A,t}(\hat{I})}\bigg(\mathrm{Res}_{\varphi^{B,t}(\hat{I})\circ B(\mu_{\hat{I}})}\Big(t(\otimes_{\mathscr{C}})\big(\hat{M}(\hat{I})\big)\Big)\bigg) \tag{4.5}$$

in ${}_{B(I)}\mathbf{Modt}(\mathscr{C})$. Since the constructions in the right member of (4.5) are functorial in $\hat{I} = (I_1, \ldots, I_n)$ and $\mu_{\hat{I}}$, they also define the corresponding morphism

$$\mathcal{t}_t(\hat{M}, I)(\hat{f}) : \mathcal{t}_t(\hat{M}, I)(\hat{I}, \mu_{\hat{I}}) \longrightarrow \mathcal{t}_t(\hat{M}, I)(\hat{I}', \mu_{\hat{I}'}) \tag{4.6}$$

in ${}_{A(I)}\mathbf{Mod}({}_{B(I)}\mathbf{Modt}(\mathscr{C}))$, given a morphism $\hat{f} = (f_1, \ldots, f_n)$ in $\mathcal{f}_t(I)$ from the object $(\hat{I}, \mu_{\hat{I}})$ to $(\hat{I}', \mu_{\hat{I}'})$. In consequence, (4.5) and (4.6) define the functor (4.3).

We now construct (4.4) and prove that (4.2) is well-defined on objects.

Lemma 4.1. *Let $t \in \Upsilon_{n-1}$ and $\hat{M} = (M_1, \ldots, M_n)$ be a tuple of objects in ${}_{A}\mathbf{Mod}(\mathscr{A})$. Let $f : I \to J$ be a morphism in $\mathscr{F}$. Then, there is a natural isomorphism (4.4) such that $\mathcal{t}_t(\hat{M}, \mathrm{id}_I)$ is the identity natural transformation and*

$$\mathcal{t}_t(\hat{M}, g\circ f)(-) \circ \phi^{\mathcal{g}}(g,f)(-) = \mathcal{t}_t(\hat{M}, g)\big(\mathcal{f}_t(f)(-)\big) \circ \mathcal{g}(g)\big(\mathcal{t}_t(\hat{M}, f)(-)\big),$$

for every morphism $g : J \to K$ in $\mathscr{F}$, where $\phi^{\mathcal{g}}(g,f)$ was defined in (3.4).

Proof. Let $(\hat{I}, \mu_{\hat{I}})$ be an object in $\mathcal{f}_t(I)$, with $\hat{I} = (I_1, \ldots, I_n)$. Then,

$$\begin{aligned}\big(\mathcal{t}_t(\hat{M}, J) \circ \mathcal{f}_t(f)\big)(\hat{I}, \mu_{\hat{I}}) &= \mathcal{t}_t(\hat{M}, J)(\hat{I}, f\circ\mu_{\hat{I}}) \\ &= \mathrm{Ind}_{A(f\circ\mu_{\hat{I}})\circ\varphi^{A,t}(\hat{I})}\bigg(\mathrm{Res}_{\varphi^{B,t}(\hat{I})\circ B(f\circ\mu_{\hat{I}})}\Big(t(\otimes_{\mathscr{C}})\big(\hat{M}(\hat{I})\big)\Big)\bigg)\end{aligned}$$

and

$$\begin{aligned}&\big(\mathcal{g}(f)\circ\mathcal{t}_t(\hat{M}, I)\big)(\hat{I}, \mu_{\hat{I}}) \\ &\quad = \mathrm{Ind}_{A(f)}\Bigg(\mathrm{Res}_{B(f)}\bigg(\mathrm{Ind}_{A(\mu_{\hat{I}})\circ\varphi^{A,t}(\hat{I})}\Big(\mathrm{Res}_{\varphi^{B,t}(\hat{I})\circ B(\mu_{\hat{I}})}\big(t(\otimes_{\mathscr{C}})(\hat{M}(\hat{I}))\big)\Big)\bigg)\Bigg)\end{aligned}$$

are naturally isomorphic by means of the isomorphism ϕ^{I} in Fact 2.3. The identities for the natural isomorphisms $\mathcal{t}_t(\hat{M}, f)$ follow immediately from Fact 2.3 and (2.5) for the homomorphism of bicategories I in Fact 2.3. $\square$

We finally define (4.2) on morphisms and show that it is indeed a functor. Let $\hat{M}' = (M'_1, \ldots, M'_n)$ and $\hat{M} = (M_1, \ldots, M_n)$ be n-tuples of objects in $\mathscr{C} = {}_A\mathbf{Mod}(\mathscr{A})$, and let $F_i : M'_i \to M_i$ be morphisms of A-modules in $\mathscr{A}$. Then, using that the expression in the right of (4.5) is natural in $\hat{M}$ together with Lemma 3.11, the analogous expression to (4.5) with $\hat{F} = (F_1, \ldots, F_n)$ instead of $\hat{M} = (M_1, \ldots, M_n)$ gives a natural transformation

$$t_t(\hat{F}, I) : t_t(\hat{M}', I) \longrightarrow t_t(\hat{M}, I)$$

for every object I of $\mathscr{F}$. The identity

$$t_t(\hat{M}, f)(-) \circ g(f)\big(t_t(\hat{F}, I)(-)\big) = t_t(\hat{F}, J)\big(f(f)(-)\big) \circ t_t(\hat{M}', f)(-)$$

for all morphisms $f : I \to J$ in $\mathscr{F}$ follows from the naturality of the isomorphism ϕ^{I} in Fact 2.3. This shows that (4.2) is well-defined on morphisms. Moreover, the identities

$$\begin{aligned} &t_t(\hat{G}, I) \circ t_t(\hat{F}, I) = t_t(G_1 \circ F_1, \ldots, G_n \circ F_n, I), \\ &t_t(\mathrm{id}_{M_1}, \ldots, \mathrm{id}_{M_n}, I) = \mathrm{id}_{t_t(\hat{M}, I)}. \end{aligned}$$

are immediate, and we thus conclude that (4.2) is indeed a functor.

Definition 4.2. *Given an integer $n \geq 2$ and $t \in \mathrm{Y}_{n-1}$, define the functor **singular tensor product** $t(\boxtimes_{\mathscr{C}}) : \mathscr{C}^n \to \mathscr{C}$ of shape t as the composition of (4.2) and (2.11). Note that we have used Lemma 3.11 to identify $\mathscr{C}$ and g-$\mathbf{Mod}$. If $t \in \mathrm{Y}_1$, we will simply write $(M_1 \boxtimes_{\mathscr{C}} M_2)(I)$ instead of $t(\boxtimes_{\mathscr{C}})(M_1, M_2)(I)$.*

Remark 4.3. *The object $t(\boxtimes_{\mathscr{C}})(\hat{M})(I)$ in Definition 4.2 was chosen in purpose as the colimit of (4.3). Indeed, the careful reader might complain that $t(\boxtimes_{\mathscr{C}})(\hat{M})$ should be defined recursively in terms of the bifunctor $\boxtimes_{\mathscr{C}}$ as we did in Section 2.2. However, since colimits commute with the tensor product $\otimes_{\mathscr{C}}$ on both sides, the referred recursive definition of $t(\boxtimes_{\mathscr{C}})(\hat{M})$ is given by the composition of a functor that is naturally isomorphic to the functor (4.2), and (2.11).*

The fact that B is oplax and contravariant, together with (2.4), implies that

$$a^{\mathscr{C}}_{t \to t'}\big(B(\hat{I})\big) \circ \varphi^{B,t}(\hat{I}) = \varphi^{B,t'}(\hat{I}) \circ B\big(a^{\mathscr{F}}_{t' \to t}(\hat{I})\big), \tag{4.7}$$

for all $t, t' \in \mathrm{Y}_{n-1}$ and any n-tuple $\hat{I} = (I_1, \ldots, I_n)$ of objects in $\mathscr{F}$, where we are using the terminology of Subsection 2.2. Analogously, A being lax and (2.4) tell us that

$$A\big(a^{\mathscr{F}}_{t \to t'}(\hat{I})\big) \circ \varphi^{A,t}(\hat{I}) = \varphi^{A,t'}(\hat{I}) \circ a^{\mathscr{C}}_{t \to t'}\big(A(\hat{I})\big). \tag{4.8}$$

Given $t, t' \in \mathrm{Y}_{n-1}$, let

$$f_{t \to t'} : f_t \longrightarrow f_{t'} \tag{4.9}$$

be the natural transformation defined as follows. Given an object I of $\mathscr{F}$, let $f_{t \to t'}(I) : f_t(I) \to f_{t'}(I)$ be the functor sending $(\hat{I}, \mu_{\hat{I}})$ to $(\hat{I}, \mu_{\hat{I}} \circ a^{\mathscr{F}}_{t' \to t}(\hat{I}))$, and that is the identity on morphisms. It is clear that $f_{t'}(f) \circ f_{t \to t'}(I) = f_{t \to t'}(J) \circ f_t(f)$ for all morphisms $f : I \to J$ in $\mathscr{F}$, which tells us that (4.9) is a natural transformation. In the language of Subsection 2.3, $f_{t \to t'}$ is a strict transformation from f_t to $f_{t'}$. We note moreover that $f_{t' \to t''} \circ f_{t \to t'} = f_{t \to t''}$, for all $t, t', t'' \in \mathrm{Y}_{n-1}$, which implies in particular that every $f_{t \to t'}$ is a natural isomorphism, since $f_{t \to t'} \circ f_{t' \to t}$ is the identity natural transformation of f_t.

We also have the following result.

Lemma 4.4. *Let $t, t' \in \mathrm{Y}_{n-1}$. Then, there exists a natural isomorphism*

$$\mathscr{a}^{\mathscr{C}}_{t\to t'} : \mathscr{t}_t \longrightarrow \mathbf{Trans_{st}}(\mathscr{f}_{t\to t'}, \mathscr{g}) \circ \mathscr{t}_{t'}$$

of functors induced by the morphisms

$$\mathscr{a}^{\mathscr{C}}_{t\to t'}\big(\hat{M}(\hat{I})\big) : t(\otimes_{\mathscr{C}})\big(\hat{M}(\hat{I})\big) \to t'(\otimes_{\mathscr{C}})\big(\hat{M}(\hat{I})\big)$$

for every n-tuples $\hat{I} = (I_1, \ldots, I_n)$ of objects in $\mathscr{F}$ and n-tuples $\hat{M} = (M_1, \ldots, M_n)$ of objects in ${}_A\mathbf{Mod}(\mathscr{A})$.

Proof. For every n-tuple $\hat{M} = (M_1, \ldots, M_n)$ of objects in ${}_A\mathbf{Mod}(\mathscr{A})$, we will first construct a modification $\mathscr{a}^{\mathscr{C}}_{t\to t'}(\hat{M}) : \mathscr{t}_t(\hat{M}) \to \mathscr{t}_{t'}(\hat{M}) \circ \mathscr{f}_{t\to t'}$, *i.e.* a collection of natural transformations $\mathscr{a}^{\mathscr{C}}_{t\to t'}(\hat{M})(I) : \mathscr{t}_t(\hat{M}, I) \to \mathscr{t}_{t'}(\hat{M}, I)\circ\mathscr{f}_{t\to t'}(I)$ for all objects I of $\mathscr{F}$ satisfying (2.7). We first note that the morphism $\mathscr{a}^{\mathscr{C}}_{t\to t'}(\hat{M}(\hat{I}))$ induces an isomorphism of modules

$$\mathscr{a}^{\mathscr{C}}_{t\to t'}\big(\hat{M}(\hat{I})\big) : t(\otimes_{\mathscr{C}})\big(\hat{M}(\hat{I})\big) \longrightarrow \mathrm{Res}_{\mathscr{a}^{\mathscr{C}}_{t\to t'}(B(\hat{I}))}\Big(t'(\otimes_{\mathscr{C}})\big(\hat{M}(\hat{I})\big)\Big) \tag{4.10}$$

over the bialgebra $t(\otimes_{\mathscr{C}})(B(\hat{I}))$, where we considered the isomorphism

$$\mathscr{a}^{\mathscr{C}}_{t\to t'}\big(B(\hat{I})\big) : t(\otimes_{\mathscr{C}})\big(B(\hat{I})\big) \longrightarrow t'(\otimes_{\mathscr{C}})\big(B(\hat{I})\big)$$

of bialgebras in $\mathscr{C}$. Applying $\mathrm{Res}_{\varphi^{B,t}(\hat{I})\circ B(\mu_{\hat{I}})}$ to (4.10) and using identity (4.7) we get an isomorphism

$$\begin{aligned}\mathscr{a}^{\mathscr{C}}_{t\to t'}\big(\hat{M}(\hat{I})\big) : &\mathrm{Res}_{\varphi^{B,t}(\hat{I})\circ B(\mu_{\hat{I}})}\Big(t(\otimes_{\mathscr{C}})\big(\hat{M}(\hat{I})\big)\Big)\\ &\longrightarrow \mathrm{Res}_{\varphi^{B,t'}(\hat{I})\circ B(\mu_{\hat{I}}\circ \mathscr{a}^{\mathscr{F}}_{t'\to t}(\hat{I}))}\Big(t'(\otimes_{\mathscr{C}})\big(\hat{M}(\hat{I})\big)\Big)\end{aligned} \tag{4.11}$$

of $B(I)$-modules in $\mathscr{C}$. The same argument tells us that

$$\begin{aligned}\mathscr{a}^{\mathscr{C}}_{t\to t'}\big(A(\hat{I})\big) : &\mathrm{Res}_{\varphi^{B,t}(\hat{I})\circ B(\mu_{\hat{I}})}\Big(t(\otimes_{\mathscr{C}})\big(A(\hat{I})\big)\Big)\\ &\longrightarrow \mathrm{Res}_{\varphi^{B,t'}(\hat{I})\circ B(\mu_{\hat{I}}\circ \mathscr{a}^{\mathscr{F}}_{t'\to t}(\hat{I}))}\Big(t'(\otimes_{\mathscr{C}})\big(A(\hat{I})\big)\Big)\end{aligned} \tag{4.12}$$

is an isomorphism of $B(I)$-module algebras in $\mathscr{C}$. Note that the (co)domain of (4.11) is a module over the (co)domain of (4.12). Moreover, identity (4.8) tells us that $\mathrm{id}_{A(I)} \otimes_{\mathscr{C}} \mathscr{a}^{\mathscr{C}}_{t\to t'}(\hat{M}(\hat{I}))$ induces an isomorphism $\mathscr{a}^{\mathscr{C}}_{t\to t'}(\hat{M})(I)$ of the form

$$\begin{aligned}&\mathrm{Ind}_{A(\mu_{\hat{I}})\circ\varphi^{A,t}(\hat{I})}\Big(\mathrm{Res}_{\varphi^{B,t}(\hat{I})\circ B(\mu_{\hat{I}})}\big(t(\otimes_{\mathscr{C}})(\hat{M}, \hat{I})\big)\Big)\\ &\longrightarrow \mathrm{Ind}_{A(\mu_{\hat{I}}\circ\mathscr{a}^{\mathscr{F}}_{t'\to t}(\hat{I}))\circ\varphi^{A,t'}(\hat{I})}\Big(\mathrm{Res}_{\varphi^{B,t'}(\hat{I})\circ B(\mu_{\hat{I}}\circ\mathscr{a}^{\mathscr{F}}_{t'\to t}(\hat{I}))}\big(t'(\otimes_{\mathscr{C}})(\hat{M}, \hat{I})\big)\Big)\end{aligned} \tag{4.13}$$

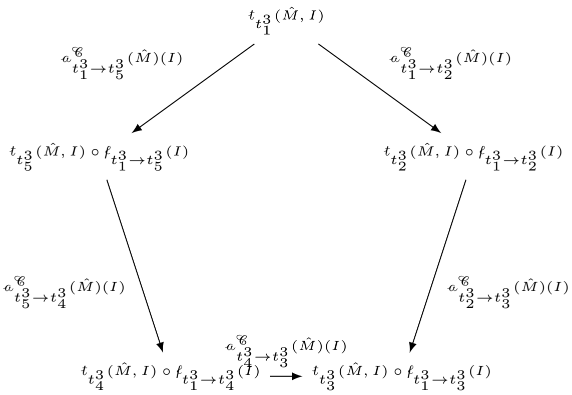
in ${}_{A(I)}\mathbf{Mod}({}_{B(I)}\mathbf{Modt}(\mathscr{C}))$. Identity (2.7) for (4.13) follows immediately from the definitions. Finally, we note that (4.13) is natural in $\hat{M}$, *i.e.* $\mathscr{a}^{\mathscr{C}}_{t\to t'}(\hat{M}) : \mathscr{t}_t(\hat{M}) \to \mathscr{t}_{t'}(\hat{M}) \circ \mathscr{f}_{t\to t'}$ is natural in $\hat{M}$, which implies that $\mathscr{a}^{\mathscr{C}}_{t\to t'} : \mathscr{t}_t \to \mathscr{t}_{t'}$ is a natural isomorphism of functors. □

4.2.2 The monoidal structure

Our first main result is the monoidal structure of the singular tensor product, which is somehow implicit in [8].

Theorem 4.5. *The category $\mathscr{C} = {}_A\mathbf{Mod}(\mathscr{A})$ introduced in Subsection 4.1, endowed with the singular tensor product $\boxtimes_{\mathscr{C}}$ and the unit given by A regarded as a module over itself with the regular action, has a structure of monoidal category with strict unit.*

Proof. We will first prove the pentagon axiom. Let I be an object of $\mathscr{F}$ and let $\hat{M} = (M_1, \dots, M_4)$ be a tetrad of objects in $\mathscr{C} = {}_A\mathbf{Mod}(\mathscr{A})$. The pentagon axiom for ${}_{A(I)}\mathbf{Modt}({}_{B(I)}\mathbf{Modt}(\mathscr{C}))$ gives us the commutative pentagon



where $\Upsilon_3 = \{t_1^3, \dots, t_5^3\}$ are given in (2.1). This induces the same commutative pentagon for the functors $\mathbf{Trans_{st}}(f_{t_1^3 \to t_i^3}, g) \circ t_{t_i^3}(\hat{M})$ instead of $t_{t_i^3}(\hat{M}, I) \circ f_{t_1^3 \to t_i^3}(I)$, for $i \in [\![1, 5]\!]$. By taking the colimit functor (2.11) and using Proposition 2.2, we obtain the analogous commutative pentagon for $t_i^3(\boxtimes_{\mathscr{C}})(\hat{M})$ instead of $t_{t_i^3}(\hat{M}) \circ f_{t_1^3 \to t_i^3}(I)$, for $i \in [\![1, 5]\!]$, as we wanted to show.

We finally prove that A is a strict unit for the tensor product $\boxtimes_{\mathscr{C}}$, *i.e.* $A \boxtimes_{\mathscr{C}} M = M = M \boxtimes_{\mathscr{C}} A$, for all $M \in \mathscr{C}$ such that $a^{\mathscr{C}}(M, A, N) : (M \boxtimes_{\mathscr{C}} A) \boxtimes_{\mathscr{C}} N \to M \boxtimes_{\mathscr{C}} (A \boxtimes_{\mathscr{C}} N)$ is the identity for all $N \in \mathscr{C}$. We first show that $A \boxtimes_{\mathscr{C}} M = M$, and leave the analogous identity $M = M \boxtimes_{\mathscr{C}} A$ to the reader. In order to show this, we note first that, if $t \in \Upsilon_1$, given I in $\mathscr{F}$, then there is an isomorphism

$$\hat{\theta}(I) : \operatorname{colim} t_t\big((A, M), I\big) \longrightarrow M(I) \tag{4.14}$$

of $A(I)$-modules in ${}_{B(I)}\mathbf{Modt}(\mathscr{C})$ defined as follows. Given $\hat{I} = (I_1, I_2)$ and $\mu_{\hat{I}} : I_1 \square I_2 \to I$, let $\mu_{I_j} = \mu_{\hat{I}} \circ \iota_{I_j}$, for $j \in \{1, 2\}$, where $\iota_{I_j} : I_j \to I_1 \square I_2$ was introduced in the penultimate paragraph of Subsection 2.2. The definition of $t_t((A, M), I)$ tells us that the image of the morphism

$$(i_{I_1}, \iota_{I_2}) : \big((\mathbf{I}_{\mathscr{F}}, I_2), \mu_{I_2} \circ \ell^{\mathscr{F}}(I_2)\big) \longrightarrow (\hat{I}, \mu_{\hat{I}}) \tag{4.15}$$

in $f_t(I)$ under the functor $t_t((A, M), I)$ is an isomorphism, where $i_{I_1} : \mathbf{I}_{\mathscr{F}} \to I_1$ is the unique morphism in $\mathscr{F}$ recalled in the penultimate paragraph of Subsection 2.2. On the other hand, using (the proof of) Lemma 3.11, $M(\mu_{I_2})$ induces a morphism

$$\operatorname{Ind}_{A(\mu_{I_2})}\Big(\operatorname{Res}_{B(\mu_{I_2})}\big(M(I_2)\big)\Big) \longrightarrow M(I)$$

of $A(I)$-modules in ${}_{B(I)}\mathbf{Modt}(\mathscr{C})$, which is equivalent to a morphism

$$\theta(I)(\hat{I}, \mu_{\hat{I}}) : t_t\big((A, M), I\big)(\hat{I}, \mu_{\hat{I}}) \longrightarrow M(I)$$

of $A(I)$-modules in ${}_{B(I)}\mathbf{Modt}(\mathscr{C})$, that is natural in $(\hat{I}, \mu_{\hat{I}})$. This induces the morphism (4.14). To prove it is an isomorphism, we recall once more that the image of (4.15) under $t_t((A, M), I)$ is an isomorphism, and consider the morphism

$$(\mathrm{id}_{\mathbf{I}_{\mathscr{F}}}, \mu_{I_2}) : \big((\mathbf{I}_{\mathscr{F}}, I_2), \mu_{I_2} \circ \ell^{\mathscr{F}}(I_2)\big) \longrightarrow \big((\mathbf{I}_{\mathscr{F}}, I), \ell^{\mathscr{F}}(I)\big)$$

in $\mathcal{f}_t(I)$. The fact that $\theta(I)((\mathbf{I}_{\mathscr{F}}, I), \ell^{\mathscr{F}}(I))$ is an isomorphism together with the definition of colimit tell us that $\hat{\theta}(I)$ is an isomorphism as well.

Let $f : I \to J$ be a morphism in $\mathscr{F}$. By comparing the morphism $M(f)$ with the explicit expression of the natural transformation $(\mathrm{colim}(t_t(A, M)))_f$ obtained from (2.12), we see that $(\mathrm{colim}(t_t(A, M)))_f \circ \hat{\theta}(I) = \hat{\theta}(J) \circ M(f)$. As a consequence, $A \boxtimes_{\mathscr{C}} M = M$, as was to be shown. We finally note that Lemma 4.4 tells us that $a^{\mathscr{C}}(M, A, N) : (M \boxtimes_{\mathscr{C}} A) \boxtimes_{\mathscr{C}} N \to M \boxtimes_{\mathscr{C}} (A \boxtimes_{\mathscr{C}} N)$ is the identity for all $M, N \in \mathscr{C}$, as was to be shown. □

Remark 4.6. *Note that the semicocartesian property for $\mathscr{F}$ was only used to prove that A is a (strict) unit of $\mathscr{C}$ for the tensor product $\boxtimes_{\mathscr{C}}$. Moreover, note that we have chosen a fixed colimit of $t_t((A, M), I)$ by means of $\hat{\theta}(I)$ in (4.14), which in turn translates in A being a strict unit for $\boxtimes_{\mathscr{C}}$. This freedom comes from the well-known but implicit fact that the functor (2.11) given by taking colimit is not really uniquely defined, but any choice of colimit for each strong transformation induces such a functor.*

4.2.3 The symmetric braiding

Assume the hypothesis (VA.6) for the rest of the article. We will complete Theorem 4.5 with the braiding property of the singular tensor product.

Given $t \in \mathrm{Y}_{n-1}$ and $\sigma \in \mathbb{S}_n$, define the natural isomorphism

$$\hat{\tau}(\sigma, t) : \mathcal{f}_t \longrightarrow \mathcal{f}_t \tag{4.16}$$

as follows. Given and object I of $\mathscr{F}$, we define the autofunctor $\hat{\tau}(\sigma, t)(I) : \mathcal{f}_t(I) \longrightarrow \mathcal{f}_t(I)$ sending an object $(\hat{I}, \mu_{\hat{I}})$ to $(\sigma \cdot \hat{I}, \mu_{\hat{I}} \circ \tau^{\mathscr{F}}(\sigma, t)(\hat{I})^{-1})$ for $\hat{I} = (I_1, \dots, I_n)$, where we write $\sigma \cdot \hat{I} = (I_{\sigma^{-1}(1)}, \dots, I_{\sigma^{-1}(n)})$, and a morphism $\hat{f} = (f_1, \dots, f_n)$ to $\sigma \cdot \hat{f} = (f_{\sigma^{-1}(1)}, \dots, f_{\sigma^{-1}(n)})$. Recall we are using the notation of Subsection 2.2 for the morphism $\tau^{\mathscr{F}}(\sigma, t)(\hat{I}) : t(\square)(\hat{I}) \to t(\square)(\sigma \cdot \hat{I})$. It is easy to see that

$$\hat{\tau}(\sigma, t)(J) \circ \mathcal{f}_t(f) = \mathcal{f}_t(f) \circ \hat{\tau}(\sigma, t)(I) \text{ and } \hat{\tau}(\sigma, t)(I) \circ \hat{\tau}(\varsigma, t)(I) = \hat{\tau}(\sigma \circ \varsigma, t)(I), \tag{4.17}$$

for any morphism $f : I \to J$ in $\mathscr{F}$ and $\sigma, \varsigma \in \mathbb{S}_n$. The first of (4.17) tells us that (4.16) is a natural transformation. In the language of Subsection 2.3, $\hat{\tau}(\sigma, t)$ is a strict transformation from $\mathcal{f}_t$ to itself. Moreover, the axioms of the braided monoidal category $\mathscr{F}$ tell us that

$$\hat{\tau}(\sigma, t')(I) \circ \mathcal{f}_{t \to t'}(I) = \mathcal{f}_{t \to t'}(I) \circ \hat{\tau}(\sigma, t)(I), \tag{4.18}$$

for $t, t' \in \mathrm{Y}_{n-1}$, I an object of $\mathscr{F}$ and $\sigma \in \mathbb{S}_n$. We will denote by $\hat{\tau}(\sigma, t \to t')$ the natural transformation $\mathcal{f}_t \to \mathcal{f}_{t'}$ whose value at an object I of $\mathscr{F}$ is (4.18).

We have the following interesting property.

Lemma 4.7. *Let $t \in \mathrm{Y}_{n-1}$ and $\sigma \in \mathbb{S}_n$. Then, there exists a natural isomorphism of functors*

$$\tilde{\tau}(\sigma, t) : t_t \longrightarrow \mathbf{Trans_{st}}\big(\hat{\tau}(\sigma, t), g\big) \circ t_t \circ c(\sigma), \tag{4.19}$$

where $c(\sigma) : \mathscr{C}^n \to \mathscr{C}^n$ is the permutation functor recalled in Subsection 2.2.

Proof. Given an n-tuple $\hat{M} = (M_1, \dots, M_n)$ of objects in ${}_A\mathbf{Mod}(\mathscr{A})$, we will define a modification

$$\tilde{\tau}(\sigma, t)(\hat{M}) : t_t(\hat{M}) \longrightarrow \mathbf{Trans_{st}}\big(\hat{\tau}(\sigma, t), g\big) \circ t_t(\sigma \cdot \hat{M}), \tag{4.20}$$

that is natural in $\hat{M}$. We will start by defining a natural isomorphism of functors

$$\tilde{\tau}(\sigma, t)(\hat{M}, I) : t_t(\hat{M}, I) \longrightarrow \mathbf{Trans_{st}}\big(\hat{\tau}(\sigma, t), g\big) \circ t_t(\sigma \cdot \hat{M}, I), \tag{4.21}$$

for every object I of $\mathscr{F}$. Let $(\hat{I}, \mu_{\hat{I}})$ be an object of $\not{f}_t(I)$, with $\hat{I} = (I_1, \dots, I_n)$. Recall first that the morphism of algebras $\tau^{\mathscr{C}}(\sigma, t)(A(\hat{I})) : t(\otimes_{\mathscr{C}})(A(\hat{I})) \to t(\otimes_{\mathscr{C}})(A(\sigma \cdot \hat{I}))$ in $\mathscr{C}$ induces an isomorphism

$$\tau^{\mathscr{C}}(\sigma, t)\big(A(\hat{I})\big) : t(\otimes_{\mathscr{C}})\big(A(\hat{I})\big) \longrightarrow \mathrm{Res}_{\tau^{\mathscr{C}}(\sigma,t)(B(\hat{I}))}\Big(t(\otimes_{\mathscr{C}})\big(A(\sigma \cdot \hat{I})\big)\Big)$$

of algebras in ${}_{t(\otimes_{\mathscr{C}})(B(\hat{I}))}\mathbf{Modt}(\mathscr{C})$. Moreover, $\tau^{\mathscr{C}}(\sigma, t)(\hat{M}(\hat{I})) : t(\otimes_{\mathscr{C}})(\hat{M}(\hat{I})) \to t(\otimes_{\mathscr{C}})((\sigma \cdot \hat{M})(\sigma \cdot \hat{I}))$ induces an isomorphism

$$t(\otimes_{\mathscr{C}})\big(\hat{M}(\hat{I})\big) \longrightarrow \mathrm{Res}_{\tau^{\mathscr{C}}(\sigma,t)(A(\hat{I}))}\left(\mathrm{Res}_{\tau^{\mathscr{C}}(\sigma,t)(B(\hat{I}))}\Big(t(\otimes_{\mathscr{C}})\big((\sigma \cdot \hat{M})(\sigma \cdot \hat{I})\big)\Big)\right) \tag{4.22}$$

of $t(\otimes_{\mathscr{C}})(A(\hat{I}))$-modules in ${}_{t(\otimes_{\mathscr{C}})(B(\hat{I}))}\mathbf{Modt}(\mathscr{C})$. By the same arguments as those at the beginning of Subsection 4.2.1 for the definition of the functor $\not{t}_t(\hat{M}, I)$, we see that applying the induction functor $\mathrm{Ind}_{A(\mu_{\hat{I}}) \circ \varphi^{A,t}(\hat{I})}$ to (4.22) gives the isomorphism

$$\tilde{\tau}(\sigma, t)\big(\hat{M}, I\big)\big(\hat{I}, \mu_{\hat{I}}\big) : \not{t}_t\big(\hat{M}, I\big)\big(\hat{I}, \mu_{\hat{I}}\big) \longrightarrow \not{t}_t\big(\sigma \cdot \hat{M}, I\big) \circ \hat{\tau}(\sigma, t)(I)\big(\hat{I}, \mu_{\hat{I}}\big) \tag{4.23}$$

of $A(I)$-modules in ${}_{B(I)}\mathbf{Modt}(\mathscr{C})$. Since the expressions in (4.23) are natural in $\hat{I}$ and $\mu_{\hat{I}}$, this indeed defines a natural transformation $\tilde{\tau}(\sigma, t)(\hat{M}, I)$. It is straightforward to check that (4.21) satisfies (2.7), and thus (4.20) is a modification. Moreover, if $\hat{F} = (F_1, \dots, F_n)$ is an n-tuple of morphisms $F_i : M_i \to M_i'$ in ${}_A\mathbf{Mod}(\mathscr{A})$, then

$$\not{t}_t(\sigma \cdot \hat{F}, I)\big(\hat{\tau}(\sigma, t)(I)(-)\big) \circ \tilde{\tau}(\sigma, t)(\hat{M}, I)(-) = \tilde{\tau}(\sigma, t)(\hat{M}', I)(-) \circ \not{t}_t(\hat{F}, I)(-),$$

which tells us that $\tilde{\tau}(\sigma, t)$ is a natural transformation. □

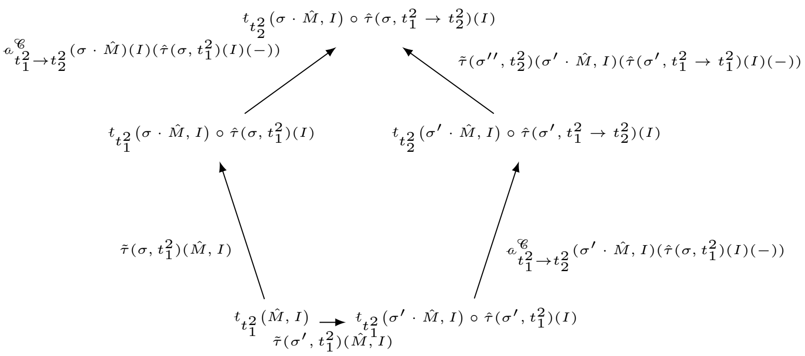
Theorem 4.8. *Given $t \in \mathrm{Y}_{n-1}$ and $\sigma \in \mathbb{S}_n$, applying the functor* colim *given in* (2.11) *to the natural isomorphism $\tilde{\tau}(\sigma, t)$ in* (4.19)*, and using the last part of Proposition* 2.2*, we get a natural isomorphism*

$$\tau^{\mathcal{C}}(\sigma, t) : \not{t}_t(\boxtimes_{\mathcal{C}}) \longrightarrow \not{t}_t(\boxtimes_{\mathcal{C}}) \circ \not{c}(\sigma).$$

This defines a symmetric braiding on the monoidal category $(\mathcal{C}, \boxtimes_{\mathcal{C}}, A)$.

Proof. Given an n-tuple $\hat{M} = (M_1, \dots, M_n)$ of objects in $\mathcal{C}$, we note first that $\tau^{\mathcal{C}}(\sigma, t)(\hat{M})(I)$ is the isomorphism of $A(I)$-modules in ${}_{B(I)}\mathbf{Modt}(\mathscr{C})$ induced by $\tilde{\tau}(\sigma, t)(\hat{M}, I)$ in Lemma 4.7. It is clear $\tau^{\mathcal{C}}(\sigma, t)(\hat{M})$ has inverse $\tau^{\mathcal{C}}(\sigma^{-1}, t)(\sigma \cdot \hat{M})$, since $\tau^{\mathscr{C}}$ and $\tau^{\mathscr{F}}$ are symmetric braidings.

We will finally prove that the natural morphisms $\tau^{\mathcal{C}}(\sigma, t)$ give a symmetric braiding of $\mathcal{C}$. Let $\hat{M} = (M_1, M_2, M_3)$ be a triple of objects in $\mathcal{C}$. The hexagon axiom for the braiding of ${}_{A(I)}\mathbf{Mod}({}_{B(I)}\mathbf{Modt}(\mathscr{C}))$ (see [14], XIII.1.1, (1.3)) gives us the commutative diagram



where $\sigma' \in \mathbb{S}_3$ is the transposition (12), $\sigma'' \in \mathbb{S}_3$ is the transposition (23) and $\sigma = \sigma'' \circ \sigma'$. Note that, by using our notation, we have replaced two edges of the usual hexagon (see [14], XIII.1.1, (1.3)) by their composition given by the simple edge $\tilde{\tau}(\sigma, t_1^2)(\hat{M}, I)$. By taking the colimit, we obtain the analogous commutative pentagon, whose vertices are the corresponding tensor products $t_i^2(\boxtimes_{\mathscr{C}})(\varsigma \cdot \hat{M})(I)$, with $i \in [\![1, 2]\!]$ and $\varsigma \in \{\mathrm{id}_{[\![1,3]\!]}, \sigma', \sigma'', \sigma\}$, which implies that $\tau^{\mathscr{C}}$ is a symmetric braiding. □

Remark 4.9. *The previous theorem does not hold a priori if $\mathscr{F}$ and $\mathscr{C}$ are just braided monoidal categories, since in that case the tensor product of bialgebras is not necessarily a bialgebra, and the braiding morphism of a tensor product of algebras is not a morphism of algebras (see Remark 2.5). These two results, which do hold for symmetric monoidal categories, were used to be even define the braiding of $\mathscr{C}$ obtained in Lemma 4.7 and Theorem 4.8. This implies at least that a positive answer to Problem 5.7 in [8] cannot be given by the simple extension of the previous arguments to braided categories.*

4.3 The double monoidal structure

Recall the functor $\mathcal{g}$ defined in (3.3). Given an integer $m \geq 2$, define the homomorphism of bicategories

$$\mathcal{g}^m : \mathscr{F} \longrightarrow \mathbf{Cat}$$

given by the map $I \mapsto \mathcal{g}(I)^m$ sending an object I of $\mathscr{F}$ to the product category of m copies of $\mathcal{g}(I)$, the map $f \mapsto \mathcal{g}(f)^m$ sending a morphism $f : I \to J$ of $\mathscr{F}$ to the diagonal functor $\mathcal{g}(f)^m : \mathcal{g}(I)^m \to \mathcal{g}(J)^m$ each of which component is $\mathcal{g}(f)$, and, given morphisms $f : I \to J$ and $g : J \to K$ in $\mathscr{F}$, the natural isomorphism $\phi^{\mathcal{g}^m}(g, f) : \mathcal{g}(g)^m \circ \mathcal{g}(f)^m \to \mathcal{g}(g \circ f)^m$ such that the i-th component of $\phi^{\mathcal{g}^m}(g, f)(X_1, \dots, X_m)$ is $\phi^{\mathcal{g}}(g, f)(X_i)$, for $X_1, \dots, X_m$ objects of $\mathcal{g}(I)$. It is trivial to verify that $\mathcal{g}^m$ is indeed a homomorphism of bicategories.

Given $t \in \mathrm{Y}_{m-1}$, we now define a strong transformation

$$t(\otimes_{\mathcal{g}}) : \mathcal{g}^m \longrightarrow \mathcal{g} \tag{4.24}$$

of homomorphisms of bicategories. For an object I of $\mathscr{F}$, let $t(\otimes_{\mathcal{g}})(I) : \mathcal{g}(I)^m \to \mathcal{g}(I)$ be the functor $t(\otimes_{A(I)})$, and if $f : I \to J$ be a morphism in $\mathscr{F}$, let $t(\otimes_{\mathcal{g}})(f) : \mathcal{g}(f) \circ t(\otimes_{\mathcal{g}})(I) \to t(\otimes_{\mathcal{g}})(J) \circ \mathcal{g}(f)^m$ be the unique natural isomorphism given by (2.18). It is straightforward to prove that (4.24) is indeed a strong transformation.

Recall the notation on trees introduced in Subsection 2.2 and at the beginning of Subsection 2.5. Let $n \geq 2$ be an integer and $\mathfrak{t} = (t, \lambda) \in \mathrm{Y}^{\mathrm{lab}}_{n-1}$ such that there exist thin trees $t_\bullet \in \mathrm{Y}_{n_\bullet - 1}$ and $t_\blacksquare \in \mathrm{Y}_{n_\blacksquare - 1}$ with t obtained by grafting the root of a copy of $t_\blacksquare$ to each leaf of $t_\bullet$, $\lambda(v) = \blacksquare$ if v is a vertex in a copy of $t_\blacksquare$ and $\lambda(v) = \bullet$ if v is a vertex in $t_\bullet$. We will say in any of these cases that $\mathfrak{t}$ is **extremal**.

Assume that $\mathfrak{t} = (t, \lambda) \in \mathrm{Y}^{\mathrm{lab}}_{n-1}$ is extremal, with associated trees $t_\bullet \in \mathrm{Y}_{n_\bullet - 1}$ and $t_\blacksquare \in \mathrm{Y}_{n_\blacksquare - 1}$. We will consider the functor

$$\mathcal{f}_{\mathfrak{t}} : \mathscr{F} \longrightarrow \mathbf{Cat}$$

defined as follows. First, given an object I of $\mathscr{F}$, define the category $\mathcal{f}_{\mathfrak{t}}(I) = \prod_{i=1}^{n_\bullet} \mathcal{f}_{t_\blacksquare}(I)$. If $f : I \to J$ is a morphism in $\mathscr{F}$, let $\mathcal{f}_{\mathfrak{t}}(f) : \mathcal{f}_{\mathfrak{t}}(I) \to \mathcal{f}_{\mathfrak{t}}(J)$ be the functor sending an $n_\bullet$-tuple $(X_1, \dots, X_{n_\bullet})$ of objects (resp., morphisms) of $\mathcal{f}_{t_\blacksquare}(I)$ to $(\mathcal{f}_{t_\blacksquare}(f)(X_1), \dots, \mathcal{f}_{t_\blacksquare}(f)(X_{n_\bullet}))$. Note that, given $t \in \mathrm{Y}_{n-1}$, if $\mathfrak{t} = (t, \lambda)$ with λ constant equal to $\blacksquare$, then $\mathfrak{t}$ is extremal with associated trees $t_\bullet \in \mathrm{Y}_0$ and $t_\blacksquare = t$. Moreover, in this case $\mathcal{f}_{\mathfrak{t}}$ coincides with the functor $\mathcal{f}_t$ defined in (4.1). As in that case, if we consider $\mathscr{F}$ as a 2-category where every morphism space $\mathrm{Hom}_{\mathscr{F}}(I, J)$ is a discrete category, then the functor $\mathcal{f}_{\mathfrak{t}}$ is a strict homomorphism of bicategories.

Let $\mathfrak{t} = (t, \lambda) \in \mathrm{Y}^{\mathrm{lab}}_{n-1}$ be extremal, with associated trees $t_\bullet \in \mathrm{Y}_{n_\bullet - 1}$ and $t_\blacksquare \in \mathrm{Y}_{n_\blacksquare - 1}$. Define the canonical functor

$$\prod_{i=1}^{n_\bullet} \mathbf{Trans_{st}}(\mathscr{f}_{t_\blacksquare}, \mathscr{g}) \longrightarrow \mathbf{Trans_{st}}\big(\mathscr{f}_{\mathfrak{t}}, \mathscr{g}^{n_\bullet}\big) \tag{4.25}$$

as follows. Given an $n_\bullet$-tuple $\hat{\zeta} = (\zeta_1, \dots, \zeta_{n_\bullet})$ with ζ_i an object of $\mathbf{Trans_{st}}(\mathscr{f}_{t_\blacksquare}, \mathscr{g})$, (4.25) sends it to $\bar{\zeta}$ such that $\bar{\zeta}(I)(X_1, \dots, X_{n_\bullet}) = (\zeta_1(I)(X_1), \dots, \zeta_{n_\bullet}(I)(X_{n_\bullet}))$ and $\bar{\zeta}(f) : \mathscr{g}(f)^{n_\bullet} \circ \bar{\zeta}(I) \to \bar{\zeta}(J) \circ \mathscr{f}_{\mathfrak{t}}(f)$ is $(\zeta_1(f), \dots, \zeta_{n_\bullet}(f))$, for objects I, J and morphisms $f : I \to J$ of $\mathscr{F}$, and X_i object of $\mathscr{f}_{t_\blacksquare}(I)$ for $i \in [\![1, n_\bullet]\!]$. Moreover, given a morphism $\hat{\Gamma} = (\Gamma_1, \dots, \Gamma_{n_\bullet})$ from $\hat{\zeta} = (\zeta_1, \dots, \zeta_{n_\bullet})$ to $\hat{\zeta}' = (\zeta'_1, \dots, \zeta'_{n_\bullet})$, *i.e.* $\Gamma_i : \zeta_i \to \zeta'_i$ is a modification for every $i \in [\![1, n_\bullet]\!]$, then (4.25) sends Γ to $\bar{\Gamma} : \bar{\zeta} \to \bar{\zeta}'$, where $\bar{\Gamma}(I)(X_1, \dots, X_{n_\bullet}) = (\Gamma_1(I)(X_1), \dots, \Gamma_{n_\bullet}(I)(X_{n_\bullet}))$, where I is an object of $\mathscr{F}$ and X_i an object of $\mathscr{f}_{t_\blacksquare}(I)$ for $i \in [\![1, n_\bullet]\!]$.

We will also consider the functor

$$\mathscr{t}_{\mathfrak{t}} : \mathscr{C}^n \longrightarrow \mathbf{Trans_{st}}(\mathscr{f}_{\mathfrak{t}}, \mathscr{g})$$

as the composition of

$$\prod_{i=1}^{n_\bullet} \mathscr{t}_{t_\blacksquare} : \prod_{i=1}^{n_\bullet} \mathscr{C}^{n_\blacksquare} \longrightarrow \prod_{i=1}^{n_\bullet} \mathbf{Trans_{st}}(\mathscr{f}_{t_\blacksquare}, \mathscr{g}),$$

(4.25) and

$$\mathbf{Trans_{st}}\big(\mathscr{f}_{\mathfrak{t}}, t_\bullet(\otimes_{\mathscr{g}})\big) : \mathbf{Trans_{st}}\big(\mathscr{f}_{\mathfrak{t}}, \mathscr{g}^{n_\bullet}\big) \longrightarrow \mathbf{Trans_{st}}(\mathscr{f}_{\mathfrak{t}}, \mathscr{g}).$$

Let $m, n \geq 2$ be integers with nm even, and $\mathfrak{t} = (t, \lambda) \in \mathrm{Y}^{\mathrm{lab}}_{mn-1}$ be the unique extremal object with associated trees $t' = t_\blacksquare \in \mathrm{Y}_{n-1}$ and $t'' = t_\bullet \in \mathrm{Y}_{m-1}$. Define the natural transformation

$$\mathscr{s}\mathscr{f}_{\mathfrak{t}} : \mathscr{f}_{t'} \longrightarrow \mathscr{f}_{\mathfrak{t}}$$

as follows. Given an object I of $\mathscr{F}$, let $\mathscr{s}\mathscr{f}_{\mathfrak{t}}(I) : \mathscr{f}_{t'}(I) \to \mathscr{f}_{\mathfrak{t}}(I)$ be the functor sending the object (resp., morphism) X in $\mathscr{f}_{t'}(I)$ to the object (resp., morphism) in $\mathscr{f}_{\mathfrak{t}}$ whose i-th component is X for $i \in [\![1, n]\!]$. It is clear that $\mathscr{s}\mathscr{f}_{\mathfrak{t}}$ is natural in I, and in particular it is a strict transformation of homomorphisms of bicategories. Moreover, it is easy to see that

$$\mathbf{Trans_{st}}(\mathscr{s}\mathscr{f}_{\mathfrak{t}}, \mathscr{g}) \circ \mathscr{t}_{\mathfrak{t}} = \mathscr{t}_{t'} \circ t''(\otimes_{\mathscr{C}})^n \circ \mathscr{c}(\sigma), \tag{4.26}$$

where $\mathscr{c}(\sigma) : \mathscr{C}^{mn} \to \mathscr{C}^{mn}$ is the functor induced by the permutation $\sigma \in \mathbb{S}_{mn}$ given by $\sigma(k+pm) = 1+p+(k-1)n$, for $p \in [\![0, n-1]\!]$ and $k \in [\![1, m]\!]$, and $t''(\otimes_{\mathscr{C}})^n : \mathscr{C}^{mn} \to \mathscr{C}^n$ is the functor sending $(\hat{X}_1, \dots, \hat{X}_n)$ to $(t''(\otimes_{\mathscr{C}})(\hat{X}_1), \dots, t''(\otimes_{\mathscr{C}})(\hat{X}_n))$, where $\hat{X}_i$ is an m-tuple of objects (resp., morphisms) of $\mathscr{C}$ for $i \in [\![1, n]\!]$. In particular, by taking $n = m = 2$, the last part of Proposition 2.2 applied to (4.26) gives us the natural morphism

$$\mathrm{sh}(M_1, M'_1, M_2, M'_2) : (M_1 \otimes_{\mathscr{C}} M'_1) \boxtimes_{\mathscr{C}} (M_2 \otimes_{\mathscr{C}} M'_2) \longrightarrow (M_1 \boxtimes_{\mathscr{C}} M_2) \otimes_{\mathscr{C}} (M'_1 \boxtimes_{\mathscr{C}} M'_2) \tag{4.27}$$

in $\mathscr{C}$, for all objects M_1, M'_1, M_2, M'_2 in $\mathscr{C}$.

Theorem 4.10. *Consider the category $\mathscr{C}$ endowed with the symmetric monoidal structure $(\mathscr{C}, \otimes_{\mathscr{C}}, \mathbf{I}_\otimes) = {}_A\mathbf{Mods}(\mathscr{A})$ as well as the symmetric monoidal structure $(\mathscr{C}, \boxtimes_{\mathscr{C}}, A)$ given in Theorems 4.5 and 4.8. Let $\mu_\otimes = \Delta_\boxtimes = \nu = \mathrm{id}_A$ and the morphism (4.27). Then, $\mathscr{C}$ is a symmetric 2-monoidal category.*

Proof. Since the units for both monoidal structures are strict and coincide, condition (i) in Definition 2.6 is automatically verified. Moreover, it is straightforward to verify that (4.27) is the identity morphism if two arguments are given by the unit A of either of the monoidal structures. Indeed, $\mathrm{sh}(M_1, M_1', M_2, M_2')$ with either $M_1 = M_1' = A$ or $M_2 = M_2' = A$ is immediately seen to be the identity, whereas the case with either either $M_1 = M_2 = A$ or $M_1' = M_2' = A$ follows from a slightly more involved argument that is similar to the proof of the unit in Theorem 4.5. Since the left and right unit constraints are the identity, the four square diagrams in condition (ii) in Definition 2.6 are commutative.

We will prove that the hexagonal diagrams in condition (iii) of Definition 2.6 commute. Let $\hat{X} = (X_1, Y_1, X_2, Y_2, X_3, Y_3) \in \mathscr{C}^6$. Concerning the first diagram, consider $\mathfrak{t} = (t, \lambda) \in \mathrm{Y}_3^{\mathrm{lab}}$ and $\mathfrak{t}' = (t', \lambda) \in \mathrm{Y}_3^{\mathrm{lab}}$ be the unique extremal objects with $t_{\blacksquare} = t_1^2$ and $t_{\bullet} \in \mathrm{Y}_1$, and $t'_{\blacksquare} = t_2^2$ and $t'_{\bullet} \in \mathrm{Y}_1$, respectively. Then, $\mathfrak{t}(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}})(\hat{X})$ is the upper vertex and the next one to its right is $\mathfrak{t}'(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}})(\hat{X})$. Moreover, the composition of the two morphisms to the left of $\mathfrak{t}(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}})(\hat{X})$ is the natural morphism given by the last part of Proposition 2.2 applied to the identity (4.26) for $\mathfrak{t}$ in this case. In the same manner, the composition of the two morphisms to the right of $\mathfrak{t}'(\otimes_{\mathscr{C}}, \boxtimes_{\mathscr{C}})(\hat{X})$ is the natural morphism given by the last part of Proposition 2.2 applied to the identity (4.26) for $\mathfrak{t}'$. The commutativity of the hexagonal diagram now follows from the fact that the remaining arrows of the hexagon are natural isomorphisms, so the corresponding colimits commute.

The proof of the commutativity of the second hexagonal diagram in condition (iii) of Definition 2.6 follows from the same argument applied to the unique extremal objects $\mathfrak{t} = (t, \lambda) \in \mathrm{Y}_3^{\mathrm{lab}}$ and $\mathfrak{t}' = (t', \lambda) \in \mathrm{Y}_3^{\mathrm{lab}}$ with $t_{\bullet} = t_1^2$ and $t_{\blacksquare} \in \mathrm{Y}_1$, and $t'_{\bullet} = t_2^2$ and $t'_{\blacksquare} \in \mathrm{Y}_1$, respectively.

Finally, the $\otimes_{\mathscr{C}}$-braiding property of (4.27) follows immediately from the symmetric property of the monoidal product $\otimes_{\mathscr{C}}$ and the universal property of colimits, whereas the $\boxtimes_{\mathscr{C}}$-braiding property of (4.27) follows from the previous arguments together with the symmetric property of the monoidal product $\square$ of $\mathscr{F}$. $\square$

5 Application to vertex algebras (after R. Borcherds)

5.1 Categorical vertex algebras

5.1.1 A general framework

We say that an element I of $\mathscr{F}$ is **irreducible** if $I \simeq J \square K$ implies that $J \simeq \mathbf{I}_{\mathscr{F}}$ or $K \simeq \mathbf{I}_{\mathscr{F}}$. Let $\mathbf{Irr}(\mathscr{F})$ be the full subcategory of $\mathscr{F}$ formed by all irreducible objects of $\mathscr{F}$. We say that $\mathscr{F}$ is **factorial** if given any noninitial object I, there exist $t \in Y_{n-1}$, an n-tuple $\hat{I} = (I_1, \dots, I_n)$ of objects in $\mathbf{Irr}(\mathscr{F})$ and an isomorphism $e : t(\square)(\hat{I}) \to I$, with $\hat{I}$ unique up to permutation and isomorphisms of $\mathscr{F}$, *i.e.* given $t' \in Y_{n'-1}$ and an n'-tuple $\hat{I}' = (I_1', \dots, I_{n'}')$ of objects in $\mathbf{Irr}(\mathscr{F})$ with an isomorphism $e' : t'(\square)(\hat{I}') \to I$, then $n = n'$ and there exist a unique $\sigma \in \mathbb{S}_n$ and a unique n-tuple $\hat{e} = (e_1, \dots, e_n)$ of isomorphisms $e_j : I_j' \to I_{\sigma^{-1}(j)}$ such that e' is the composition of $t'(\square)(\hat{e}) : t'(\square)(\hat{I}') \to t'(\square)(\sigma \cdot \hat{I})$, the unique isomorphism $t'(\square)(\sigma \cdot \hat{I}) \to t(\square)(\hat{I})$ obtained from the braiding and the associativity constraint of $\mathscr{F}$, and e. By abuse of notation, we call the n-tuple $\hat{I} = (I_1, \dots, I_n)$ the **irreducible decomposition** of I.

Remark 5.1. *We did not include an analogous natural condition on the morphisms in the definition of factorial category for the simple reason we will not make use of it.*

Recall that $\mathbf{Irr}(\mathscr{F})$ is **coreflective** if the inclusion functor $L : \mathbf{Irr}(\mathscr{F}) \to \mathscr{F}$ has a right adjoint U. We recall that this is tantamount to the counit $\epsilon(I) : L(U(I)) \to I$ of the adjunction being an isomorphism (see [4], Prop. 3.4.1). In this case, let Q be the endofunctor of $\mathscr{F}$ given by the composition of U and L, which is a **comonad** (*i.e.* an algebra in the monoidal category of endofunctors of $\mathscr{F}$ for the tensor product given by composition and the unit given by the identity functor) for the coproduct $\Delta_Q(I) = L(\eta(U(I)))$ and the counit $\epsilon_Q(I) = \epsilon(I)$, where $\epsilon(I) : L(U(I)) \to I$ is the counit of the adjunction and $\eta(I) : I \to U(L(I))$ is the unit. By (the dual of) [5], Prop. 4.2.3, the coreflective condition on $\mathbf{Irr}(\mathscr{F})$ is equivalent to the comonad Q being **idempotent**, *i.e.* $\Delta_Q(I)$ is an isomorphism for every object I of $\mathscr{F}$.

Given a noninitial object I of $\mathscr{F}$ with irreducible decomposition $\hat{I} = (I_1, \dots, I_n)$ and isomorphism $e : t(\square)(\hat{I}) \to I$. Let $\iota_{i,t}(\hat{I}) : I_i \to t(\square)(\hat{I})$ be the morphism defined in the penultimate paragraph of Subsection 2.2. Define the n-tuple $\hat{\kappa} = (\kappa_1, \dots, \kappa_n)$ of morphisms $\kappa_i : I_i \to Q(I)$, where κ_i is given as the composition of the inverse of the counit $\epsilon_Q(I_i) : Q(I_i) \to I_i$ of the endofunctor Q evaluated at I_i and $Q(e \circ \iota_{i,t}(\hat{I}))$. It is trivial to see that $\epsilon_Q(I) \circ \kappa_i = e \circ \iota_{i,t}(\hat{I})$, where $\epsilon_Q(I) : Q(I) \to I$ is the counit of Q evaluated at I.

Assume that $\mathscr{F}$ is factorial, $\mathbf{Irr}(\mathscr{F})$ is coreflective with associated endofunctor Q. Let I be a noninitial object in $\mathscr{F}$ with irreducible decomposition $\hat{I} = (I_1, \dots, I_n)$. Note that there are a morphism of bialgebras $B(\kappa_i) : B(Q(I)) \to B(I_i)$ and a morphism $A(\kappa_i) : \mathrm{Res}_{B(\kappa_i)}(A(I_i)) \to A(Q(I))$ of $B(Q(I))$-module algebras in $\mathscr{C}$. Let

$$\kappa(t, A, I) : t(\otimes_{\mathscr{C}})\Big(\mathrm{Res}_{B(\kappa_1)}\big(A(I_1)\big), \dots, \mathrm{Res}_{B(\kappa_n)}\big(A(I_n)\big)\Big) \longrightarrow A\big(Q(I)\big) \tag{5.1}$$

be the morphism given as the composition of $t(\otimes_{\mathscr{C}})(\hat{\kappa})$ and $\mu^t_{A(Q(I))}$. We will say that A is **admissible** if $\kappa(t, A, I)$ is an epimorphism of commutative $B(Q(I))$-module algebras in $\mathscr{C}$ for all noninitial objects I of $\mathscr{F}$. For simplicity, we will denote the domain of $\kappa(t, A, I)$ by $t(\otimes_{\mathscr{C}})(A(\hat{I}))$.

Given $M \in \mathscr{C} = {}_A\mathbf{Mod}(\mathscr{A})$, we say that M is **admissible** if given any noninitial object I of $\mathscr{F}$, the morphism $M(\epsilon_Q(I)) : M(Q(I)) \to M(I)$ obtained from the counit $\epsilon_Q(I) : Q(I) \to I$ of the endofunctor Q induces an isomorphism

$$\mathrm{Ind}_{A(f) \circ \varphi^{A,t}(\hat{I})}\Big(\mathrm{Res}_{\kappa(t,A,I)}\big(M\big(Q(I)\big)\big)\Big) \longrightarrow M(I), \tag{5.2}$$

where $f : t(\square)(\hat{I}) \to I$ is the isomorphism given by the irreducible decomposition.

The previous notions verify the following nice property.

Proposition 5.2. *Assume that $\mathscr{F}$ is factorial and A is admissible. Then, the full subcategory ${}_A\mathbf{Adm}$ of $\mathscr{C} = {}_A\mathbf{Mods}(\mathscr{A})$ formed by all admissible submodules is a (symmetric) monoidal subcategory.*

Proof. We have to show that $A \in {}_A\mathbf{Adm}$ and that, given M' and M' in ${}_A\mathbf{Adm}$, then $M' \otimes_A M' \in {}_A\mathbf{Adm}$. For the first, Lemma 2.4 tells us that the map (5.2) for $M = A$ is an isomorphism. Analogously, given M' and M'' in ${}_A\mathbf{Adm}$, the fact that (5.2) is an isomorphism for $M = M'$ and $M = M''$ gives us the canonical isomorphism

$$M'(I) \otimes_{A(I)} M''(I) \simeq \mathrm{Ind}_{A(f) \circ \varphi^{A,t}(\hat{I})}\Big(M'\big(Q(I)\big) \otimes_{t(\otimes_{\mathscr{C}})(A(\hat{I}))} M''\big(Q(I)\big)\Big). \tag{5.3}$$

Since (5.1) is an epimorphism, Lemma 2.4 gives us in turn the canonical isomorphism

$$M'\big(Q(I)\big) \otimes_{t(\otimes_{\mathscr{C}})(A(\hat{I}))} M''\big(Q(I)\big) \longrightarrow M'\big(Q(I)\big) \otimes_{A(Q(I))} M''\big(Q(I)\big),$$

which together with (5.3) implies that (5.2) is an isomorphism for $M = M' \otimes_A M'$. □

5.1.2 Categorical vertex algebras

We finally present in this paragraph the definition of a categorical (quantum) vertex algebra. This extends the notion of (quantum) vertex algebras introduced in [8], Def. 3.12, which are a particular case of our notion, but satisfies all the properties mentioned by Borcherds for (quantum) vertex algebras. Incidentally, the hypothesis of admissibility on A does not appear in [8], but it is required to have that the classical tensor product of vertex algebras is also a vertex algebra (see Proposition 5.4).

Definition 5.3. *Recall that we suppose the data (VA.1)-(VA.6). Assume further that the algebra $A \in \mathscr{A}$ is admissible. A* ***categorical vertex algebra*** *is an admissible module $M \in \mathscr{C}$ together with the structure of commutative algebra with respect to the singular tensor structure $(\mathscr{C}, \boxtimes_{\mathscr{C}}, A)$. Moreover, a* ***categorical quantum vertex algebra*** *is an admissible module $M \in \mathscr{C}$ together with the structure of braided algebra with respect to the singular tensor structure $(\mathscr{C}, \boxtimes_{\mathscr{C}}, A)$ (see [8], Def. 2.4).*

Proposition 5.4. *Given two categorical vertex algebras M and N in $\mathscr{C}$, the algebra structure in Proposition 2.9 for tensor product $M \otimes_A N$ is a categorical vertex algebra.*

Proof. This follows directly from Propositions 2.9 and 5.2. □

5.2 The definition of $(\mathscr{C}, H, A)$-vertex algebra by Borcherds

5.2.1 Some finitary categories

We first collect here some basic definitions on categories, completing [8]. We recall that $\mathbf{Set}$ denotes the category whose objects are given by all sets, the morphisms are given by usual maps and the composition is the usual one of maps. We denote by $\mathbf{FinSet}$ the full subcategory of $\mathbf{Set}$ whose objects are given by all finite sets. Let $\mathbf{FinSet^{eq}}$ be the category whose objects are pairs $(X, \sim_X)$, where X is a finite set and $\sim_X$ is an equivalence relation on X, whose set of morphisms from $(X, \sim_X)$ to $(Y, \sim_Y)$ is formed by all maps $f : X \to Y$ satisfying that $x \sim_X x'$ whenever $f(x) \sim_Y f(x')$, and the composition is the usual one of maps. We have the following immediate result.

Fact 5.5. *The assignment sending $(X, \sim_X)$ to X and any map to itself defines a faithful functor $U : \mathbf{FinSet^{eq}} \to \mathbf{FinSet}$. It has a fully faithful left adjoint $L : \mathbf{FinSet} \to \mathbf{FinSet^{eq}}$ that sends X to $(X, X \times X)$ and is the identity on morphisms.*

The category $\mathbf{FinSet}$ is finitely cocomplete, *i.e.* finite coproducts exist, since they are described by disjoint unions, denoted by $\sqcup$. In consequence, $(\mathbf{FinSet}, \sqcup, \emptyset)$ is a symmetric monoidal category, usually called **cocartesian**. On the other hand, given $(X, \sim_X)$ and $(Y, \sim_Y)$ in $\mathbf{FinSet^{eq}}$, let $(X, \sim_X)\square(Y, \sim_Y)$ be the set $X \sqcup Y$ with the equivalence relation $\sim_X \sqcup \sim_Y$, and $\mathcal{I} = L(\emptyset)$. We have the following immediate result.

Fact 5.6. *There is a unique structure of braided monoidal category on $(\mathbf{FinSet^{eq}}, \square, \mathcal{I})$ such that U is a braided strong monoidal functor. Equivalently, the associativity (resp., left unit, right unit, braiding) isomorphisms of $(\mathbf{FinSet^{eq}}, \square, \mathcal{I})$ are given by the morphisms whose image under the functor U are the associativity (resp., left unit, right unit, braiding) isomorphisms of $(\mathbf{FinSet}, \sqcup, \emptyset)$.*

The braiding of $(\mathbf{FinSet^{eq}}, \square, \mathcal{I})$ is symmetric, and the functor L is braided oplax monoidal, with structure morphisms given by the identity maps. Moreover, $\mathbf{FinSet^{eq}}$ is clearly semicocartesian.

We also have the following result.

Fact 5.7. *The subcategory* $\mathrm{Irr}(\mathbf{FinSet}^{\mathbf{eq}})$ *of* $\mathbf{FinSet}^{\mathbf{eq}}$ *is precisely the image of the fully faithful functor* $L : \mathbf{FinSet} \to \mathbf{FinSet}^{\mathbf{eq}}$*, whose right adjoint is the forgetful functor* U*, so it is coreflective. Moreover, the semicocartesian category* $(\mathbf{FinSet}^{\mathbf{eq}}, \square, \mathscr{I}, \tau)$ *is factorial.*

Proof. It is clear that $L(X)$ is irreducible for all $X \in \mathbf{FinSet}$. Conversely, given $(X, \sim_X) \neq L(X)$ with $X \neq \emptyset$, then there exist $x, y \in X$ such that x and y are in different equivalence classes. Let X' be the equivalence class of x and $X'' = X \setminus X'$. Then, $(X', \sim_X |_{X' \times X'}) \square (X'', \sim_X |_{X'' \times X''}) = (X, \sim_X)$, which implies that X is not irreducible.

Moreover, given any $(X, \sim_X)$ in $\mathbf{FinSet}^{\mathbf{eq}}$ with $X \neq \emptyset$, let $\{X_i : i \in [\![1, n]\!]\}$ be the set of equivalence classes of $\sim_X$. Then, we immediately get an isomorphism $e : (X, \sim_X) \to L(X_1) \square \dots \square L(X_n)$. The uniqueness condition is a straightforward exercise. $\square$

5.2.2 Some basic examples of (co)algebras

The following is one of the main examples of (co)algebras in the category of functors considered in [8] (see Def. 3.3 and Example 3.4).

Let $\mathscr{C}$ be a symmetric monoidal category. We assume for simplicity that there is a faithful strong monoidal functor $\mathscr{C} \to {}_k\mathbf{Mod}$ inside the category of modules over a commutative unitary ring k, but the general case for the definitions below follows from considering the usual notion of (categorical) element x of an object X in a category given by a morphism $x : X' \to X$ (see [22]).

Let A be a commutative algebra in $\mathscr{C}$. Define $T_*(A) \in \mathbf{Fun}(\mathbf{FinSet}, \mathscr{C})$ as the unique functor such that

$$T_*(A)(I) = \bigotimes_{i \in I} A_i \qquad \text{for all finite sets } I \text{ and } A_i = A \text{ for } i \in I,$$

and, given I, J finite sets and $f : I \to J$ a map, let $T_*(A)(f) : T_*(A)(I) \to T_*(A)(J)$ be given by

$$T_*(A)(f)\Big(\bigotimes_{i \in I} x_i\Big) = \bigotimes_{j \in J} y_j, \text{ with } y_j = \mu^{[\#(f^{-1}(\{j\}))]}\Big(\bigotimes_{i \in f^{-1}(\{j\})} x_i\Big).$$

Given a cocommutative coalgebra C in $\mathscr{C}$, set $T^*(C) \in \mathbf{Fun}(\mathbf{FinSet}^{\mathrm{op}}, \mathscr{C})$ as the unique functor such that

$$T^*(C)(I) = \bigotimes_{i \in I} C_i \qquad \text{for all finite sets } I \text{ and } C_i = C \text{ for } i \in I,$$

and, given I, J finite sets and $f : I \to J$ a map, let $T^*(C)(f) : T^*(C)(J) \to T^*(C)(I)$ be given by

$$T^*(C)(f)\Big(\bigotimes_{j \in J} x_j\Big) = \prod_{j \in J \setminus f(I)} \epsilon(x_j) \Big(\bigotimes_{i \in I} y_i\Big), \text{ with} \bigotimes_{i \in f^{-1}(\{j\})} y_i = \Delta^{[\#(f^{-1}(\{j\}))]}(x_j),$$

for $j \in f(I)$.

Notice that the (co)commutativity is essential for the good definition. Moreover, a direct application of Fact 3.1 tells us that $T_*(A)$ (resp., $T^*(C)$) is a (co)commutative (co)algebra in the category $\mathbf{Funi}(\mathbf{FinSet}, \mathscr{C})$ (resp., $\mathbf{Funi}(\mathbf{FinSet}^{\mathrm{op}}, \mathscr{C})$). We also note that $T_* A$ is a lax algebra, whereas $T^* C$ is an oplax coalgebra, for the structure morphisms of Lemma 3.2. Furthermore, applying the same result we also conclude that if A is further assumed to be a (resp., cocommutative) coalgebra or a bialgebra, then $T_*(A)$ is also. Dually, if C is further assumed to be a (resp., commutative) algebra or a bialgebra, then $T^*(C)$ is also.

Furthermore, Fact 3.4 applied to the previous families of (co)algebras together with the forgetful functor $U : \mathbf{FinSet^{eq}} \to \mathbf{FinSet}$ given in Fact 5.5 produces examples of (co)algebras in the category $\mathbf{Funi}(\mathbf{FinSet^{eq}}, \mathscr{C})$.

5.2.3 The definition of $(\mathscr{C}, H, A)$-vertex algebra by Borcherds and the link with usual vertex algebras

For comparison, we finally present the notion of vertex algebra in [8], Def. 3.12.

Definition 5.8. *A (resp., **quantum**) $(\mathscr{C}, H, A)$-**vertex algebras** is a categorical (resp., quantum) vertex algebra under the assumptions (VA.1)-(VA.6), where $\mathscr{F}$ is the symmetric monoidal category $\mathbf{FinSet^{eq}}$ introduced in Subsection 5.2.1, B is given by the bialgebra $T^*H \circ U$ in $\mathbf{Funi}(\mathscr{F}^{\mathrm{op}}, \mathscr{C})$ for H a cocommutative bialgebra in $\mathscr{C}$, and $A \in \mathscr{A}$ is an admissible algebra.*

Borcherds showed the following result in [8], Thm. 4.3, which we cite for completeness.

Theorem 5.9. *Let R be a commutative ring, $\mathscr{C} = {}_R\mathbf{Mods}$, H be the commutative and cocommutative Hopf algebra in $\mathscr{C}$ given by the free R-module $H = \oplus_{p \in \mathbb{N}_0} R.D^{(p)}$ with product and coproduct*

$$D^{(p)} D^{(q)} = \binom{p+q}{p} D^{(p+q)} \text{ and } \Delta(D^{(p)}) = \sum_{q=0}^{p} D^{(q)} \otimes_R D^{(p-q)},$$

respectively, and A the algebra in $\mathscr{A} = {}_{T^*H\circ U}\mathbf{Modt}(\mathscr{F}, \mathscr{C})$ that sends $I = (X, \sim_X)$ in $\mathbf{FinSet^{eq}}$ to the R-algebra

$$R[(X_i - X_j)^{\pm 1} : i, j \in X \text{ such that } i \nsim_X j],$$

where the $D^{(p)}$ in the i-th tensor factor of H on $T^*H(I)$ acts by the usual divided power differential operator, and A acts in the obvious manner on morphisms of $\mathbf{FinSet^{eq}}$. Then, A is clearly admissible and if $M \in {}_A\mathbf{Adm}$ is a $(\mathscr{C}, H, A)$-vertex algebra, then $M(L(\{*\}))$ is an ordinary vertex algebra.

For a more comprehensive comparison between $(\mathscr{C}, H, A)$-vertex algebras and classical vertex algebras, see [10].

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